AN ℓ^1-PENALTY SCHEME FOR THE OPTIMAL CONTROL OF ELLIPTIC VARIATIONAL INEQUALITIES

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Abstract. An ℓ_1-penalty scheme in function space for the optimal control of elliptic variational inequalities (VIs) is proposed. In an L^2-tracking context, an iterative algorithm is proven to generate a sequence which converges to a C-stationary point and, under certain conditions, even to a strongly stationary point of the original problem. In the case of point tracking control, where the objective contains pointwise function evaluations of the state variable, a modified model problem with constraints on the dual variable associated with the VI constraint is introduced and an auxiliary problem that penalizes not only the complementarity, but also the state constraint, is analyzed. Passing to the limit with the penalty parameter in the stationarity system of the auxiliary problem yields a C-stationarity system for the original problem if the additional dual constraints are not active. Finally, numerical results obtained by the new algorithms are documented.

1. Introduction

In this work we analyze a penalty scheme for the optimal control of variational inequalities, i.e. for problems of the type

$$\text{Minimize } J(y,u) = j(y) + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \tag{1a}$$

$$\text{over } (y,u) \in Y \times U_{ad} \tag{1b}$$

$$\text{subject to} \quad \forall z \in K, \quad \langle Ay - u - f, z - y\rangle_{H^{-1}(\Omega)} \geq 0, \tag{1c}$$

where $U_{ad} \subset L^2(\Omega)$ is the set of feasible controls u, the set K is convex and closed, A is linear, bounded and coercive such that the variational inequality (1c) admits a unique solution for every $u \in U_{ad}$ and given f. Here, Y is a suitable space for the state variable y such that the solution operator of (1c) maps from U_{ad} into Y, and the summand $j(y)$ in the objective may, for instance, implement the difference of the state variable y to a given desired state, either in the L^2-norm, or in a finite set of tracking points. The cost of the control action is $\nu > 0$.

The constraint (1c) is prototypical for a broad range of problems, such as energy minimization problems or free boundary problems, that can be modeled by variational inequalities. In rather abstract form they have been analyzed since the

2010 *Mathematics Subject Classification.* 49M30, 65K05, 90C33 .

Key words and phrases. optimal control of variational inequalities in function space, complementarity constraints, C-stationarity, strong stationarity, point tracking, elastic mode algorithm.

M.H. acknowledges support by the German Research Fund (DFG) through the Research Center MATHEON Project C28 and the SPP 1253 "Optimization with Partial Differential Equations", and the Austrian Science Fund (FWF) through the START Project Y 305-N18 "Interfaces and Free Boundaries" and the SFB F-32 "Mathematical Optimization and Its Application in Biomedical Sciences". C.L. was supported by the DFG through the SPP 1253 "Optimization with Partial Differential Equations".

1960s. A comprehensive study and a survey on the literature on this subject can be found, for instance, in [19, 9, 28].

In optimal control problems, resulting, e.g., from engineering sciences, one typically can influence a system by a control mechanism, which aims to minimize an objective depending on the state of the system as well as on the control action. Similarly, in inverse problems, a parameter that plays the role of a control variable, has to be determined from (defective) measurements associated with the state y, either on the (whole) domain (L^2 tracking), or in certain predefined locations (point tracking). A classical work on optimal control of systems governed by partial differential equations is [20], but, in recent years, numerous extensions on the analysis and numerical treatment of such problems have been contributed to the literature.

When one aims to control variational inequalities, the inherent non-smoothness of the solution operator associated with the VI results in constraint degeneracy. Moreover, considering the reduced control problem (written in u), one is confronted with a challenging non-smooth and non-convex minimization problem. In finite dimensions, problems of this structure are termed mathematical programs with equilibrium contraints (MPECs); compare [22, 27]. For characterizing stationarity of a feasible point of the MPEC, the classical Karush-Kuhn-Tucker theory cannot be applied due to the aforementioned constraint degeneracy. Depending on the problem instance, but also depending on the utilized mathematical tools alternative stationarity systems have been derived. A hierarchy of such stationarity conditions in finite dimensions can, e.g., be found in [29]. These concepts were transferred to function space in [15] and later also in [11, 12, 17]. Concerning the choice of mathematical tools, we mention that penalty and smoothing methods as well as concepts from convex or set-valued analysis and generalized differentiation have been used in order to derive stationarity conditions for MPECs in function space; see, e.g., [4, 23, 24, 25] in addition to the aforementioned references.

In the literature one often finds classical L^2-tracking type control problems. In this paper, however, inspired by point measurements (e.g., obtained by sensors mounted at fixed locations within a region of interest), we study point tracking of solutions of variational inequalities. We mention that point measurements naturally occur, for instance, in mathematical finance; see, e.g., [1]. Mathematically this requires higher regularity of the VI-solution in order to justify evaluations of the state at isolated points within a domain of interest. Such an increased regularity then has various analytical implications on the regularity of associated dual quantities and requires new analytical as well as numerical considerations. For VIs involving second-order elliptic partial differential operators, a stationarity system for control problems with point tracking objectives has been derived in [7].

Since stationarity systems of MPECs are typically (upon proper reformulation) non-smooth and, thus, hard to solve numerically, the algorithmic treatment of MPECs is delicated. In function space, penalty and smoothing approaches have been analyzed in [16, 30, 7], a relaxation method can found in [15], and a descent method has been implemented in [18]. All of these approaches apply some type of relaxation and/or smoothing of the control problem or the VI constraint. Consequently, the solution process depends on parameters which need to be taken to their limits in order to approach some type of stationary point of the original problem. Only, [18] may operate without smoothing depending on certain properties of the iterates generated by the associated algorithm; at biactive iterates, however, smoothing needs to be applied again.

In this paper, we extend the so-called elastic mode algorithm of [3] to the function space setting. One of the interesting aspects of this algorithm is related to the fact that it relies on an ℓ_1-type penalty approach which, under appropriate conditions, acts as an exact penalty method. Thus, a finite penalty parameter suffices to obtain a solution of the original problem. As, upon discretization, the condition number of the underlying stationarity system typically scales adversely with respect to increasing penalty parmeter, the exactness of the penalization is attractive as it allows to keep this parameter (and hence conditioning) bounded. Here we extend and study this method for L^2-tracking as well as for the point-tracking case, which require separate analyses, respectively.

The rest of the text is structured as follows: Section 2 treats the L^2-type control of variational inequalities. In order to develop an efficient solution algorithm for this problem class, we begin with the analysis of a penalty method in function space in Section 2.1. In contrast to other available penalty schemes for MPECs, this method does not smoothen the original problem but directly penalizes the critical complementarity condition in the variational inequality. In particular, we prove solvability of the auxiliary problem and consistency of the penalty scheme. Section 2.2 contains stationarity conditions for the auxiliary problem as well as a limiting stationarity system for the MPEC. In [3], the authors treat MPECs in a finite dimensional context, which is more general than a finite dimensional version of the problem treated here. Under certain conditions, strong stationarity of a solution obtained after a finite number of iterations (i.e. γ updates) is proven. The notion of strong stationarity is hereby based on the definition of active and biactive sets. In contrast to the finite dimensional case, where these are specified according to the set of indices where certain solution vectors are zero, it is not straightforward to define the zero set of objects in $H_0^1(\Omega)$ or $H^{-1}(\Omega)$. We give a definition that allows us to prove strong stationarity of feasible first order points of the auxiliary problem in Section 2.3.

In Section 3, we consider point tracking subject to variational inequalities, i.e., the functional $j : Y \to \mathbb{R}$ is given by $j(y) = \frac{1}{2}\sum_{w \in I}(y(w) - y_w)^2$ where $I \subset \Omega$ is finite and for all $w \in I$, $y_w \in \mathbb{R}$. Although the smoothing method of [7] can be understood as an iterative algorithm, that finds limiting ϵ-almost C-stationary points (or, in the finite dimensional world, C-stationary points) in the limit, we construct a method that penalized the critical complementarity constraint. In contrast to the analysis in the first part of this paper, we have to account for regularity questions that arise from the function evaluations in the objective functional. In particular, in Section 3.1 we modify the problem class and prove consistency of a penalty scheme. We show that a weak version of C-stationarity holds for limits of first order points of the auxiliary problem in Section 3.2.

Finally, Section 4 documents numerical results obtained by an algorithm associated with our analytic approach. We start with the description of the solution algorithm in Section 4.1 and provide two examples in Sections 4.2 and 4.3, respectively.

Notation. For a measurable subset $\omega \subset \Omega \subset \mathbb{R}$ we denote the characteristic function $\chi_\omega : \Omega \to \{0,1\}$, $\chi_\omega(w) = 1$ if $w \in \omega$ and else $\chi_\omega(w) = 0$, and the averaged characteristic function by $\bar{\chi}_\omega : \Omega \to \{0,1\}$, $\bar{\chi}_\omega(w) = \frac{1}{|\omega|}$ if $w \in \omega$ and else $\chi_\omega(w) = 0$. Here, we assume that the Lebesgue measure $|\omega|$ of ω is positive. If Ω is a Lipschitz domain, we denote the usual Lebesgue, Hilbert and Sobolev spaces by $L^2(\Omega), H_0^1(\Omega), W_0^{1,q}(\Omega)$, where $q \geq 1$, and the dual spaces by $H^{-1}(\Omega) = (H_0^1(\Omega))^*$, $W^{-1,q'}(\Omega) = (W_0^{1,q}(\Omega))^*$, where $q' = \frac{q}{q-1}$. The scalar product in a Hilbert space X is denoted by round brackets $(\cdot,\cdot)_X$. If X is a Banach space, then the duality pairing of an object $x^* \in X^*$ with an object $x \in X$ is denoted by $\langle x^*, x\rangle_{X^*}$. We set $\left(H_0^1(\Omega)\right)_+ := \{v \in H_0^1(\Omega) \,|\, v \geq 0 \text{ almost everywhere (a.e.) on } \Omega\}$ and $\left(H^{-1}(\Omega)\right)_+ := \{\psi \in H^{-1}(\Omega) \,|\, \forall v \in K : \langle \psi, v\rangle_{H^{-1}(\Omega)} \geq 0\}$. For a bounded linear operator $A : X \to Y$, where X and Y are Banach spaces, the corresponding adjoint operator is denoted by $A^* : Y^* \to X^*$.

