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Toda 3-Point Functions From Topological Strings

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Abstract

We consider the long-standing problem of obtaining the 3-point functions of Toda CFT. Our main tools are topological strings and the AGT-W relation between gauge theories and 2D CFTs. In [1] we computed the partition function of 5D T_N theories on $S^4 \times S^1$ and suggested that they should be interpreted as the three-point structure constants of q-deformed Toda. In this paper, we provide the exact AGT-W dictionary for this relation and rewrite the 5D T_N partition function in a form that makes taking the 4D limit possible. Thus, we obtain a prescription for the computation of the partition function of the 4D T_N theories on S^4, or equivalently the undeformed 3-point Toda structure constants. Our formula, has the correct symmetry properties, the zeros that it should and, for $N = 2$, gives the known answer for Liouville CFT.

Contents

1 Introduction

The AGT(-W) correspondence [2–4] is a relationship between, on one side, the 2D Liouville (Toda) CFT on a Riemann surface of genus g with n punctures and, on the other side, the 4D $\mathcal{N} = 2$ SU(2) (SU(N)) quiver gauge theories obtained by compactifying the 6D (2,0) SCFT on that same surface . The correlation functions of the 2D Toda $\mathbf{W}_N$ conformal field theories are obtained from by the partition functions of the corresponding 4D $\mathcal{N} = 2$ gauge theories as

$$\mathcal{Z}^{S^4} = \int [da] \left| \mathcal{Z}^{\rm 4D}_{\rm Nek}(a, m, \epsilon_{1,2}) \right|^2 \propto \langle V_{\boldsymbol{\alpha}_1}(z_1) \cdots V_{\boldsymbol{\alpha}_n}(z_n) \rangle_{\rm Toda} \,. \tag{1}$$

The conformal blocks of the 2D CFTs are given by the appropriate instanton partition functions, while the three point structure constants should be obtained by the S^4 partition functions of the T_N superconformal theories. These partition functions were until recently [1, 5] unknown, with the sole exception of the $\mathbf{W}_2$ case, *i.e.* the Liouville case, whose three point structure constants are given by the famous DOZZ formula [6, 7]. The AGT(-W) relation (1) holds after the mass parameters m of the gauge theory, the UV coupling constants and the vacuum expectation values a of the scalars in the vector multiplet (the Coulomb moduli) are appropriately identified with, respectively, the external momenta $\boldsymbol{\alpha}$ of the primary fields, the moduli z_i of the 2D surface (*i.e.* the sewing parameters) and the internal momenta over which we integrate. Finally, the IR regulators of the gauge theory, which are given by the Omega deformation parameters $\epsilon_{1,2}$, are identified with the Toda dimensionless coupling constant via $b = \epsilon_1 = \epsilon_2^{-1}$. The AGT conjecture, *i.e.* the $N = 2$ case, was recently proven in [8–13], while a lot of evidence and even proofs for specific cases exist [14–16] in support of the AGT-W correspondence for $N > 2$.

Similarly, there exists a 5D version of the AGT(-W) relation[1] [18, 19] (see also [1, 20–27]) which relates the 5D Nekrasov partition functions on $S^4 \times S^1$ to correlation functions of q-deformed Liouville (Toda) field theory:

$$\mathcal{Z}^{S^4\times S^1} = \int [da] \left| \mathcal{Z}^{\text{5D}}_{\text{Nek}}(a, m, \beta, \epsilon_{1,2}) \right|^2 \propto \langle V_{\boldsymbol{\alpha}_1}(z_1) \cdots V_{\boldsymbol{\alpha}_n}(z_n) \rangle_{q\text{-Toda}} \,, \tag{2}$$

where $\beta = -\log q$ is the circumference of the S^1. Importantly, the integral of the norm squared of 5D Nekrasov partition function is the 5D superconformal index $\mathcal{Z}^{S^4\times S^1}$, which as discussed recently in [28] can be computed using the topological string partition function

$$\mathcal{Z}^{S^4\times S^1} = \int [da]\, |\mathcal{Z}^{\text{5D}}_{\text{Nek}}(a)|^2 \propto \int [da]\, |\mathcal{Z}_{\text{top}}(a)|^2 \,. \tag{3}$$

From both the 4D and the 5D AGT-W relations a very important element is missing: the three point functions of the $\mathbf{W}_N$ Toda CFT. Computing the three point functions of the $\mathbf{W}_N$ Toda CFT has been a long standing unsolved problem. From the the CFT side, the state of the art is due to Fateev and Litvinov, who in [29–31], were able to compute the 3-point functions of Toda primaries for the special case in which one of the fields is semi-degenerate, using [32]. On the gauge theory side, the 3-point functions correspond to the partition functions of the T_N theories, but since these theories lack any known Lagrangian description, the usual methods of computing the partition functions are not applicable.

In [1] we computed the partition functions of the 5D T_N theories on $S^4 \times S^1$ by using the web diagram provided by [33] and by employing the refined topological vertex formalism

[1] Originally suggested in [17].

of [34, 35]. We further argued that these partition functions should give the three point functions of q-deformed Toda, which was also proposed earlier in [36]. Our results were checked by computing the 5D superconformal index, *i.e.* the partition function on $S^4 \times S^1$, using the prescription in [28] and comparing it to the result obtained via localization in [37]. The same partition functions were also obtained in [5] and the two computations agree. More comparisons with the superconformal index were given in the recent work [38].

In this paper we show how to, in principle[2], take the 4D limit, thus obtaining the 4D T_N partition functions. Through the AGT(-W) relation, they are identified with the usual, undeformed Toda three point functions. Our formula has the correct symmetry properties, zeros and reproduces the known answer for the Liouville CFT. Furthermore, we carefully study the 5D AGT-W dictionary. For that, it was very important to examine the known q-Liouville case [23, 36] for which for the first time we were able to write the formula with the complete factors, thanks to the exact definition of the functions Υ_q, see appendix C.2.

Our method of attacking the problem of solving Toda, even though indirect, is very powerful for the following reasons. For 2D CFTs with only Virasoro symmetry the multipoint correlation functions of Virasoro descendants can be obtained from the ones containing only Virasoro primary fields [40]. On the other hand, for the $\mathbf{W}_N$ Toda CFTs with $N > 2$ complete knowledge of the correlation functions of $\mathbf{W}_N$ primary fields is not enough to obtain the correlation functions of descendents. Fully solving Toda means being able to construct the complete set of correlation functions *both* of primaries and descendants. Obtaining the three point functions with descendants is very naturally done using topological strings and is work in progress [41].

Since this article relates two somewhat disjointed fields, each used to its own notations, we wish to include a reader's guide to the other sections. We begin in section 2 with a presentation of the parametrizations and the precise relations between the partition functions of section 4 and the correlators of section 3. In the following section 3, we review shortly the Toda CFT, introduce the associated notation and make some observations regarding the symmetries of the correlation functions that to our knowledge are not available in the literature. We finish section 3 by a discussion of the pole structures and the q-deformations of the correlation functions. In section 4, we give a short review of the derivation of the partition functions of the T_N theories, rewrite them using the functions Υ_q that in our opinion are the appropriate tools to use in this context. We then discuss their 4D limit. In sections 5 and 6 we illustrate our claims for the two simplest cases with $N = 2$ and $N = 3$. The reader

[2]The specification "in principle" refers to the fact that there is still a missing ingredient which is to perform the sums in (69). This work will appear in a separate [39] publication, where we will show that some of the sums can be computed.

can find a collection of useful formulas, notations and parametrizations in the in appendices. Finally, the exact definition of the functions Υ_q is given in appendix C.2 together with a discussion of their properties.

2 The AGT dictionary

The main goal of this section is to provide the dictionary needed to relate the topological string amplitudes of section 4 to the Toda CFT correlation functions of section 3. First, we review the parameters of the Omega deformation. The circumference of the 5D circle is $\beta > 0$ and the Ω background parameters are ϵ_1 and ϵ_2 from which we derive

$$\mathfrak{q} := e^{-\beta\epsilon_1}, \qquad \mathfrak{t} := e^{\beta\epsilon_2}. \tag{4}$$

Furthermore, we need to also define[3]

$$q := e^{-\beta}, \qquad x := \sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}} = q^{\frac{\epsilon_+}{2}}, \qquad y := \sqrt{\mathfrak{q}\mathfrak{t}} = q^{\frac{\epsilon_-}{2}}, \tag{5}$$

with $\epsilon_\pm := \epsilon_1 \pm \epsilon_2$, and q the q-deformation parameter. The combinations x and y are the natural variables, fugacities of the 5D superconformal index. When we need to relate the topological string partition functions to the Toda CFT correlators, the Ω background parameters need to be specialized as

$$\epsilon_1 = b, \qquad \epsilon_2 = b^{-1}, \tag{6}$$

which implies in particular that $|q| < 1$, $|\mathfrak{q}| < 1$, $|\mathfrak{t}| > 1$ and $|x| < 1$ since we take b to be positive.

On the Toda CFT side, see section 3, one uses the weights $\boldsymbol{\alpha}_i$ parametrized by (18) to label the primary fields, while on the T_N theory side, one uses the positions of the exterior branes, see section 4 and appendix A, as parameters. The rough relationship is illustrated in figure 1 and the precise identifications are

$$\begin{aligned} m_i &= (\boldsymbol{\alpha}_1 - \mathcal{Q}, h_i) = N\sum_{j=i}^{N-1}\alpha_1^j - \sum_{j=1}^{N-1} j\alpha_1^j - \frac{N+1-2i}{2}Q, \\ n_i &= -(\boldsymbol{\alpha}_2 - \mathcal{Q}, h_i) = -N\sum_{j=i}^{N-1}\alpha_2^j + \sum_{j=1}^{N-1} j\alpha_2^j + \frac{N+1-2i}{2}Q, \\ l_i &= (\boldsymbol{\alpha}_3 - \mathcal{Q}, h_i) = N\sum_{j=i}^{N-1}\alpha_3^j - \sum_{j=1}^{N-1} j\alpha_3^j - \frac{N+1-2i}{2}Q, \end{aligned} \tag{7}$$

[3] The combinations $\beta\,\epsilon_i$ are dimensionless, but not β or ϵ_i separately. In this paper we rescale them by the dimensionful constant $\sqrt{\epsilon_1\epsilon_2}$ while keeping their product $\beta\,\epsilon_i$ fixed so that each one of them β and ϵ_i are separately dimensionless.

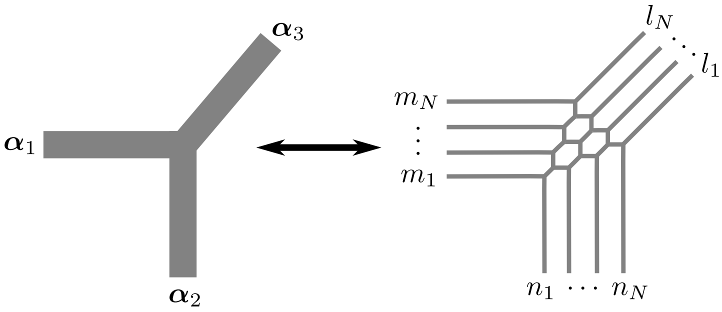

Figure 1: *This figure depicts the identification of the $\boldsymbol{\alpha}$ weights appearing on the Toda CFT side with the position of the flavor branes on the T_N side, here drawn for the case $N = 4$.*

where h_i are the weights of the fundamental representation of SU(N). In appendix B the reader can find all the group theory conventions. In particular, for $N = 2$, we have

$$m_1 = -m_2 = \alpha_1^1 - \frac{Q}{2}, \qquad n_1 = -n_2 = -\alpha_2^1 + \frac{Q}{2}, \qquad l_1 = -l_2 = \alpha_3^1 + \frac{Q}{2}, \tag{8}$$

while for $N = 3$ we have

$$m_1 = 2\alpha_1^1 + \alpha_1^2 - Q, \qquad m_2 = -\alpha_1^1 + \alpha_1^2, \qquad m_3 = -\alpha_1^1 - 2\alpha_1^2 + Q, \tag{9}$$

with similar expressions for the n_i and l_i.

Having set up the parametrization, we are ready to present our full claim. For that it is important to stress that from the Toda CFT 3-point structure constants C, see (24), we can extract the *Weyl-invariant structure constants* $\mathfrak{C}$ as

$$C(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = \left(\left[\pi\mu\gamma(b^2) b^{2-2b^2} \right]^{\frac{(2\mathcal{Q},\rho)}{b}} \prod_{i=1}^{3} Y(\boldsymbol{\alpha}_i) \right) \times \mathfrak{C}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3), \tag{10}$$

with the functions $Y(\alpha)$ defined in (25) encoding all the information about the Weyl transformation. All the details needed are introduced in section 3. We claim that the exact AGT-W dictionary relates the Weyl-invariant structure constants $\mathfrak{C}$ to the 4D T_N partition function on S^4 ($\mathcal{Z}_N^{S^4}$) as

$$\mathfrak{C}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = \text{const} \times \mathcal{Z}_N^{S^4} \tag{11}$$

where the constant part can be a function of N and of the Omega deformation parameters but *cannot* be a function of the masses. The partition function on S^4 itself is obtained from the partition function on $S^4 \times S^1$, also called the 5D *superconformal index*, by taking the appropriate limit when the circumference β of the S^1 goes to zero:

$$\mathcal{Z}_N^{S^4} = \text{const} \times \lim_{\beta \to 0} \beta^{-\frac{\chi_N}{\epsilon_1 \epsilon_2}} \mathcal{Z}_N^{S^4 \times S^1}. \tag{12}$$

The partition function $\mathcal{Z}_N^{S^4\times S^1}$ is contained in (81) and the power χ_N of the divergence in (83). Moreover, as far as the 5D AGT-W dictionary is concerned, we need (6) to set $b=\epsilon_1=\epsilon_2^{-1}$ and obtain

$$\mathfrak{C}_q(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)=\frac{C_q(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)}{J_q(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)}=\text{const}\times\frac{C_q(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)}{\prod_{j=1}^3 Y_q(\boldsymbol{\alpha}_j)}=\text{const}\times(1-q)^{-\chi_N}\mathcal{Z}_N^{S^4\times S^1}\,,$$
$$\prod_{j=1}^3 Y_q(\boldsymbol{\alpha}_j)=\text{const}\times\left[\left(1-q^b\right)^{2b^{-1}}\left(1-q^{b^{-1}}\right)^{2b}\right]^{-\sum_{k=1}^3(\boldsymbol{\alpha}_k,\rho)}(1-q)^{\chi_N}|\mathcal{Z}_N^{\text{dec}}|^2\,, \tag{13}$$

where again the constant parts can only depend on N and of the Omega deformation parameters but *cannot* be functions of the parameters that define the theory, *i.e.* the masses. The $\mathfrak{C}_q$ are the q-deformed Weyl-invariant structure constants (55), J_q the q-deformation of the Weyl-covariant part of the structure constants (54) and $\mathcal{Z}_N^{\text{dec}}$ the partition function of some extra degrees of freedom (71) that are included in the topological string calculation but then decouple from the 5D theory. In [1] we refer to them as *non-full spin content*. Note that the second line of (13) is the same as equation (72), where the constant factor is explicitly written.

Finally, putting (12) and (13) together, we obtain the final identification

$$\boxed{C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)=\text{const}\times\left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{\left(2Q-\sum_{i=1}^3\alpha_i,\rho\right)}{b}}\lim_{\beta\to 0}\frac{\left|\mathcal{Z}_N^{\text{dec}}\right|^2\mathcal{Z}_N^{S^4\times S^1}}{\beta^{2Q\sum_{i=1}^3(\boldsymbol{\alpha}_i,\rho)}}} \tag{14}$$

where the limit is well defined up to an overall divergent term that only depends on β and b. The above equation gives the complete relationship between the Toda 3-point structure constants and the partition functions of the T_N theories.

3 Toda 3-point functions

We begin this section with a review of known facts about Toda 3-point functions of three primaries that we will need in later sections. We follow [29–31] whenever possible. We then discuss the symmetry enhancement of the Weyl invariant part of the 3-point functions as well as it's pole structure. We conclude the section with a generalization of these facts for the q-deformed Toda.

3.1 Review

The Lagrangian of the Toda CFT theory is given by

$$L=\frac{1}{8\pi}\left(\partial_\nu\varphi,\partial^\nu\varphi\right)+\mu\sum_{k=1}^{N-1}e^{b(e_k,\varphi)}, \tag{15}$$

where $\varphi := \sum_{i=1}^{N-1} \varphi_i \omega_i$ and e_k, respectively ω_k are the simple roots, respectively fundamental weights of SU(N). We have collected all useful definitions and notations in in appendix B for the convenience of the reader. The parameter μ is called the *cosmological constant.* The theory defined by (15) is invariant under the exchange $b \leftrightarrow b^{-1}$, which sends the cosmological constant to its dual $\tilde{\mu}$, defined in such a way that

$$\left(\pi\tilde{\mu}\gamma(b^{-2})\right)^{b} \stackrel{!}{=} \left(\pi\mu\gamma(b^2)\right)^{\frac{1}{b}} \Longrightarrow \tilde{\mu} = \frac{\left(\pi\mu\gamma(b^2)\right)^{1/b^2}}{\pi\gamma(1/b^2)}. \tag{16}$$

The Toda CFT has a $\mathbf{W}_N$ higher spin chiral symmetry generated by the spin k fields $W_2 \equiv T$, $W_3, \ldots, W_N$. The fields that are primary under W_N are denoted by $V_{\boldsymbol{\alpha}}$, are labeled by a weight of SU(N), *i.e.* an $(N-1)$-component vector $\boldsymbol{\alpha}$ and are given explicitly by

$$V_{\boldsymbol{\alpha}} := e^{(\boldsymbol{\alpha},\varphi)}. \tag{17}$$

For the sake of avoiding some fractions, we shall parametrize the weights $\boldsymbol{\alpha}$ of the fields $V_{\boldsymbol{\alpha}_i}$ entering the correlation functions as follows

$$\boldsymbol{\alpha}_i = N \sum_{j=1}^{N-1} \alpha_i^j \omega_j. \tag{18}$$

The central charge of the Toda CFT and the conformal dimension of the primary fields are

$$c = N - 1 + 12\,(\mathcal{Q},\mathcal{Q}) = (N-1)\left(1 + N(N+1)Q^2\right), \qquad \Delta(\alpha) = \frac{(2\mathcal{Q} - \boldsymbol{\alpha}, \boldsymbol{\alpha})}{2}, \tag{19}$$

where $\mathcal{Q} := Q\rho = (b + b^{-1})\rho$ with the Weyl vector ρ defined in (126). The conformal dimension, as well as the eigenvalues of all the other higher spin currents W_k are invariant under the affine[4] Weyl transformations (132) of the weights $\boldsymbol{\alpha}_i$. Furthermore, the primary fields of Toda CFT transform under an affine Weyl transformations $\boldsymbol{\alpha} \to \mathsf{w} \circ \boldsymbol{\alpha}$ as follows

$$V_{\mathsf{w}\circ\boldsymbol{\alpha}} = \mathsf{R}^{\mathsf{w}}(\boldsymbol{\alpha}) V_{\boldsymbol{\alpha}} \tag{20}$$

with the reflection amplitude R given by the expression

$$\mathsf{R}^{\mathsf{w}}(\boldsymbol{\alpha}) := \frac{\mathsf{A}(\boldsymbol{\alpha})}{\mathsf{A}(\mathsf{w}\circ\boldsymbol{\alpha})}\,. \tag{21}$$

Here, as in [31], we define the function

$$\mathsf{A}(\boldsymbol{\alpha}) := \left(\pi\mu\gamma(b^2)\right)^{\frac{(\boldsymbol{\alpha}-\mathcal{Q},\rho)}{b}} \prod_{e>0} \Gamma\left(1 - b\,(\boldsymbol{\alpha} - \mathcal{Q}, e)\right) \Gamma\left(-b^{-1}\,(\boldsymbol{\alpha} - \mathcal{Q}, e)\right). \tag{22}$$

[4]One should not confuse the affine Weyl tranformation, *i.e.* Weyl reflections accompanied by two translations, with Weyl reflections belonging to the Weyl group of the affine Lie algebra.

The 2-point correlation functions of primary fields are fixed by conformal invariance and by the normalization (17). They read

$$\langle V_{\boldsymbol{\alpha}_1}(z_1,\bar{z}_1)V_{\boldsymbol{\alpha}_2}(z_2,\bar{z}_2)\rangle = \frac{(2\pi)^{N-1}\delta(\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2-2\mathcal{Q}) + \text{Weyl-reflections}}{|z_1-z_2|^{4\Delta(\boldsymbol{\alpha}_1)}}, \tag{23}$$

where "Weyl-reflections" stands for additional δ-contributions that come from the field identifications (20).

In this article, we shall be mostly interested in the three point functions of primary fields. Their coordinate dependence is fixed by conformal symmetry up to an overall coefficient $C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)$ called the 3-point structure constants as

$$\langle V_{\boldsymbol{\alpha}_1}(z_1,\bar{z}_1)V_{\boldsymbol{\alpha}_2}(z_2,\bar{z}_2)V_{\boldsymbol{\alpha}_3}(z_3,\bar{z}_3)\rangle = \frac{C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)}|z_{13}|^{2(\Delta_1+\Delta_3-\Delta_2)}|z_{23}|^{2(\Delta_2+\Delta_3-\Delta_1)}}, \tag{24}$$

where $z_{ij} := z_i - z_j$.

