Three-loop SM beta-functions for matrix Yukawa couplings

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Abstract

We present the extension of our previous results for three-loop Yukawa coupling beta-functions to the case of complex Yukawa matrices describing the flavour structure of the SM. The calculation is carried out in the context of unbroken phase of the SM with the help of the `MINCER` program in a general linear gauge, and cross-checked by means of `MATAD`/`BAMBA` codes. In addition, ambiguities in Yukawa matrix beta-functions are studied.

Keywords: Standard Model, Renormalization Group

It is important property of the SM that all the particle masses are related to the corresponding couplings of the Higgs boson, which was discovered recently in the LHC experiments [1, 2]. Careful experimental investigation of the Higgs decay modes shows consistency with the prediction of the SM. This kind of studies are complemented by the experiments aimed to the Flavour problem of the SM. In spite of the fact that the SM provides a consistent description of the processes involving transitions between different fermion generations, the origin of flavour physics is still unclear. Most of the observable flavour effects in the SM are encoded in the Cabibbo-Kobayashi-Maskawa (CKM) matrix which enters into the tree-level charged quark currents[1]. The CKM matrix originates from matrix Yukawa couplings after transition from weak (interaction) eigenstates to the mass basis. It is obvious that general complex matrices involve a lot of unphysical parameters which can be “rotated” away by unitary transformations. However, it is sometimes convenient to study this general structure in view of possible New Physics which can potentially explain the observed hierarchy in quark masses and mixing (see, e.g., [3])).

This article concludes the series of our papers on three-loop beta-functions in the SM with complex Yukawa matrices [4, 5]. One- and two-loop results for SM beta-functions have been known for quite a long time [6–19] and are summarized in Ref. [20], in which the renormalization group equations (RGE) are given in the matrix form. The three-loop gauge-coupling beta-functions with the full flavour structure were calculated for the first time in Ref. [21] and confirmed later by our group [4]. The beta-functions for the parameters of the Higgs potential in the case of complex Yukawa matrices were considered in Ref. [5]. It is interesting to note that the results for three-loop RGE in the Minimal Supersymmetric Standard Model were found by means of the supergraph formalism [22] about ten years ago [23].

In order to obtain the results for matrix beta-functions we use `FeynArts` [24] and `DIANA` [25]. In both cases, we extend the corresponding model files to account for explicit flavour indices and develop a simple routines for dealing with them. Since the matrix couplings do not pose any additional problems in γ_5 treatment, we will not discuss it here and only refer to Refs. [26, 27].

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[1] In this paper, we do not consider mixing of massive neutrinos described by Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix.

As in our previous studies the calculation is carried out in an almost automatic way with the help of different infra-red rearrangement (IRR) [28] procedures implemented in our codes. The utilized IRR prescriptions consist either in setting all but two external momenta to zero or in the introduction of an auxiliary mass parameter M in every propagator and the subsequent expansion in all external momenta [29, 30]. In the first case one uses `MINCER` [31, 32] to compute massless propagator-type integral[2]. In the second approach the fully massive vacuum integrals are evaluated by means of the `MATAD` package [33] or `BAMBA` code developed by V.N. Velizhanin. For the color algebra the FORM package `COLOR` [34] is utilized.

Let us briefly specify our notation and renormalization procedure. The full Lagrangian of the "unbroken" (= massless) SM which was used in this calculation is given in our previous paper [4]. For the reader's convenience we present here only the terms describing the fermion-Higgs interactions

$$\mathcal{L}_{\text{Yukawa}} = -\left(Y_u^{ij} (Q_i^L \Phi^c) u_j^R + Y_d^{ij} (Q_i^L \Phi) d_j^R + Y_l^{ij} (L_i^L \Phi) l_j^R + \text{h.c.} \right), \tag{1}$$

Here $Y_{u,d,l}$ are Yukawa matrices. The Higgs doublet Φ and his charge-conjugated counterpart $\Phi^c = i\sigma^2 \Phi^\dagger$ have hypercharge $Y_W = 1$ and $Y_W = -1$, respectively. The left-handed quark and lepton SU(2) doublets, Q_i^L, and L_i^L, carry flavour indices $i = 1, 2, 3$. The same is true for the SU(2) singlets corresponding to the right-handed SM fermions u_i^R, d_i^R, and l_i^R. It is worth mentioning that the matrix element Y_u^{ij} describes the transition of right-handed up-type quark of j-th generation to the left-handed quark (either up- or down-type) of generation i. Conversely, the matrix element $Y_u^{\dagger,ij}$ corresponds to the transition of the left-handed quark from the doublet Q_j to the right-handed up-type quark u_i.