2. The L^2-Tracking Case

In the present section we analyze the optimal control problem (1) under the following assumptions:

Assumption 2.1. For $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ is an open bounded domain; the functional $j : L^2(\Omega) \to \mathbb{R}$ is weakly lower semi-continuous, bounded from below and continuously Fréchet differentiable; the set of feasible controls is given by the box constraint

$$U_{ad} = \{v \in L^2(\Omega) \,|\, \underline{u} \leq v \leq \overline{u}\},$$

where $\underline{u}, \overline{u} \in L^2(\Omega) \cup \{-\infty, \infty\}$ satisfy $\underline{u} < \overline{u}$ a.e. in Ω; $\nu \geq 0$ and, if U_{ad} is not bounded in $L^2(\Omega)$, then $\nu > 0$. The feasible set K in the variational inequality is given by $K = \left(H_0^1(\Omega)\right)_+$; the operator $A : H_0^1(\Omega) \to H_0^1(\Omega)$ is linear, bounded and coercive; and $f \in L^2(\Omega)$.

The canonical example for j is the L^2-tracking objective $j(y) = \frac{1}{2}\left\|y - y_d\right\|^2_{L^2(\Omega)}$ where $y_d \in L^2(\Omega)$ is a given desired state. We restate problem (1) with the complementarity formulation of the variational inequality in the constraint, as follows:

$$\text{(2a)} \qquad \text{Minimize } J(y,u) := j(y) + \frac{\nu}{2}\left\|u\right\|^2_{L^2(\Omega)}$$

$$\text{(2b)} \qquad \text{over } (y,u,\xi) \in H_0^1(\Omega) \times U_{ad} \times H^{-1}(\Omega)$$

$$\text{(2c)} \qquad \text{subject to} \quad Ay - u - \xi = f \quad \text{in } H^{-1}(\Omega),$$

$$\text{(2d)} \qquad y \geq 0 \quad \text{in } H_0^1(\Omega), \quad \xi \geq 0 \quad \text{in } H^{-1}(\Omega), \text{ and } \langle \xi, y\rangle_{H^{-1}(\Omega)} = 0.$$

Below, whenever it is clear from the context, we leave off $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ within the inequalities in (2d).

2.1. **Solvability and Consistency of the Penalization Scheme.**

For a penalty parameter $\gamma > 0$, we define the following auxiliary problem:

$$\text{(3a)} \qquad \text{Minimize } J_\gamma(y,u,\xi) := j(y) + \frac{\nu}{2}\left\|u\right\|^2_{L^2(\Omega)} + \gamma\langle \xi, y\rangle_{H^{-1}(\Omega)}$$

$$\text{(3b)} \qquad \text{over } (y,u,\xi) \in H_0^1(\Omega) \times U_{ad} \times H^{-1}(\Omega)$$

$$\text{(3c)} \qquad \text{subject to } Ay - u - \xi = f,\ y \geq 0,\ \xi \geq 0.$$

Given the non-negativity of y and ξ, the term $\gamma\langle \xi, y\rangle_{H^{-1}(\Omega)}$ penalizes the ℓ_1-norm (i.e., the absolute value) of the constraint $\langle \xi, y\rangle_{H^{-1}(\Omega)} = 0$. Note that the auxiliary problem is in general non-convex.

The following lemma is needed to prove solvability and consistency of the penalty scheme.

Lemma 2.2. *We consider a bounded sequence* $(u_k)_{k\in\mathbb{N}} \subset L^2(\Omega)$ *and* $(y_k, \xi_k)_{k\in\mathbb{N}} \subset H_0^1(\Omega) \times H^{-1}(\Omega)$ *such that for all* $k \in \mathbb{N}$,

$$(4) \quad Ay_k - \xi_k = u_k + f \quad \text{in } H^{-1}(\Omega), \quad y_k \geq 0, \quad \xi_k \geq 0, \quad \langle \xi_k, y_k \rangle_{H^{-1}(\Omega)} \leq C,$$

where Ω*,* A *and* f *satisfy Assumption 2.1. Then, there exists a subsequence (also denoted by* $(y_k, \xi_k)_{k\in\mathbb{N}}$*) such that*

$$u_k \rightharpoonup u \quad \text{in } L^2(\Omega), \quad \xi_k \rightharpoonup \xi \quad \text{in } H^{-1}(\Omega), \quad y_k \rightharpoonup y \quad \text{in } H_0^1(\Omega),$$

and the limit $(y, u, \xi) \in H_0^1(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$ *satisfies*

$$Ay - \xi = u + f \quad \text{in } H^{-1}(\Omega), \quad y \geq 0, \quad \xi \geq 0,$$
$$\liminf \{ \langle \xi_k, y_k \rangle_{H^{-1}(\Omega)} \,|\, k \in \mathbb{N} \} \geq \langle \xi, y \rangle_{H^{-1}(\Omega)} \geq 0.$$

If in particular $\lim_{k\to\infty} \langle \xi_k, y_k \rangle_{H^{-1}(\Omega)} = 0$*, then* (y, u, ξ) *is the solution of the complementarity problem* (2c), (2d) *and we have the following strong convergences,*

$$y_k \to y \quad \text{in } H_0^1(\Omega), \quad \xi_k \to \xi \quad \text{in } H^{-1}(\Omega).$$

Proof. We test the equation in (4) with y_k and use the coercivity of A to obtain the estimate

$$\begin{aligned} C \|y_k\|_{H_0^1(\Omega)}^2 &\leq \langle Ay_k, y_k \rangle_{H^{-1}(\Omega)} = (u_k + f, y_k)_{L^2(\Omega)} + \langle \xi_k, y_k \rangle_{H^{-1}(\Omega)} \\ &\leq (\|u_k\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}) \|y_k\|_{H_0^1(\Omega)} + \langle \xi_k, y_k \rangle_{H^{-1}(\Omega)}. \end{aligned} \tag{5}$$

The bounds on $\langle \xi_k, y_k \rangle_{H^{-1}(\Omega)}$ and on u_k thus yield a uniform bound on $\|y_k\|_{H_0^1(\Omega)}$. Moreover it holds that

$$\|\xi_k\|_{H^{-1}(\Omega)} = \|Ay_k - u_k - f\|_{H^{-1}(\Omega)} \leq C. \tag{6}$$

We now consider a subsequence still denoted by (y_k, u_k, ξ_k) with weak limit (y, u, ξ) in $H_0^1(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$. The limit satisfies $y \geq 0$, $\xi \geq 0$ and it holds that

$$0 = Ay_k - \xi_k - u_k - f \rightharpoonup Ay - \xi - u - f \text{ in } H^{-1}(\Omega). \tag{7}$$

We thus have $Ay - \xi - u - f = 0$ in $H^{-1}(\Omega)$. The compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ yields strong convergence of $(y_k)_{k\in\mathbb{N}}$ to its limit y in $L^2(\Omega)$. Hence, the product $(u_k, y_k)_{L^2(\Omega)}$ converges to $(u, y)_{L^2(\Omega)}$ and the weak lower semi-continuity of $z \mapsto \langle Az, z \rangle_{H^{-1}(\Omega)}$ in $H_0^1(\Omega)$ yields that

$$\begin{aligned} \liminf_{k\in\mathbb{N}} \langle \xi_k, y_k \rangle_{H^{-1}(\Omega)} &= \liminf_{k\in\mathbb{N}} \langle Ay_k, y_k \rangle_{H^{-1}(\Omega)} - (u_k, y_k)_{L^2(\Omega)} - (f, y_k)_{L^2(\Omega)} \\ &\geq \langle Ay, y \rangle_{H^{-1}(\Omega)} - (u, y)_{L^2(\Omega)} - (f, y)_{L^2(\Omega)} = \langle \xi, y \rangle_{H^{-1}(\Omega)} \geq 0. \end{aligned}$$

This proves the first part of the assertion. We know from (4) and (7) that

$$0 = Ay - u - \xi - f - (Ay_k - u_k - \xi_k - f) = A(y - y_k) - (u - u_k) - (\xi - \xi_k).$$

Using $y_k - y$ as test function, and the coercivity of A, we obtain that

$$C \|y - y_k\|_{H_0^1(\Omega)}^2 \leq (u - u_k, y - y_k)_{L^2(\Omega)} - \langle \xi - \xi_k, y - y_k \rangle_{H^{-1}(\Omega)}.$$

The first product in the last term converges to zero as $k \to \infty$ owing to the weak convergence of $u - u_k$ to zero, and the strong convergence of $y - y_k$ to zero in $L^2(\Omega)$. The second product can be expressed as

$$\langle \xi - \xi_k, y - y_k \rangle_{H^{-1}(\Omega)} = \langle \xi, y \rangle_{H^{-1}(\Omega)} - \langle \xi_k, y \rangle_{H^{-1}(\Omega)} - \langle \xi, y_k \rangle_{H^{-1}(\Omega)} + \langle \xi_k, y_k \rangle_{H^{-1}(\Omega)}.$$

The mixed terms in the middle both converge to $\langle \xi, y\rangle_{H^{-1}(\Omega)}$. If we have the additional assumption that $\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)} \to 0$ for $k \to \infty$, then we have $\langle \xi, y\rangle_{H^{-1}(\Omega)} = 0$ by the first part of the lemma, and thus, $\langle \xi - \xi_k, y - y_k\rangle_{H^{-1}(\Omega)} \to 0$ for $k \to \infty$. This shows that $\|y - y_k\|^2_{H^1_0(\Omega)} \to 0$, i.e. the strong convergence of $(y_k)_{k\in\mathbb{N}}$ to y in $H^1_0(\Omega)$ and thus also the convergence of $(\xi_k)_{k\in\mathbb{N}}$ to $\xi = Ay - u - f$ in $H^{-1}(\Omega)$. □

Note that although in the solution of the variational inequality the slack variable ξ satisfies $\xi \in L^2(\Omega)$, we can not guarantee this regularity for ξ_k in the auxiliary problem. An attempt to prove convergence of the slack variables in $L^2(\Omega)$ thus fails in our setting.