Due to the property (20), the 3-point structure constants are not invariant under affine Weyl reflections of the weights α_i, but are instead *covariant* and transform like the primaries themselves. As [31], we will find it advantageous to talk about the Weyl invariant part of the 3-point structure constants. For that purpose, it is useful to define the functions[5] Y as

$$\begin{aligned} Y(\boldsymbol{\alpha}) &:= \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{-\frac{(\boldsymbol{\alpha},\rho)}{b}} \prod_{e>0} \Upsilon\left((\mathcal{Q}-\boldsymbol{\alpha},e)\right) \\ &= \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{-\frac{N}{2b}\sum_{j=1}^{N-1}\alpha^j j(N-j)} \prod_{k=1}^{N-1}\prod_{i=1}^{N-k} \Upsilon\left(kQ - N(\alpha^i+\cdots+\alpha^{i+k-1})\right), \end{aligned} \tag{25}$$

where the product in the first line goes over all $\frac{N(N-1)}{2}$ positive roots of SU(N). These functions obeys the same reflection property as the primary fields, *i.e.*

$$Y(\mathsf{w}\circ\boldsymbol{\alpha}) = \mathsf{R}^{\mathsf{w}}(\boldsymbol{\alpha})Y(\boldsymbol{\alpha}). \tag{26}$$

The transformation property (26) under affine Weyl transformation can be easily derived for reflections on the simple roots e_i by noting that for any function f

$$\prod_{e>0} f((\mathcal{Q}-\boldsymbol{\alpha},e)) \mapsto \prod_{e>0} f((\mathcal{Q}-\boldsymbol{\alpha},e-e_j\,(e_j,e))) = \prod_{\substack{e>0\\ e\neq e_j}} f((\mathcal{Q}-\boldsymbol{\alpha},e)) \times f(-(\mathcal{Q}-\boldsymbol{\alpha},e_j)), \tag{27}$$

where the transformation acts as $w_i \circ \boldsymbol{\alpha} = \boldsymbol{\alpha} - (\boldsymbol{\alpha}-\mathcal{Q},e_i)\,e_i$. After that one uses $\Upsilon(-x) = \Upsilon(x+Q)$ as well as equation (136) to show (26). As a final remark on $Y(\boldsymbol{\alpha})$, we observe that this function is zero if $\boldsymbol{\alpha}$ is a multiple of a fundamental weight and in particular it has a zero of order $\frac{(N-1)(N-2)}{2}$ if we set $\boldsymbol{\alpha} = \kappa\omega_1$ or $\boldsymbol{\alpha} = \kappa\omega_{N-1}$.

[5] For the Liouville case, these functions are also introduced by AGT [2] and labeled by $f(\alpha)$.

Now, we can introduce the Weyl invariant part of the structure constants

$$\mathfrak{C}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) := \frac{C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)}{\left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{(2\mathcal{Q},\rho)}{b}}\prod_{i=1}^3 Y(\boldsymbol{\alpha}_i)}. \tag{28}$$

by dividing out the piece that transforms non-trivially under Weyl transformations. The function $\mathfrak{C}$ of the weights $\boldsymbol{\alpha}$ is independent of the cosmological constant μ and is invariant under affine Weyl reflections in the $\boldsymbol{\alpha}$. Anticipating a bit, we will show in the later sections that the Weyl invariant part of the 3-point structure constants has a higher symmetry than the naive affine Weyl symmetry of $\mathrm{SU}(N)^3$. In particular, for $N=2$ it is invariant under the SU(4) Weyl group, while for $N=3$ it is invariant under the E_6 Weyl group.

While the general formula for the 3-point structure constants of Toda CFT is not known, they have been computed in special cases. The formula for the structure constants of $\mathbf{W}_N$ for the *degenerate case* in which one of the three weights becomes proportional to the first or the last fundamental weight, *i.e.* $\boldsymbol{\alpha}_3 = \varkappa\omega_1$ or $\boldsymbol{\alpha}_3 = \varkappa\omega_{N-1}$ is known from [29] and reads[6]

$$\begin{aligned} C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\varkappa\omega_{N-1}) = &\left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{\left(2\mathcal{Q}-\sum_{i=1}^3\boldsymbol{\alpha}_i,\rho\right)}{b}} \times \\ &\times \frac{\Upsilon'(0)^{N-1}\Upsilon(\varkappa)\prod_{e>0}\Upsilon((\mathcal{Q}-\boldsymbol{\alpha}_1,e))\Upsilon((\mathcal{Q}-\boldsymbol{\alpha}_2,e))}{\prod_{i,j=1}^N\Upsilon(\frac{\varkappa}{N}+(\boldsymbol{\alpha}_1-\mathcal{Q},h_i)+(\boldsymbol{\alpha}_2-\mathcal{Q},h_j))}. \end{aligned} \tag{29}$$

We remark that in the limit in which the degenerate field becomes the identity, *i.e.* $\varkappa\to 0$ one can show that the 3-point structure constants (29) converge to (23).

In the $N=2$ case, the degeneration doesn't matter since there is only one fundamental weight anyway and (29) reduces to (we set $\varkappa = 2\alpha_3$) the famous DOZZ formula[7]

$$C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{Q-\sum_{i=1}^3\alpha_i}{b}}\frac{\Upsilon'(0)\prod_{i=1}^3\Upsilon(2\alpha_i)}{\Upsilon(\sum_{i=1}^3\alpha_i-Q)\prod_{j=1}^3\Upsilon(\sum_{i=1}^3\alpha_i-2\alpha_j)}, \tag{30}$$

which was conjecture by [6,7] and derived by [42,43].

3.2 Enhanced symmetry of the Weyl invariant part

In this subsection, we shall make a couple of observations on the symmetries of the Weyl invariant part of the structure constants that to our knowledge are not found in the literature.

In the Liouville case ($N=2$) the Weyl invariant piece of the structure constants (28) take the form

$$\mathfrak{C}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) = \frac{\Upsilon'(0)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_1-\alpha_2+\alpha_3)\Upsilon(-\alpha_1+\alpha_2+\alpha_3)}. \tag{31}$$

[6] In [31] a more general formula was derived for $N=3$ for the case of semi-degenerate fields $\boldsymbol{\alpha}_3 = \varkappa\omega_2 - mb\omega_1$ with m integer. We will not need it here.

[7] For $N=2$ we set $\boldsymbol{\alpha}_i = 2\alpha_i\omega_1$, *i.e.* we omit the unnecessary second index and set $\alpha_i^1 \equiv \alpha_i$.

At this point, we use (8) and replace the α_i by the m_1, n_1 and l_1 as

$$\alpha_1 = m_1 + \frac{Q}{2}\,, \qquad \alpha_2 = -n_1 + \frac{Q}{2}\,, \qquad \alpha_3 = l_1 + \frac{Q}{2}\,. \tag{32}$$

Setting then

$$m_1 = \frac{u_1 + u_3}{2}\,, \quad n_1 = \frac{u_2 + u_3}{2}\,, \quad l_1 = \frac{u_1 + u_2}{2} \tag{33}$$

and using the symmetries of the Υ functions leads to the following compact expression for the Weyl invariant structure constants of the Liouville CFT

$$\mathfrak{C}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = \frac{\Upsilon'(0)}{\prod_{i=1}^{4} \Upsilon(u_i + \frac{Q}{2})}, \qquad \text{where} \qquad \sum_{i=1}^{4} u_i = 0. \tag{34}$$

We observe that the above is invariant under the SU(4) Weyl group that acts as the permutation group S^4 on the variables u_i. We have thus uncovered the presence of a "hidden" symmetry group.

The $N = 3$ case is considerably more involved. For reasons that will become apparent shortly, we will label by an index $j = 1, 2, 3$ the weights $h_i^{(j)}$ of the three different SU(3)s that appear, *i.e.* each $\boldsymbol{\alpha}_j$ lives in its own copy of the SU(3) weight space labeled by j. Using [31], we know that $\mathfrak{C}$ is invariant not only under SU(3) affine Weyl reflections of the $\boldsymbol{\alpha}_j$'s, but also under the 27 new transformations

$$\boldsymbol{\alpha}_1 \to \boldsymbol{\alpha}_1 - \varsigma_{ijk} h_i^{(1)}, \qquad \boldsymbol{\alpha}_2 \to \boldsymbol{\alpha}_2 - \varsigma_{ijk} h_j^{(2)}, \qquad \boldsymbol{\alpha}_3 \to \boldsymbol{\alpha}_3 - \varsigma_{ijk} h_k^{(3)}, \tag{35}$$

where i, j and k are fixed and we have defined

$$\varsigma_{ijk} := \left(\boldsymbol{\alpha}_1 - \mathcal{Q}, h_i^{(1)}\right) + \left(\boldsymbol{\alpha}_2 - \mathcal{Q}, h_j^{(2)}\right) + \left(\boldsymbol{\alpha}_3 - \mathcal{Q}, h_k^{(3)}\right). \tag{36}$$

We can now make the following set of observations. First, the affine SU(3) Weyl transformations in the $\boldsymbol{\alpha}_i$ become the usual SU(3) Weyl reflections when expressed in the variables m_i, n_i and l_i defined via (7), *i.e.* they act as the S^3 permutations. Using the parametrization (7), we then observe that

$$\varsigma_{ijk} = m_i - n_j + l_k, \qquad \text{where} \qquad \sum_{i=1}^{3} m_i = \sum_{i=1}^{3} n_i = \sum_{i=1}^{3} l_i = 0. \tag{37}$$

Therefore, the transformation (35) for a given choice of i, j and k acts of the variables m_a, n_b and l_c as

$$\begin{aligned} m_a &\to m_a - (m_i - n_j + l_k)\left(\delta_{ai} - \frac{1}{3}\right), \\ n_b &\to n_b + (m_i - n_j + l_k)\left(\delta_{bj} - \frac{1}{3}\right), \\ l_c &\to l_c - (m_i - n_j + l_k)\left(\delta_{ck} - \frac{1}{3}\right), \end{aligned} \tag{38}$$

where no sum over i, j, k is to be taken. We now want to interpret the new transformations (35) as being the result of the (non-affine) action of the Weyl group of E_6. Since the Weyl group is generated by the Weyl reflections associated to the simple roots, we only need to consider those. We have 9 weights $h_i^{(j)}$ subject to the three constraints $\sum_{i=1}^3 h_i^{(j)} = 0$ and we can build the E_6 root system from them as

$$\begin{aligned} e_1^{E_6} &= h_1^{(1)} - h_2^{(1)}, & e_2^{E_6} &= h_2^{(1)} - h_3^{(1)}, & e_3^{E_6} &= h_3^{(1)} + h_3^{(2)} + h_1^{(3)}, \\ e_4^{E_6} &= -h_1^{(3)} + h_2^{(3)}, & e_5^{E_6} &= -h_2^{(3)} + h_3^{(3)}, & e_6^{E_6} &= h_2^{(2)} - h_3^{(2)}, \end{aligned} \tag{39}$$

where we refer to figure 2 for the numbering of the E_6 simple roots. We observe that

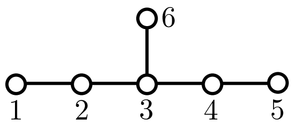

Figure 2: *The figure shows the E_6 Dynkin diagram together with our labeling of the simple roots.*

$\left(e_i^{E_6}, e_j^{E_6}\right)$ is the Cartan matrix of E_6, if we require $\left(h_a^{(k)}, h_b^{(l)}\right) = 0$ if $k \neq l$. Therefore, we have constructed the E_6 root system within the space spanned by the $h_i^{(j)}$. Furthermore, we can obtain all the variables m_i, n_i and l_i by taking the scalar products $\left(\sum_{i=1}^3 (\boldsymbol{\alpha}_i - \mathcal{Q}), e_j^{E_6}\right)$, where each $\boldsymbol{\alpha}_k - \mathcal{Q}$ is expressed only through the $h_i^{(k)}$. We find that Weyl reflections for the simple roots $e_i^{E_6}$ with $i \neq 3$ correspond to permutations of the m's, n's and l's among themselves. However, the Weyl reflection corresponding to $e_3^{E_6}$ transforms the variables as

$$\begin{aligned} m_1 &\to m_1 + \lambda, & m_2 &\to m_2 + \lambda, & m_3 &\to m_3 - 2\lambda, \\ n_1 &\to n_1 - \lambda, & n_2 &\to n_2 - \lambda, & n_3 &\to n_3 + 2\lambda, & (40) \\ l_1 &\to l_1 - 2\lambda, & l_2 &\to l_2 + \lambda, & l_3 &\to l_3 + \lambda, & (41) \end{aligned}$$

where $3\lambda = m_3 - n_3 + l_1$. We easily see that this transformation corresponds to (35) for $i = 3$, $j = 3$ and $k = 1$. The transformations corresponding to the other choices of i, j and k can be obtained by acting with some other $e_l^{E_6}$ first. Hence, the Weyl transformations of the three SU(3) can be combined with (35) to generate the Weyl group of the entire E_6.

For the cases $N \geq 4$ the full enhanced symmetry of the Weyl invariant structure constants is not completely known. We shall argue in the conclusions that the enhanced symmetry should contain E_7 in the case $N = 4$ and E_8 for $N = 6$.

3.3 Pole structure of the Weyl invariant part

We see from (34), that the poles for the $N=2$ Liouville case are all captured by the expression

$$\left[\prod_{i=1}^{4}\Upsilon(u_i+\frac{Q}{2})\right]^{-1}=\left[\prod_{h\in\mathbf{4}\oplus\overline{\mathbf{4}}}\mathbf{G}\left(\frac{Q}{2}+\left(\sum_{i=1}^{3}(\boldsymbol{\alpha}_i-\mathcal{Q}),h\right)\right)\right]^{-1}, \tag{42}$$

where we used the function $\mathbf{G}(x)=\frac{1}{\Gamma_b(x)}$ with $\Upsilon(x)=\mathbf{G}(x)\mathbf{G}(Q-x)$ introduced in [31], see (143). The weights h are SU(4) weights and in the fundamental representation $\mathbf{4}$ they are[8]

$$\begin{aligned} h_1^{\mathrm{SU}(4)}&=h_1^{(1)}-h_1^{(2)}+h_1^{(3)}, & h_2^{\mathrm{SU}(4)}&=-h_1^{(1)}+h_1^{(2)}+h_1^{(3)},\\ h_3^{\mathrm{SU}(4)}&=h_1^{(1)}+h_1^{(2)}-h_1^{(3)}, & h_4^{\mathrm{SU}(4)}&=-h_1^{(1)}-h_1^{(2)}-h_1^{(3)}, \end{aligned} \tag{43}$$

with the weights of $\overline{\mathbf{4}}$ being the negatives of the above.

Moving on to the case with $N=3$, it was argued in [31] that the pole structure of the full correlation function $C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)$ is given by

$$C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)=\mathfrak{F}\left[\prod_{i_1,i_2,i_3=1}^{3}\mathbf{\mathfrak{Z}}\left(\sum_{k=1}^{3}(\boldsymbol{\alpha}_k-\mathcal{Q},h_{i_k}^{(k)})\right)\right]^{-1}=\mathfrak{F}\left[\prod_{i,j,k=1}^{3}\mathfrak{Z}(m_i-n_j+l_k)\right]^{-1}, \tag{44}$$

where $\mathfrak{F}$ is some unknown entire function and the function $\mathfrak{Z}$ is defined in (144), $\mathfrak{Z}(x):=\mathbf{G}(Q+x)\mathbf{G}(Q-x)$. Using the E_6 Weyl symmetry of $\mathfrak{C}$, it follows that the poles of $\mathfrak{C}$ are contained in

$$\mathfrak{C}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)\sim\left[\mathfrak{Z}(0)^3\prod_{i,j,k=1}^{3}\mathfrak{Z}(m_i-n_j+l_k)\prod_{i<j=1}^{3}\mathfrak{Z}(m_i-m_j)\mathfrak{Z}(n_i-n_j)\mathfrak{Z}(l_i-l_j)\right]^{-1}, \tag{45}$$

where $\mathfrak{Z}(0)^3$ is just convenient normalization. We recognize in this expression the weights of the 78-dimensional adjoint representation of E_6 expressed using the weights of $\mathrm{SU}(3)^3\subset E_6$,

$$\mathfrak{C}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)\sim\frac{1}{\prod_{h\in\mathbf{78}}\mathbf{G}\left(Q+\left(\sum_{i=1}^{3}(\boldsymbol{\alpha}_i-\mathcal{Q}),h\right)\right)}. \tag{46}$$

The additional poles introduced in (45) are completely canceled by the Weyl covariant part in the formula (28) relating them to the 3-point structure constants, because

$$\prod_{k=1}^{3}Y(\boldsymbol{\alpha}_k)\propto\frac{\prod_{i<j=1}^{3}\mathfrak{Z}(m_i-m_j)\mathfrak{Z}(n_i-n_j)\mathfrak{Z}(l_i-l_j)}{\prod_{k=1}^{3}\prod_{e>0}(\mathcal{Q}-\boldsymbol{\alpha}_k,e)\,\Gamma\big(b\,(\mathcal{Q}-\boldsymbol{\alpha}_k,e)\big)\Gamma\big(b^{-1}\,(\mathcal{Q}-\boldsymbol{\alpha}_k,e)\big)} \tag{47}$$

[8] Note that in order to get the suitably normalized scalar product for SU(4), we need to define $(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)^{\mathrm{SU}(4)}:={}^{(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)^{\mathrm{SU}(2)^3}}\!/2$, *i.e.* we compute the scalar products as before and divide the answer by two.

where we have used (25) and (144). The proportionality factor in (47) depends only on μ and b and has no zeroes or poles while the additional factors of Γ in the denominator of (47) lead only to more zeroes of $\prod_{k=1}^{3} Y(\boldsymbol{\alpha}_k)$. Thus, multiplying (45) with the Weyl covariant part, see (28), in order to get the full 3-point structure constants will cancel the extra poles that we introduced.

Finally, it is compelling to conjecture that for any N the poles of the Weyl invariant structure constants should behave as

$$\mathfrak{C}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) \sim \frac{1}{\prod_{h\in\mathbf{R}} \mathbf{G}\left(\frac{N-1}{2}Q + \left(\sum_{i=1}^{3}(\boldsymbol{\alpha}_i - \mathcal{Q}), h\right)\right)} \tag{48}$$

for an appropriate representation $\mathbf{R}$ of $\mathrm{SU}(N)^3$.

3.4 The q-deformed Toda field theory

One of our goals in this paper is to show how to use the topological string formalism to solve the Toda field theory. This will require a careful study of the q-deformed Toda correlation functions which topological strings naturally provide and to then learn how to take the $q \to 1$ limit. For this purpose here we generalize some of the formulas that we discussed in the previous sections. An incomplete list of references includes [1, 18–27] . This section goes hand in hand with appendix C.2, where we define the q-deformed version of the Υ functions and discuss in detail its symmetry properties as well as its zeros. To our knowledge these formulas do not exist in the literature.

We begin by stressing some defining properties that all the q-deformed formulas must have:

- They must reproduce the exact undeformed formula in the $q \to 1$ limit. With no further prefactor, unless stated otherwise. That will be the case of the C_q (49).
- For the N=2 case, they must give the known answers, insofar they are available [23].
- They must have exactly the same symmetries and transformation properties as the undeformed ones under the (affine) Weyl, as well as the enhanced symmetry group.
- They must have their poles and zeros in the same place with the undeformed ones. To be more precise, the q-deformed functions have more zeroes/poles, specifically a whole tower of zeroes/poles for each zero/pole of the undeformed function as discussed in (163). The tower is generated by beginning with the undeformed zero/pole and translating it by $m\frac{2\pi i}{\log q} = -m\frac{2\pi i}{\beta}$, where m is a positive integer.

We moreover want to stress that the q-deformed version of Toda field theory does not have a known Lagrangian description. Everything is defined algebraically in analogy to the usual case via a deformation of the $\mathbf{W}_N$ algebra. Since no Lagrangian description is known for the q-deformed Toda field theory, we can compute everything up to overall factors that in the $q \to 1$ limit give the cosmological constant. Thus, we define the 5D correlation functions up to the $\pi\mu\gamma(b^2)$ term, since they together form the $b \to b^{-1}$ invariant combination. Explicitly, we have for the q-deformed 3-point structure constants

$$C_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) \xrightarrow{q\to 1} \left(\pi\mu\gamma(b^2)\right)^{-\frac{\mathcal{Q}-\sum_i \alpha_i}{b}} C(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3)\,. \tag{49}$$

Obviously, after the $q \to 1$ limit is taken and the undeformed answer is obtained, it is clear how one can put back the appropriate $\pi\mu\gamma(b^2)$ factors for a given correlation function, thus obtaining the full result with all the factors.