In the Lagrangian (1) we assume that the bare fermion fields and Yukawa couplings are expressed in terms of the corresponding running quantities defined in the $\overline{\text{MS}}$ renormalization scheme by means of ($f = u, d, l$)

$$\left(Q^L\right)_{\text{Bare}} = \left[Z_Q^{1/2}\right] Q^L \mu^{-\epsilon}, \quad \left(u^R\right)_{\text{Bare}} = \left[Z_u^{1/2}\right] u^R \mu^{-\epsilon}, \quad \left(d^R\right)_{\text{Bare}} = \left[Z_d^{1/2}\right] d^R \mu^{-\epsilon}, \tag{2}$$

$$\left(L^L\right)_{\text{Bare}} = \left[Z_L^{1/2}\right] L^L \mu^{-\epsilon}, \quad \left(l^R\right)_{\text{Bare}} = \left[Z_l^{1/2}\right] l^R \mu^{-\epsilon}, \quad \left(Y_f\right)_{\text{Bare}} = \left(Y_f + \Delta Y_f\right) \mu^{\epsilon}, \tag{3}$$

where the generation indices are suppressed. The renormalization constants $Z = (Z_Q^{1/2}, Z_L^{1/2}, Z_u^{1/2}, Z_d^{1/2}, Z_l^{1/2})$ and ΔY_f are 3×3 matrices in flavour space and can be decomposed as

$$Z = 1 + \sum_{l=1}^{\infty} \delta Z^{(l)}, \qquad \delta Z^{(l)} = \sum_{n=1}^{l} \frac{c^{(l,n)}}{\epsilon^n}, \tag{4}$$

$$\Delta Y_f = \sum_{l=1}^{\infty} \left[\Delta Y_f^{(l)}\right], \qquad \left[\Delta Y_f^{(l)}\right] = \sum_{k=1}^{l} \frac{1}{\epsilon^k} \left[\Delta Y_f^{(l,k)}\right] \tag{5}$$

with $\delta Z^{(l)}$ and $\Delta Y_f^{(l)}$ being l - loop contributions and $\epsilon = (D-4)/2$ corresponding to the parameter of dimensional regularization.

As it was mentioned above, we used two approaches to IRR. Let us consider the first one, when all Feynman integrals are converted to massless propagators and it is convenient to use multiplicative renormalization of Green functions. Contrary to the second case, when new auxiliary mass is introduced, no new parameter appears in the problem, so that the matrix renormalization constants can be recursively obtained via the relations

$$\Gamma^{(2)}_{f,\text{Ren}}\left(\frac{k^2}{\mu^2}, a_i, Y_f\right) = \left[Z_f^{1/2}\right]^\dagger \Gamma^{(2)}_{f,\text{Bare}}\left(k^2, a_{i,\text{Bare}}, Y_{f,\text{Bare}}, \epsilon\right) \left[Z_f^{1/2}\right] \tag{6}$$

$$\Gamma^{(3)}_{\bar{f}' f \phi,\text{Ren}}\left(\frac{k_i^2}{\mu^2}, a_i, Y_f\right) = \left[Z_{f'}^{1/2}\right]^\dagger \Gamma^{(3)}_{\bar{f}' f \phi,\text{Bare}}\left(k_i^2, a_{i,\text{Bare}}, Y_{f,\text{Bare}}, \epsilon\right) \left[Z_f^{1/2}\right] Z_\phi^{1/2}. \tag{7}$$

[2]No spurious IR divergences are generated in Yukawa vertices if one neglects the momentum entering the scalar leg.

In these equations $\Gamma_f^{(2)}$ corresponds to the one-particle irreducible (1PI) two-point Green functions for the fermion f (left-handed or right-handed). The three-point 1PI vertex $\Gamma^{(3)}_{\bar{f}' f \phi}$ describes the transition of right-handed fermion f to the left-handed fermion f' due to interactions with the Higgs boson ϕ, either neutral $\phi = h, \chi$ or charged $\phi = \phi^{\pm}$. The Green functions are normalized in such a way, that at tree-level $\Gamma_f^{(2)} = 1$ and $\Gamma^{(3)}_{\bar{f}' f \phi} = Y_f$. As in our previous papers [4, 27] we define the following quantities

$$a_i = \left(\frac{5}{3}\frac{g_1^2}{16\pi^2}, \frac{g_2^2}{16\pi^2}, \frac{g_s^2}{16\pi^2}, \frac{\lambda}{16\pi^2}, \xi_G, \xi_W, \xi_B\right) \tag{8}$$