Given Ω, A and f due to Assumption 2.1 we denote the feasible set of (3) by

$$\mathcal{F} := \{(y, u, \xi) \in H^1_0(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega) \,|\, Ay - u - \xi = f, \quad y \geq 0, \quad \xi \geq 0\}.$$

Note that $\mathcal{F}$ does not depend on the penalty parameter γ.

With this preparation, we can state the following existence result:

Proposition 2.3. *If Assumption 2.1 holds, then for every penalty parameter $\gamma > 0$, problem* (3) *has a solution $(y_\gamma, u_\gamma, \xi_\gamma)$.*

Proof. The objective J_γ is bounded from below on the feasible set $\mathcal{F}$. Assume that $(y_k, u_k, \xi_k)_{k\in\mathbb{N}}$ is an infimizing sequence, i.e. for all $k \in \mathbb{N}$, $(y_k, u_k, \xi_k) \in \mathcal{F}$ is feasible and

$$\lim_{k\to\infty} J_\gamma(y_k, u_k, \xi_k) = \inf\{J_\gamma(y, u, \xi) \,|\, (y, u, \xi) \in \mathcal{F}\} =: M.$$

Since the sequence of objective values $\big(J_\gamma(y_k, u_k, \xi_k)\big)_{k\in\mathbb{N}}$ is bounded from above owing to its convergence and since the first summands $j(y_k) + \frac{\nu}{2}\|u_k\|^2_{L^2(\Omega)}$ are bounded from below, we infer that $\big(\gamma\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)}\big)_{k\in\mathbb{N}}$ is bounded from above. The sequence $(u_k)_{k\in\mathbb{N}}$ is bounded by Assumption 2.1. The feasibility of $(y_k, u_k, \xi_k) \in \mathcal{F}$ then guarantees that (4) is satisfied such that we can apply the first part of Lemma 2.2. This yields a subsequence with weak limit (y, u, ξ) which is feasible and ensures that $\liminf\{\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)} \,|\, k \in \mathbb{N}\} \geq \langle \xi, y\rangle_{H^{-1}(\Omega)}$. The weak lower semi-continuity of j and the norm $\|\cdot\|_{L^2(\Omega)}$ imply the optimality of the limit as follows,

$$M \geq \liminf_{k\in\mathbb{N}} j(y_k) + \liminf_{k\in\mathbb{N}} \frac{\nu}{2}\|u_k\|^2_{L^2(\Omega)} + \liminf_{k\in\mathbb{N}} \gamma\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)} \geq J_\gamma(y, u, \xi). \quad \square$$

The next proposition states consistency of the penalty scheme.

Proposition 2.4. *Assume that $(\gamma_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ with $\gamma_k > 0$ and $\gamma_k \to \infty$ for $k \to \infty$, and that for every $k \in \mathbb{N}$, (y_k, u_k, ξ_k) solves* (3) *with $\gamma = \gamma_k$ and Ω, A and f from Assumption 2.1. Then there exists a subsequence, still denoted by $(y_k, u_k, \xi_k)_{k\in\mathbb{N}}$ such that*

$$y_k \to y^\star \quad \text{in } H^1_0(\Omega), \qquad u_k \rightharpoonup u^\star \quad \text{in } L^2(\Omega), \qquad \xi_k \to \xi^\star \quad \text{in } H^{-1}(\Omega)$$

and $(y^\star, u^\star, \xi^\star)$ solves the optimal control problem (2).

Proof. For all $k \in \mathbb{N}$ and a tuple (y, u, ξ) satisfying the complementarity problem (2d), optimality of (y_k, u_k, ξ_k), feasibility of $(y, u, \xi) \in \mathcal{F}$ and the definition of J_{γ_k} imply that

$$J_{\gamma_k}(y_k, u_k, \xi_k) \leq J_{\gamma_k}(y, u, \xi) = J(y, u).$$

Given the boundedness of j and $\frac{\nu}{2}\|\cdot\|^2_{L^2(\Omega)}$ from below, one derives the uniform boundedness of $\gamma_k\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)}$. Additionally, $\|u_k\|_{L^2(\Omega)} \le C$ by Assumption 2.1. Lemma 2.2 thus yields a subsequence with weak limit $(y^\star, u^\star, \xi^\star)$. Since U_{ad} is closed and convex, it is weakly closed and thus contains the weak limit $u^\star \in U_{ad}$. The convergence of $\gamma_k \to \infty$ implies that $\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)} \to 0$ and the second part of Lemma 2.2 proves feasibility of $(y^\star, u^\star, \xi^\star)$ in (2) and the strong convergence of y_k and ξ_k. Moreover, using the weak lower semi-continuity of J, the non-negativity of the penalty term and optimality of (y_k, u_k, ξ_k) for the auxiliary problem, we obtain that for any feasible (y, u, ξ) it holds that

$$\begin{aligned} J(y^\star, u^\star) &\le \liminf\{J(y_k, u_k) \mid k \in \mathbb{N}\} \le \liminf\{J_{\gamma_k}(y_k, u_k, \xi_k) \mid k \in \mathbb{N}\} \\ &\le \liminf\{J_{\gamma_k}(y, u, \xi) \mid k \in \mathbb{N}\} = J(y, u). \end{aligned}$$

Therefore $(y^\star, u^\star)$ is feasible and optimal and thus a solution of (2). □

2.2. **First Order Stationarity.** In order to derive first order stationarity conditions for the penalized problem (3) we aim to apply [33, Thm.3.1] and thus have to guarantee the respective constraint qualification (regularity of solutions in the sense of [33]).

We define the following spaces and mappings:

$$\begin{aligned} &X = H^1_0(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega), && Y = H^{-1}(\Omega) \times H^1_0(\Omega) \times H^{-1}(\Omega), \\ &\mathcal{C} = H^1_0(\Omega) \times U_{ad} \times H^{-1}(\Omega), && \mathcal{K} = \{0\} \times K \times \left(H^{-1}(\Omega)\right)_+, \\ &g : X \to Y, && g(y, u, \xi) = (Ay - u - \xi - f, y, \xi). \end{aligned}$$

With these denotations, we can formulate the following statement on regularity of feasible points (and in particular of solutions) of the auxiliary problem.

Proposition 2.5. *Under Assumption 2.1, every feasible point $\bar{x} = (y_\gamma, u_\gamma, \xi_\gamma)$ of problem* (3) *with penalty parameter $\gamma > 0$ is regular in the sense of* [33].

Proof. Assume that $z = (z_1, z_2, z_3) \in Y$. We aim to show that there exist $c \in \mathcal{C}(\bar{x})$ and $k \in \mathcal{K}(g(\bar{x}))$ such that $g'(\bar{x})c - k = z$. Note that $\mathcal{C}(\bar{x}) = H^1_0(\Omega) \times U_{ad}(u_\gamma) \times H^{-1}(\Omega)$, where $U_{ad}(u_\gamma) = \{\beta(\tilde{c}_u - u_\gamma) \mid \beta \ge 0, \tilde{c}_u \in U_{ad}\}$. Due to the feasibility of $\bar{x}$ for problem (3), it holds that $g(\bar{x}) = (0, y_\gamma, \xi_\gamma)$. We therefore have

$$\mathcal{K}(g(\bar{x})) = \Big\{(0, k_y - \beta y_\gamma, k_\xi - \beta \xi_\gamma) \mid k_y \in K, k_\xi \in \left(H^{-1}(\Omega)\right)_+, \beta \ge 0\Big\}. \tag{8}$$

The Fréchet derivative of g in $\bar{x}$ applied to $c = (c_y, c_u, c_\xi) \in C(\bar{x})$ reads

$$g'(y_\gamma, u_\gamma, \xi_\gamma)(c_y, c_u, c_\xi) = (Ac_y - c_u - c_\xi, c_y, c_\xi). \tag{9}$$

We choose $c_u = 0 \in U_{ad}(u_\gamma)$ and define $(k_y, k_\xi) \in K \times \left(H^{-1}(\Omega)\right)_+$ as the solution to the complementarity problem

$$Ak_y - k_\xi = z_1 + c_u - Az_2 + z_3 \in H^{-1}(\Omega), \quad k_y \ge 0, \quad k_\xi \ge 0, \quad \langle k_\xi, k_y\rangle_{H^{-1}(\Omega)} = 0.$$

Then, with $\beta = 0$, $c_y = z_2 + k_y$ and $c_\xi = z_3 + k_\xi$, it holds that

$$g'(\bar{x})\begin{pmatrix} c_y \\ c_u \\ c_\xi \end{pmatrix} - \begin{pmatrix} 0 \\ k_y - \beta y_\gamma \\ k_\xi - \beta \xi_\gamma \end{pmatrix} = \begin{pmatrix} Ak_y + Az_2 - c_u - k_\xi - z_3 \\ k_y + z_2 - k_y \\ k_\xi + z_3 - k_\xi \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}. \qquad \square$$

Proposition 2.5 yields the following proposition on necessary first order conditions for optimal points of the auxiliary problem.