As we already said in section 3.1 the Weyl invariant part $\mathfrak{C}$ is independent of the cosmological constant and thus it's q-deformed version should be straightforward. However, the Weyl covariant part with which we need to multiply in order to obtain the full C_q will converge to its undeformed version, up to an $\pi\mu\gamma(b^2)$ factor. In particular, the q-deformed version of the functions Y defined in (25) is

$$Y_q(\boldsymbol{\alpha}) := \left[\frac{\left(1-q^b\right)^{2b^{-1}}\left(1-q^{b^{-1}}\right)^{2b}}{(1-q)^{2\mathcal{Q}}}\right]^{-(\boldsymbol{\alpha},\rho)} \prod_{e>0} \Upsilon_q\left((\mathcal{Q}-\boldsymbol{\alpha}, e)\right), \tag{50}$$

where the functions Υ_q are introduced in (158). Using (162), we find that this function behaves under affine Weyl transformations as

$$Y_q(\mathsf{w}\circ\boldsymbol{\alpha}) = \mathsf{R}_q^{\mathsf{w}}(\boldsymbol{\alpha}) Y_q(\boldsymbol{\alpha}) \tag{51}$$

with the q-deformed version of the reflection amplitude

$$\mathsf{R}_q^{\mathsf{w}}(\boldsymbol{\alpha}) := \frac{\mathsf{A}_q(\boldsymbol{\alpha})}{\mathsf{A}_q(\mathsf{w}\circ\boldsymbol{\alpha})} \tag{52}$$

being composed out of

$$\mathsf{A}_q(\boldsymbol{\alpha}) := \prod_{e>0} \Gamma_{q^{b^{-1}}}\left(1 - b\left(\boldsymbol{\alpha}-\mathcal{Q}, e\right)\right)\Gamma_{q^b}\left(-b^{-1}\left(\boldsymbol{\alpha}-\mathcal{Q}, e\right)\right). \tag{53}$$

Note that also the q-deformed version of the reflection amplitude in the $q \to 1$ limit gives R^{w} up to an overall $\pi\mu\gamma(b^2)$ factor. The q-deformed factor that we need to divide by in order to get the Weyl invariant structure constants is

$$J_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = \left[\frac{\left(1-q^b\right)^2\left(1-q^{b^{-1}}\right)^{2b^2}}{(1-q)^{2(1+b^2)}}\right]^{\frac{(2\mathcal{Q},\rho)}{b}} \prod_{i=1}^{3} Y_q(\boldsymbol{\alpha}_i) = \text{const} \times \prod_{i=1}^{3} Y_q(\boldsymbol{\alpha}_i), \tag{54}$$

so that like in (28)

$$\mathfrak{C}_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) := \frac{C_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3)}{J_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3)}. \tag{55}$$

The q-deformation version of the Fateev and Litvinov formula (29) for the 3-point correlation functions with one degenerate insertion reads

$$\begin{aligned} C_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \varkappa\omega_{N-1}) = &\left(\frac{\left(1-q^b\right)^2 \left(1-q^{b^{-1}}\right)^{2b^2}}{(1-q)^{2(1+b^2)}} \right)^{\frac{\left(2\mathcal{Q}-\sum_{i=1}^3 \alpha_i, \rho\right)}{b}} \times \\ &\times \frac{\Upsilon_q'(0)^{N-1} \Upsilon_q(\varkappa) \prod_{e>0} \Upsilon_q((\mathcal{Q}-\boldsymbol{\alpha}_1, e)) \Upsilon_q((\mathcal{Q}-\boldsymbol{\alpha}_2, e))}{\prod_{i,j=1}^N \Upsilon_q(\frac{\varkappa}{N} + (\boldsymbol{\alpha}_1 - \mathcal{Q}, h_i) + (\boldsymbol{\alpha}_2 - \mathcal{Q}, h_j))}. \end{aligned} \tag{56}$$

This formula to our knowledge does not appear anywhere else in the literature. We write it down as the unique formula that has the properties mentioned at the beginning of the section. First, it has its poles and zeros in the correct positions, see (137). Second, it has the correct covariance properties under the affine Weyl symmetries of the non-degenerate fields (52). Finally, for the $N = 2$ case, (56) reduces to the q-deformation of the DOZZ formula (up to the μ dependence)

$$\begin{aligned} C_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = &\left(\frac{\left(1-q^b\right)^2 \left(1-q^{b^{-1}}\right)^{2b^2}}{(1-q)^{2(1+b^2)}} \right)^{\frac{Q-\sum_{i=1}^3 \alpha_i}{b}} \\ &\times \frac{\Upsilon_q'(0) \prod_{i=1}^3 \Upsilon_q(2\alpha_i)}{\Upsilon_q(\sum_{i=1}^3 \alpha_i - Q) \prod_{j=1}^3 \Upsilon_q(\sum_{i=1}^3 \alpha_i - 2\alpha_j)}, \end{aligned} \tag{57}$$

derived in [36]. From it we can extract the q-deformed version of the Weyl invariant part using equation (55),

$$\mathfrak{C}_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = \frac{\Upsilon_q'(0)}{\Upsilon_q(\sum_{i=1}^3 \alpha_i - Q) \prod_{j=1}^3 \Upsilon_q(\sum_{i=1}^3 \alpha_i - 2\alpha_j)} = \frac{\Upsilon_q'(0)}{\prod_{i=1}^4 \Upsilon_q(u_i + \frac{Q}{2})}, \tag{58}$$

which immediately gives the he correct undeformed $\mathfrak{C}$

$$\mathfrak{C}_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) \quad \stackrel{q\to 1}{\longrightarrow} \quad \mathfrak{C}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) \tag{59}$$

as it is in equations (31) and (34) with no further factors.

4 The T_N partition function from topological strings

In this section we introduce the formula for the 5D T_N partition functions that we computed in [1] and we discuss how they can be brought to a form that allows us to take the 4D limit ($\beta \to 0$) in order to obtain the T_N partition functions on S^4. Since the parametrization is crucial, we begin by carefully discussing it and the way it is read off from the web diagrams. Some details of the computations are presented in appendix D.

The T_N theories are isolated strongly coupled fixed points that one can discover by taking the strong coupling limit of the $SU(N)^{N-2}$ or of the $U(N-1) \times U(N-2) \times \cdots \times U(1)$ linear quivers. The calculation of the T_N partition function is not possible using any purely field theoretic method currently known, because the T_N theories have no known Lagrangian description. The only applicable method is string theory and in particular 5-brane webs [44, 45] from which the answer is derived using topological strings.

4.1 The 5-brane webs

A very short review of 5-brane webs is in order. First, 5D $\mathcal{N} = 1$ gauge theories can be embedded in string theory by using type IIB (p, q) 5-brane webs [44, 45]. All the information needed to describe the low energy effective theory on the Coulomb branch is encoded in the web diagrams, through which the 5D SW curves can be easily derived [22, 44–46]. Furthermore, 5D $\mathcal{N} = 1$ gauge theories can also be realized using geometric engineering [47, 48], in particular M-theory compactified on Calabi-Yau threefolds. This alternative description provides an efficient way of computing the Nekrasov partition functions of the gauge theories by computing the partition functions of topological strings living on these backgrounds. Recent reviews on the subject can be found in [49, 50]. In particular, the dual to the Calabi-Yau toric diagram is exactly equal to the web diagram of the type IIB (p, q) 5-brane systems [51].

The SW curves and the Nekrasov partition functions are parametrized by the Coulomb moduli a as well as the UV masses m and coupling constants τ of the gauge theory. These parameters are encoded in the web diagrams as follows. On the one hand, deformations of the webs that do not change the asymptotic form of the 5-branes correspond to the Coulomb moduli a and their number is the number of *faces* of the web diagram. On the other hand, deformations of the webs that do change the asymptotic form of the 5-branes correspond to parameters that define the theory, namely masses and coupling constants and they are equal to the number of external branes minus three. Note that at each vertex there is a no-force condition (D5/NS5 (p, q) charge conservation) that serves to preserve 8 supersymmetries.

Having said all the above, we can now return to the T_N theories. The first step towards being able to calculate the T_N partition functions was taken by Benini, Benvenuti and Tachikawa, who gave in [33] the web diagrams of the 5D T_N theories. Subsequently, in [1] we tested their proposal by deriving the corresponding SW curves and Nekrasov partition functions. Most importantly, we were able to cross-check our results for the partition functions against the 5D superconformal index that was recently calculated in [37]. For similar work see also [5, 38].

We now turn to the parametrization of the T_N web diagrams. The general parametrization in contained in the appendix, see figure 6 and here we just give a short introduction. We have one parameter $a_i^{(j)}$ for each face, or hexagon, of the diagram, that will also appear as $\tilde{A}_i^{(j)} = e^{-\beta a_i^{(j)}}$. They can be thought of as Coulomb moduli that will be integrated over and are called *breathing modes*. The number of faces in the web diagram of the T_N theory is $\frac{(N-1)(N-2)}{2}$. In addition, we have $3N$ parameters m_i, n_i, l_i labeling the positions of the exterior flavor branes for the branes on the, respectively, left, lower and upper right side of the diagram. From them, we define the fugacities

$$\tilde{M}_i := e^{-\beta m_i}, \qquad \tilde{N}_i := e^{-\beta n_i}, \qquad \tilde{L}_i := e^{-\beta l_i}, \tag{60}$$

that are subject to the relation

$$\prod_{k=1}^{N} \tilde{M}_k = \prod_{k=1}^{N} \tilde{N}_k = \prod_{k=1}^{N} \tilde{L}_k = 1 \Longleftrightarrow \sum_{k=1}^{N} m_k = \sum_{k=1}^{N} n_k = \sum_{k=1}^{N} l_k = 0. \tag{61}$$

From the mass parameters, we also define the "boundary" Coulomb parameters. They are the $\tilde{A}_i^{(j)}$ with $i + j = N$, with $i = 0$ or with $j = 0$ and are given as functions of the positions of the flavor branes in (116).

In the dual, geometric engineering description, the parameters above correspond to the Kähler parameters of the Calabi-Yau threefold. On the web diagram, the Kähler parameters correspond to the horizontal, the diagonal and the vertical lines and are labeled by $Q_{n;i}^{(j)}$, $Q_{m;i}^{(j)}$ and $Q_{l;i}^{(j)}$ respectively. They are derived quantities through the equations (119) and are useful because they are the ones that enter in the computation of the partition function via the topological vertex.

In order to familiarize the reader with the parametrization, we shall illustrate the simplest cases $N = 2$ and $N = 3$ with some examples. The parametrization in those cases is contained in figure 3. For $N = 2$, we see that we have no Coulomb moduli and the Kähler parameters obey the relation

$$Q_{m;1}^{(1)} Q_{l;1}^{(1)} = \frac{\tilde{M}_1}{\tilde{M}_2}, \qquad Q_{m;1}^{(1)} Q_{n;1}^{(1)} = \frac{\tilde{N}_1}{\tilde{N}_2}, \qquad Q_{n;1}^{(1)} Q_{l;1}^{(1)} = \frac{\tilde{L}_1}{\tilde{L}_2}. \tag{62}$$

Using (61), we find $Q_{m;1}^{(1)} = \frac{\tilde{M}_1 \tilde{N}_1}{\tilde{L}_1}$, $Q_{n;1}^{(1)} = \frac{\tilde{M}_1 \tilde{L}_1}{\tilde{N}_1}$ and $Q_{l;1}^{(1)} = \frac{\tilde{N}_1 \tilde{L}_1}{\tilde{M}_1}$. For $N = 3$ we have seven independent parameters: one Coulomb modulus $\mathsf{A} \equiv \tilde{A}_1^{(1)}$ and $3 \times (3 - 1)$ independent brane positions. A straightforward computation gives the nine Kähler parameters of the web

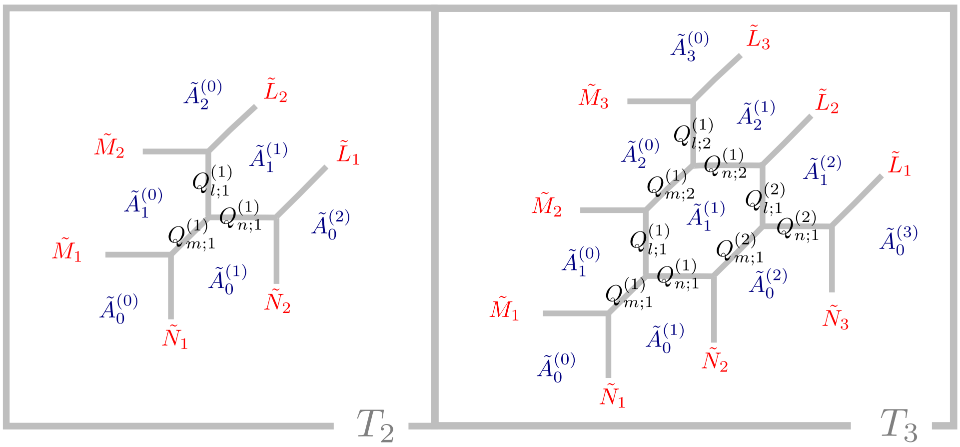

Figure 3: *The parametrization and Kähler parameters of the T_2 and T_3 junctions. The external "mass" parameters are shown in red, the "face" moduli in blue and the "edge" ones in black.*

diagram as

$$
\begin{aligned}
Q_{m;1}^{(1)} &= \mathsf{A}^{-1}\tilde{M}_1\tilde{N}_1, & Q_{m;2}^{(1)} &= \mathsf{A}\tilde{M}_2\tilde{L}_3, & Q_{m;1}^{(2)} &= \mathsf{A}\tilde{N}_2\tilde{L}_1^{-1}, \\
Q_{n;1}^{(1)} &= \mathsf{A}\tilde{M}_1^{-1}\tilde{N}_2^{-1}, & Q_{n;2}^{(1)} &= \mathsf{A}\tilde{M}_3\tilde{L}_2, & Q_{n;1}^{(2)} &= \mathsf{A}^{-1}\tilde{N}_3^{-1}\tilde{L}_1, \\
Q_{l;1}^{(1)} &= \mathsf{A}\tilde{M}_2^{-1}\tilde{N}_1^{-1}, & Q_{l;2}^{(1)} &= \mathsf{A}^{-1}\tilde{M}_3^{-1}\tilde{L}_3^{-1}, & Q_{l;1}^{(2)} &= \mathsf{A}\tilde{N}_3\tilde{L}_2^{-1}.
\end{aligned} \tag{63}
$$

It is easy to check that the above solutions obey the set of equations (121) relating them to the brane position parameters and that they furthermore satisfy the two constraints coming from matching the height and widths of the hexagon of figure 3

$$
Q_{m;1}^{(2)}Q_{n;1}^{(1)} = Q_{m;2}^{(1)}Q_{n;2}^{(1)}, \qquad Q_{m;2}^{(1)}Q_{l;1}^{(1)} = Q_{m;1}^{(2)}Q_{l;1}^{(2)}. \tag{64}
$$

4.2 The topological vertex computation

Now that we have gained some understanding of the parametrization of the T_N web diagram, we would like to compute its refined topological string amplitude. For this, we use the refined topological vertex, choose the preferred direction to be the horizontal one and cut the toric diagram diagonally into sub-diagrams called *strips*. The calculation was carried out in [1], here we just reproduce the results for the reader's convenience. We consider the strip diagram of arbitrary length $L \geq 0$, drawn on the left in figure 4. The corresponding partition function depends on the external horizontal partitions $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_{L+1})$, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_L)$ as well as

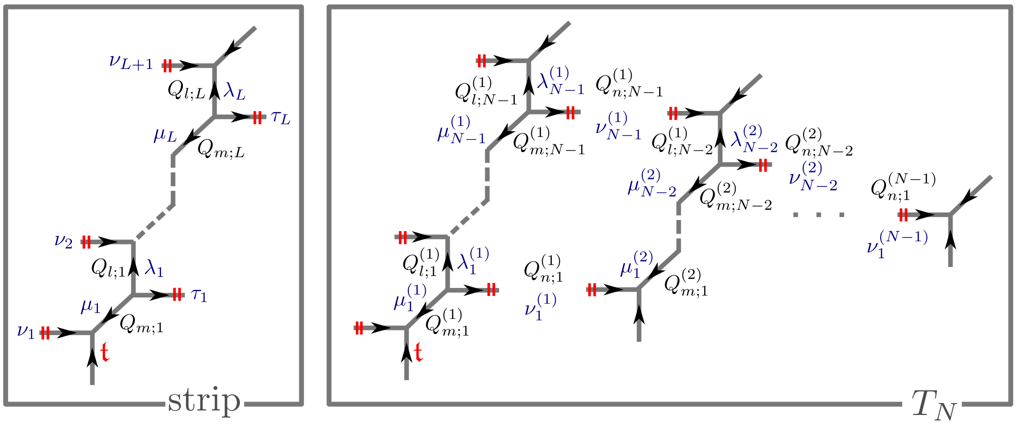

Figure 4: *The left part of the figure shows the strip diagram, while the right one depicts the dissection of the T_N diagram into N strips. The partitions associated with the horizontal, diagonal and vertical lines are $\nu_i^{(j)}$, $\mu_i^{(j)}$ and $\lambda_i^{(j)}$ with $j = 1, \ldots, N-1$, $i = 1, \ldots, N-j$ respectively. The Kähler parameters of the horizontal, diagonal and vertical lines are $Q_{n;i}^{(j)}$, $Q_{m;i}^{(j)}$, $Q_{l;i}^{(j)}$ respectively with the same range of indices.*

the parameters $\boldsymbol{Q}_m = (Q_{m;1}, \ldots, Q_{m;L})$ and $\boldsymbol{Q}_l = (Q_{l;1}, \ldots, Q_{l;L})$. It takes the form

$$\mathcal{Z}^{\rm strip}_{\boldsymbol{\nu\tau}}(\boldsymbol{Q}_m, \boldsymbol{Q}_l; \mathfrak{t}, \mathfrak{q}) = \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}} \prod_{i=1}^{L} (-Q_{m;i})^{|\mu_i|}(-Q_{l;i})^{|\lambda_i|} \prod_{j=1}^{L+1} C_{\mu_j^t \lambda_{j-1}^t \nu_j^t}(\mathfrak{q}, \mathfrak{t}) \prod_{k=1}^{L} C_{\mu_k \lambda_k \tau_k}(\mathfrak{t}, \mathfrak{q}), \tag{65}$$

where $\mu_{L+1} = \lambda_0 = \emptyset$. We refer to [35] for a definition of the topological vertex $C_{\lambda\mu\nu}$. The full topological string partition function is then given by

$$\mathcal{Z}_N^{\rm top} = \sum_{\boldsymbol{\nu}} \prod_{r=1}^{N} \Big(-\boldsymbol{Q}_n^{(r)} \Big)^{|\boldsymbol{\nu}^{(r)}|} \mathcal{Z}^{\rm strip}_{\boldsymbol{\nu}^{(r-1)}\boldsymbol{\nu}^{(r)}}(\boldsymbol{Q}_m^{(r)}, \boldsymbol{Q}_l^{(r)}; \mathfrak{t}, \mathfrak{q}). \tag{66}$$

The strip partition function (65) was computed in [1]. In appendix D, we show that it is useful to redefine the strip slightly, *i.e.* to “cut” the T_N junction in a different way by moving some factors from one strip to its neighbors. These redefinitions do not change the full topological string partition function of the T_N junction. The technical details are left to appendix D. Combining everything, we obtain

$$\mathcal{Z}_N^{\rm top} = \mathcal{Z}_N^{\rm pert} \mathcal{Z}_N^{\rm inst}, \tag{67}$$

where we have defined the “perturbative” partition function

$$\mathcal{Z}_N^{\rm pert} := \prod_{r=1}^{N-1} \prod_{i\le j=1}^{N-r} \frac{\mathcal{M}\Big(\frac{\tilde{A}_i^{(r-1)}\tilde{A}_j^{(r-1)}}{\tilde{A}_{i-1}^{(r-1)}\tilde{A}_{j+1}^{(r-1)}}\Big)}{\mathcal{M}\Big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{A}_i^{(r)}\tilde{A}_{j-1}^{(r)}}{\tilde{A}_{i-1}^{(r-1)}\tilde{A}_j^{(r)}}\Big)\mathcal{M}\Big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{A}_i^{(r)}\tilde{A}_j^{(r-1)}}{\tilde{A}_{i-1}^{(r)}\tilde{A}_{j+1}^{(r-1)}}\Big)} \prod_{i\le j=1}^{N-r-1} \mathcal{M}\Big(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{A}_i^{(r)}\tilde{A}_j^{(r)}}{\tilde{A}_{i-1}^{(r)}\tilde{A}_{j+1}^{(r)}}\Big), \tag{68}$$

and the "instanton" one

$$
\begin{aligned}
\mathcal{Z}_N^{\mathrm{inst}} :=& \sum_{\boldsymbol{\nu}} \prod_{r=1}^{N} \prod_{i=1}^{N-r} \left(\frac{\tilde{N}_r \tilde{L}_{N-r}}{\tilde{N}_{r+1} \tilde{L}_{N-r+1}} \right)^{\frac{|\nu_i^{(r)}|}{2}} \\
& \times \prod_{r=1}^{N} \prod_{i \leq j=1}^{N-r} \left[\frac{\mathsf{N}^{\beta}_{\nu_i^{(r-1)} \nu_j^{(r)}} \left(a_i^{(r-1)} + a_{j-1}^{(r)} - a_{i-1}^{(r-1)} - a_j^{(r)} - \epsilon_+/2 \right)}{\mathsf{N}^{\beta}_{\nu_i^{(r-1)} \nu_{j+1}^{(r-1)}} \left(a_i^{(r-1)} + a_j^{(r-1)} - a_{i-1}^{(r-1)} - a_{j+1}^{(r-1)} \right)} \right. \\
& \times \left. \frac{\mathsf{N}^{\beta}_{\nu_i^{(r)} \nu_{j+1}^{(r-1)}} \left(a_i^{(r)} + a_j^{(r-1)} - a_{i-1}^{(r)} - a_{j+1}^{(r-1)} - \epsilon_+/2 \right)}{\mathsf{N}^{\beta}_{\nu_i^{(r)} \nu_j^{(r)}} \left(a_i^{(r)} + a_{j-1}^{(r)} - a_{i-1}^{(r)} - a_j^{(r)} - \epsilon_+ \right)} \right],
\end{aligned}
\tag{69}
$$

where the $a_i^{(j)}$ are defined via $\tilde{A}_i^{(j)} = e^{-\beta a_j^{(r)}}$. We put the words "perturbative" and "instanton" inside quotation marks because for the T_N there is not really a notion of instanton expansion. There is no coupling constant, since there is no gauge group. We recall that the boundary $a_i^{(j)}$ are related to the masses via (116). In writing (68) and (69) we have introduced the notation[9]

$$
\begin{aligned}
\mathcal{M}(u; \mathfrak{t}, \mathfrak{q}) \equiv \mathcal{M}(u) &= \prod_{i,j=1}^{\infty} (1 - u \mathfrak{t}^{-i} \mathfrak{q}^j), \\
\mathsf{N}^{\beta}_{\lambda\mu}(m; \mathfrak{t}, \mathfrak{q}) \equiv \mathsf{N}^{\beta}_{\lambda\mu}(m) &= \prod_{(i,j) \in \lambda} 2 \sinh \frac{\beta}{2} \left[m + \epsilon_1 (\lambda_i - j + 1) + \epsilon_2 (i - \mu_j^t) \right] \\
& \times \prod_{(i,j) \in \mu} 2 \sinh \frac{\beta}{2} \left[m + \epsilon_1 (j - \mu_i) + \epsilon_2 (\lambda_j^t - i + 1) \right].
\end{aligned}
\tag{70}
$$

We refer to appendix C.2, respectively C.3 for more details concerning $\mathcal{M}$, respectively $\mathsf{N}^{\beta}_{\lambda\mu}$.