with g_s, g_2, g_1 being $SU(3) \times SU(2) \times U(1)$ gauge couplings, λ — Higgs self-coupling and ξ_G, ξ_W, ξ_B corresponding to the gauge-fixing parameters in the unbroken SM. In addition, the following abbreviations ($f = u, d, l$)

$$\begin{aligned}
\mathcal{Y}_f &\equiv \frac{Y_f Y_f^\dagger}{16\pi^2}, & \mathcal{Y}_{ff} &\equiv \frac{Y_f Y_f^\dagger Y_f Y_f^\dagger}{(16\pi^2)^2}, & \mathcal{Y}_{fff} &\equiv \frac{Y_f Y_f^\dagger Y_f Y_f^\dagger Y_f Y_f^\dagger}{(16\pi^2)^3}, \\
\mathcal{Y}_{ud} &= \frac{Y_u Y_u^\dagger Y_d Y_d^\dagger}{(16\pi^2)^2}, & \mathcal{Y}_{du} &\equiv \frac{Y_d Y_d^\dagger Y_u Y_u^\dagger}{(16\pi^2)^2}, & & \\
\mathcal{Y}_{uud} &\equiv \frac{Y_u Y_u^\dagger Y_u Y_u^\dagger Y_d Y_d^\dagger}{(16\pi^2)^3}, & \mathcal{Y}_{udu} &\equiv \frac{Y_u Y_u^\dagger Y_d Y_d^\dagger Y_u Y_u^\dagger}{(16\pi^2)^3}, & \mathcal{Y}_{duu} &\equiv \frac{Y_d Y_d^\dagger Y_u Y_u^\dagger Y_u Y_u^\dagger}{(16\pi^2)^3}, \\
\mathcal{Y}_{udd} &\equiv \frac{Y_u Y_u^\dagger Y_d Y_d^\dagger Y_d Y_d^\dagger}{(16\pi^2)^3}, & \mathcal{Y}_{ddu} &\equiv \frac{Y_d Y_d^\dagger Y_d Y_d^\dagger Y_u Y_u^\dagger}{(16\pi^2)^3}, & \mathcal{Y}_{dud} &\equiv \frac{Y_d Y_d^\dagger Y_u Y_u^\dagger Y_d Y_d^\dagger}{(16\pi^2)^3}
\end{aligned} \tag{9}$$

will be used for the Yukawa matrix products.

It is worth pointing that in the absence of Yukawa interactions (1) the SM Lagrangian is invariant under accidental global flavour symmetry[3] $U(3)_Q \times U(3)_u \times U(3)_d \times U(3)_L \times U(3)_l$. Due to this, we have an equivalence relation

$$(Y_u, Y_d, Y_l) \Leftrightarrow (Y_u', Y_d', Y_l') = \left(V_Q Y_u V_u^\dagger, V_Q Y_d V_d^\dagger, V_L Y_l V_l^\dagger\right), \qquad V_f \in U(3)_f, \quad f = Q, L, u, d, l, \tag{10}$$

implying that the Lagrangians (1) with couplings (Y_u, Y_d, Y_l) and (Y_u', Y_d', Y_l') lead to the same physics [35], since one can always compensate the factors V_f by appropriate basis change $f \to f' = V_f f$ without affecting the rest of the SM Lagrangian. It is easy to notice that Eq. (10) entails equivalence relations for the matrix products (9), i.e, $\mathcal{Y}_{q\ldots q'} \Leftrightarrow \mathcal{Y}'_{q\ldots q'} = V_Q \mathcal{Y}_{q\ldots q'} V_Q^\dagger$. This property of the Lagrangian will be important in our discussion of the ambiguities in Yukawa matrix beta-functions.

The left-hand (LHS) side of Eqs.(6)-(7) is finite when the parameter ϵ tends to zero. This allows us to find the expressions for certain combinations of $\overline{\text{MS}}$-renormalization constants by canceling recursively the poles in ϵ, which appear in the right-hand side (RHS) of the same equations. From the fermion self-energies $\Gamma_f^{(2)}$ we extract a hermitian combination $Z_{2,f} = \left[Z_f^{1/2}\right]^\dagger Z_f^{1/2}$. Taking the square root operation in perturbation theory one fixes the hermitian part (denoted by $\tilde{Z}_f^{1/2}$) of $Z_f^{1/2}$:

$$\begin{aligned}
\tilde{Z}_f^{1/2} &= 1 + \frac{1}{2}\delta Z_{2,f}^{(1)} + \frac{1}{2}\left(\delta Z_{2,f}^{(2)} - \frac{1}{4}\delta Z_{2,f}^{(1)} \delta Z_{2,f}^{(1)}\right) \\
&\quad + \frac{1}{2}\left(\delta Z_{2,f}^{(3)} - \frac{1}{4}\left(\delta Z_{2,f}^{(1)}\delta Z_{2,f}^{(2)} + \delta Z_{2,f}^{(2)}\delta Z_{2,f}^{(1)}\right) + \frac{1}{8}\delta Z_{2,f}^{(1)}\delta Z_{2,f}^{(1)}\delta Z_{2,f}^{(1)}\right) + \ldots,
\end{aligned} \tag{11}$$

where only the terms relevant for our three-loop calculations are retained. It turns out that at three loops hermitian factors $\tilde{Z}_f^{1/2}$ used in place of $Z_f^{1/2}$ give rise to infinite expressions (in the limit $\epsilon \to 0$) for the

[3]corresponding to SU(2)-compatible rotations of left-handed (Q and L) and right-handed (u, d, and l) fermion fields

matrix anomalous dimension γ_f of renormalized quark fields $\mathfrak{F}_f = Q^L, u^R, d^R$ defined as

$$\gamma_f \cdot \mathfrak{F}_f(\mu) \equiv \frac{d}{d\ln\mu^2}\mathfrak{F}_f(\mu,\epsilon)\Big|_{\epsilon=0} = -\left(Z_f^{-1/2}\frac{d}{d\ln\mu^2}Z_f^{1/2}\right)\cdot\mathfrak{F}_f. \tag{12}$$

In other words, pure hermitian renormalization constants do not satisfy the pole equations [36] for the u- and d-quark. To circumvent this problem we introduced the following unitary factors:

$$\bar{Z}_{Q_L}^{1/2} = 1 - \frac{a_1 h^3}{320}\left(\frac{1}{6\epsilon^2}-\frac{1}{\epsilon^3}\right)\Big[\mathcal{Y}_u,\mathcal{Y}_d\Big] + \frac{h^3}{64}\left(\frac{1}{6\epsilon^2}+\frac{1}{\epsilon^3}\right)\Big\{\mathcal{Y}_u-\mathcal{Y}_d,\Big[\mathcal{Y}_u,\mathcal{Y}_d\Big]\Big\}, \tag{13}$$

$$\bar{Z}_{u_R}^{1/2} = 1 - \frac{h^3}{32}\left(\frac{1}{6\epsilon^2}-\frac{1}{\epsilon^3}\right)Y_u^\dagger\Big[\mathcal{Y}_u,\mathcal{Y}_d\Big]Y_u, \quad \bar{Z}_{d_R}^{1/2} = 1 + \frac{h^3}{32}\left(\frac{1}{6\epsilon^2}-\frac{1}{\epsilon^3}\right)Y_d^\dagger\Big[\mathcal{Y}_u,\mathcal{Y}_d\Big]Y_d, \tag{14}$$

where the commutator $[\mathcal{Y}_u,\mathcal{Y}_d]$ is an anti-hermitian matrix[4] and h^l is used to indicate l-loop contribution. Due to the unbroken SU(2) invariance we have $\bar{Z}_{u_L}^{1/2} = \bar{Z}_{d_L}^{1/2} = \bar{Z}_{Q_L}^{1/2}$. The factors $\tilde{Z}^{1/2}$ and $\bar{Z}^{1/2}$ combine to form $Z^{1/2}$

$$Z_f^{1/2} = \bar{Z}_f^{1/2}\tilde{Z}_f^{1/2}, \qquad \left[Z_f^{1/2}\right]^\dagger = \tilde{Z}_f^{1/2}\left[\bar{Z}_f^{1/2}\right]^\dagger, \qquad \left[\bar{Z}_f^{1/2}\right]^\dagger\bar{Z}_f^{1/2} = 1 \qquad Z_{2,f} = \left[\tilde{Z}_f^{1/2}\right]^2. \tag{15}$$