Proposition 2.6. *Every solution $(y_\gamma, u_\gamma, \xi_\gamma)$ of problem (3) with penalty parameter $\gamma > 0$ and Assumption 2.1 is a first order point for problem (3), i.e., there exists a multiplier tuple $(p_\gamma, \vartheta_\gamma, \tau_\gamma) \in Y^\star$ such that the following conditions hold,*

$$Ay_\gamma - u_\gamma - \xi_\gamma - f = 0 \quad \text{in } H^{-1}(\Omega), \tag{10a}$$

$$y_\gamma \geq 0, \quad \xi_\gamma \geq 0, \tag{10b}$$

$$A^\star p_\gamma + j'(y_\gamma) + \gamma\xi_\gamma - \vartheta_\gamma = 0 \quad \text{in } H^{-1}(\Omega), \tag{10c}$$

$$u_\gamma - \mathrm{Proj}_{U_{ad}}\left(\frac{1}{\nu}p_\gamma\right) = 0 \quad \text{in } L^2(\Omega), \tag{10d}$$

$$\gamma y_\gamma - p_\gamma - \tau_\gamma = 0 \quad \text{in } H_0^1(\Omega), \tag{10e}$$

$$\vartheta_\gamma \geq 0, \quad \langle \vartheta_\gamma, y_\gamma\rangle_{H^{-1}(\Omega)} = 0, \quad \tau_\gamma \geq 0, \quad \langle \xi_\gamma, \tau_\gamma\rangle_{H^{-1}(\Omega)} = 0. \tag{10f}$$

For a first order point $(y_\gamma, u_\gamma, \xi_\gamma)$ of the auxiliary problem (3) with the multiplier vector $(p_\gamma, \vartheta_\gamma, \tau_\gamma)$ we define

$$\lambda_\gamma := \vartheta_\gamma - \gamma\xi_\gamma \quad \text{and} \quad \mu_\gamma := \tau_\gamma - \gamma y_\gamma. \tag{11}$$

We will show that for $\gamma_k \to \infty$, there exist accumulation points of $(\lambda_{\gamma_k})_{k\in\mathbb{N}}$ and $(\mu_{\gamma_k})_{k\in\mathbb{N}}$ which play the role of the multipliers λ and μ in the C-stationarity system for the MPEC (1). The following lemma provides the required uniform bounds and thus prepares the proof of this convergence result.

Proposition 2.7. *Assume that besides the standard Assumption 2.1, $\gamma > 0$ is given and that $(y_\gamma, u_\gamma, \xi_\gamma) \in X$ is a first order point of problem (3) with multiplier tuple $(p_\gamma, \vartheta_\gamma, \tau_\gamma) \in Y^\star$. If $\|u_\gamma\|_{L^2(\Omega)} \leq C$ and $\langle \xi_\gamma, y_\gamma\rangle_{H^{-1}(\Omega)} \leq C$, then the following estimates hold with a constant $C > 0$ that does not depend on γ:*

$$\langle \lambda_\gamma, \mu_\gamma\rangle_{H^{-1}(\Omega)} \geq 0, \qquad \|\mu_\gamma\|_{H_0^1(\Omega)} \leq C, \qquad 0 \leq \gamma^2\langle \xi_\gamma, y_\gamma\rangle_{H^{-1}(\Omega)} \leq C, \tag{12a}$$

$$\|\lambda_\gamma\|_{H^{-1}(\Omega)} \leq C, \qquad -C \leq \gamma\langle \xi_\gamma, \mu_\gamma\rangle_{H^{-1}(\Omega)} \leq 0. \tag{12b}$$

Proof. Given the feasibility of first order points and the uniform bound on $\|u_\gamma\|_{L^2(\Omega)}$ and on $\langle \xi_\gamma, y_\gamma\rangle_{H^{-1}(\Omega)}$, we can utilize (5) and (6) in the proof of Lemma 2.2 and derive uniform bounds on $\|y_\gamma\|_{H_0^1(\Omega)}$ and on $\|\xi_\gamma\|_{H^{-1}(\Omega)}$. Multiply the adjoint equation (10c) by p_γ and use the coercivity of A^* and the definition of λ_γ to obtain the estimate

$$c\,\|p_\gamma\|^2_{H_0^1(\Omega)} \leq \langle A^* p_\gamma, p_\gamma\rangle_{H^{-1}(\Omega)} = \langle \lambda_\gamma, p_\gamma\rangle_{H^{-1}(\Omega)} - (j'(y_\gamma), p_\gamma)_{L^2(\Omega)}\,.$$

The fact that $j : L^2(\Omega) \to \mathbb{R}$ is continuously Fréchet-differentiable by Assumption 2.1 yields that

$$|\,(j'(y_\gamma), p_\gamma)_{L^2(\Omega)}\,| \leq \|j'(y_\gamma)\|_{\mathcal{L}(L^2(\Omega),\mathbb{R})}\,\|p_\gamma\|_{L^2(\Omega)} \leq C\,\|p_\gamma\|_{L^2(\Omega)}\,.$$

This yields the estimate

$$c\,\|p_\gamma\|^2_{H_0^1(\Omega)} - \langle \lambda_\gamma, p_\gamma\rangle_{H^{-1}(\Omega)} \leq C\,\|p_\gamma\|_{H_0^1(\Omega)}\,.$$

Since $p_\gamma = -\mu_\gamma$ by (10e) and the definition of μ_γ, we can write

$$c\,\|\mu_\gamma\|^2_{H_0^1(\Omega)} + \langle \lambda_\gamma, \mu_\gamma\rangle_{H^{-1}(\Omega)} \leq C\,\|\mu_\gamma\|_{H_0^1(\Omega)}\,. \tag{13}$$

The definition of λ_γ and μ_γ in (11) yields, together with the complementarity and sign conditions in (10f), that

$$\langle \lambda_\gamma, \mu_\gamma \rangle_{H^{-1}(\Omega)} = \gamma^2 \langle \xi_\gamma, y_\gamma \rangle_{H^{-1}(\Omega)} + \langle \vartheta_\gamma, \tau_\gamma \rangle_{H^{-1}(\Omega)} \geq 0.$$

This is the first estimate in (12a). Together with (13) it additionally guarantees the second bound in (12a). We plug the expression for the dual pairing $\langle \lambda_\gamma, \mu_\gamma \rangle_{H^{-1}(\Omega)}$ into (13) and obtain

$$c \, \|\mu_\gamma\|^2_{H^1_0(\Omega)} + \gamma^2 \langle \xi_\gamma, y_\gamma \rangle_{H^{-1}(\Omega)} + \langle \vartheta_\gamma, \tau_\gamma \rangle_{H^{-1}(\Omega)} \leq C \, \|\mu_\gamma\|_{H^1_0(\Omega)} \leq C,$$

and in particular, the last bound in (12a). We once again employ the adjoint equation (10c) to bound $\|\lambda_\gamma\|_{H^{-1}(\Omega)} = \|A^* p_\gamma + y_\gamma - y_d\|_{H^{-1}(\Omega)} \leq C$. Finally, making use of (10e) and the complementarity $\langle \tau_\gamma, \xi_\gamma \rangle_{H^1_0(\Omega)} = 0$, we have

$$\gamma \langle \xi_\gamma, \mu_\gamma \rangle_{H^{-1}(\Omega)} = \gamma \langle \xi_\gamma, -p_\gamma \rangle_{H^{-1}(\Omega)} = \gamma \langle \xi_\gamma, \tau_\gamma - \gamma y_\gamma \rangle_{H^{-1}(\Omega)} = -\gamma^2 \langle \xi_\gamma, y_\gamma \rangle_{H^{-1}(\Omega)}$$

and thus the same estimates hold for both terms and we proved (12b). □

Assume that $(\gamma_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ with $\gamma_k > 0$ and $\gamma_k \to \infty$ for $k \to \infty$, and that for every $k \in \mathbb{N}$, $(y_k, u_k, \xi_k, p_k, \vartheta_k, \tau_k)$ satisfies (10) with $\gamma = \gamma_k$. If $(\|u_k\|_{L^2(\Omega)})_{k\in\mathbb{N}}$ and $\langle \xi_k, y_k \rangle_{H^{-1}(\Omega)}$ are bounded, then we can apply Proposition 2.7 to obtain the uniform boundedness of the sequences

$$(\|p_k\|_{H^1_0(\Omega)})_{k\in\mathbb{N}}, \quad (\|\lambda_k\|_{H^{-1}(\Omega)})_{k\in\mathbb{N}}, \quad (\|\mu_k\|_{H^1_0(\Omega)})_{k\in\mathbb{N}} \quad \text{and} \quad \gamma_k^2 \langle \xi_k, y_k \rangle_{H^{-1}(\Omega)}.$$

Then, one can extract subsequences with weak limits $p^\star$, $\lambda^\star$ and $\mu^\star$. Additionally, the last bound in (12a) implies that $\langle \xi_k, y_k \rangle_{H^{-1}(\Omega)} \to 0$ for $k \to \infty$. The second part of Lemma 2.2 thus yields the strong convergences of $y_k \to y^\star$ in $H^1_0(\Omega)$ and $\xi_k \to \xi^\star$ in $H^{-1}(\Omega)$. Using the compact embedding of $H^{-1}(\Omega)$ into $L^2(\Omega)$ we find that $p_k \to p^\star$ in $L^2(\Omega)$ and so, the continuity of the projection operator $\mathrm{Proj}_{U_{ad}} : L^2(\Omega) \to L^2(\Omega)$ gives that $u_k = \mathrm{Proj}_{U_{ad}}\left(\frac{1}{\nu} p_k\right) \to \mathrm{Proj}_{U_{ad}}\left(\frac{1}{\nu} p^\star\right) = u^\star$ in $L^2(\Omega)$. This yields the following corollary:

Corollary 2.8. *With the notation and under the assumptions of the previous paragraph and in particular, under the condition that $(u_k)_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, there exists a subsequence of $(y_k, u_k, \xi_k, p_k, \vartheta_k, \tau_k)_{k\in\mathbb{N}}$ (denoted the same) such that*

$$\begin{aligned} &y_k \to y^\star \quad \text{in } H^1_0(\Omega), && u_k \to u^\star \quad \text{in } L^2(\Omega), && \xi_k \to \xi^\star \quad \text{in } H^{-1}(\Omega) \\ &p_k \rightharpoonup p^\star \quad \text{in } H^1_0(\Omega), && \vartheta_k - \gamma_k \xi_k \rightharpoonup \lambda^\star \quad \text{in } H^{-1}(\Omega), && \tau_k - \gamma_k y_k \rightharpoonup \mu^\star \quad \text{in } H^1_0(\Omega). \end{aligned}$$

We are now ready to prove limiting stationarity conditions.