As in [1], we define the non-full spin content (also called U(1) factor in [5])

$$
\mathcal{Z}_N^{\mathrm{dec}} := \prod_{i<j=1}^{N} \mathcal{M}(\tilde{M}_i \tilde{M}_j^{-1}) \mathcal{M}(\mathfrak{t}/\mathfrak{q} \tilde{N}_i \tilde{N}_j^{-1}) \mathcal{M}(\tilde{L}_i \tilde{L}_j^{-1}).
\tag{71}
$$

We remark that for $b = \epsilon_1 = -\epsilon_2$, we can write

$$
\begin{aligned}
\left| \mathcal{Z}_N^{\mathrm{dec}} \right|^2 =& \Lambda^{\frac{3N(N-1)}{2}} (1-q)^{\frac{N(N-1)(2N^2-5)}{8}} Q^2 \times \\
& \times \prod_{k=1}^{3} (1-q)^{N(\boldsymbol{\alpha}_k, \boldsymbol{\alpha}_k - 2\mathcal{Q})} \left(\left(1 - q^b\right)^{2b^{-1}} \left(1 - q^{b^{-1}}\right)^{2b} \right)^{(\boldsymbol{\alpha}_k, \rho)} Y_q(\boldsymbol{\alpha}_k)
\end{aligned}
\tag{72}
$$

where we have used (7), the identity (130) and the q-deformed function (50). Thus, up to some ambiguities, there is a clear identification of the decoupled part $\left| \mathcal{Z}_N^{\mathrm{dec}} \right|^2$ with the Weyl covariant part (54) of the correlation functions, see (13).

[9] We often drop the explicit dependence of these functions on the parameters $\mathfrak{t}$ and $\mathfrak{q}$.

The contributions (71) *decouple* from the gauge theory and need to be removed in order to obtain the $S^4 \times S^1$ partition function. In particular, using (118), we find for the quotient

$$\begin{aligned} \frac{\mathcal{Z}_N^{\text{pert}}}{\mathcal{Z}_N^{\text{dec}}} = \prod_{r=1}^{N-1} & \frac{\prod_{i\leq j=1}^{N-r-1} \mathcal{M}\Big(\frac{\tilde{A}_i^{(r)}\tilde{A}_j^{(r)}}{\tilde{A}_{i-1}^{(r)}\tilde{A}_{j+1}^{(r)}}\Big)\mathcal{M}\Big(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{A}_i^{(r)}\tilde{A}_j^{(r)}}{\tilde{A}_{i-1}^{(r)}\tilde{A}_{j+1}^{(r)}}\Big)}{\prod_{i\leq j=1}^{N-r} \mathcal{M}\Big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{A}_i^{(r-1)}\tilde{A}_{j-1}^{(r)}}{\tilde{A}_{i-1}^{(r-1)}\tilde{A}_j^{(r)}}\Big)\mathcal{M}\Big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{A}_i^{(r)}\tilde{A}_j^{(r-1)}}{\tilde{A}_{i-1}^{(r)}\tilde{A}_{j+1}^{(r-1)}}\Big)} \\ & \times \left[\prod_{i<j=1}^{N} \mathcal{M}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{A}_0^{(i)}\tilde{A}_0^{(j-1)}}{\tilde{A}_0^{(i-1)}\tilde{A}_0^{(j)}}\right)\mathcal{M}\left(\frac{\tilde{A}_i^{(N-i)}\tilde{A}_{j-1}^{(N-j+1)}}{\tilde{A}_{i-1}^{(N-i+1)}\tilde{A}_j^{(N-j)}}\right)\right]^{-1}. \end{aligned} \tag{73}$$

We now want to compute the norm squared of the above expression and write it in a way that would make the 4D limit more accessible. First, from the definition (158) of the q-deformed Υ function, we see that

$$|\mathcal{M}(e^{-\beta x};\mathfrak{t},\mathfrak{q})|^2 = |\mathcal{M}(q^{-\frac{\epsilon_+}{2}};\mathfrak{t},\mathfrak{q})|^2(1-q)^{\frac{1}{\epsilon_1\epsilon_2}\left(x+\frac{\epsilon_+}{2}\right)^2}\Upsilon_q(-x|\epsilon_1,\epsilon_2). \tag{74}$$

Here and elsewhere, we shall use the notation

$$|f(u_1,\ldots,u_r;\mathfrak{t},\mathfrak{q})|^2 := f(u_1,\ldots,u_r;\mathfrak{t},\mathfrak{q})f(u_1^{-1},\ldots,u_r^{-1};\mathfrak{t}^{-1},\mathfrak{q}^{-1}). \tag{75}$$

For the remainder of the section we shall write $\Upsilon_q(x)$ instead of $\Upsilon_q(x|\epsilon_1,\epsilon_2)$. Since it will appear often, it is convenient to define

$$\Lambda := |\mathcal{M}(q^{-\frac{\epsilon_+}{2}};\mathfrak{t},\mathfrak{q})|^2. \tag{76}$$

Furthermore, we need to carefully define the norm squared of the refined McMahon function in order to avoid a trivial zero. We follow [28] and define

$$|M(\mathfrak{t},\mathfrak{q})|^2 := \lim_{u\to 1}\frac{|\mathcal{M}(u;\mathfrak{t},\mathfrak{q})|^2}{1-u^{-1}} = |\mathcal{M}(\mathfrak{q}^{-1};\mathfrak{t},\mathfrak{q})|^2 = (1-q)^{\frac{(\epsilon_1-\epsilon_2)^2}{4\epsilon_1\epsilon_2}}\Lambda\Upsilon_q(\epsilon_1). \tag{77}$$

The advantage of using the functions Υ_q is the fact that they have a well defined 4D limit $\beta\to 0$ or $q\to 1$, while the $\mathcal{M}$ do not. We can apply this to the norm squared of (73) with the result

$$\begin{aligned} \left|\frac{\mathcal{Z}_N^{\text{pert}}}{\mathcal{Z}_N^{\text{dec}}}\right|^2 = \Lambda^{-2N(N-1)}(1-q)^{\frac{\chi_N'}{\epsilon_1\epsilon_2}}\prod_{r=1}^{N-1} & \left[\frac{\prod_{i\leq j=1}^{N-r-1}\Upsilon_q\left(a_{i-1}^{(r)}+a_{j+1}^{(r)}-a_i^{(r)}-a_j^{(r)}\right)}{\prod_{i\leq j=1}^{N-r}\Upsilon_q\left(\frac{\epsilon_+}{2}+a_{i-1}^{(r-1)}+a_j^{(r)}-a_i^{(r-1)}-a_{j-1}^{(r)}\right)}\right. \\ \times & \left.\frac{\prod_{i\leq j=1}^{N-r-1}\Upsilon_q\left(a_i^{(r)}+a_j^{(r)}-a_{i-1}^{(r)}-a_{j+1}^{(r)}\right)}{\prod_{i\leq j=1}^{N-r}\Upsilon_q\left(\frac{\epsilon_+}{2}+a_i^{(r)}+a_j^{(r-1)}-a_{i-1}^{(r)}-a_{j+1}^{(r-1)}\right)}\right]\left[\prod_{i<j=1}^{N}\Upsilon_q\left(n_i-n_j\right)\Upsilon_q\left(l_j-l_i\right)\right]^{-1}, \end{aligned} \tag{78}$$

with the exponent

$$\begin{aligned}\chi'_N = \sum_{r=1}^{N-1} \Big[\sum_{i\le j=1}^{N-r} &\left(a_i^{(r-1)} + a_j^{(r-1)} - a_{i-1}^{(r-1)} - a_{j+1}^{(r-1)} + \frac{\epsilon_+}{2}\right)^2 \\ &- \left(a_i^{(r-1)} + a_{j-1}^{(r)} - a_{i-1}^{(r-1)} - a_j^{(r)}\right)^2 - \left(a_i^{(r)} + a_j^{(r-1)} - a_{i-1}^{(r)} - a_{j+1}^{(r-1)}\right)^2 \\ &+ \sum_{i\le j=1}^{N-r-1} \left(a_i^{(r)} + a_j^{(r)} - a_{i-1}^{(r)} - a_{j+1}^{(r)} - \frac{\epsilon_+}{2}\right)^2 \Big] \\ &- \sum_{i<j=1}^{N} \left[\left(m_i - m_j + \frac{\epsilon_+}{2}\right)^2 + \left(n_j - n_i + \frac{\epsilon_+}{2}\right)^2 + \left(l_i - l_j + \frac{\epsilon_+}{2}\right)^2\right],\end{aligned} \tag{79}$$

that miraculously depends only on the boundary parameters

$$\begin{aligned}\chi'_N = &-(N-1)\sum_{i=1}^{N} m_i^2 - \sum_{i<j=1}^{N}\left[\left(n_j - n_i + \frac{\epsilon_+}{2}\right)^2 + \left(l_i - l_j + \frac{\epsilon_+}{2}\right)^2\right] \\ &- \frac{1}{N}\sum_{i<j=1}^{N}\left[(n_j-n_i)^2 + (l_i-l_j)^2\right] - 2\sum_{i=1}^{N} n_i l_{N+1-i} + \frac{N(N-1)(N-2)}{12}\epsilon_+^2.\end{aligned} \tag{80}$$

Now we have all the ingredients in order to compute the partition function on $S^4 \times S^1$. First, we should remember that we need [1, 5, 28] to add a copy $|M(\mathfrak{t},\mathfrak{q})|^2$ of the norm squared of the refined McMahon function for each one of the $\frac{(N-1)(N-2)}{2}$ faces of the diagram and integrate over all the Coulomb moduli. Then, the partition function on $S^4 \times S^1$ for the T_N superconformal theory reads

$$\mathcal{Z}_N^{S^4\times S^1} := \int_{-\frac{i\pi}{\beta}}^{\frac{i\pi}{\beta}} \prod_{k=1}^{N-2} \prod_{l=1}^{N-1-k} \frac{\beta da_k^{(l)}}{2\pi i} \, |M(\mathfrak{t},\mathfrak{q})|^{(N-1)(N-2)} \left|\frac{\mathcal{Z}_N^{\text{pert}}}{\mathcal{Z}_N^{\text{dec}}}\right|^2 \left|\mathcal{Z}_N^{\text{inst}}\right|^2, \tag{81}$$

where we need to plug in (78) for the perturbative part $\left|\mathcal{Z}_N^{\text{pert}}/\mathcal{Z}_N^{\text{dec}}\right|^2$, while we use (69) for the instanton part. The integrals over the $a_k^{(l)}$ originate as contour integrals $\oint \frac{d\tilde{A}_k^{(l)}}{2\pi i \tilde{A}_k^{(l)}}$ after the substitution $\tilde{A}_k^{(l)} = e^{-\beta a_k^{(l)}}$. Observe that there are $\frac{(N-1)(N-2)}{2}$ integrals to be done which is equal to the number of faces of the web diagram and that in the simplest T_2 case no integrals have to be done. Furthermore, in order to compute the final expression for the partition function, we still need to perform $\frac{N(N-1)}{2}$ sums over the partitions $\nu_i^{(j)}$. This can unfortunately for now only be done exactly in the $N = 2$ case. Finally, the derivation of (67) depended strongly on a choice of a preferred direction for the refined topological vertex. It is conjectured [35, 52], under a principle called *slicing invariance*, that the final answer will not depend on the choice of the preferred direction. We can make three different choices of preferred direction for the T_N web diagram and in section 6, we shall do it for T_3. In the Toda field theory interpretation, each choice puts one of the primary fields on a special footing.

4.3 The 4D limit

Naively, taking the 4D limit requires simply taking $\beta \to 0$. However, as we show in the previous subsection for the perturbative part for $N > 2$ and for the full partition function for $N = 2$ (see also section 5), in the limit most quantities diverge, but thankfully only with an overall factor of $(1 - q)$ raised to the appropriate power. We conjecture that this will also be the case for the full partition function for every N, even after the instantons are accounted for. Our conjecture is supported by symmetry arguments, a careful study of the $N = 2$ case and from the lessons we extracted form section 3.4, in particular equation (56). What is more, it is supported by [35], where it was conjectured that the refined topological string partition function read off using the refined topological vertex from any web diagram should always at the end be possible to be written as a product of $\mathcal{M}$'s [10]. Thus, we define the partition function of the T_N theory on S^4 to be

$$\mathcal{Z}_N^{S^4} = \text{const} \times \lim_{\beta \to 0} \left(\beta^{-\frac{\chi_N}{\epsilon_1 \epsilon_2}} \, \mathcal{Z}_N^{S^4 \times S^1} \right) , \tag{82}$$

where by definition the power χ_N is taken so that the limit is convergent. The constant factor cannot depend on the parameters of the theory, *i.e.* the masses, though it can, and in the cases checked does, depend on the Omega background parameters.

In what follows we want to use symmetries and the known limits for the partition function to argue that the exponent χ_N of β is given in terms of the quadratic Casimir of $\mathrm{SU}(N)^3$

$$\chi_N = - \sum_{i<j=1}^{N} \left[(m_i - m_j)^2 + (n_j - n_i)^2 + (l_i - l_j)^2 \right] = -N \sum_{i=1}^{3} (\boldsymbol{\alpha}_i - \mathcal{Q}, \boldsymbol{\alpha}_i - \mathcal{Q}) \ . \tag{83}$$

First, for the $N = 2$ case, we can explicitly calculate the exponent and we find

$$\chi_2 = - \sum_{i<j=1}^{2} \left[(m_i - m_j)^2 + (n_j - n_i)^2 + (l_i - l_j)^2 \right] = -2 \sum_{i=1}^{3} (\boldsymbol{\alpha}_i - \mathcal{Q}, \boldsymbol{\alpha}_i - \mathcal{Q}) \tag{84}$$

where we have made use of formulas (7) and (130). Moreover, for the perturbative part (80) we can also explicitly calculate χ'_N and we find that it is quadratic in the masses. What is more, we know the answer for the case with one degenerate insertion (56), it is expressed in terms of Υ_q-functions, which when combined with (74) tells us that the power χ_N is a quadratic function in the masses. Furthermore, both $\mathcal{Z}_N^{S^4}$ and $\mathcal{Z}_N^{S^4 \times S^1}$ are invariant under (affine) Weyl reflections of $\mathrm{SU}(N)^3$ and since the constant term is independent of the

[10]One might worry that the product would be infinite, but our symmetry argument that χ_N should be given by the quadratic Casimir suggests that cancellations will always happen so that the degree of divergence χ_N is finite!

parameters, the power χ_N has to be Weyl invariant as well. Therefore, we have to have[11]

$$\chi_N = c_1 \sum_{i=1}^{3} (\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i - 2\mathcal{Q}) + c_2, \tag{85}$$

where the c_i are constants that symmetry cannot fix. The second constant $c_2 = -3N\,(\mathcal{Q},\mathcal{Q})$ in (84) can in any case be reabsorbed in the constant prefactor of (82) as it does not depend on the masses. For c_1 we compare with (56). When the l-parameters are degenerate[12]

$$C_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \varkappa\omega_{N-1}) = \text{const} \times \left| \frac{\mathcal{M}(\frac{\tilde{L}_N}{\tilde{L}_{N-1}}) \prod_{i<j=1}^{N} \mathcal{M}\left(\frac{\tilde{M}_i}{\tilde{M}_j}\right) \mathcal{M}\left(\frac{\mathfrak{t}}{\mathfrak{q}} \frac{\tilde{N}_i}{\tilde{N}_j}\right)}{\prod_{i,j=1}^{N} \mathcal{M}\left(\tilde{M}_i^{-1} \tilde{N}_j \tilde{L}_1^{-1} \left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{N-1}{2}}\right)} \right|^2 . \tag{86}$$

Since the l-part is degenerate and some zeroes from the non-full spin content have canceled some poles from the index in order to obtain (86), we don't expect to get the correct l-dependence in χ_N. Thus, if we ignore l, subtract the remaining non-full spin content in the numerator for the m and n parts and compute the power of the β divergence using (74), we obtain

$$-\sum_{i,j=1}^{N} \left(m_i - n_j + \frac{\epsilon_+}{2}\right)^2 = -\sum_{i<j=1}^{N} \left[(m_i - m_j)^2 + (n_i - n_j)^2\right] + \text{const} \tag{87}$$

which sets $c_1 = -N$ and supports our claim (83).

We would like to conclude this section by stressing that even though in the present paper we do not show how to do the sums, we know that their outcome will be a product of functions $\mathcal{M}$, exactly as in (86), but of course for the general non-degenerate case with more $\mathcal{M}$s. That was already conjectured in [35] for any topological partition function coming from a toric diagram, see [53] for a more recent discussion. This statement is just the refinement of the Gopakumar-Vafa formula [54, 55]. This is fully in agreement with our claim that the power χ_N has to be at most quadratic in the masses.

5 Liouville from topological strings

In this section we show in detail how one can start from the partition function of T_2 that we computed in section 4 and derive the known Liouville 3-point function. This exercise allows us to draw experience and learn some tricks that we shall be able to use for $N > 2$, fix our conversions and test the dictionary we presented in section 2.

[11] Usually the eigenvalue of the quadratic Casimir is written $(\boldsymbol{\alpha}, \boldsymbol{\alpha} + 2\rho)$, where ρ is the Weyl vector. After a rescaling of the weight $\boldsymbol{\alpha}$, this is the same as (85).

[12] In that case $l_i = \frac{N-i}{N}\varkappa - \frac{N+1-2i}{2}Q$ for $i < N$ an $l_N = -\frac{N-1}{N}\varkappa + \frac{N-1}{2}Q$, implying $\varkappa = l_{N-1} - l_N + Q$.

For $N = 2$ there are no Coulomb moduli. The perturbative part (68) is

$$\mathcal{Z}_2^{\text{pert}} = \frac{\mathcal{M}\left(\left(\tilde{A}_1^{(0)}\right)^2\right)}{\mathcal{M}\big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{A}_0^{(1)}\tilde{A}_1^{(0)}}{\tilde{A}_1^{(1)}}\big)\mathcal{M}\big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{A}_1^{(0)}\tilde{A}_1^{(1)}}{\tilde{A}_0^{(1)}}\big)} \tag{88}$$

while the instanton one (69) reads

$$\mathcal{Z}_2^{\text{inst}} = \sum_\nu \left(\frac{\tilde{L}_1\tilde{N}_1}{\tilde{L}_2\tilde{N}_2}\right)^{\frac{|\nu|}{2}} \frac{\mathsf{N}_{\nu\emptyset}^\beta(a_1^{(0)} + a_1^{(1)} - a_0^{(1)} - \frac{\epsilon_+}{2})\mathsf{N}_{\emptyset\nu}^\beta(a_0^{(1)} + a_1^{(0)} - a_1^{(1)} - \frac{\epsilon_+}{2})}{\mathsf{N}_{\nu\nu}^\beta(0)} \tag{89}$$

so that (67) becomes after replacing the $\tilde{A}$'s with the mass parameters via (116)

$$\begin{aligned}\mathcal{Z}_2^{\text{top}} =& \frac{\mathcal{M}(\tilde{M}_1^2)}{\mathcal{M}\big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{L}_1\tilde{M}_1}{\tilde{N}_1}\big)\mathcal{M}\big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{N}_1\tilde{M}_1}{\tilde{L}_1}\big)} \\ &\times \sum_\nu \left(\tilde{L}_1\tilde{N}_1\right)^{|\nu|} \frac{\mathsf{N}_{\nu\emptyset}^\beta(l_1 + m_1 - n_1 - \frac{\epsilon_+}{2})\mathsf{N}_{\emptyset\nu}^\beta(n_1 + m_1 - l_1 - \frac{\epsilon_+}{2})}{\mathsf{N}_{\nu\nu}^\beta(0)}.\end{aligned} \tag{90}$$

We can use the identity of equation (172) to perform the sum over partitions and get

$$\mathcal{Z}_2^{\text{top}} = \frac{\mathcal{M}(\tilde{M}_1^2)\mathcal{M}(\tilde{N}_1^2\frac{\mathfrak{t}}{\mathfrak{q}})\mathcal{M}(\tilde{L}_1^2)}{\mathcal{M}(\frac{\tilde{M}_1\tilde{L}_1}{\tilde{N}_1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}})\mathcal{M}(\frac{\tilde{N}_1\tilde{L}_1}{\tilde{M}_1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}})\mathcal{M}(\frac{\tilde{M}_1\tilde{N}_1}{\tilde{L}_1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}})\mathcal{M}(\tilde{M}_1\tilde{N}_1\tilde{L}_1\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}})}. \tag{91}$$

Setting $b = \epsilon_1 = \epsilon_2^{-1}$ and using the definition (158) of the Υ_q functions as well as the parametrization (7), we get from (91) the following expression for $|\mathcal{Z}_2^{\text{top}}|^2$

$$|\mathcal{Z}_2^{\text{top}}|^2 = \Lambda^{-1}(1-q)^{2Q(\sum_{i=1}^3 \alpha_i - Q) - \frac{Q^2}{4}} \frac{\prod_{i=1}^3 \Upsilon_q(2\alpha_i)}{\Upsilon_q(\sum_k \alpha_k - Q)\prod_{i=1}^3 \Upsilon_q(\sum_k \alpha_k - 2\alpha_i)} \tag{92}$$

where we have used the symmetry $\Upsilon_q(x) = \Upsilon_q(Q-x)$. Up to an infinite constant prefactor, the same formula was obtained in equation (3.73) of [23] as well as equation (5.10) of [36]. We can use the expression for the derivative (164) found in the appendix as well as the McMahon function (77) to combine some factors into $\Upsilon_q'(0)$ leading to

$$\left|\mathcal{Z}_2^{\text{top}}\right|^2 = \frac{(1-q)^{2Q(\sum_{i=1}^3 \alpha_i - Q)}}{\beta |M(\mathfrak{t},\mathfrak{q})|^2} \frac{\Upsilon_q'(0)\prod_{i=1}^3 \Upsilon_q(2\alpha_i)}{\Upsilon_q(\sum_k \alpha_k - Q)\prod_{i=1}^3 \Upsilon_q(\sum_k \alpha_k - 2\alpha_i)}. \tag{93}$$

This result is almost the q-deformed structure constants. In fact, we see by looking at (57) that

$$C_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = \left[\beta|M(\mathfrak{t},\mathfrak{q})|^2\left(\left(1-q^b\right)^{2b^{-1}}\left(1-q^{b^{-1}}\right)^{2b}\right)^{Q-\sum_i \alpha_i}\right]\left|\mathcal{Z}_2^{\text{top}}\right|^2, \tag{94}$$

which is the q-deformed version of (14) for T_2.