The renormalization constants for other SM quantum fields, e.g., $Z_\phi^{1/2}$ required to define (7), can be easily obtained from the corresponding two-point functions. In order to find the renormalization constant for Yukawa matrix Y_f we use (7). After two-loop renormalization of RHS we are left with the divergence, which should be canceled by the three-loop part of the vertex counter-term

$$Z_{\bar{f}'f\phi}Y_f \equiv \left[Z_{f'}^{1/2}\right]^\dagger(Y_f+\Delta Y_f)\left[Z_f^{1/2}\right]Z_\phi^{1/2}, \tag{16}$$

originating from the Lagrangian (1). It is worth noticing that it is always possible to factorize Y_f ($Y_f^\dagger$) from the right (left) of the considered matrix three-point Green functions involving incoming (outgoing) right-handed fermion f. From Eq. (16) one can deduce that

$$Y_f + \Delta Y_f = \left[Z_{f'}^{-1/2}\right]^\dagger Z_{\bar{f}'f\phi}Y_f\left[Z_f^{-1/2}\right]Z_\phi^{-1/2} = \bar{Z}_{f'}^{1/2}\left[\tilde{Z}_{f'}^{-1/2}Z_{\bar{f}'f\phi}Y_f\tilde{Z}_f^{-1/2}Z_\phi^{-1/2}\right]\left[\bar{Z}_f^{1/2}\right]^\dagger \tag{17}$$

In our calculation we used $\Gamma^{(3)}_{\bar{f}f\phi}$ with $f = u,d,l$ and $\phi = h,\chi$ for extraction of ΔY_f. Moreover, the vertex $\Gamma^{(3)}_{\bar{u}d\phi^+}$ was also considered for additional verification of the correctness of the results. The Green function with charged would-be goldstone ϕ^+ allows us to find the counter-terms both for $Y_u^\dagger$ ($\bar{u}_R d_L\phi^+$) and Y_d ($\bar{u}_L d_R\phi^+$). Irrespectively of the considered vertex the results for Y_f turn out to be the same.

The matrix Yukawa beta-functions β_{Y_f} are defined by

$$\beta_{Y_f}Y_f \equiv \frac{dY_f(\mu,\epsilon)}{d\ln\mu^2}\Big|_{\epsilon=0} \tag{18}$$

and can be found from the relation between bare and renormalized couplings by differentiation

$$0 = \frac{d}{d\ln\mu^2}(Y_f)_{\text{Bare}} = \left(-\frac{\epsilon}{2}Y_f + \beta_{Y_f}Y_f + \frac{d}{d\ln\mu^2}(\Delta Y_f)\right)\mu^\epsilon + \frac{\epsilon}{2}(Y_f+\Delta Y_f)\mu^\epsilon, \tag{19}$$

[4]It is interesting to note that $\det[\mathcal{Y}_u,\mathcal{Y}_d] = -2T(y_u)B(y_d)J$, where $T(y_u) = (y_t^2-y_u^2)(y_t^2-y_c^2)(y_c^2-y_u^2)$, $B(y_d) = (y_b^2-y_d^2)(y_b^2-y_s^2)(y_s^2-y_d^2)$, with y_f and J being the Yukawa coupling for quark mass eigenstate f and the Jarlskog invariant [37], correspondingly.

from which one deduces

$$\beta_{Y_f} Y_f = \sum_{l=1}^{\infty} \left[a_i \frac{\partial}{\partial a_i} + \frac{1}{2} \sum_{f'=u,d,l} \left(Y_{f'}^{ij} \frac{\partial}{\partial Y_{f'}^{ij}} + Y_{f'}^{\dagger,ij} \frac{\partial}{\partial Y_{f'}^{\dagger,ij}} \right) - \frac{1}{2} \right] \Delta Y_f^{(l,1)} = \sum_{l=1}^{\infty} l \cdot \Delta Y_f^{(l,1)}, \tag{20}$$

assuming the validity of the corresponding pole equations.

It is interesting to mention that the expressions for the matrix renormalization constants discussed in this article were obtained for the first time by means of the described procedure. The computer setup (`FeynArts`[24] for diagram generation + `MINCER` for integral computations) was the same that we used in our first two papers [4, 27] on three-loop SM beta-function. The obtained expressions for ΔY_f were free from gauge-parameter dependence. However, in this calculation we did not take into account the unitary factors $\bar{Z}_f^{1/2}$ and were unable to satisfy the pole equations for the quark fields and Yukawa couplings.

In order to figure out the problem and to cross-check the obtained results we made use of another established setup (`DIANA`+ `MATAD` / `BAMBA`), which is based on the second mentioned approach to IRR. The crucial difference in the renormalization procedure is that one calculates renormalization constants for Green function Γ via

$$Z_\Gamma = 1 - \mathcal{K}\mathcal{R}'\Gamma, \tag{21}$$

with $\mathcal{R}'$ being incomplete $\mathcal{R}$-operation without last subtraction, and $\mathcal{K}$ extracts the singular part in ϵ. The implementation of the method requires introduction of explicit counter-term insertions corresponding to all SM fields and parameters and, in addition, to the auxiliary mass M. By means of this procedure the same result for renormalization constants of the considered two- and three-point Green functions was obtained. This ensures the correctness of the corresponding expressions.