Theorem 2.9. *With the notation and under the assumptions of Corollary 2.8, the limit point $(y^\star, u^\star, \xi^\star, p^\star, \lambda^\star, \mu^\star)$ satisfies the following conditions:*

$$Ay^\star - u^\star - \xi^\star = f \quad \text{in } H^{-1}(\Omega), \tag{14a}$$

$$\xi^\star \geq 0, \quad y^\star \geq 0, \quad \langle \xi^\star, y^\star \rangle_{H^{-1}(\Omega)} = 0, \tag{14b}$$

$$A^* p^\star + j'(y^\star) - \lambda^\star = 0 \quad \text{in } H^{-1}(\Omega), \tag{14c}$$

$$u^\star - \mathrm{Proj}_{U_{ad}}\left(\frac{1}{\nu} p^\star\right) = 0 \quad \text{in } H^1_0(\Omega), \tag{14d}$$

$$\mu^\star + p^\star = 0 \quad \text{in } H^1_0(\Omega), \tag{14e}$$

$$\langle \lambda^\star, y^\star \rangle_{H^{-1}(\Omega)} = 0, \ \langle \mu^\star, \xi^\star \rangle_{H^{-1}(\Omega)} = 0, \ \langle \lambda^\star, \mu^\star \rangle_{H^{-1}(\Omega)} \geq 0. \tag{14f}$$

Proof. The second part of Lemma 2.2 yields the feasibility of $(y^\star, u^\star, \xi^\star, p^\star, \lambda^\star, \mu^\star)$ for problem (2) as stated in (14a)-(14b). Moreover the limit point satisfies the adjoint equation (14c) owing to weak continuity of A^*, the continuity of j' and the definition of λ_k. The continuity of the projection operator and the strong convergence of p_k to p in $L^2(\Omega)$ as well as the definition of μ_k and (10e) directly resolve to (14d) and (14e). The dual pairing $\langle \xi_k, p_k\rangle_{H^{-1}(\Omega)}$ converges to $-\langle \xi^\star, \mu^\star\rangle_{H^{-1}(\Omega)} = \langle \xi^\star, p^\star\rangle_{H^{-1}(\Omega)}$. On the other hand, the uniform bound on $\big(\gamma_k\langle \xi_k, p_k\rangle_{H^{-1}(\Omega)}\big)_{k\in\mathbb{N}}$ from (12b) in Proposition 2.7 implies that

$$-\langle \xi^\star, \mu^\star\rangle_{H^{-1}(\Omega)} = \lim_{k\to\infty} \langle \xi_k, p_k\rangle_{H^{-1}(\Omega)} = 0.$$

Similarly, we infer from the complementarity of ϑ_k and y_k in (10f) and from the uniform bound on $(\gamma_k^2\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)})_{k\in\mathbb{N}}$ in (12b) that

$$\langle \lambda^\star, y^\star\rangle_{H^{-1}(\Omega)} = \lim_{k\to\infty} \langle \vartheta_k, y_k\rangle_{H^{-1}(\Omega)} - \gamma_k\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)} = 0.$$

Finally, (12a) in Proposition 2.7 implies that $\liminf\{\langle \lambda_k, \mu_k\rangle_{H^{-1}(\Omega)} \,|\, k \in \mathbb{N}\} \geq 0$ and we employ the adjoint equation in order to show that

$$\langle \lambda^\star, \mu^\star\rangle_{H^{-1}(\Omega)} \geq \liminf\{\langle \lambda_k, \mu_k\rangle_{H^{-1}(\Omega)} \,|\, k \in \mathbb{N}\} \geq 0 \tag{15}$$

as it is postulated in (14f): Firstly, basic considerations ensure the following estimate,

$$\begin{aligned}\liminf\{-\langle Ap_k, p_k\rangle_{H^{-1}(\Omega)} \,|\, k \in \mathbb{N}\} &= -\limsup\{\langle Ap_k, p_k\rangle_{H^{-1}(\Omega)} \,|\, k \in \mathbb{N}\}\\ &\leq -\liminf\{\langle Ap_k, p_k\rangle_{H^{-1}(\Omega)} \,|\, k \in \mathbb{N}\}.\end{aligned}$$

The weak lower semi-continuity of the mapping $v \mapsto \langle Av, v\rangle_{H^{-1}(\Omega)}$ then results in

$$-\liminf\{\langle Ap_k, p_k\rangle_{H^{-1}(\Omega)} \,|\, k \in \mathbb{N}\} \leq -\langle Ap^\star, p^\star\rangle_{H^{-1}(\Omega)}.$$

Secondly we infer the convergence $(j'(y_k), p_k)_{L^2(\Omega)} \to (j'(y^\star), p^\star)_{L^2(\Omega)}$ from the continuity of j' and the strong convergence of y_k to y in $L^2(\Omega)$. Now we examine the limit inferior in (15):

$$\begin{aligned}\liminf\{\langle \lambda_k, \mu_k\rangle_{H^{-1}(\Omega)} \,|\, k \in \mathbb{N}\} &= \liminf\{-\langle Ap_k, p_k\rangle_{H^{-1}(\Omega)} - (j'(y_k), p_k)_{L^2(\Omega)} \,|\, k \in \mathbb{N}\}\\ &= \liminf\{-\langle Ap_k, p_k\rangle_{H^{-1}(\Omega)} \,|\, k \in \mathbb{N}\} - (j'(y^\star), p^\star)_{L^2(\Omega)}\\ &\leq -\langle Ap^\star, p^\star\rangle_{H^{-1}(\Omega)} - (j'(y^\star), p^\star)_{L^2(\Omega)} = \langle \lambda^\star, \mu^\star\rangle_{H^{-1}(\Omega)}.\end{aligned}$$

This yields the sign condition in (14f). □

The conditions stated in Theorem 2.9 are weaker than the C- or strong stationarity conditions that are known from the literature, cf. [23, 15, 32]. But, as e.g. in [15], the analysis here is constructive in the sense that it suggests an iterative solution algorithm for the MPEC. We show in the subsequent section, that under certain conditions, such solutions are strongly stationary.

2.3. **Remarks on Exactness of the Penalty Scheme and on Strong Stationarity.** In [3] the authors assume an algorithm that finds second order points of the penalized problem for a sequence $(\gamma_k)_{k\in\mathbb{N}}$ of penalty parameters with $\lim_{k\to\infty}\gamma_k = \infty$. It terminates as soon as the penalized complementarity is satisfied exactly, i.e., if $\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)} = 0$. Theorem 4 in [3] says that if this algorithm does not terminate after a finite number of steps then every accumulation point of the generated sequence of solutions $(x_k)_{k\in\mathbb{N}}$ is either infeasible for the original MPEC or fails to satisfy MPEC-LICQ (which requires that an MPEC satisfies the linear independence constraint qualification (LICQ) when the product condition $\langle \xi, y\rangle_{H^{-1}(\Omega)} = 0$ is omitted, cf. [3, Def.2]). If no control constraints are active on the biactive set where $y = 0$ and $\xi = 0$ simultaneously, then an analogue to the MPEC-LICQ in function space is satisfied in every feasible point. In particular, when there are no control constraints, this constraint qualification is satisfied. This means that in the special case discussed here, and if the elastic mode penalty method in the sense of [3] computes second order points of the auxiliary problem, then after a finite number of steps the iterate is feasible for the original problem. Theorem 2 in [3] proves that any first order point of the auxiliary problem which is feasible for the MPEC is in fact strongly stationary. In total, the two theorems thus indicate exactness of the penalty method (if second order points are computed), which means that for every finite-dimensional restriction of the MPEC (2), it computes a strongly stationary point after a finite number of iterations.

This fact provokes the question whether it is possible to prove exactness of the penalty scheme also in function space and, indeed, we prove here the counterpart of [3, Thm.2] in function space. It is not clear how the second ingredient, namely the convergence of a sequence of second order points, can be utilized to prove a result that is analog to [3, Thm.4]. Moreover, the non-convexity in the objective of the auxiliary problem seems to preclude second order conditions.

We define the zero set of a function y in $H_0^1(\Omega)$ in the same way as e.g. in [26]: Utilizing a quasi-continuous representative $\tilde{y}$ of y, see e.g. [6], we set

$$A^f(y) := \{\tilde{y} = 0\} = \{x \in \Omega \,|\, \tilde{y}(x) = 0\}. \tag{16}$$

Since the quasi-continuous representative is unique up to capacity zero, cf. [6, Lemma 6.55], this definition is also unique up to a set of capacity zero. In general we abbreviate $\{y =^f 0\} := \{x \in \Omega \,|\, \tilde{y}(x) = 0\}$, $\{y >^f 0\} := \{x \in \Omega \,|\, \tilde{y}(x) > 0\}$, etc. as sets defined up to set of capacity zero. We may thus understand $A^f(y)$ without specifying $\tilde{y}$, as the set where any representative is zero quasi everywhere (q.e.), such that in contrast to the set $\{y = 0\}$ which is defined in the sense of almost everywhere, the set $A^f(y)$ is defined in the sense of quasi everywhere.