Already in [1] we computed the superconformal index for the T_2 theory. It is obtained from $|\mathcal{Z}_2^{\text{top}}|^2$ by dividing with the non-full spin content $|\mathcal{Z}_2^{\text{dec}}|^2$ that corresponds to degrees of freedom that decouple from the 5D theory.

$$\begin{aligned}
\mathcal{Z}_2^{S^4\times S^1} =& \left|\left[\mathcal{M}\Big(\frac{\tilde{M}_1\tilde{N}_1}{\tilde{L}_1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\Big)\mathcal{M}\Big(\frac{\tilde{M}_1\tilde{L}_1}{\tilde{N}_1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\Big)\mathcal{M}\Big(\frac{\tilde{N}_1\tilde{L}_1}{\tilde{M}_1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\Big)\mathcal{M}\Big(\tilde{M}_1\tilde{N}_1\tilde{L}_1\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\Big)\right]^{-1}\right|^2 \\
=& \Bigg[\Lambda^4(1-q)^{4(m_1^2+n_1^2+l_1^2)}\Upsilon_q\left(\frac{Q}{2}+m_1+n_1+l_1\right)\Upsilon_q\left(\frac{Q}{2}+m_1+n_1-l_1\right) \\
&\times\Upsilon_q\left(\frac{Q}{2}+m_1-n_1+l_1\right)\Upsilon_q\left(\frac{Q}{2}-m_1+n_1+l_1\right)\Bigg]^{-1} \\
=& \left[\Lambda^4(1-q)^{\sum_{i=1}^4 u_i^2}\prod_{i=1}^4\Upsilon_q\left(\frac{Q}{2}+u_i\right)\right]^{-1}.
\end{aligned} \tag{95}$$

In particular, we find by comparing with (58)

$$\mathfrak{C}_q(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) = \Lambda^4\Upsilon_q'(0)(1-q)^{-\chi_2}\mathcal{Z}_2^{S^4\times S^1}, \tag{96}$$

as promised in (11). The index (95) of the T_2 theory can be expanded in powers of $x=\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}$ with coefficients that can be interpreted either as sums of characters of SU(2)×SU(2)×SU(2) or of SU(4). Specifically, we find:

$$\begin{aligned}
\mathcal{Z}_2^{S^4\times S^1} =& 1+\chi_{(\mathbf{2},\mathbf{2},\mathbf{2})}x+\Big[\chi_{(\mathbf{3},\mathbf{1},\mathbf{1})}+\chi_{(\mathbf{1},\mathbf{3},\mathbf{1})}+\chi_{(\mathbf{1},\mathbf{1},\mathbf{3})}+\chi_{(\mathbf{3},\mathbf{3},\mathbf{3})}+\chi_{\mathbf{2}}(y)\chi_{(\mathbf{2},\mathbf{2},\mathbf{2})}\Big]x^2+\mathcal{O}(x^3) \\
=& 1+\Big[\chi_{\mathbf{4}}^{\mathrm{SU}(4)}+\chi_{\bar{\mathbf{4}}}^{\mathrm{SU}(4)}\Big]x+\Big[1+\chi_{\mathbf{10}}^{\mathrm{SU}(4)}+\chi_{\mathbf{15}}^{\mathrm{SU}(4)}+\chi_{\overline{\mathbf{10}}}^{\mathrm{SU}(4)} \\
&+\chi_{\mathbf{2}}(y)\big(\chi_{\mathbf{4}}^{\mathrm{SU}(4)}+\chi_{\bar{\mathbf{4}}}^{\mathrm{SU}(4)}\big)\Big]x^2+\mathcal{O}(x^3),
\end{aligned} \tag{97}$$

where we have the $\mathrm{SU}(2)^3$ characters $\chi_{(\mathbf{m_1},\mathbf{m_2},\mathbf{m_3})}\equiv\chi_{\mathbf{m_1}}(\tilde{M}_1)\chi_{\mathbf{m_2}}(\tilde{N}_1)\chi_{\mathbf{m_3}}(\tilde{L}_1)$ and the SU(4) characters depend on the four variables U_i with $\prod_{i=1}^4 U_i=1$ that are given by

$$\tilde{M}_1=\sqrt{U_1U_3},\qquad \tilde{N}_1=\sqrt{U_2U_3},\qquad \tilde{L}_1=\sqrt{U_1U_2}. \tag{98}$$

We thus also see by comparing with (34) that the index $\mathcal{Z}_2^{S^4\times S^1}$ has the same symmetry as the Weyl-invariant structure constants of the Liouville CFT (34), as was expected.

6 $\mathbf{W}_3$ from topological strings

In this section, we want to review our result for the 3-point structure constants of primaries for the case $N=3$ in its full glory and to investigate its symmetries and structure.

6.1 Slicing invariance

We begin with slicing invariance. In figure 5, we depict the three possible ways of choosing the preferred direction. Each one is labeled by the mass parameters that become prominent for that choice. The one we have used in section 4 for the determination of the strip partition functions is $\tilde{M}$. For the choice $\tilde{M}$ of the preferred direction, we can compute the sum over the

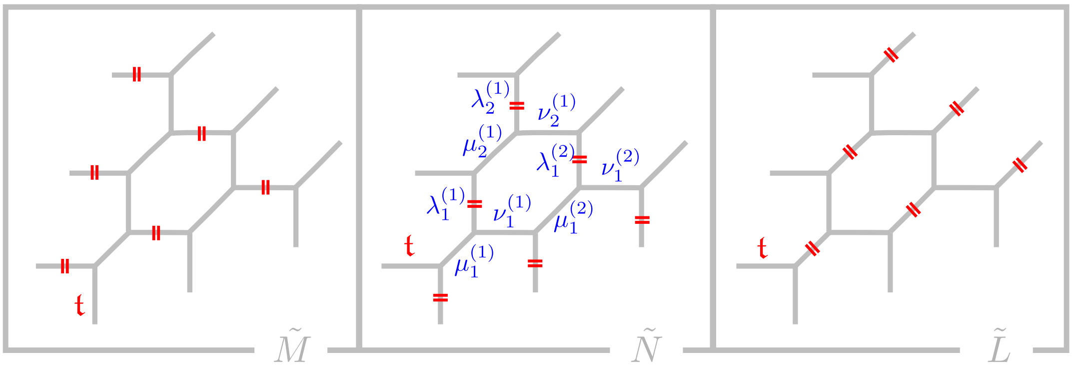

Figure 5: *This figure shows the three different possible preferred directions for the T_3 junction. Each one is labeled by the Kähler moduli of the non-full spin content that is factorized. We also indicate the names of the partitions entering the instanton sums and to avoid clutter, we only do it for the middle one.*

partitions $\lambda_i^{(j)}$ and $\mu_i^{(j)}$, but not over $\nu_i^{(j)}$. Similarly, for the choice $\tilde{N}$, we cannot perform the sum over the $\lambda_i^{(j)}$ and for the choice $\tilde{L}$ we cannot do it for the $\mu_i^{(j)}$. From equations (68) and (69), we can read off the partition function for the $\tilde{M}$ choice. After some rearrangements, we find the cumbersome expression

$$\begin{aligned}
\mathcal{Z}_3^{\text{top}} \quad = \quad & \frac{\mathcal{M}\left(\mathbf{A}^2\tilde{N}_1^{-1}\tilde{L}_3\right)}{\mathcal{M}\left(\sqrt{\tfrac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}^{-1}\tilde{M}_1\tilde{N}_1\right)\mathcal{M}\left(\sqrt{\tfrac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{M}_2^{-1}\tilde{N}_1^{-1}\right)\mathcal{M}\left(\sqrt{\tfrac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{M}_3^{-1}\tilde{N}_1^{-1}\right)\mathcal{M}\left(\sqrt{\tfrac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{N}_3\tilde{L}_2^{-1}\right)} \\
& \times\frac{\mathcal{M}\left(\tfrac{\mathfrak{t}}{\mathfrak{q}}\mathbf{A}^2\tilde{N}_1^{-1}\tilde{L}_3\right)}{\mathcal{M}\left(\sqrt{\tfrac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{M}_1\tilde{L}_3\right)\mathcal{M}\left(\sqrt{\tfrac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{M}_2\tilde{L}_3\right)\mathcal{M}\left(\sqrt{\tfrac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}^{-1}\tilde{M}_3^{-1}\tilde{L}_3^{-1}\right)\mathcal{M}\left(\sqrt{\tfrac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{N}_2\tilde{L}_1^{-1}\right)} \\
& \times\mathcal{M}\left(\tilde{M}_1\tilde{M}_2^{-1}\right)\mathcal{M}\left(\tilde{M}_1\tilde{M}_3^{-1}\right)\mathcal{M}\left(\tilde{M}_2\tilde{M}_3^{-1}\right)\sum_{\boldsymbol{\nu}}\left(\frac{\tilde{N}_2\tilde{L}_1}{\tilde{N}_3\tilde{L}_2}\right)^{\frac{|\nu_1^{(2)}|}{2}}\left(\frac{\tilde{N}_1\tilde{L}_2}{\tilde{N}_2\tilde{L}_3}\right)^{\frac{|\nu_1^{(1)}|+|\nu_2^{(1)}|}{2}} \\
& \times\frac{\mathsf{N}^{\beta}_{\nu_1^{(2)}\nu_2^{(1)}}\left(\mathbf{a}+n_3-l_2-\epsilon_+/2\right)\mathsf{N}^{\beta}_{\nu_1^{(1)}\nu_1^{(2)}}\left(\mathbf{a}+n_2-l_1-\epsilon_+/2\right)}{\mathsf{N}^{\beta}_{\nu_1^{(2)}\nu_1^{(2)}}(0)} \\
& \times\frac{\prod_{k=1}^{3}\mathsf{N}^{\beta}_{\nu_1^{(1)}\emptyset}\left(\mathbf{a}-n_1-m_k-\epsilon_+/2\right)\mathsf{N}^{\beta}_{\emptyset\nu_2^{(1)}}\left(\mathbf{a}+l_3+m_k-\epsilon_+/2\right)}{\mathsf{N}^{\beta}_{\nu_1^{(1)}\nu_1^{(1)}}(0)\,\mathsf{N}^{\beta}_{\nu_2^{(1)}\nu_2^{(1)}}(0)\,\mathsf{N}^{\beta}_{\nu_1^{(1)}\nu_2^{(1)}}\left(2\mathbf{a}-n_1+l_3\right)\mathsf{N}^{\beta}_{\nu_2^{(1)}\nu_1^{(1)}}\left(-2\mathbf{a}+n_1-l_3\right)},
\end{aligned} \tag{99}$$

where $\mathbf{A} = e^{-\beta \mathbf{a}} = \tilde{A}_1^{(1)}$ is the relabeled Coulomb modulus.

A direct computation using the topological vertex shows that the topological amplitude $\mathcal{Z}_3^{\text{top}}$ for the choice $\tilde{N}$ of preferred direction can be obtained from (99) after the substitution

$$m_k \to n_{4-k}, \qquad n_k \to m_k, \qquad l_k \to -l_{4-k}, \tag{100}$$

after exchanging $\mathfrak{t} \leftrightarrow \mathfrak{q}$. Furthermore, the amplitude $\mathcal{Z}_3^{\text{top}}$ for the last remaining possible choice of preferred direction is obtained by setting in (99)

$$m_k \to -l_{4-k}, \qquad n_k \to -m_{4-k}, \qquad l_k \to n_k, \tag{101}$$

without exchanging $\mathfrak{t} \leftrightarrow \mathfrak{q}$. Since it is thought and in some cases shown [35,52] that the choice of preferred direction is irrelevant, the transformations (100) and (101) must be symmetries of the topological amplitude, *i.e.* we conjecture that (99) is invariant under them:

$$\mathcal{Z}_3^{\text{top}}(m_k, n_k, l_k) = \mathcal{Z}_3^{\text{top}}(n_{4-k}, m_k, -l_{4-k}) = \mathcal{Z}_3^{\text{top}}(-l_{4-k}, -m_{4-k}, n_k). \tag{102}$$

6.2 Weyl invariance

The slicing invariance of the partition function can help us prove the Weyl covariance of the structure constants. Using (99) and the properties of the Υ_q functions, the index

$$\mathcal{Z}_3^{S^4\times S^1} = \oint \frac{d\mathbf{A}}{2\pi i \mathbf{A}} \left|M(\mathfrak{t},\mathfrak{q})\right|^2 \left|\frac{\mathcal{Z}_3^{\text{top}}}{\mathcal{Z}_3^{\text{dec}}}\right|^2 \tag{103}$$

reads

$$\begin{aligned}
\mathcal{Z}_3^{S^4\times S^1} = \oint \frac{d\mathbf{A}}{2\pi i \mathbf{A}} & \frac{(1-q)^{\frac{\chi_3' + \epsilon_-^2/4}{\epsilon_1\epsilon_2}}}{\Lambda^8} \frac{\Upsilon_q(\epsilon_1)\left[\prod_{i<j=1}^3 \Upsilon_q(n_i-n_j)\Upsilon_q(l_j-l_i)\right]^{-1}}{\prod_{k=1}^3 \Upsilon_q(\mathbf{a}-m_k-n_1+\epsilon_+/2)\Upsilon_q(\mathbf{a}+m_k+l_3+\epsilon_+/2)} \\
&\times \frac{\Upsilon_q(2\mathbf{a}-n_1+l_3)\Upsilon_q(-2\mathbf{a}+n_1-l_3)}{\Upsilon_q(\mathbf{a}+n_3-l_2+\epsilon_+/2)\Upsilon_q(\mathbf{a}+n_2-l_1+\epsilon_+/2)} \Bigg| \sum_{\boldsymbol{\nu}} \left(\frac{\tilde{N}_2\tilde{L}_1}{\tilde{N}_3\tilde{L}_2}\right)^{\frac{|\nu_1^{(2)}|}{2}} \left(\frac{\tilde{N}_1\tilde{L}_2}{\tilde{N}_2\tilde{L}_3}\right)^{\frac{|\nu_1^{(1)}|+|\nu_2^{(1)}|}{2}} \\
&\times \frac{\mathsf{N}^\beta_{\nu_1^{(2)}\nu_2^{(1)}}(\mathbf{a}+n_3-l_2-\epsilon_+/2)\,\mathsf{N}^\beta_{\nu_1^{(1)}\nu_1^{(2)}}(\mathbf{a}+n_2-l_1-\epsilon_+/2)}{\mathsf{N}^\beta_{\nu_1^{(2)}\nu_1^{(2)}}(0)} \\
&\times \frac{\prod_{k=1}^3 \mathsf{N}^\beta_{\nu_1^{(1)}\emptyset}(\mathbf{a}-n_1-m_k-\epsilon_+/2)\,\mathsf{N}^\beta_{\emptyset\nu_2^{(1)}}(\mathbf{a}+l_3+m_k-\epsilon_+/2)}{\mathsf{N}^\beta_{\nu_1^{(1)}\nu_1^{(1)}}(0)\,\mathsf{N}^\beta_{\nu_2^{(1)}\nu_2^{(1)}}(0)\,\mathsf{N}^\beta_{\nu_1^{(1)}\nu_2^{(1)}}(2\mathbf{a}-n_1+l_3)\,\mathsf{N}^\beta_{\nu_2^{(1)}\nu_1^{(1)}}(-2\mathbf{a}+n_1-l_3)}\Bigg|^2,
\end{aligned} \tag{104}$$

where we the exponent of $(1-q)$ is

$$\begin{aligned}
\chi_3' = &-2\sum_{i=1}^3 m_i^2 - \sum_{i<j=1}^3 \left[\left(n_j-n_i+\frac{\epsilon_+}{2}\right)^2 + \left(l_i-l_j+\frac{\epsilon_+}{2}\right)^2\right] \\
&- \frac{1}{N}\sum_{i<j=1}^3 \left[(n_j-n_i)^2+(l_i-l_j)^2\right] - 2\sum_{i=1}^3 n_i l_{4-i} + \frac{\epsilon_+^2}{2}.
\end{aligned} \tag{105}$$

agreeing with (80). The additional factor of $\epsilon_-^2/4$ in (104) comes from the factor of $|M(\mathfrak{t},\mathfrak{q})|^2$. In deriving expression (104), we have used (74) and (77).

Now the invariance of $\mathcal{Z}_3^{S^4\times S^1}$ under the Weyl reflections of $\mathrm{SU}(3)^3$ is almost trivial to check. Affine Weyl transformations on the $\boldsymbol{\alpha}_i$ act as usual Weyl transformations on the m_i, n_i and l_i, *i.e.* they simply permute them. For the choice $\tilde{M}$ of preferred direction shown in (104), we can easily see that the expression is invariant. However, while the invariance of (104) under the Weyl group of the first SU(3) is easy, the Weyl reflections of the remaining Weyl groups act non-trivially. At this point we need to use the fact that slicing invariance is a symmetry of the problem and by applying first (100) or (101) on (104) before acting with the Weyl reflection we can prove the complete invariance under Weyl reflections.

6.3 The index and E_6 symmetry

While the invariance under Weyl reflections of the SU(3)'s are easy to see, the symmetry under E_6 transformations is not. For this, we expand in $x = e^{-\frac{\beta\epsilon_+}{2}}$ before performing the integration. As shown in [1, 5] this leads to the index computed in [37], that reads

$$\begin{aligned}\mathcal{Z}_3^{S^4\times S^1} &= 1 + \chi^{E_6}_{\mathbf{78}}x^2 + \chi_{\mathbf{2}}(y)(1+\chi^{E_6}_{\mathbf{78}})x^3 + \left[1 + \chi^{E_6}_{\mathbf{2430}} + \chi_{\mathbf{3}}(y)(1+\chi^{E_6}_{\mathbf{78}})\right]x^4\\ &+ \left[\chi_{\mathbf{2}}(y)\left(1+\chi^{E_6}_{\mathbf{78}} + \chi^{E_6}_{\mathbf{2430}} + \chi^{E_6}_{\mathbf{2928}}\right) + \chi_{\mathbf{4}}(y)(1+\chi^{E_6}_{\mathbf{78}})\right]x^5 + \left[2\chi^{E_6}_{\mathbf{78}} + \chi^{E_6}_{\mathbf{2925}} + \chi^{E_6}_{\mathbf{43758}}\right.\\ &\left. +\chi_{\mathbf{3}}(y)\left(2 + 2\chi^{E_6}_{\mathbf{78}} + \chi^{E_6}_{\mathbf{650}} + 2\chi^{E_6}_{\mathbf{2430}} + \chi^{E_6}_{\mathbf{2925}}\right) + \chi_{\mathbf{5}}(y)\left(1+\chi^{E_6}_{\mathbf{78}}\right)\right]x^6 + \mathcal{O}(x^7). \end{aligned} \tag{106}$$

The fugacities $\tilde{M}$, $\tilde{N}$ and $\tilde{L}$ enter the E_6 characters as follows. We have an embedding $\mathrm{SU}(3)^3 \subset E_6$ and with the fundamental $\mathbf{3}$ representation of the first SU(3) having the character $\chi_{\mathbf{3}} = \tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3$ with similar expressions for the other SU(3) factors. The character of the 78-dimensional adjoint representation of E_6 then decomposes as

$$\chi^{E_6}_{\mathbf{78}} = \chi_{(\mathbf{8},\mathbf{1},\mathbf{1})} + \chi_{(\mathbf{1},\mathbf{8},\mathbf{1})} + \chi_{(\mathbf{1},\mathbf{1},\mathbf{8})} + \chi_{(\mathbf{3},\bar{\mathbf{3}},\mathbf{3})} + \chi_{(\bar{\mathbf{3}},\mathbf{3},\bar{\mathbf{3}})}, \tag{107}$$

where $\chi_{(\mathbf{j_1},\mathbf{j_2},\mathbf{j_3})} := \prod_{k=1}^3 \chi^{\mathrm{SU}(3)}_{\mathbf{j_k}}$. The other characters can be decomposed in a similar fashion, see appendix C of [1] for more details.

Since we wish to identify the index as the q-deformed Weyl invariant structure constants and since we showed in section 3.2 that the Weyl invariant structure constants have an E_6 symmetry, we have an additional piece of evidence in our favor. Furthermore, we can use the fact that (45) captures all the poles of the Weyl invariant structure constants and that the position of the poles does not change under q-deformation to write another formula for the index[13]. Specifically, we make a guess for the q-deformation of (45) and write

$$\mathcal{Z}_3^{S^4\times S^1} = \frac{\mathcal{F}_3}{\mathfrak{Z}_q(1)^3 \prod_{i,j,k=1}^{3} \mathfrak{Z}_q(\tilde{M}_i\tilde{N}_j^{-1}\tilde{L}_k) \prod_{i<j=1}^{3} \mathfrak{Z}_q(\tilde{M}_i\tilde{M}_j^{-1})\mathfrak{Z}_q(\tilde{N}_i\tilde{N}_j^{-1})\mathfrak{Z}_q(\tilde{L}_i\tilde{L}_j^{-1})}, \tag{108}$$

where

$$\mathfrak{Z}_q(u;\mathfrak{t},\mathfrak{q}) := \prod_{i,j=0}^{\infty} (1-u\mathfrak{t}^{-i-1}\mathfrak{q}^{j+1})(1-u^{-1}\mathfrak{t}^{-i-1}\mathfrak{q}^{j+1}) \tag{109}$$

is up to a constant the q-deformation of $\mathfrak{Z}$ and the compensating factor $\mathcal{F}_3$ is an unknown entire[14] function with the expansion in x given by

$$\begin{aligned}\mathcal{F}_3 = 1 + \chi_{\mathbf{2}}(y)x^3 + \left[\chi_{\mathbf{3}}(y) - \chi^{E_6}_{\mathbf{650}}\right]x^4 + \left[\chi_{\mathbf{4}}(y) - \left(\chi^{E_6}_{\mathbf{78}} + \chi^{E_6}_{\mathbf{650}}\right)\chi_{\mathbf{2}}(y)\right]x^5 \\ + \left[\chi_{\mathbf{5}}(y) - \left(\chi^{E_6}_{\mathbf{78}} + \chi^{E_6}_{\mathbf{650}}\right)\chi_{\mathbf{3}}(y) + \left(\chi^{E_6}_{\mathbf{5284}} + \chi^{E_6}_{\overline{\mathbf{5284}}} + \chi^{E_6}_{\mathbf{650}}\right)\right]x^6 + \mathcal{O}(x^7).\end{aligned} \tag{110}$$

So far, we have no closed expression for the function $\mathcal{F}_3$.