After a careful study of the employed procedures we have found that the square root operation, which was used to find $Z_f^{1/2}$, has an ambiguity. The latter was utilized and the unitary factors $\bar{Z}_f^{1/2}$ were introduced. It is worth mentioning that these factors themselves do not satisfy pole equations so that one can not define a finite anomalous dimensions $\bar{\gamma}_f = -\bar{Z}_f^{-1/2}\dot{\bar{Z}}_f^{1/2}$.

A comment on the remaining ambiguity is in order since one can introduce additional unitary factors involving first poles in ϵ. For example, we can multiply the obtained renormalization constant $Z_{Q_L}^{1/2}$ by a factor (A_1 is an arbitrary constant)

$$\begin{aligned} \mathcal{Z}_{Q_L} = 1 \; + \; & \frac{h^2}{\epsilon} A_1 \Big[\mathcal{Y}_u, \mathcal{Y}_d\Big] \\ + \; & \frac{h^3}{\epsilon^2} A_1 \left[\frac{2\alpha_u^u + \alpha_u^d}{3} \Big\{ \mathcal{Y}_u, \Big[\mathcal{Y}_u, \mathcal{Y}_d\Big] \Big\} + \frac{2\alpha_d^d + \alpha_d^u}{3} \Big\{ \mathcal{Y}_d, \Big[\mathcal{Y}_u, \mathcal{Y}_d\Big] \Big\} + \frac{2\alpha_0^d + 2\alpha_0^u}{3} \Big[\mathcal{Y}_u, \mathcal{Y}_d\Big] \right] \end{aligned} \tag{22}$$

without any effect on the propagator ($Z_{Q_L}^{1/2\dagger} Z_{Q_L}^{1/2} - 1$) and vertex ($Z_{\bar{f}' f\phi} - 1)Y_f$ counter-terms. The coefficient of h^3/ϵ^2 in (22) is determined from pole equations and ensures the finiteness of the corresponding anomalous dimension $\gamma'_L \equiv -\mathcal{Z}_{Q_L}^\dagger \dot{\mathcal{Z}}_{Q_L} = 2A_1 h^2 (\mathcal{Y}_{ud} - \mathcal{Y}_{du})$ up to three loops. The coefficients α_i^u and α_i^d enter one-loop beta-functions for Y_u and Y_d:

$$\beta_{Y_f} \; = \; \alpha_u^f \cdot \mathcal{Y}_u + \alpha_d^f \cdot \mathcal{Y}_d + \alpha_0^f, \qquad f = u, d, \tag{23}$$

and in the SM

$$\alpha_u^u \; = \; \alpha_d^d = -\alpha_d^u = -\alpha_u^d = \frac{3}{4}, \quad \begin{pmatrix} \alpha_0^u \\ \alpha_0^d \end{pmatrix} = -4a_s - \frac{9}{8} a_2 + \frac{3\,\mathrm{tr}[\mathcal{Y}_u] + 3\,\mathrm{tr}[\mathcal{Y}_d] + \mathrm{tr}[\mathcal{Y}_l]}{2} - \begin{pmatrix} \frac{17}{40} \\ \frac{1}{8} \end{pmatrix} a_1 \tag{24}$$

It is clear that the substitution $Z_{Q_L}^{1/2} \to \mathcal{Z}_{Q_L} Z_{Q_L}^{1/2}$ in (2) will modify the anomalous dimension for the left-handed quarks and the Yukawa coupling beta-functions in the following way[5]

$$\gamma_Q \to \gamma'_Q = \gamma_Q + \gamma'_L, \qquad \beta_{Y_f} \to \beta_{Y'_f} = \beta_{Y_f} + \gamma'_L. \tag{25}$$

[5] in general case we have $\beta_{Y_f} Y_f \to \beta_{Y_f} Y_f + \gamma'_L Y_f - Y_f \gamma'_f$ with γ'_f being an analog of γ'_L for the right-handed quark f.

With the chosen $\mathcal{Z}_{Q_L}$ the RGE functions (25) are affected already at the two-loop level. The three-loop beta-functions can also be easily modified by adding h^3/ϵ terms to $\mathcal{Z}_{Q_L}$. However, having in mind the freedom (10), it is easy to convince oneself that it is possible to get rid of $\mathcal{Z}_{Q_L}$ together with aribitrary right-handed $\mathcal{Z}_u$, $\mathcal{Z}_d$ factors by the substitution $Y_f \to Y'_f = \mathcal{Z}^\dagger_{Q_L} Y_f \mathcal{Z}_f$ accompanied by a SU(2)-compatible change of basis for the quark fields $Q^L \to Q'^L = \mathcal{Z}^\dagger_{Q_L} Q^L$, etc. As a consequence, beta-functions β_{Y_f} and $\beta_{Y'_f}$ from Eq. (25) are equivalent, leading to the same RG flow of the quark sector "observables" — six eigenvalues of $\mathcal{Y}_u$ and $\mathcal{Y}_d$ together with four independent parameters of the CKM matrix. Due to this, we restrict ourselves to the "minimal" case with all $\mathcal{Z} \equiv 1$, for which the quark anomalous dimensions are purely hermitian.