Since any feasible ξ is non-negative, it can be interpreted as a positive measure, [2] and we define the *finely active set* utilizing the fine topology ([2, Sec.6.3]) as it is done in the preprint [32]:

$$A^f(\xi) := \{\xi =^f 0\} := \bigcup \{\omega \subset \Omega \,|\, \omega \text{ finely open}, \ \xi(\omega) = 0\}. \tag{17}$$

This set is the union of finely open sets, and thus finely open.

Its complement f-supp(ξ), the *fine support of* ξ, is used to define the *fine strongly active set* A_s^f and the *finely biactive set* B^f up to a set of capacity zero as follows,

$$A_s^f(y,\xi) := \text{f-supp}(\xi) \cap A^f(y) = (\Omega \setminus A^f(\xi)) \cap A^f(y), \tag{18}$$

$$B^f(y,\xi) := A^f(y) \cap A^f(\xi). \tag{19}$$

At first, we prove two lemmas which appear in a similar form in [32].

Lemma 2.10. *Assume that $v \in H_0^1(\Omega)$ and $\zeta \in H^{-1}(\Omega)$ are both non-negative, and that the dual pairing $\langle \zeta, v\rangle_{H^{-1}(\Omega)} = 0$. Then, it holds that $v = 0$ ζ-almost everywhere on Ω and in particular,*

$$\zeta(\{x \in \Omega \,|\, v(x) > 0\}) = 0.$$

Proof. The complementarity conditions $v \geq 0$, $\zeta \geq 0$, $\langle \zeta, v \rangle_{H^{-1}(\Omega)} = 0$ are equivalent to the variational inequality

$$\forall z \in H_0^1(\Omega), \quad z \geq 0: \qquad \langle \zeta, z - v \rangle_{H^{-1}(\Omega)} \geq 0.$$

Consider a compact subset C of Ω and a smooth function $\chi_C \in C_0^\infty(\Omega)$ with compact support in Ω which takes values in the interval $[0,1]$ and is equal to 1 on C. We set $z = (1 - \chi_C)v \in H_0^1(\Omega)$. Since $z \geq 0$, the assumptions yield $\langle \zeta, z - v \rangle_{H^{-1}(\Omega)} \geq 0$. On the other hand, we can write $z - v = -\chi_C v \leq 0$ and infer that $\langle \zeta, z - v \rangle_{H^{-1}(\Omega)} \leq 0$ from the signs of ζ and $-\chi_C v$. These two inequalities imply $\langle \zeta, \chi_C v \rangle_{H^{-1}(\Omega)} = 0$. We write the dual pairing as a finite integral with respect to the measure ζ,

$$0 = \langle \zeta, \chi_C v \rangle_{H^{-1}(\Omega)} = \int_\Omega \chi_C v \, d\zeta,$$

and employ the non-negativity of $\chi_C v$ to obtain that $\chi_C v = 0$ ζ-almost-everywhere, which in turn means that $v = 0$ ζ-almost-everywhere on arbitrary compact sets $C \subset \Omega$. Finally, Ω is the countable union of compact sets, e.g. of all closed balls with rational midpoints and rational radii in Ω, and the σ-additivity of $\langle \zeta, \cdot \rangle_{H^{-1}(\Omega)}$ yields the assertion. □

Lemma 2.11. *For $\xi \in \left(H^{-1}(\Omega)\right)_+$ and $y \in K$ with $y = 0$ ξ-a.e. it holds that*

$$\{v \in H_0^1(\Omega) \,|\, v = 0 \ \xi\text{-a.e.}\} = \{v \in H_0^1(\Omega) \,|\, v = 0 \ \text{q.e. on } A_s^f(y,\xi)\}.$$

Proof. Assume that $z \in \{v \in H_0^1(\Omega) \,|\, v = 0 \ \xi\text{-a.e.}\}$. We thus know that $\xi(\{z \neq^f 0\}) = 0$, and the set $\{z \neq^f 0\}$ can be supposed to be finely open because it is quasi-open and differs only by a set of capacity zero from its fine interior (cf. [2, Thm.6.4.13]). Hence, (17) guarantees that $\{z \neq^f 0\} \subset \{\xi =^f 0\} = \Omega \setminus \text{f-supp}(\xi)$ and one can infer from (18) that

$$\text{cap}\left(\{z \neq^f 0\} \cap A_s^f(y,\xi)\right) \leq \text{cap}\left(\{z \neq^f 0\} \cap \text{f-supp}(\xi)\right) = 0.$$

This proves that the first set is included in the second one in the assertion. Now consider $z \in \{v \in H_0^1(\Omega) \,|\, v = 0 \text{ q.e. on } A_s^f(y,\xi)\}$. It holds that

$$\text{cap}\left(\{z \neq^f 0\} \cap \text{f-supp}(\xi) \cap A(y)\right) = 0$$

and thus, $\xi\left(\{z \neq^f 0\} \cap \text{f-supp}(\xi) \cap A(y)\right) = 0$. Since

$$\{z \neq^f 0\} \subset \left(\{z \neq^f 0\} \cap \text{f-supp}(\xi) \cap A(y)\right) \cup \{\xi =^f 0\} \cup \{y >^f 0\},$$

and $\xi(\{\xi =^f 0\}) = 0$ as well as $\xi(\{y >^f 0\}) = 0$ we can infer that $\xi(\{z \neq^f 0\}) = 0$. □

The next theorem is the counterpart of [3, Thm.2].

Theorem 2.12. *Assume that $(y_\gamma, u_\gamma, \xi_\gamma)$ is a first order point for* (3) *with multipliers $(p_\gamma, \vartheta_\gamma, \tau_\gamma)$ and that $(y_\gamma, u_\gamma, \xi_\gamma)$ is feasible for problem* (2) *with Assumption 2.1. Then $(y_\gamma, u_\gamma, \xi_\gamma)$ is strongly stationary for problem* (2) *in the sense that for*

$$(y, u, \xi, p) = (y_\gamma, u_\gamma, \xi_\gamma, p_\gamma), \quad \lambda = \vartheta_\gamma - \gamma\xi_\gamma, \quad \mu = \tau_\gamma - \gamma y_\gamma,$$

the assertions (14a)-(14e) *and the following complementarity and sign conditions hold:*

(20) $\forall \phi \in H_0^1(\Omega)$, $\phi = 0$ *q.e. on* $A^f(y)$: $\langle \lambda, \phi\rangle_{H^{-1}(\Omega)} = 0$,

(21) $\forall \phi \in H_0^1(\Omega)$, $\phi \geq 0$ *q.e. on* $B^f(y,\xi)$, $\phi = 0$ *q.e. on* $A_s^f(y,\xi)$: $\langle \lambda, \phi\rangle_{H^{-1}(\Omega)} \geq 0$,

(22) $\mu = 0$ *q.e. on* $A_s^f(y,\xi)$,

(23) $\mu \geq 0$ *q.e. on* $B^f(y,\xi)$.

Proof. The condition on feasibility of (y, u, ξ) for the original problem (2) directly implies (14a), (14b). Equations (14c)-(14e) result from the definition of λ and μ and the first order stationarity conditions (10c)-(10e). For $\phi \in H_0^1(\Omega)$ with $\phi = 0$ q.e. on $A^f(y)$ we have

$$\langle \lambda, \phi\rangle_{H^{-1}(\Omega)} = \langle \vartheta_\gamma, \phi\rangle_{H^{-1}(\Omega)} - \gamma\langle \xi_\gamma, \phi\rangle_{H^{-1}(\Omega)}. \tag{24}$$

The non-negativity of ϑ_γ permits us to interpret it as a measure and we can split the dual pairing $\langle \vartheta_\gamma, \phi\rangle_{H^{-1}(\Omega)}$ into the following integrals,

$$\langle \vartheta_\gamma, \phi\rangle_{H^{-1}(\Omega)} = \int_\Omega \phi d\vartheta_\gamma = \int_{A^f(y)} \phi d\vartheta_\gamma + \int_{\{y >^f 0\}} \phi d\vartheta_\gamma.$$

The first integral vanishes because $\phi = 0$ q.e. on $A^f(y)$. By Lemma 2.10, the complementarity and sign conditions on y and ϑ_γ imply that $\vartheta_\gamma(\{y >^f 0\})) = 0$, and so the second integral also vanishes. In the same way, the dual pairing $\langle \xi_\gamma, \phi\rangle_{H^{-1}(\Omega)}$ is split into

$$\langle \xi_\gamma, \phi\rangle_{H^{-1}(\Omega)} = \int_\Omega \phi d\xi_\gamma = \int_{A^f(y)} \phi d\xi_\gamma + \int_{\{y >^f 0\}} \phi d\xi_\gamma,$$

and with the same arguments as above we observe that, together with $\langle \xi_\gamma, \phi\rangle_{H^{-1}(\Omega)}$, $\langle \lambda, \phi\rangle_{H^{-1}(\Omega)}$ vanishes. This proves (20). Consider $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ q.e. on $B^f(y,\xi)$ and $\phi = 0$ q.e. on $A_s^f(y,\xi)$. We reuse (24) and again analyze the two summands separately. With the disjoint decomposition $A^f(y) = \big(A^f(y) \cap A^f(\xi)\big) \dot\cup \big(A^f(y) \cap (\Omega \setminus A^f(\xi))\big)$ and the definition of $A_s^f(y,\xi)$ and $B^f(y,\xi)$ in (18) and (19) it is possible to split the first summand into

$$\langle \vartheta_\gamma, \phi\rangle_{H^{-1}(\Omega)} = \int_{\{y >^f 0\}} \phi d\vartheta_\gamma + \int_{A_s^f(y,\xi)} \phi d\vartheta_\gamma + \int_{B^f(y,\xi)} \phi d\vartheta_\gamma.$$