We end this section with one last remark. Our claim (11) states that

$$\mathfrak{C}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) = \text{const} \times \lim_{\beta\to 0} \beta^{-\chi_3} \mathcal{Z}_3^{S^4\times S^1}. \tag{111}$$

We see in (106) that $\mathcal{Z}_3^{S^4\times S^1}$ is invariant under an E_6 symmetry and we saw in section 3.2 that $\mathfrak{C}$ is invariant under that symmetry as well. A direct computation shows that χ_3 given by (83)

$$\chi_3 = -\sum_{i<j=1}^{3} \left[(m_i-m_j)^2 + (n_j-n_i)^2 + (l_i-l_j)^2\right] \tag{112}$$

is invariant under the E_6 Weyl tranformations (38) as well.

7 Conclusions and Outlook

In [1] we calculated the 5D partition function of the non-Lagrangian T_N theories on $S^4\times S^1$ using topological strings. In this paper we take the next very important step and argue that it is possible to take the 4D limit ($\beta\to 0$ *i.e.* $q=e^{-\beta}\to 1$), thus obtaining the partition function of the 4D non-Lagrangian T_N theories on S^4. Taking the 4D limit is *not* as simple as one might naively think and it is definitely not as easy as for theories with a Lagrangian description.

The first step in overcoming this difficulty was realizing that one can bring formula (66) into the form (67) in which the individual building blocks are only the Υ_q functions and the

[13]To be more precise, as we discussed in section 3.4 and can be seen in equation (137) of the appendix C.2, after the q-deformed versions of the functions have more poles. For for each single pole of the undeformed function, they have a whole tower of poles.

[14]That the function $\mathcal{F}_3$ is entire follows from the facts that 1) the function $\mathfrak{F}$ in (5.5) of [31] is entire and 2) the Weyl covariant part has no poles.

“Nekrasov functions” $\mathsf{N}^{\beta}_{\mu\nu}$ for which the 4D limit is well defined as individual functions (161) (174). Our formula for the partition function (81) is then written as a product of the factors[15] $\left|\mathcal{Z}^{\text{pert}}_N/\mathcal{Z}^{\text{dec}}_N\right|^2$ and $\left|\mathcal{Z}^{\text{inst}}_N\right|^2$. The first factor $\left|\mathcal{Z}^{\text{pert}}_N/\mathcal{Z}^{\text{dec}}_N\right|^2$ could be explicitly brought into a form that only includes products of the Υ_q functions times a divergent factor of $(1-q)^{\chi'_N/\epsilon_1\epsilon_2}$. Thus, taking the limit is straightforward after we divide by $(1-q)^{\chi'_N/\epsilon_1\epsilon_2}$. However, for the $\left|\mathcal{Z}^{\text{inst}}_N\right|^2$ piece we have a further obstacle to overcome. The sums that contain the “Nekrasov functions” $\mathsf{N}^{\beta}_{\mu\nu}$ diverge if one naively takes the 4D limit, in contrast with the usual sums in theories with Lagrangian description. Schematically, instead of having a coupling constant $q_{UV} = e^{2\pi i\tau_{UV}}$ as for theories with a Lagrangian description, where one can commute the limit with the sum, as for example in

$$\sum_{\mu}\left(q^{5D}_{UV}\right)^{|\mu|}\frac{\mathsf{N}^{\beta}_{\mu\nu_1}(a_1)\cdots\mathsf{N}^{\beta}_{\mu\nu_L}(a_L)}{\mathsf{N}^{\beta}_{\mu\lambda_1}(b_1)\cdots\mathsf{N}^{\beta}_{\mu\lambda_L}(b_L)}\xrightarrow{\beta\to 0}\sum_{\mu}\left(q^{4D}_{UV}\right)^{|\mu|}\frac{\mathsf{N}_{\mu\nu_1}(a_1)\cdots\mathsf{N}_{\mu\nu_L}(a_L)}{\mathsf{N}_{\mu\lambda_1}(b_1)\cdots\mathsf{N}_{\mu\lambda_L}(b_L)},\tag{113}$$

for the case of the T_N theories (that are isolated non trivial fixed points) there is no of q_{UV} but rather a combination $e^{-\beta x}$ of the mass parameters ($M = e^{-\beta m}$) and instead of (113) we have

$$\sum_{\mu}\left(e^{-\beta x}\right)^{|\mu|}\frac{\mathsf{N}^{\beta}_{\mu\nu_1}(a_1)\cdots\mathsf{N}^{\beta}_{\mu\nu_L}(a_L)}{\mathsf{N}^{\beta}_{\mu\lambda_1}(b_1)\cdots\mathsf{N}^{\beta}_{\mu\lambda_L}(b_L)}\xrightarrow{\beta\to 0}\beta^{\text{power}}\times\text{finite}\tag{114}$$

that makes the sum diverge as β^{power}. Explicitly obtaining this power would require performing the sums in (69). In this paper we do not do that, except for the $N = 2$ case where the sum is given by (172). However, by carefully studying the symmetry properties of the 3-point functions for general N, the properties of the Υ_q functions and the known $N = 2$ case we manage to obtain this power of the divergence (95). Our result is tested against the q-deformed version of the Fateev-Litvinov formula with one semi-degenerate insertion (56). Combining everything, we propose that the 4D limit of the superconformal index (82), multiplied with β raised to the appropriate power $-\chi_N$, will be finite and equal to the partition function of the T_N theory on S^4. Moreover, we explicitly computed in (72) the decoupled part $\left|\mathcal{Z}^{\text{dec}}_N\right|^2$ and it is finite after extracting a divergent factor of β raised to the power $2\epsilon_+\sum_k(\boldsymbol{\alpha}_k,\rho)-\chi_N$. Finally, the full topological string partition function itself is finite after the divergent factor of β to the power $2\epsilon_+\sum_k(\boldsymbol{\alpha}_k,\rho)$ has been removed.

Via the AGT-W correspondence, we translate our formula for the T_N partition function to the 3-point structure constants of three generic primaries of the Toda field theories, both for the undeformed (4D AGT-W) as well as for the deformed (5D AGT-W) $\mathbf{W}_N$ algebra. We give explicitly the parameter identification from the topological string parameters to the gauge theory ones in appendix A and then to the 2D Toda parameters in equations (7). We

[15] As we already stress in the main text, using the worlds “perturbative” and “instanton” is an abuse of terminology.

identify the 3-point structure constants of the Toda CFT with the topological string partition function in (14). A very nice byproduct of our work is our ability to give the exact definition of the q-deformed Υ_q functions together the all their factors, which to our knowledge do not appear in the literature. This discussion appears in appendix C.2.

Moreover, we identified in (13) $\mathfrak{C}_q$, the Weyl invariant part of the q-deformed 3-point structure constants, with the 5D superconformal index $\mathcal{Z}^{S^4\times S^1}$, a powerful gauge theory object[16]. This identification allows us to predict that the Weyl invariant part of the q-deformed 3-point structure constants should have not just $\mathrm{SU(N)}^3$ symmetry but also an extended symmetry as predicted by [62–64] due to the existence of non-trivial UV fixed points for the 5D gauge theories. We have explicitly checked in the Liouville case that the Weyl invariant part of the DOZZ formula (34) enjoys SU(4) enhanced symmetry, and that for the N=3 case the Weyl invariant structure constants have E_6 enhanced symmetry. Checking that the Weyl invariant 3-point structure constants for higher N enjoy some, other than just the Weyl group of $SU(N)^3$, enhanced symmetry is an important future direction[17]

The formula we give for the 3-point functions at this point is very implicit, since there are still integrals and sums that need to be performed. In a separate publication [39], we will show how at least some of the sums can be performed. In so doing, we will be able to explicitly calculate our degree of divergence χ_N that we conjectured in (83). Moreover, beginning with our formulas and specializing some of the $\boldsymbol{\alpha}$'s we should be able to obtain the formulas of [29–31] for the cases of degenerate or semi-degenerate primaries and for the semiclassical limit $b \to 0$. Conversely, our formulas predict highly nontrivial relations for the sums of "Nekrasov functions" $\mathsf{N}^{\beta}_{\lambda\mu}$ by for example requiring that our formulas reproduces (83) in the semi-degenerate case.

In this paper we give the Toda 3-point functions with three primaries, which however is not enough to solve Toda. To achieve that, we need to also compute the correlation functions of descendants, which as we discussed in the introduction is not as immediate as in the Liouville case. However, it is straightforward to see from the point of view of the topological strings what needs to be done in order to compute them. Specifically, we need

[16]The superconformal index in any dimension is the partition function of protected operators and is independent of the coupling constants of the theory, implying that it remains invariant under S-duality. In 4D the superconformal index $S^3 \times S^1$ was proven to be equivalent to a 2D TQFT [56–60]. It is very possible that something very similar will also be proven for the 5D superconformal index $S^4 \times S^1$ (see [61] for some progress in this direction) and thus the Weyl invariant part of the q-deformed 3-point function could be discovered to obey special properties not visible from the CFT point of view, but realized only once one is using the superconformal index interpretation.

[17]The authors of [5] were able to discover that some specialization (called "Higgsing" in the gauge theory jargon) of the parameters in the T_4 theory leads to an E_7 symmetry, while in [38] a similar specialization of the parameters in T_6 leads to an E_8 symmetry. These specializations change the T_N geometry significantly and in particular reduce the number of Coulomb moduli to one. It would be very interesting to see the meaning of this on the CFT side.

to take the T_N web diagrams from figure 4 and evaluate them with the refined topological vertex without putting empty Young diagram to the external legs. This will provide the general Ding-Iohara algebra interwiners. The Ding-Iohara algebra [65] in the free boson representation (with N free bosons) is known to become

$$\mathcal{A} = \mathbf{W}_N \otimes \mathbf{H} \tag{115}$$

where $\mathbf{H}$ is the Heisenberg algebra which is exactly the algebra that is needed to describe what is obtained from AGT-W [11, 14]. In particular, it is quite easy to obtain the 3-point function of two primaries and one descendant and in fact the answer is just (67) without putting empty Young diagrams for $\boldsymbol{\nu}^{(0)}$. Such 3-point functions are already going to give us, via bootstrapping many higher point functions. Solving this problem is work in progress [41].

We would like to finish by remarking that for many reasons it seems to be much more advantageous to study the q-deformed version of the Toda field theories instead of the undeformed ones. For example, the functions Υ_q behave in a sense a bit better than the Υ ones, since for example the product formula (158) is much simpler that (149) and (142). Furthermore, in the q-deformed case, we can use the topological string formalism to compute the partition functions, tools that are not directly available in the undeformed theory.

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A Parametrization of the T_N junction

We gather in this appendix all necessary formulas for the parametrizations of the Kähler moduli. First, the "interior" Coulomb moduli $\tilde{A}_j^{(i)} = e^{-\beta a_i^{(j)}}$ are independent, while the "border" ones are given by

$$\tilde{A}_i^{(0)} = \prod_{k=1}^{i} \tilde{M}_k, \qquad \tilde{A}_0^{(j)} = \prod_{k=1}^{j} \tilde{N}_k, \qquad \tilde{A}_i^{(N-i)} = \prod_{k=1}^{i} \tilde{L}_k. \tag{116}$$

The parameters labeling the positions of the flavors branes obey the relations

$$\prod_{k=1}^{N} \tilde{M}_k = \prod_{k=1}^{N} \tilde{N}_k = \prod_{k=1}^{N} \tilde{L}_k = 1 \Longleftrightarrow \sum_{k=1}^{N} m_k = \sum_{k=1}^{N} n_k = \sum_{k=1}^{N} l_k = 0. \tag{117}$$

Therefore, $\tilde{A}_0^{(0)} = \tilde{A}_N^{(0)} = \tilde{A}_0^{(N)} = 1$ and we can invert relation (116) as

$$\tilde{M}_i = \frac{\tilde{A}_i^{(0)}}{\tilde{A}_{i-1}^{(0)}}, \qquad \tilde{N}_i = \frac{\tilde{A}_0^{(i)}}{\tilde{A}_0^{(i-1)}}, \qquad \tilde{L}_i = \frac{\tilde{A}_i^{(N-i)}}{\tilde{A}_{i-1}^{(N-i+1)}}. \tag{118}$$

All placements are illustrated in figure 6.

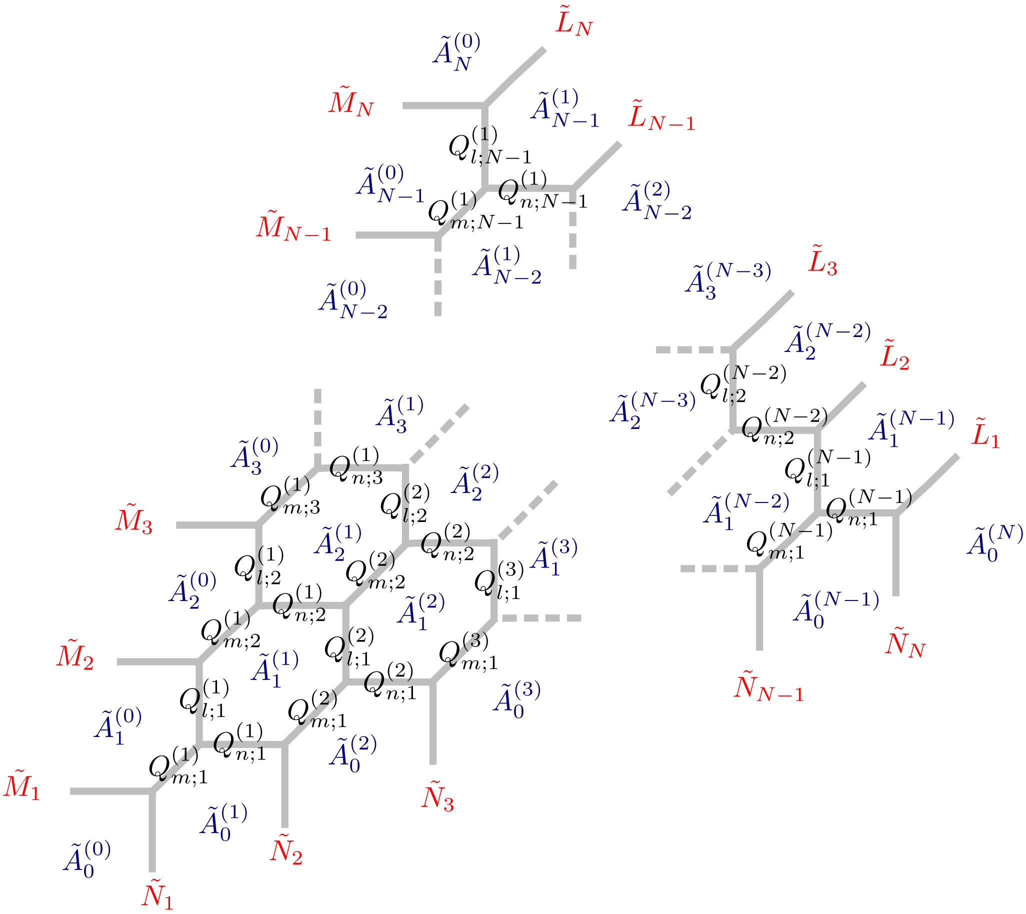

Figure 6: *Parametrization for T_N. We denote the Kähler moduli parameters corresponding to the horizontal lines as $Q_{n;i}^{(j)}$, to the vertical lines as $Q_{l;i}^{(j)}$, and to tilted lines as $Q_{m;i}^{(j)}$. We denote the breathing modes as $\tilde{A}_i^{(j)}$. The index j labels the strips in which the diagram can be decomposed.*

The Kähler parameters associated to the edges of the T_N junction are related to the $\tilde{A}_i^{(j)}$

as follows

$$Q^{(j)}_{n;i} = \frac{\tilde{A}^{(j)}_i \tilde{A}^{(j)}_{i-1}}{\tilde{A}^{(j-1)}_i \tilde{A}^{(j+1)}_{i-1}}, \qquad Q^{(j)}_{l;i} = \frac{\tilde{A}^{(j)}_i \tilde{A}^{(j-1)}_i}{\tilde{A}^{(j)}_{i-1} \tilde{A}^{(j-1)}_{i+1}}, \qquad Q^{(j)}_{m;i} = \frac{\tilde{A}^{(j-1)}_i \tilde{A}^{(j)}_{i-1}}{\tilde{A}^{(j)}_i \tilde{A}^{(j-1)}_{i-1}}. \tag{119}$$

For each inner hexagon of (6), the following two constraints are satisfied

$$Q^{(j)}_{l;i} Q^{(j)}_{m;i+1} = Q^{(j+1)}_{m;i} Q^{(j+1)}_{l;i}, \qquad Q^{(j)}_{n;i} Q^{(j+1)}_{m;i} = Q^{(j)}_{m;i+1} Q^{(j)}_{n;i+1}. \tag{120}$$

Furthermore, we find the following identities relating them to the masses:

$$Q^{(1)}_{m;i} Q^{(1)}_{l;i} = \frac{\tilde{M}_i}{\tilde{M}_{i+1}}, \qquad Q^{(i)}_{m;1} Q^{(i)}_{n;1} = \frac{\tilde{N}_i}{\tilde{N}_{i+1}}, \qquad Q^{(N-i)}_{n;i} Q^{(N-i)}_{l;i} = \frac{\tilde{L}_i}{\tilde{L}_{i+1}}. \tag{121}$$

Using the above, we find the following expressions for the products appearing in the T_N partition function:

$$\begin{aligned}
\prod_{k=i}^{j} Q^{(r)}_{l;k} Q^{(r)}_{m;k+1} &= \prod_{k=i}^{j} \frac{\left(\tilde{A}^{(r)}_k\right)^2}{\tilde{A}^{(r)}_{k-1} \tilde{A}^{(r)}_{k+1}} = \frac{\tilde{A}^{(r)}_i \tilde{A}^{(r)}_j}{\tilde{A}^{(r)}_{i-1} \tilde{A}^{(r)}_{j+1}}, \\
\prod_{k=i}^{j} Q^{(r)}_{l;k} Q^{(r)}_{m;k} &= \prod_{k=i}^{j} \frac{\left(\tilde{A}^{(r-1)}_k\right)^2}{\tilde{A}^{(r-1)}_{k-1} \tilde{A}^{(r-1)}_{k+1}} = \frac{\tilde{A}^{(r-1)}_i \tilde{A}^{(r-1)}_j}{\tilde{A}^{(r-1)}_{i-1} \tilde{A}^{(r-1)}_{j+1}}, \\
Q^{r}_{m;j} \prod_{k=i}^{j-1} Q^{(r)}_{l;k} Q^{(r)}_{m;k} &= \frac{\tilde{A}^{(r-1)}_j \tilde{A}^{(r)}_{j-1}}{\tilde{A}^{(r)}_j \tilde{A}^{(r-1)}_{j-1}} \frac{\tilde{A}^{(r-1)}_i \tilde{A}^{(r-1)}_{j-1}}{\tilde{A}^{(r-1)}_{i-1} \tilde{A}^{(r-1)}_j} = \frac{\tilde{A}^{(r-1)}_i \tilde{A}^{(r)}_{j-1}}{\tilde{A}^{(r-1)}_{i-1} \tilde{A}^{(r)}_j}, \\
Q^{r}_{l;i} \prod_{k=i+1}^{j} Q^{(r)}_{l;k} Q^{(r)}_{m;k} &= \frac{\tilde{A}^{(r)}_i \tilde{A}^{(r-1)}_i}{\tilde{A}^{(r)}_{i-1} \tilde{A}^{(r-1)}_{i+1}} \frac{\tilde{A}^{(r-1)}_{i+1} \tilde{A}^{(r-1)}_j}{\tilde{A}^{(r-1)}_i \tilde{A}^{(r-1)}_{j+1}} = \frac{\tilde{A}^{(r)}_i \tilde{A}^{(r-1)}_j}{\tilde{A}^{(r)}_{i-1} \tilde{A}^{(r-1)}_{j+1}}.
\end{aligned} \tag{122}$$

Furthermore, the following two follow directly from (121) and are used in the derivation of the "perturbative" part of the topological string partition function (68)

$$\prod_{j=1}^{i} Q^{(r)}_{m;j} = \frac{\tilde{A}^{(r)}_0 \tilde{A}^{(r-1)}_i}{\tilde{A}^{(r-1)}_0 \tilde{A}^{(r)}_i}, \qquad \prod_{k=i}^{N-r} Q^{(r)}_{l;k} = \frac{\tilde{A}^{(r-1)}_i \tilde{A}^{(r)}_{N-r}}{\tilde{A}^{(r)}_{i-1} \tilde{A}^{(r-1)}_{N-r+1}}. \tag{123}$$

B Conventions and notations for SU(N)

The purpose of this appendix is to summarize our SU(N) conventions. The weights of the fundamental representation of SU(N) are h_i with $\sum_{i=1}^{N} h_i = 0$. We remind that the scalar product is defined via $(h_i, h_j) = \delta_{ij} - \frac{1}{N}$. The simple roots are

$$e_k := h_k - h_{k+1}, \qquad k = 1, \ldots, N-1, \tag{124}$$

and the positive roots $e > 0$ are contained in the set

$$\Delta^+ := \{h_i - h_j\}_{i<j=1}^{N} = \{e_i\}_{i=1}^{N-1} \cup \{e_i + e_{i+1}\}_{i=1}^{N-2} \cup \cdots \cup \{e_1 + \cdots + e_{N-1}\}. \tag{125}$$

The Weyl vector ρ for SU(N) is given by

$$\rho := \frac{1}{2}\sum_{e>0} e = \frac{1}{2}\sum_{i<j=1}^{N}(h_i - h_j) = \sum_{i=1}^{N}\frac{N+1-2i}{2}h_i = \omega_1 + \cdots + \omega_{N-1}, \tag{126}$$

and it obeys $(\rho, e_i) = 1$ for all i. The $N-1$ fundamental weights ω_i of SU(N) are given by

$$\omega_i = \sum_{k=1}^{i} h_k, \qquad i = 1, \ldots, N-1 \tag{127}$$

and the corresponding finite dimensional representations are the i-fold antisymmetric tensor product of the fundamental representation. They obey the scalar products $(e_i, \omega_j) = \delta_{ij}$, *i.e.* they are a dual basis. Furthermore, we find the following scalar products useful

$$(\rho, h_j) = \frac{N+1}{2} - j, \qquad (\rho, \omega_i) = \frac{i(N-i)}{2}, \qquad (h_j, \omega_i) = \begin{cases} 1 - \frac{i}{N} & j \leq i \\ -\frac{i}{N} & j > i \end{cases}, \tag{128}$$

as well as

$$(\omega_i, \omega_j) = \frac{\min(i,j)\,(N - \max(i,j))}{N}, \qquad (\rho, \rho) = \frac{N(N^2-1)}{12}. \tag{129}$$

After some work, one can prove using the scalar products (128) and (129) that

$$\frac{1}{N}\sum_{e>0}(\boldsymbol{\alpha}_1, e)\,(\boldsymbol{\alpha}_2, e) = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)\,, \tag{130}$$

for any two weights $\boldsymbol{\alpha}_i$.