To conclude, by explicit calculation we obtained the three-loop RGE for general complex Yukawa matrices. The two-loop part reproduce the known expressions[6] [20]. The three-loop contributions are free from gauge-parameter dependence and coincide with our previous results in the limit of diagonal Yukawa couplings. In addition, we analyzied the ambiguity in $\overline{\text{MS}}$ Yukawa matrix beta-functions which obviously appears starting from the two-loop order.

In order to save space, we do not present the full expressions for β_{Y_f}, $f = u, d, l$ here[7]. However, in a quite reasonable limit of vanishing couplings $g_1 = g_2 = Y_l = 0$ the beta-functions are not very lengthy so we present here the result for $\beta_{Y_u} = \beta^{(1)}_{Y_u} + \beta^{(2)}_{Y_u} + \beta^{(3)}_{Y_u} + \ldots$ in this approximation.

Employing this notation the loop expansion of β_{Y_f} can be given as ($\hat{\lambda} = a_\lambda$)

$$\beta^{(1)}_{Y_u} = -4a_s + \frac{3}{2}\left(\mathrm{tr}[\mathcal{Y}_d] + \mathrm{tr}[\mathcal{Y}_u]\right) + \frac{3}{4}(\mathcal{Y}_u - \mathcal{Y}_d), \tag{26}$$

$$\begin{aligned}\beta^{(2)}_{Y_u} &= 3\hat{\lambda}^2 - 6\hat{\lambda}\mathcal{Y}_u + \frac{11}{8}\mathcal{Y}_{dd} - \frac{1}{2}\mathcal{Y}_{du} - \frac{1}{8}\mathcal{Y}_{ud} + \frac{3}{4}\left(\mathcal{Y}_{uu} + \mathrm{tr}[\mathcal{Y}_{ud}]\right) + \frac{15}{8}\mathcal{Y}_d(\mathrm{tr}[\mathcal{Y}_d] + \mathrm{tr}[\mathcal{Y}_u]) + 8a_s\left(\mathcal{Y}_u - \mathcal{Y}_d\right)\\
&+10a_s\left(\mathrm{tr}[\mathcal{Y}_d] + \mathrm{tr}[\mathcal{Y}_u]\right) - \frac{27}{8}\left(\mathrm{tr}[\mathcal{Y}_{dd}] + \mathrm{tr}[\mathcal{Y}_{uu}] + \left(\mathrm{tr}[\mathcal{Y}_d] + \mathrm{tr}[\mathcal{Y}_u]\right)\mathcal{Y}_u\right) + a_s^2\left(\frac{40}{9}n_G - \frac{202}{3}\right),\end{aligned} \tag{27}$$