The first integral vanishes with the same argument as above. The conditions on ϕ imply that the second one is zero and the third one is non-negative. Replacing ϑ_γ by ξ_γ in the representation above we immediately obtain that $\langle \xi_\gamma, \phi\rangle_{H^{-1}(\Omega)} = 0$ because

$$\xi_\gamma(\{y >^f 0\}) + \xi_\gamma(B^f(y,\xi)) = \xi_\gamma(\{y >^f 0\} \dot\cup B^f(y,\xi)) = \xi_\gamma(A^f(y)) = 0.$$

This proves (21). We now turn our attention to $\mu = \tau_\gamma - \gamma y_\gamma$. To begin with, it is clear that $y_\gamma = y$ vanishes q.e. on the fine strongly active set $A_s^f(y)$ as well as on the finely biactive set $B^f(y,\xi)$ both of which are a subset of $A^f(y)$, the fine zero set of y. Lemma 2.10 guarantees that $\mu_\gamma = 0$ ξ_γ-a.e. which by Lemma 2.11 implies that $\mu_\gamma = 0$ q.e. on $A_s^f(y)$. This yields (22). On the finely biactive set $B^f(y,\xi)$, we now know that $\mu = \tau_\gamma$ q.e. which is non-negative q.e. on Ω. We thus have the claimed sign of μ on the biactive set from (23). $\square$

3. Point Tracking Control Problem

In this section we consider point tracking subject to a variational inequality. In fact, we assume that a finite set of tracking points I and desired values $y_w \in R$ (for $w \in I$) are given and that the mapping $j : Y \to \mathbb{R}$ in the objective is defined by

$$j : C(\bar{\Omega}) \to \mathbb{R}, \quad j(y) = \frac{1}{2} \sum_{w \in I} (y(w) - y_w)^2.$$

In the lower level problem, we consider an additional constraint on the slack variable ξ which restricts, for instance in an application where the elastic deformation of a membrane is modeled by the variational inequality, the force that the elastic membrane exerts on the obstacle in the contact region. We choose the state space $W_0^{1,q}(\Omega)$ which embeds into $C(\bar{\Omega})$ if q is larger than the dimension of the computational domain Ω.

3.1. **Model Problem, Penalty Scheme: Solvability and Consistency.**

Consider the problem

$$\text{Minimize } J(y, u) = \frac{1}{2} \sum_{w \in I} (y(w) - y_w)^2 + \frac{\nu}{2} \|u\|^2_{L^2(\Omega)} \tag{25a}$$

$$\text{over } (y, u, \xi) \in W_0^{1,q}(\Omega) \times U_{ad} \times \Xi_{ad} \tag{25b}$$

$$\text{subject to } Ay - u - \xi = f, \quad y \geq 0, \quad (y, \xi)_{L^2(\Omega)} = 0, \tag{25c}$$

with the following data: For an open bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, we consider $a_{ij} \in L^\infty(\Omega)$ $(i, j \in \{1, \dots, n\})$ collected in the matrix $(a_{ij}) \in L^\infty(\Omega)^{n \times n}$ such that for all $\zeta \in \mathbb{R}^n$ and $x \in \Omega$,

$$\zeta^\top (a_{ij}(x))\zeta \geq \Sigma_A |\zeta|^2, \qquad |(a_{ij}(x))\zeta| \leq C_A |\zeta|, \tag{26}$$

where $\Sigma_A, C_A > 0$, and we define the bounded and uniformly elliptic differential operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$A = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} = -\operatorname{div}((a_{ij})\nabla \cdot). \tag{27}$$

If the dimension of the problem is at most 2, i.e., $\Omega \subset \mathbb{R}^n$, $n \in \{1, 2\}$, then we assume that Ω is a Lipschitz domain and $q \in (2, Q)$ with $Q > 2$ from [7, 21]. If the dimension of the problem is larger than 2, i.e. $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 3$, then we assume that $\partial\Omega \in C^1$, that the coefficients (a_{ij}) of the operator A defined in (27) satisfy additionally $a_{ij} \in C(\bar{\Omega})$ and that $q > n$ is given due to the regularity result in [28, Thm.5:2.5 (i)]. Then, by [5, Thm.4.2] and [10, Thm.3], respectively, the operator $A : W_0^{1,q}(\Omega) \to W^{-1,q}(\Omega)$ is invertible with continuous inverse operator $A^{-1} : W^{-1,q}(\Omega) \to W_0^{1,q}(\Omega)$. The set of feasible controls is given by the box constraint

$$U_{ad} = \{v \in L^2(\Omega) \,|\, \underline{u} \leq v \leq \overline{u}\},$$

where $\underline{u}, \overline{u} \in L^2(\Omega) \cup \{-\infty, \infty\}$ satisfy $\underline{u} < \overline{u}$ a.e. on Ω, $\nu \geq 0$ and, if U_{ad} is not bounded in $L^2(\Omega)$, then $\nu > 0$, and $f \in L^2(\Omega)$.

We collect these definitions in the following assumption.

Assumption 3.1. The quantities Ω, A, U_{ad}, I, $(y_w)_{w\in I}$, $q > n$ are given as specified in the previous paragraph and in the beginning of Section 3, and it holds that

(28)
$$-f \in U_{ad}, \qquad \Xi_{ad} = \{v \in L^2(\Omega) \,|\, 0 \le v \le \phi\} \text{ with } \phi \in L^2(\Omega),\ \phi > 0 \text{ a.e. on } \Omega.$$

By use of an infimizing sequence argument, the solvability of the problem class stated above can be argued: Given the weak closedness of the (non-empty) set $U_{ad} \times \Xi_{ad}$ in $L^2(\Omega) \times L^2(\Omega)$ and the compact embedding of $L^2(\Omega)$ into $W^{-1,q}(\Omega)$, the continuity of the solution operator $A^{-1} : W^{-1,q}(\Omega) \to W_0^{1,q}(\Omega)$ yields the feasibility of an accumulation point of an infimizing sequence of J over the feasible set. The fact that the point evaluation mapping $\delta_w : W_0^{1,q}(\Omega) \to \mathbb{R}, \delta_w(z) := z(w)$ in the first term of J is linear and bounded and the weak lower semi-continuity of the norm mapping in the second part gives the optimality of such accumulation points. This proves the following proposition.

Proposition 3.2. *Under Assumption 3.1, problem* (25) *has a solution.*

We propose the following regularized and penalyzed version of problem (25):

$$\text{(29a)} \qquad \text{Minimize } \tilde{J}_{\gamma,r}(y,u,\xi) = \frac{1}{2}\sum_{w\in I} ⨍_{B_r(w)} (y - y_w)^2\, dx + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)}$$

$$\text{(29b)} \qquad + \frac{\gamma}{2}\left\|(y)_-\right\|^2_{L^2(\Omega)} + \delta(\gamma)\,(y,\xi)_{L^2(\Omega)}$$

$$\text{(29c)} \qquad \text{over } (y,u,\xi) \in H_0^1(\Omega) \times U_{ad} \times \Xi_{ad}$$

$$\text{(29d)} \qquad \text{subject to } Ay - u - \xi = f.$$

Here, $r > 0$ is an averaging parameter that serves in the same way as in [7]: We define $B_r(w) = \{x \in \Omega \,|\, |x - w| < r\}$ and approximate the point tracking term in the original problem by the integrals in the first summand of $\tilde{J}_{\gamma,r}$. In difference to the pure ℓ_1-penalty from Section 2, we also penalize the constraint $y \ge 0$ here to avoid a constraint degeneracy. A lack of complementarity in $(y,\xi)_{L^2(\Omega)}$ contributes in the term $\delta(\gamma)\,(y,\xi)_{L^2(\Omega)}$ to the objective, involving a mapping $\delta : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ with $\delta(\gamma) \to \infty$ for $\gamma \to \infty$.

The following two lemmas guarantee boundedness and convergence properties that are necessary to prove solvability and consistency of the auxiliary problem (29).

Lemma 3.3. *For all* $\gamma, \delta = \delta(\gamma), r > 0$ *and* $y \in W_0^{1,q}(\Omega)$, $u \in L^2(\Omega)$ *and* $\xi \in \Xi_{ad}$ *from Assumption 3.1, the objective functional* $\tilde{J}_{\gamma,r}$ *satisfies*

$$\tilde{J}_{\gamma,r}(y,u,\xi) \ge -\frac{\delta(\gamma)^2}{2\gamma}\|\phi\|^2_{L^2(\Omega)}.$$

Proof. The lower boundedness of $\tilde{J}_{\gamma,r}$ is non-trivial only on the account of its last term. Since $\xi \in \Xi_{ad}$ is non-negative, the product $(y,\xi)_{L^2(\Omega)}$ satisfies

(30)
$$(y,\xi)_{L^2(\Omega)} \ge \int_{\{y<0\}} y\xi\, dx \ge -\left\|(y)_-\right\|_{L^2(\Omega)}\|\xi\|_{L^2(\Omega)} \ge -\left\|(y)_-\right\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)}.$$

For all feasible (y,u,ξ) we thus have

$$\text{(31)} \qquad \tilde{J}_{\gamma,r}(y,u,\xi) \ge \frac{\gamma}{2}\left\|(y)_-\right\|^2_{L^2(\Omega)} - \delta(\gamma)\left\|(y)_-\right\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)}.$$

The analysis of the right hand side, which is a quadratic function in $\left\|(y)_-\right\|_{L^2(\Omega)}$, then yields the assertion: For $v \geq 0$ it holds that

$$\frac{\gamma}{2}v^2 - \delta(\gamma)\,\|\phi\|_{L^2(\Omega)}\,v \geq -\frac{\delta(\gamma)^2}{2\gamma}\,\|\phi\|^2_{L^2(\Omega)}\,. \tag{32}$$