The Weyl group of SU(N) is isomorphic to S_N and is generated by the $N-1$ Weyl reflections associated to the simple roots. If $\boldsymbol{\alpha}$ is a weight, we define the Weyl reflections with respect to the simple root e_i

$$\mathsf{w}_i \cdot \boldsymbol{\alpha} := \boldsymbol{\alpha} - 2\frac{(e_i, \boldsymbol{\alpha})}{(e_i, e_i)}e_i = \boldsymbol{\alpha} - (e_i, \boldsymbol{\alpha})\,e_i. \tag{131}$$

Furthermore, we define the affine Weyl reflections with respect to e_i as follows

$$\mathsf{w}_i \circ \boldsymbol{\alpha} := \mathcal{Q} + \mathsf{w}_i \cdot (\boldsymbol{\alpha} - \mathcal{Q}) = \mathsf{w}_i \cdot \boldsymbol{\alpha} + Qe_i = \boldsymbol{\alpha} - (\boldsymbol{\alpha} - \mathcal{Q}, e_i)\,e_i, \tag{132}$$

where $\mathcal{Q} := Q\rho = (b + b^{-1})\rho$.

C Special functions

For the reader's convenience, we gather here the definitions and properties of all special functions used in the main text.

C.1 The Υ function.

The purpose of this part of the appendix is to summarize the known identities for the functions used in the undeformed Liouville and Toda CFT. We begin with the function $\Upsilon(x)$ which is defined for $0 < \Re(x) < Q = b + b^{-1}$ as the integral

$$\log \Upsilon(x) := \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left[\left(\frac{Q}{2} - x \right) \frac{t}{2} \right]}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right]. \tag{133}$$

It is clear from the definition that

$$\Upsilon(x) = \Upsilon(Q - x), \qquad \Upsilon\left(\frac{Q}{2}\right) = 1. \tag{134}$$

One can show from the alternative definition below that the following shift identities are obeyed

$$\Upsilon(x + b) = \gamma(xb) b^{1-2bx} \Upsilon(x), \qquad \Upsilon(x + b^{-1}) = \gamma(xb^{-1}) b^{2xb^{-1}-1} \Upsilon(x). \tag{135}$$

where $\gamma(x) := \frac{\Gamma(x)}{\Gamma(1-x)}$. An useful implication is

$$\Upsilon(x + Q) = b^{2(b^{-1}-b)x} \frac{\Gamma\big(1 + bx\big)\Gamma\big(b^{-1}x\big)}{\Gamma\big(1 - bx\big)\Gamma\big(-b^{-1}x\big)} \Upsilon(x), \tag{136}$$

which is used in the derivation of the reflection amplitude (21). It follows from (135) that Υ is an entire function with zeroes at

$$x = -n_1 b - n_2 b^{-1}, \qquad \text{or} \qquad x = (n_1 + 1)b + (n_2 + 1)b^{-1}, \tag{137}$$

where $n_i \in \mathbb{N}_0$.

The function Υ can be connected to the Barnes Double Gamma function $\Gamma_2(x|\omega, \omega_2)$. First, we define $\Gamma_2(x|\omega_1, \omega_2)$ via the *analytic continuation* (the sum is only well-defined if $\Re(t) > 2$) of

$$\log \Gamma_2(s|\omega_1, \omega_2) = \left[\frac{\partial}{\partial t} \sum_{n_1, n_2 = 0}^\infty (s + n_1\omega_1 + n_2\omega_2)^{-t} \right]_{t=0}. \tag{138}$$

From this definition, one can prove (see A.54 of [66]) the *difference property*

$$\frac{\Gamma_2(s + \omega_1|\omega_1, \omega_2)}{\Gamma_2(s|\omega_1, \omega_2)} = \frac{\sqrt{2\pi}}{\omega_2^{\frac{s}{\omega_2} - \frac{1}{2}} \Gamma\left(\frac{s}{\omega_2}\right)}, \qquad \frac{\Gamma_2(s + \omega_2|\omega_1, \omega_2)}{\Gamma_2(s|\omega_1, \omega_2)} = \frac{\sqrt{2\pi}}{\omega_1^{\frac{s}{\omega_1} - \frac{1}{2}} \Gamma\left(\frac{s}{\omega_1}\right)}. \tag{139}$$

In order to express the Υ function using the Barnes double Gamma function, we have to first define the *normalized* function

$$\Gamma_b(x) := \frac{\Gamma_2(x|b, b^{-1})}{\Gamma_2(\frac{Q}{2}|b, b^{-1})}. \tag{140}$$

The log of the function $\Gamma_b(x)$ has an integral representation as

$$\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t}\left(\frac{e^{-xt}-e^{-\frac{Qt}{2}}}{(1-e^{-tb})(1-e^{-tb^{-1}})} - \frac{\left(\frac{Q}{2}-x\right)^2}{2}e^{-t} - \frac{\frac{Q}{2}-x}{t}\right). \tag{141}$$

Then, using (140) we can express the $\Upsilon(x)$ as

$$\Upsilon(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}. \tag{142}$$

This, together with the difference properties of Γ_2 proves the shift identities (135). Also of interest is the function $\mathbf{G}(x)$ introduced in [31] with the properties

$$\mathbf{G}(x+b) = \frac{b^{1/2-bx}}{\sqrt{2\pi}}\Gamma(bx)\mathbf{G}(x), \qquad \mathbf{G}(x+b^{-1}) = \frac{b^{b^{-1}x-1/2}}{\sqrt{2\pi}}\Gamma(b^{-1}x)\mathbf{G}(x). \tag{143}$$

The zeroes of this function are $x=-mb-nb^{-1}$ for m, $n \in \mathbb{N}_0$. If we normalize it by setting $\mathbf{G}(\frac{Q}{2})=1$, then we have $\mathbf{G}(x)=\frac{1}{\Gamma_b(x)}$. Furthermore, [31] also introduce the function $\mathfrak{Z}$ as

$$\mathfrak{Z}(x) = \mathbf{G}(Q+x)\mathbf{G}(Q-x) = \frac{b^{b^{-1}x-bx}}{2\pi}x\Gamma(bx)\Gamma(b^{-1}x)\Upsilon(x). \tag{144}$$

One very often encounters a product formula for the function $\Gamma_2 = \prod_{n_1,n_2}(x+\omega_1 n_1 + \omega_2 n_2)^{-1}$ that is unfortunately not quite correct. To get the product formula for $\Gamma_2(x)$ working, one has to use (A.62) of [66]. Specifically, we set for $\Re(s)>2$

$$\chi(s|\omega_1,\omega_2) := \sum_{n_1,n_2\geq 0}{}^{'} \frac{1}{(\omega_1 n_1+\omega_2 n_2)^s}, \tag{145}$$

where the prime removes the value $(n_1,n_2)=(0,0)$ from the sum. The function $\chi(s|\omega_1,\omega_2)$ can be analytically continued for all $s\in\mathbb{C}$ except for $s=1$ and $s=2$ where there are poles. We have the residues

$$\mathrm{Res}(\chi(s|\omega_1,\omega_2), s=1) = \frac{1}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right), \qquad \mathrm{Res}(\chi(s|\omega_1,\omega_2), s=2) = \frac{1}{\omega_1\omega_2} \tag{146}$$

and the finite parts

$$\begin{aligned}
\mathrm{Res}\Big(\frac{\chi(s|\omega_1,\omega_2)}{s-1}, s=1\Big) &= -\frac{\log\omega_1}{\omega_1} + \frac{1}{2}\left(\frac{1}{\omega_1}-\frac{1}{\omega_2}\right)\log\omega_2 + \frac{\gamma}{\omega_1} + \frac{\gamma}{2\omega_2} - \frac{1}{2\omega_1}\log 2\pi \\
&\quad -\frac{i}{b}\int_0^\infty \frac{\psi(i\frac{\omega_1}{\omega_2}y+1)-\psi(-i\frac{\omega_1}{\omega_2}y+1)}{e^{2\pi y}-1}dy \\
\mathrm{Res}\Big(\frac{\chi(s|\omega_1,\omega_2)}{s-2}, s=2\Big) &= \frac{\zeta(2)}{\omega_1^2} + \frac{\zeta(2)}{2\omega_2^2} + \frac{1}{\omega_1\omega_2}(\gamma-1-\log\omega_2) \\
&\quad -\frac{i}{\omega_2}\int_0^\infty \frac{\zeta_H(2,i\frac{\omega_1}{\omega_2}y+1)-\zeta_H(2,-i\frac{\omega_1}{\omega_2}y+1)}{e^{2\pi y}-1}dy,
\end{aligned} \tag{147}$$

where ψ is the digamma function, γ is the Euler - Mascheroni constant and $\zeta_H(s,q)$ is the Hurwitz-ζ function with ($\Re(s) > 1$ and $\Re(q) > 0$)

$$\zeta_H(s,q) := \sum_{n=0}^{\infty} \frac{1}{(q+n)^s}. \tag{148}$$

Finally, using the shorthands $\alpha := \mathrm{Res}(\frac{\chi(s|\omega_1,\omega_2)}{s-1}, s=1)$ and $\beta := \mathrm{Res}(\frac{\chi(s|\omega_1,\omega_2)}{s-2}, s=2) + \mathrm{Res}(\chi(s|\omega_1,\omega_2), s=2)$ we obtain

$$\Gamma_2(x|\omega_1,\omega_2) = \frac{e^{-\alpha x + \frac{\beta x^2}{2}}}{x} \prod_{n_1,n_2 \geq 0}{}' \frac{e^{\frac{x}{\omega_1 n_1 + \omega_2 n_2} - \frac{x^2}{2(\omega_1 n_1 + \omega_2 n_2)^2}}}{1 + \frac{x}{\omega_1 n_1 + \omega_2 n_2}}. \tag{149}$$

C.2 The q-deformed Υ function.

In this subsection, we wish to summarize some results involving the q-deformed Υ functions. First we begin by defining the shifted factorials[18] (we require for convergence that $|q_i| < 1$ for all i)

$$(x; q_1, \ldots, q_r)_\infty := \prod_{i_1=0,\ldots,i_r=0}^{\infty} (1 - x q_1^{i_1} \cdots q_r^{i_r}). \tag{150}$$

We can extend the definition of the shifted factorial for all values of q_i by imposing the relations

$$(x; q_1, \ldots, q_i^{-1}, \ldots, q_r)_\infty = \frac{1}{(xq_i; q_1, \ldots, q_r)_\infty}. \tag{151}$$

Furthermore, they obey the following shifting properties

$$(q_j x; q_1, \ldots, q_r)_\infty = \frac{(x; q_1, \ldots, q_r)_\infty}{(x; q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_r)_\infty}. \tag{152}$$

We then define the function $\mathcal{M}(u; \mathfrak{t}, \mathfrak{q})$ as

$$\mathcal{M}(u;\mathfrak{t},\mathfrak{q}) := (u\mathfrak{q};\mathfrak{t},\mathfrak{q})_\infty^{-1} = \begin{cases} \prod_{i,j=1}^{\infty}(1 - u\mathfrak{t}^{i-1}\mathfrak{q}^{j})^{-1} & \text{for } |\mathfrak{t}|<1, |\mathfrak{q}|<1 \\ \prod_{i,j=1}^{\infty}(1 - u\mathfrak{t}^{i-1}\mathfrak{q}^{1-j}) & \text{for } |\mathfrak{t}|<1, |\mathfrak{q}|>1 \\ \prod_{i,j=1}^{\infty}(1 - u\mathfrak{t}^{-i}\mathfrak{q}^{j}) & \text{for } |\mathfrak{t}|>1, |\mathfrak{q}|<1 \\ \prod_{i,j=1}^{\infty}(1 - u\mathfrak{t}^{-i}\mathfrak{q}^{1-j})^{-1} & \text{for } |\mathfrak{t}|>1, |\mathfrak{q}|>1 \end{cases}, \tag{153}$$

converging for all u. This function can be written as a plethystic exponential

$$\mathcal{M}(u;\mathfrak{t},\mathfrak{q}) = \exp\left[\sum_{m=1}^{\infty} \frac{u^m}{m} \frac{\mathfrak{q}^m}{(1-\mathfrak{t}^m)(1-\mathfrak{q}^m)}\right], \tag{154}$$

which converges for all $\mathfrak{t}$ and all $\mathfrak{q}$ provided that $|u| < \mathfrak{q}^{-1+\theta(|\mathfrak{q}|-1)}\mathfrak{t}^{\theta(|\mathfrak{t}|-1)}$. Here and elsewhere $\theta(x) = 1$ if $x > 0$ and is zero otherwise. The following identity is obvious from the definition

$$\mathcal{M}(u;\mathfrak{q},\mathfrak{t}) = \mathcal{M}(u\mathfrak{t}/\mathfrak{q};\mathfrak{t},\mathfrak{q}). \tag{155}$$

[18] A good source for the properties of the shifted factorials is [67].

From the analytic properties of the shifted factorials (151), we read the identities

$$\mathcal{M}(u;\mathsf{t}^{-1},\mathsf{q})=\frac{1}{\mathcal{M}(u\mathsf{t};\mathsf{t},\mathsf{q})},\qquad \mathcal{M}(u;\mathsf{t},\mathsf{q}^{-1})=\frac{1}{\mathcal{M}(u\mathsf{q}^{-1};\mathsf{t},\mathsf{q})},\tag{156}$$

while from (152) we take the following shifting identities

$$\mathcal{M}(u\mathsf{t};\mathsf{t},\mathsf{q})=(u\mathsf{q};\mathsf{q})_\infty\mathcal{M}(u;\mathsf{t},\mathsf{q}),\qquad \mathcal{M}(u\mathsf{q};\mathsf{t},\mathsf{q})=(u\mathsf{q};\mathsf{t})_\infty\mathcal{M}(u;\mathsf{t},\mathsf{q}).\tag{157}$$

We define the q-deformed Υ function as

$$\begin{aligned}\Upsilon_q(x|\epsilon_1,\epsilon_2)=&(1-q)^{-\frac{1}{\epsilon_1\epsilon_2}\left(x-\frac{\epsilon_+}{2}\right)^2}\prod_{n_1,n_2=0}^{\infty}\frac{(1-q^{x+n_1\epsilon_1+n_2\epsilon_2})(1-q^{\epsilon_+-x+n_1\epsilon_1+n_2\epsilon_2})}{(1-q^{\epsilon_+/2+n_1\epsilon_1+n_2\epsilon_2})}\\=&(1-q)^{-\frac{1}{\epsilon_1\epsilon_2}\left(x-\frac{\epsilon_+}{2}\right)^2}\left|\frac{\mathcal{M}(q^{-x};\mathsf{t},\mathsf{q})}{\mathcal{M}(\sqrt{\frac{\mathsf{t}}{\mathsf{q}}};\mathsf{t},\mathsf{q})}\right|^2,\end{aligned}\tag{158}$$

where we have used the definition (75) for the norm squared. If follows from the definition that $\Upsilon_q(\epsilon_+/2|\epsilon_1,\epsilon_2)=1$, that $\Upsilon_q(x|\epsilon_1,\epsilon_2)=\Upsilon_q(\epsilon_+-x|\epsilon_1,\epsilon_2)$ and that $\Upsilon_q(x|\epsilon_1,\epsilon_2)=\Upsilon_q(x|\epsilon_2,\epsilon_1)$. Furthermore, from the shifting identities for $\mathcal{M}$, we can easily prove that

$$\Upsilon_q(x+\epsilon_1|\epsilon_1,\epsilon_2)=\left(\frac{1-q}{1-q^{\epsilon_2}}\right)^{1-2\epsilon_2^{-1}x}\gamma_{q^{\epsilon_2}}(x\epsilon_2^{-1})\Upsilon_q(x|\epsilon_1,\epsilon_2),\tag{159}$$

together with a similar equation for the shift with ϵ_2. Here, we have used the definition of the q-deformed Γ and γ functions

$$\Gamma_q(x):=(1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty},\qquad \gamma_q(x):=\frac{\Gamma_q(x)}{\Gamma_q(1-x)}=(1-q)^{1-2x}\frac{(q^{1-x};q)_\infty}{(q^x;q)_\infty},\tag{160}$$

valid for $|q|<1$. They obey $\Gamma_q(x+1)=\frac{1-q^x}{1-q}\Gamma_q(x)$, implying $\gamma_q(x+1)=\frac{(1-q^x)(1-q^{-x})}{(1-q)^2}\gamma_q(x)$. Because of the normalization of $\Upsilon_q(x|\epsilon_1,\epsilon_2)$ and since the factors of the right hand side of (159) have a well defined limit for $q\to1$, we find by comparing functional identities that[19]

$$\Upsilon_q(x+\epsilon_1|\epsilon_1,\epsilon_2)\xrightarrow{q\to1}\Upsilon(x|\epsilon_1,\epsilon_2):=\frac{\Gamma_2\left(\frac{\epsilon_+}{2}|\epsilon_1,\epsilon_2\right)^2}{\Gamma_2\left(x|\epsilon_1,\epsilon_2\right)\Gamma_2\left(\epsilon_+-x|\epsilon_1,\epsilon_2\right)}.\tag{161}$$

In particular, the function $\Upsilon(x)$ defined in subsection C.1 is equal to $\Upsilon(x|b,b^{-1})$. We shall often just write $\Upsilon_q(x)$ instead of $\Upsilon_q(x|\epsilon_1,\epsilon_2)$ and indicate in the text whether the ϵ_i parameters are arbitrary or whether $b=\epsilon_1=\epsilon_2^{-1}$.