$$\begin{aligned}\beta^{(3)}_{Y_u} &= -18\hat{\lambda}^3 - a_s^3\left[1249 - \left(\frac{4432}{27} - \frac{320}{3}\zeta_3\right)n_G - \frac{560}{81}n_G^2\right]\\
&+a_s^2\left[\mathrm{tr}[\mathcal{Y}_d]\left(\frac{457}{3} - 16n_G - 108\zeta_3\right) + \mathrm{tr}[\mathcal{Y}_u]\left(\frac{505}{3} - 16n_G - 12\zeta_3\right)\right.\\
&\qquad\left.+\mathcal{Y}_u\left(\frac{2779}{12} - 11n_G - 102\zeta_3\right) - \mathcal{Y}_d\left(\frac{2659}{12} - \frac{41}{3}n_G - 86\zeta_3\right)\right]\\
&+a_s\left[\mathcal{Y}_{dd}\left(13 + 40\zeta_3\right) + \mathcal{Y}_{ud}(14 + 4\zeta_3) - \mathcal{Y}_{du}(9 - 4\zeta_3) - 38\mathcal{Y}_{uu} + 8\hat{\lambda}\mathcal{Y}_u + \mathrm{tr}[\mathcal{Y}_{ud}]\left[\frac{57}{2} - 24\zeta_3\right]\right.\\
&\left.+(\mathrm{tr}[\mathcal{Y}_d] + \mathrm{tr}[\mathcal{Y}_u])\left[\mathcal{Y}_d\left(\frac{97}{4} - 36\zeta_3\right) - \left(\frac{177}{4} - 36\zeta_3\right)\mathcal{Y}_u\right] + \left[\frac{15}{4} - 36\zeta_3\right](\mathrm{tr}[\mathcal{Y}_{dd}] + \mathrm{tr}[\mathcal{Y}_{uu}])\right]\\
&+\hat{\lambda}\left(\frac{3}{2}\mathcal{Y}_{ud} - 15\mathcal{Y}_{dd} + \frac{63}{2}\mathcal{Y}_{uu} + 45\mathcal{Y}_u(\mathrm{tr}[\mathcal{Y}_d] + \mathrm{tr}[\mathcal{Y}_u]) + \frac{45}{2}(\mathrm{tr}[\mathcal{Y}_{dd}] + \mathrm{tr}[\mathcal{Y}_{uu}])\right)\\
&+\hat{\lambda}^2\left(\frac{285}{8}\mathcal{Y}_u - \frac{21}{8}\mathcal{Y}_d - \frac{135}{4}(\mathrm{tr}[\mathcal{Y}_d] + \mathrm{tr}[\mathcal{Y}_u])\right) + \left(\frac{9}{8} - \frac{9}{4}\zeta_3\right)\mathcal{Y}_{ddd} - \left(\frac{345}{32} - \frac{9}{4}\zeta_3\right)\mathcal{Y}_{uuu}\\
&+\frac{43}{16}\mathcal{Y}_{udu} + \frac{75}{32}\mathcal{Y}_{uud} + \frac{83}{32}\mathcal{Y}_{duu} - \frac{37}{16}\mathcal{Y}_{dud} - \left(\frac{95}{16} - 6\zeta_3\right)\mathcal{Y}_{udd} - \left(\frac{183}{32} - 6\zeta_3\right)\mathcal{Y}_{ddu}\\
&+\left(\frac{3}{8}\mathcal{Y}_{du} - \frac{69}{8}\mathcal{Y}_{dd} - \frac{9}{8}\mathcal{Y}_{uu} + \frac{21}{8}\mathcal{Y}_{ud}\right)(\mathrm{tr}[\mathcal{Y}_d] + \mathrm{tr}[\mathcal{Y}_u])\\
&+\left(\frac{135}{16}\mathcal{Y}_u - \frac{9}{4}\mathcal{Y}_d\right)(\mathrm{tr}[\mathcal{Y}_{uu}] + \mathrm{tr}[\mathcal{Y}_{dd}]) + \left(\left(\frac{147}{4} - 36\zeta_3\right)\mathcal{Y}_d - \frac{81}{8}\mathcal{Y}_u\right)\mathrm{tr}[\mathcal{Y}_{ud}]\end{aligned}$$

[6] One should identify Y_u, Y_d and Y_l with $\mathbf{H}^+$, $\mathbf{F}^+_{\mathbf{d}}$ and $\mathbf{F}^+_{\mathbf{L}}$ of Refs. [15, 20], respectively.

[7] The results in a computer-readable form are available as the ancillary files of the arXiv version of the paper.

$$+\left(\frac{789}{32}+\frac{9}{2}\zeta_3\right)(\mathrm{tr}[\mathcal{Y}_{ddd}]+\mathrm{tr}[\mathcal{Y}_{uuu}])+\frac{831}{32}(\mathrm{tr}[\mathcal{Y}_{udd}]+\mathrm{tr}[\mathcal{Y}_{uud}]) \tag{28}$$

where $n_G = 3$ is the number of fermion generations. It is worth pointing that the corresponding expressions for $\beta^{(l)}_{Y_u^\dagger}$ can be deduced from (26), (27) and (28) by the substitutions

$$\mathcal{Y}_{ud} \leftrightarrow \mathcal{Y}_{du}, \quad \mathcal{Y}_{udd} \leftrightarrow \mathcal{Y}_{ddu}, \quad \mathcal{Y}_{uud} \leftrightarrow \mathcal{Y}_{duu}. \tag{29}$$

The obtained expressions can be applied to RGE studies of different BSM models aimed to unveil the physics behind the observed SM flavour pattern. It is also worth mentioning that from Y_u and Y_d it is possible to deduce the three-loop RGE for the CKM matrix elements (see, e.g., Ref.[38, 39]) or Quark Flavour invariants (see Ref. [40, 41]) in the $\overline{\mathrm{MS}}$-scheme.

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