For the rest of the section, we set $b=\epsilon_1=\epsilon_2^{-1}$. One very useful implication of (159) for the derivation of the reflection amplitude (52)

$$\Upsilon_q(x+Q)=\left[\frac{\left(1-q^{b^{-1}}\right)^b\left(1-q^b\right)^{b^{-1}}}{(1-q)^Q}\right]^{2x}\frac{\Gamma_{q^{b^{-1}}}(1+bx)\Gamma_{q^b}(b^{-1}x)}{\Gamma_{q^{b^{-1}}}(1-bx)\Gamma_{q^b}(-b^{-1}x)}\Upsilon_q(x),\tag{162}$$

[19]The $q\to1$ limit has also been checked numerically for the case $b=\epsilon_1=\epsilon_2^{-1}$.

which reduces to (136) in the limit $q \to 1$. We finish this part of the appendix with two small remarks. First, the zeroes of Υ_q are given by

$$x = -n_1\epsilon_1 - n_2\epsilon_2 + \frac{2\pi i}{\log q}m, \qquad x = (n_1+1)\epsilon_1 + (n_2+1)\epsilon_2 + \frac{2\pi i}{\log q}m', \tag{163}$$

where $n_i \in \mathbb{N}_0$ and m and m' are integer. We thus see by comparing with (137) that for each zero of Υ we have a whole tower, Kaluza-Klein like, of zeroes of Υ_q. The new zeroes are q-dependent, but the ones that are also zeroes of Υ, *i.e.* those with $m = 0$ are q-independent. The tower of zeros is obtained by beginning with the q-independent $m = 0$ zero and shifting by multiples of $\frac{2\pi i}{\log q} = -\frac{2\pi i}{\beta}$. Second, we will need to evaluate the derivative of $\Upsilon_q(x)$ at $x = 0$. Since the zero of $\Upsilon_q(x)$ at $x = 0$ is due to the factor $(1 - q^x)$ in the numerator of (158), we find that the only piece of the derivative that survives is

$$\Upsilon_q'(0) = \frac{\beta}{1-q}\Upsilon_q(b). \tag{164}$$

C.3 The finite product functions

In this subsection ϵ_1 and ϵ_2 are general. In the definition of the topological string amplitudes, we often need to use the following two functions given by finite products

$$\begin{aligned}
\tilde{Z}_\nu(\mathsf{t},\mathsf{q}) &:= \prod_{i=1}^{\ell(\nu)}\prod_{j=1}^{\nu_i}\left(1 - \mathsf{t}^{\nu_j^t - i + 1}\mathsf{q}^{\nu_i - j}\right)^{-1}, \\
\mathcal{N}_{\lambda\mu}(Q;\mathsf{t},\mathsf{q}) &:= \prod_{i,j=1}^{\infty}\frac{1 - Q\mathsf{t}^{i-1-\lambda_j^t}\mathsf{q}^{j-\mu_i}}{1 - Q\mathsf{t}^{i-1}\mathsf{q}^j} \\
&= \prod_{(i,j)\in\lambda}(1 - Q\mathsf{t}^{\mu_j^t - i}\mathsf{q}^{\lambda_i - j + 1})\prod_{(i,j)\in\mu}(1 - Q\mathsf{t}^{-\lambda_j^t + i - 1}\mathsf{q}^{-\mu_i + j}).
\end{aligned} \tag{165}$$

We shall use in the following $|\lambda| := \sum_{i=1}^{\ell(\lambda)}\lambda_i$ and $||\lambda||^2 := \sum_{i=1}^{\ell(\lambda)}\lambda_i^2$, where $\ell(\lambda)$ is the number of rows of the partition λ. We observe that in some cases the function $\mathcal{N}_{\lambda\mu}$ behaves like a delta function, for instance $\mathcal{N}_{\lambda\emptyset}(\frac{\mathsf{t}}{\mathsf{q}}) = \mathcal{N}_{\emptyset\lambda}(1) = \delta_{\lambda\emptyset}$. Furthermore, we find a relation allowing us to express the product of two $\tilde{Z}_\mu$ through

$$\mathcal{N}_{\mu\mu}(1;\mathsf{t},\mathsf{q}) = \left(-\sqrt{\frac{\mathsf{q}}{\mathsf{t}}}\right)^{|\mu|}\mathsf{t}^{-\frac{||\mu^t||^2}{2}}\mathsf{q}^{-\frac{||\mu||^2}{2}}\left(\tilde{Z}_\mu(\mathsf{t},\mathsf{q})\tilde{Z}_{\mu^t}(\mathsf{q},\mathsf{t})\right)^{-1}. \tag{166}$$

Using the identities

$$\sum_{(i,j)\in\lambda} i = \frac{1}{2}\left(||\lambda^t||^2 + |\lambda|\right), \qquad \sum_{(i,j)\in\lambda}\mu_i = \sum_{i=1}^{\min\{\ell(\lambda),\ell(\mu)\}}\lambda_i\mu_i, \tag{167}$$

we find the exchange identities

$$\begin{aligned}
\mathcal{N}_{\lambda\mu}(Q;\mathfrak{q},\mathfrak{t}) &= \mathcal{N}_{\mu^t\lambda^t}(Q\frac{\mathfrak{t}}{\mathfrak{q}};\mathfrak{t},\mathfrak{q}),\\
\mathcal{N}_{\lambda\mu}(Q^{-1};\mathfrak{t},\mathfrak{q}) &= \left(-Q^{-1}\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\right)^{|\lambda|+|\mu|}\mathfrak{t}^{\frac{-||\lambda^t||^2+||\mu^t||^2}{2}}\mathfrak{q}^{\frac{||\lambda||^2-||\mu||^2}{2}}\mathcal{N}_{\mu\lambda}(Q\frac{\mathfrak{t}}{\mathfrak{q}};\mathfrak{t},\mathfrak{q}).
\end{aligned} \tag{168}$$

From [1, 36] we take the following summation formula

$$\sum_\mu\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}Q_3\right)^{|\mu|}\frac{\mathcal{N}_{\mu\emptyset}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}Q_1\right)\mathcal{N}_{\emptyset\mu}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}Q_2\right)}{\mathcal{N}_{\mu\mu}(1)} = \frac{\mathcal{M}\left(Q_1Q_3\right)\mathcal{M}\left(\frac{\mathfrak{t}}{\mathfrak{q}}Q_2Q_3\right)}{\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}Q_3\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}Q_1Q_2Q_3\right)}. \tag{169}$$

In the 4D limit, it is often useful to use the rescaled $\mathcal{N}$ functions that we refer to as "Nekrasov" functions ($Q = e^{-\beta m}$)

$$\mathcal{N}_{\lambda\mu}(Q;\mathfrak{t},\mathfrak{q}) = \left(Q\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\right)^{\frac{|\lambda|+|\mu|}{2}}\mathfrak{t}^{\frac{||\mu^t||^2-||\lambda^t||^2}{4}}\mathfrak{q}^{\frac{||\lambda||^2-||\mu||^2}{4}}\mathsf{N}^\beta_{\lambda\mu}(m;\epsilon_1,\epsilon_2), \tag{170}$$

where, using the parametrization (4), the new functions are given by

$$\begin{aligned}
\mathsf{N}^\beta_{\lambda\mu}(m;\epsilon_1,\epsilon_2) = &\prod_{(i,j)\in\lambda}2\sinh\frac{\beta}{2}\left[m+\epsilon_1(\lambda_i-j+1)+\epsilon_2(i-\mu^t_j)\right]\\
&\times\prod_{(i,j)\in\mu}2\sinh\frac{\beta}{2}\left[m+\epsilon_1(j-\mu_i)+\epsilon_2(\lambda^t_j-i+1)\right].
\end{aligned}$$

The new function obeys the simpler exchange identities

$$\begin{aligned}
\mathsf{N}^\beta_{\lambda\mu}(m;-\epsilon_2,-\epsilon_1) &= \mathsf{N}^\beta_{\mu^t\lambda^t}(m-\epsilon_1-\epsilon_2;\epsilon_1,\epsilon_2),\\
\mathsf{N}^\beta_{\lambda\mu}(-m;\epsilon_1,\epsilon_2) &= (-1)^{|\lambda|+|\mu|}\mathsf{N}^\beta_{\mu\lambda}(m-\epsilon_1-\epsilon_2;\epsilon_1,\epsilon_2),\\
\mathsf{N}^\beta_{\lambda\mu}(m;\epsilon_2,\epsilon_1) &= \mathsf{N}^\beta_{\lambda^t\mu^t}(m;\epsilon_1,\epsilon_2),
\end{aligned} \tag{171}$$

as well as the summation formula

$$\begin{aligned}
\sum_\mu e^{-\beta(\frac{m_1}{2}+\frac{m_2}{2}+m_3)|\mu|}&\frac{\mathsf{N}^\beta_{\mu\emptyset}\left(m_1-\frac{\epsilon_+}{2}\right)\mathsf{N}^\beta_{\emptyset\mu}\left(m_2-\frac{\epsilon_+}{2}\right)}{\mathsf{N}^\beta_{\mu\mu}(0)} =\\
&= \frac{\mathcal{M}\left(e^{-\beta(m_1+m_3)}\right)\mathcal{M}\left(e^{-\beta(m_2+m_3-\epsilon_+)}\right)}{\mathcal{M}\left(e^{-\beta(m_3-\frac{\epsilon_+}{2})}\right)\mathcal{M}\left(e^{-\beta(m_1+m_2+m_3-\frac{\epsilon_+}{2})}\right)}.
\end{aligned} \tag{172}$$

We finish this section by remarking that in the limit $\beta\to 0$, the functions $\mathsf{N}^\beta_{\lambda\mu}$ behave as

$$\mathsf{N}^\beta_{\lambda\mu}\xrightarrow{\beta\to 0}\beta^{|\lambda|+|\mu|}\mathsf{N}_{\lambda\mu}, \tag{173}$$

where we have defined

$$\begin{aligned}
\mathsf{N}_{\lambda\mu}(m;\epsilon_1,\epsilon_2) = &\prod_{(i,j)\in\lambda}\left[m+\epsilon_1(\lambda_i-j+1)+\epsilon_2(i-\mu^t_j)\right]\\
&\times\prod_{(i,j)\in\mu}\left[m+\epsilon_1(j-\mu_i)+\epsilon_2(\lambda^t_j-i+1)\right].
\end{aligned} \tag{174}$$

Thus for instance ratios of $\mathsf{N}^{\beta}_{\lambda\mu}$ that are balanced in the sense that the same partitions appear in the numerator and in the denominator have proper limits for $\beta \to 0$.

D Computation of the T_N partition function

In this part of the appendix, we wish to put together the computations that bring us from equations (65) and (66) to (67), (68) and (69). We define the function

$$\mathcal{R}_{\lambda\mu}(Q;\mathsf{t},\mathsf{q}) := \prod_{i,j=1}^{\infty}\left(1 - Q\mathsf{t}^{i-\frac{1}{2}-\lambda_j}\mathsf{q}^{j-\frac{1}{2}-\mu_i}\right) = \mathcal{M}(Q\sqrt{\frac{\mathsf{t}}{\mathsf{q}}};\mathsf{t},\mathsf{q})^{-1}\mathcal{N}_{\lambda^t\mu}(Q\sqrt{\frac{\mathsf{t}}{\mathsf{q}}};\mathsf{t},\mathsf{q}), \quad (175)$$

and, after using some Cauchy identities, we rewrite (65) as equation (4.67) of [1]:

$$\begin{aligned}\mathcal{Z}^{\text{strip}}_{\boldsymbol{\nu}\boldsymbol{\tau}}(\boldsymbol{Q}_m, \boldsymbol{Q}_l;\mathsf{t},\mathsf{q}) = &\prod_{j=1}^{L+1}\mathsf{t}^{\frac{||\nu_j^t||^2}{2}}\tilde{Z}_{\nu_j^t}(\mathsf{q},\mathsf{t})\prod_{j=1}^{L}\mathsf{q}^{\frac{||\tau_j||^2}{2}}\tilde{Z}_{\tau_j}(\mathsf{t},\mathsf{q}) \\ &\times\prod_{i\le j=1}^{L}\frac{\mathcal{R}_{\nu_i^t\tau_j}\Big(Q_{m;j}\prod_{k=i}^{j-1}Q_{m;k}Q_{l;k}\Big)\mathcal{R}_{\tau_i^t\nu_{j+1}}\Big(Q_{l;i}\prod_{k=i+1}^{j}Q_{m;k}Q_{l;k}\Big)}{\mathcal{R}_{\nu_i^t\nu_{j+1}}\Big(\prod_{k=i}^{j}Q_{l;k}Q_{m;k}\sqrt{\frac{\mathsf{q}}{\mathsf{t}}}\Big)} \\ &\times\prod_{i\le j=1}^{L-1}\mathcal{R}_{\tau_i^t\tau_{j+1}}\Big(\prod_{k=i}^{j}Q_{l;k}Q_{m;k+1}\sqrt{\frac{\mathsf{t}}{\mathsf{q}}}\Big)^{-1}. \end{aligned}\quad (176)$$

The complete T_N diagram is made out of N such strips, as depicted in figure 4 and written down in equation (66). We remind that $\nu_j^{(0)} = \emptyset$ for all j, see the right part of figure 4. We can redefine the strip partition function without affecting the topological string partition function (66), by moving half of the $\tilde{Z}$ from one strip to the another. Specifically, we move the $\tilde{Z}_{\nu_i^t}(\mathsf{q},\mathsf{t})\mathsf{t}^{\frac{||\nu_i^t||^2}{2}}$ of the left lines to the right ones, so that they become $\tilde{Z}_{\tau_i^t}(\mathsf{q},\mathsf{t})\mathsf{t}^{\frac{||\tau_i^t||^2}{2}}$ for the strip on the left. This redefinition doesn't change $\mathcal{Z}_N$, since the partitions to the extreme left of the T_N diagram are all empty. Putting it all together, we get a new strip partition function,

$$\begin{aligned}\mathcal{Z}^{\text{strip}\,\prime}_{\boldsymbol{\nu}\boldsymbol{\tau}}(\boldsymbol{Q}_m, \boldsymbol{Q}_l;\mathsf{t},\mathsf{q}) = &\prod_{j=1}^{L}\mathsf{t}^{\frac{||\tau_j^t||^2}{2}}\mathsf{q}^{\frac{||\tau_j||^2}{2}}\tilde{Z}_{\tau_j}(\mathsf{t},\mathsf{q})\tilde{Z}_{\tau_j^t}(\mathsf{q},\mathsf{t}) \\ &\times\prod_{i\le j=1}^{L}\frac{\mathcal{R}_{\nu_i^t\tau_j}\Big(Q_{m;j}\prod_{k=i}^{j-1}Q_{m;k}Q_{l;k}\Big)\mathcal{R}_{\tau_i^t\nu_{j+1}}\Big(Q_{l;i}\prod_{k=i+1}^{j}Q_{m;k}Q_{l;k}\Big)}{\mathcal{R}_{\nu_i^t\nu_{j+1}}\Big(\prod_{k=i}^{j}Q_{l;k}Q_{m;k}\sqrt{\frac{\mathsf{q}}{\mathsf{t}}}\Big)} \\ &\times\prod_{i\le j=1}^{L-1}\mathcal{R}_{\tau_i^t\tau_{j+1}}\Big(\prod_{k=i}^{j}Q_{l;k}Q_{m;k+1}\sqrt{\frac{\mathsf{t}}{\mathsf{q}}}\Big)^{-1}. \end{aligned}\quad (177)$$

We can get rid of the $\tilde{Z}$ functions using (166). Putting things together in the products and replacing the $\mathcal{R}$ functions by using (175), we get

$$\begin{aligned}
\mathcal{Z}^{\text{strip}\,\prime}_{\boldsymbol{\nu}\boldsymbol{\tau}}(\boldsymbol{Q}_m,\boldsymbol{Q}_l;\mathsf{t},\mathsf{q}) &= \prod_{i\leq j=1}^{L-1}\mathcal{M}\Big(\frac{\mathsf{t}}{\mathsf{q}}\prod_{k=i}^{j}Q_{l;k}Q_{m;k+1}\Big)\\
&\times\prod_{i\leq j=1}^{L}\frac{\mathcal{M}\Big(\prod_{k=i}^{j}Q_{l;k}Q_{m;k}\Big)}{\mathcal{M}\Big(\sqrt{\frac{\mathsf{t}}{\mathsf{q}}}Q_{m;j}\prod_{k=i}^{j-1}Q_{m;k}Q_{l;k}\Big)\mathcal{M}\Big(\sqrt{\frac{\mathsf{t}}{\mathsf{q}}}Q_{l;i}\prod_{k=i+1}^{j}Q_{m;k}Q_{l;k}\Big)}\prod_{k=1}^{L}\Big(-\sqrt{\frac{\mathsf{t}}{\mathsf{q}}}\Big)^{|\tau_k|}\\
&\times\prod_{i\leq j=1}^{L}\frac{\mathcal{N}_{\nu_i\tau_j}\Big(\sqrt{\frac{\mathsf{t}}{\mathsf{q}}}Q_{m;j}\prod_{k=i}^{j-1}Q_{m;k}Q_{l;k}\Big)\mathcal{N}_{\tau_i\nu_{j+1}}\Big(\sqrt{\frac{\mathsf{t}}{\mathsf{q}}}Q_{l;i}\prod_{k=i+1}^{j}Q_{m;k}Q_{l;k}\Big)}{\mathcal{N}_{\nu_i\nu_{j+1}}\Big(\prod_{k=i}^{j}Q_{l;k}Q_{m;k}\Big)\mathcal{N}_{\tau_i\tau_j}\Big(\frac{\mathsf{t}}{\mathsf{q}}\prod_{k=i}^{j-1}Q_{l;k}Q_{m;k+1}\Big)}. \qquad (178)
\end{aligned}$$

We can straightforwardly obtain (68) by taking only the $\mathcal{M}$ dependent terms of the strip partition functions, plugging them in (66) and replacing the Kähler parameters Q_m, and Q_l by the $\tilde{A}$'s using the formulas (123) of appendix A. Thus, we get the "perturbative part" of the topological string T_N partition (68), *i.e.* the part that is independent of the partitions entering the sum.

Using the functions N^β defined in (171), the relations (170) and performing a shift of the factors from one strip to the one standing on its left, which implies the following change:

$$\prod_{i\leq j=1}^{L}\left(\frac{\mathsf{t}}{\mathsf{q}}\right)^{\frac{|\nu_i|+|\nu_{j+1}|}{4}}\longrightarrow\prod_{i\leq j=1}^{L-1}\left(\frac{\mathsf{t}}{\mathsf{q}}\right)^{\frac{|\tau_i|+|\tau_{j+1}|}{4}}, \qquad (179)$$

we can write the "instanton" part of the redefined strip as

$$\begin{aligned}
\mathcal{Z}^{\text{strip, inst}\,\prime\prime}_{\boldsymbol{\nu}\boldsymbol{\tau}}(\boldsymbol{Q}_m,\boldsymbol{Q}_l;\mathsf{t},\mathsf{q}) &= \prod_{k=1}^{L}(-1)^{|\tau_k|}\prod_{i\leq j=1}^{L}Q_{m;i}^{\frac{|\tau_j|-|\nu_{j+1}|}{2}}Q_{l;j}^{\frac{|\tau_i|-|\nu_i|}{2}}\\
&\times\frac{\mathsf{N}^\beta_{\nu_i\tau_j}(\sum_{k=i}^{j-1}q_{l;k}+\sum_{k=i}^{j}q_{m;k}-\frac{\epsilon_+}{2})\mathsf{N}^\beta_{\tau_i\nu_{j+1}}(\sum_{k=i}^{j}q_{l;k}+\sum_{k=i+1}^{j}q_{m;k}-\frac{\epsilon_+}{2})}{\mathsf{N}^\beta_{\nu_i\nu_{j+1}}(\sum_{k=i}^{j}(q_{l;k}+q_{m;k}))\mathsf{N}^\beta_{\tau_i\tau_j}(\sum_{k=i}^{j-1}q_{l;k}+\sum_{k=i+1}^{j}q_{m;k}-\epsilon_+)}, \qquad (180)
\end{aligned}$$

where we have used $Q^{(i)}_{m;j}=e^{-\beta q^{(i)}_{m;j}}$, $Q^{(i)}_{l;j}=e^{-\beta q^{(i)}_{l;j}}$ and $Q^{(i)}_{n;j}=e^{-\beta q^{(i)}_{n;j}}$. Before we move on, let us remark that

$$\prod_{i\leq j=1}^{L}Q_{m;i}^{\frac{|\tau_j|-|\nu_{j+1}|}{2}}Q_{l;j}^{\frac{|\tau_i|-|\nu_i|}{2}}=\prod_{i=1}^{L}\left(\prod_{j=1}^{i}Q_{m;j}\prod_{k=i}^{L}Q_{l;k}\right)^{\frac{|\tau_i|}{2}}\prod_{i=1}^{L+1}\left(\prod_{j=1}^{i-1}Q_{m;j}\prod_{k=i}^{L}Q_{l;k}\right)^{-\frac{|\nu_i|}{2}}. \qquad (181)$$

Armed with (180), we can compute (69). We have

$$\mathcal{Z}^{\text{inst}}_N=\prod_{r=1}^{N}\Big(-\boldsymbol{Q}^{(r)}_n\Big)^{|\boldsymbol{\nu}^{(r)}|}\mathcal{Z}^{\text{strip, inst}\,\prime\prime}_{\boldsymbol{\nu}^{(r-1)}\boldsymbol{\nu}^{(r)}}(\boldsymbol{Q}^{(r)}_m,\boldsymbol{Q}^{(r)}_l;\mathsf{t},\mathsf{q}). \qquad (182)$$

First, we consider the part of the sum of $\mathcal{Z}_N^{\text{inst}}$ that doesn't involve the N^β functions. It consists solely of the $\prod_{r=1}^N \Big(- \boldsymbol{Q}_n^{(r)} \Big)^{|\boldsymbol{\nu}^{(r)}|}$ term (The minus sign will be canceled by the $\prod_{k=1}^L (-1)^{|\tau_k|}$ part of (180)) of (182) and of the product of (181) over all the strips, where $\boldsymbol{\nu}$ is to be replaced by $\boldsymbol{\nu}^{(r-1)}$ and $\boldsymbol{\tau}$ by $\boldsymbol{\nu}^{(r)}$. Explicitly, this part of the summand has the form (the length L of the strip is given by $N-r$, where r numbers the strips from left to right)

$$
\begin{aligned}
&\prod_{r=1}^{N}\prod_{i=1}^{N-r}\left(Q_{n;i}^{(r)}\right)^{|\nu_i^{(r)}|}\prod_{i=1}^{N-r}\left(\prod_{j=1}^{i}Q_{m;j}^{(r)}\prod_{k=i}^{N-r}Q_{l;k}^{(r)}\right)^{\frac{|\nu_i^{(r)}|}{2}}\prod_{i=1}^{N-r+1}\left(\prod_{j=1}^{i-1}Q_{m;j}^{(r)}\prod_{k=i}^{N-r}Q_{l;k}^{(r)}\right)^{-\frac{|\nu_i^{(r-1)}|}{2}}\\
&=\prod_{r=1}^{N}\prod_{i=1}^{N-r}\left(Q_{n;i}^{(r)}\right)^{|\nu_i^{(r)}|}\left(\prod_{j=1}^{i}Q_{m;j}^{(r)}\prod_{k=i}^{N-r}Q_{l;k}^{(r)}\right)^{\frac{|\nu_i^{(r)}|}{2}}\left(\prod_{j=1}^{i-1}Q_{m;j}^{(r+1)}\prod_{k=i}^{N-r-1}Q_{l;k}^{(r+1)}\right)^{-\frac{|\nu_i^{(r)}|}{2}}\\
&=\prod_{r=1}^{N}\prod_{i=1}^{N-r}\left[\left(Q_{n;i}^{(r)}\right)^2\frac{\prod_{j=1}^{i}Q_{m;j}^{(r)}\prod_{k=i}^{N-r}Q_{l;k}^{(r)}}{\prod_{j=1}^{i-1}Q_{m;j}^{(r+1)}\prod_{k=i}^{N-r-1}Q_{l;k}^{(r+1)}}\right]^{\frac{|\nu_i^{(r)}|}{2}}\\
&=\prod_{r=1}^{N}\prod_{i=1}^{N-r}\left[\frac{\left(\tilde{A}_0^{(r)}\tilde{A}_{N-r}^{(r)}\right)^2}{\tilde{A}_0^{(r-1)}\tilde{A}_0^{(r+1)}\tilde{A}_{N-r+1}^{(r-1)}\tilde{A}_{N-r-1}^{(r+1)}}\right]^{\frac{|\nu_i^{(r)}|}{2}}=\prod_{r=1}^{N}\prod_{i=1}^{N-r}\left[\frac{\tilde{N}_r\tilde{L}_{N-r}}{\tilde{N}_{r+1}\tilde{L}_{N-r+1}}\right]^{\frac{|\nu_i^{(r)}|}{2}}
\end{aligned}
\tag{183}
$$

where in the second line we have used the fact that $\boldsymbol{\nu}^{(0)}$ consists entirely of empty partitions, in the fourth we have used the very useful formulas (123) and in the last equation have used (116).

Taking now the last line of (183) and adding the remaining N^β parts, we get the full instanton part of the T_N partition function that we wrote in (69).

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