□

Lemma 3.4. *Let $(u_k)_{k\in\mathbb{N}}$ and $(\xi_k)_{k\in\mathbb{N}}$ be sequences in $L^2(\Omega)$ that converge weakly to u and $\xi \in L^2(\Omega)$, respectively. Then, the sequence $(y_k)_{k\in\mathbb{N}}$ defined by $y_k = A^{-1}(u_k + \xi_k + f)$ converges strongly in $W_0^{1,q}(\Omega)$ to $y = A^{-1}(u + \xi + f)$ and for every sequence $(r_k)_{k\in\mathbb{N}} \in \mathbb{R}^{>0}$ with $r_k \to 0$, we have that*

$$\sum_{w\in I} \fint_{B_{r_k}(w)} (y_k(x) - y_w)^2\,dx \to \sum_{w\in I}(y_k(w) - y_w)^2.$$

Proof. The compact embedding of $L^2(\Omega)$ into $W^{-1,q}(\Omega)$ and the continuity of A^{-1} as a mapping from $W^{-1,q}(\Omega)$ to $W_0^{1,q}(\Omega)$ shows the first assumption. We then use the embedding of $W_0^{1,q}(\Omega)$ into $C(\bar{\Omega})$ and [7, L.2.10] to prove the second assertion. □

Lemma 3.3 guarantees the existence of a feasible infimizing sequence of $\tilde{J}_{\gamma,r}$, and Lemma 3.4 allows to derive the existence of a solution for the auxiliary problem by use of the typical Weierstraß-argument. This yields the following proposition.

Proposition 3.5. *Under Assumption 3.1, problem* (29) *has a solution for any set of parameters $r, \gamma, \delta(\gamma) > 0$.*

The following assumption on the parameters for the auxiliary problem is motivated by the dependence of the lower bound on the objective from Lemma 3.3.

Assumption 3.6. In our penalty scheme we use positive sequences $(r_k)_{k\in\mathbb{N}}$, $(\gamma_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ with $r_k \to 0$, $\gamma_k \to \infty$ for $k \to \infty$, and a mapping $\delta : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ which satisfies $\lim_{\gamma\to\infty} \delta(\gamma) = \infty$ and $\lim_{\gamma\to\infty} \frac{\delta(\gamma)^2}{\gamma} = 0$.

The following lemma on a strong-weak lower semi-continuity property will be helpful in the proof of consistency of the penalty scheme.

Lemma 3.7. *Let $(r_k)_{k\in\mathbb{N}}$, $(\gamma_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ and $\delta : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ satisfy Assumption 3.6 and for each $k \in \mathbb{N}$ assume that (y_k, u_k, ξ_k) solves the auxiliary problem* (29) *where Assumption 3.1 holds true. If $y_k \to y^\star$ in $W_0^{1,q}(\Omega)$ and $u_k \rightharpoonup u^\star$ in $L^2(\Omega)$, then*

$$J(y^\star, u^\star) \leq \liminf\{\tilde{J}_{\gamma_k,r_k}(y_k, u_k, \xi_k)\,|\,k \in \mathbb{N}\}. \tag{33}$$

Proof. We examine the summands of $\tilde{J}_{\gamma_k,r_k}(y_k, u_k, \xi_k)$ separately. Lemma 3.4 yields that

$$\lim_{k\to\infty} \frac{1}{2}\sum_{w\in I} \fint_{B_{r_k}(w)} (y_k - y_w)^2\,dx = \frac{1}{2}\sum_{w\in I}(y^\star(w) - y_w)^2.$$

From the weak lower semi-continuity of the norm $\|\cdot\|_{L^2(\Omega)} : L^2(\Omega) \to \mathbb{R}$ we infer that

$$\liminf\{\frac{\nu}{2}\,\|u_k\|^2_{L^2(\Omega)}\,\,|\,k \in \mathbb{N}\} \geq \frac{\nu}{2}\,\|u^\star\|^2_{L^2(\Omega)}\,.$$

Now we use the estimate (32) in the proof of Lemma 3.3 to see that for all $k \in \mathbb{N}$,

$$\frac{\gamma_k}{2}\,\left\|(y_k)_-\right\|^2_{L^2(\Omega)} + \delta(\gamma_k)\,(y_k, \xi_k)_{L^2(\Omega)} \geq -\frac{\delta(\gamma_k)^2}{2\gamma_k}\,\|\phi\|^2_{L^2(\Omega)}\,.$$

Hence, Assumption 3.6 leads to

$$\begin{aligned}&\liminf \left\{ \frac{\gamma_k}{2} \left\|(y_k)_-\right\|^2_{L^2(\Omega)} + \delta(\gamma_k)\,(y_k,\xi_k)_{L^2(\Omega)} \mid k\in\mathbb{N}\right\}\\ &\geq \liminf\{-\frac{\delta(\gamma_k)^2}{2\gamma_k}\left\|\phi\right\|^2_{L^2(\Omega)} \mid k\in\mathbb{N}\} = \lim_{k\to\infty} -\frac{\delta(\gamma_k)^2}{2\gamma_k}\left\|\phi\right\|^2_{L^2(\Omega)} = 0.\end{aligned}$$

Summing up the terms yields the assertion. □

Note that for $s,t>0$ and the function $h:\mathbb{R}\to\mathbb{R}$ defined by $h(v)=sv^2-tv$ it holds that if $v,\kappa>0$ are given such that $h(v)\leq\kappa$, then

$$v \leq \sqrt{\frac{\kappa}{s}+\frac{t^2}{4s^2}}+\frac{t}{2s}. \tag{34}$$

Now we prove consistency of the penalty scheme.

Proposition 3.8. *Let $(r_k)_{k\in\mathbb{N}}$, $(\gamma_k)_{k\in\mathbb{N}}\subset\mathbb{R}$ and $\delta:\mathbb{R}^{>0}\to\mathbb{R}^{>0}$ satisfy Assumption 3.6, and for each $k\in\mathbb{N}$ assume that (y_k,u_k,ξ_k) solves the auxiliary problem* (29) *with Assumption 3.1. Then, there exists a subsequence such that for $k\to\infty$, it holds that $y_k\to y^\star$ in $W^{1,q}_0(\Omega)$, $u_k\rightharpoonup u^\star$ in $L^2(\Omega)$ and $\xi_k\rightharpoonup\xi^\star$ in $L^2(\Omega)$ and the limit $(y^\star,u^\star,\xi^\star)\in W^{1,q}_0(\Omega)\times U_{ad}\times\Xi_{ad}$ solves problem* (25).

Proof. The weak convergence of a subsequence of $(u_k)_{k\in\mathbb{N}}$ and $(\xi_k)_{k\in\mathbb{N}}$ is due to the boundedness of U_{ad} (resp. the term $\frac{\nu}{2}\left\|u_k\right\|^2_{L^2(\Omega)}$ in the objective $\tilde{J}_{\gamma_k,r_k}$, cf. (35) below) and Ξ_{ad}. We denote the weak limits by $u^\star$ and $\xi^\star$, respectively, and note that $y_k := A^{-1}(u_k+\xi_k+f)$ converges to $y^\star = A^{-1}(u^\star+\xi^\star+f)$ in $W^{1,q}_0(\Omega)$ due to the compact embedding of $L^2(\Omega)$ into $W^{-1,q}(\Omega)$ and the continuity of $A^{-1}: W^{-1,q}(\Omega)\to W^{1,q}_0(\Omega)$. The weak limits $u^\star$ and $\xi^\star$ are feasible owing to the weak closedness of U_{ad} and Ξ_{ad} in $L^2(\Omega)$ and $y^\star$ satisfies the partial differential equation in (25c). For all $k\in\mathbb{N}$ it holds that

$$\frac{\gamma_k}{2}\left\|(y_k)_-\right\|^2_{L^2(\Omega)} - \delta(\gamma_k)\left\|(y_k)_-\right\|_{L^2(\Omega)}\left\|\phi\right\|_{L^2(\Omega)} \leq \tilde{J}_{\gamma_k,r_k}(y_k,u_k,\xi_k).$$

The fact that $(y,u,\xi)=(0,-f,0)$ is feasible for the auxiliary problem for all penalty and averaging parameters yields the uniform bound

$$\tilde{J}_{\gamma_k,r_k}(y_k,u_k,\xi_k) \leq \tilde{J}_{\gamma_k,r_k}(0,-f,0) = \frac{1}{2}\sum_{w\in I} y_w^2 + \frac{\nu}{2}\left\|f\right\|^2_{L^2(\Omega)} =: \kappa. \tag{35}$$

We apply formula (34) for $s=\frac{\gamma_k}{2}$, $t=\delta(\gamma_k)\left\|\phi\right\|_{L^2(\Omega)}$, $v=\left\|(-y_k)_-\right\|_{L^2(\Omega)}$ and κ as defined above to derive that

$$\left\|(y_k)_-\right\|_{L^2(\Omega)} \leq \sqrt{\frac{2\kappa}{\gamma_k}+\frac{\delta(\gamma_k)^2}{\gamma_k^2}\left\|\phi\right\|^2_{L^2(\Omega)}} + \frac{\delta(\gamma_k)}{\gamma_k}\left\|\phi\right\|_{L^2(\Omega)}.$$

By Assumption 3.6 we thus have $\lim_{k\to\infty}\left\|(y_k)_-\right\|_{L^2(\Omega)}\leq 0$, i.e., $y^\star\geq 0$. To complete feasibility of the limit point we prove the complementarity of $y^\star$ and $\xi^\star$. The strong convergence of $y_k\to y^\star$ in $L^2(\Omega)$ and the weak convergence of $\xi_k\rightharpoonup\xi^\star$ as well as feasibility of $y^\star$ and $\xi^\star$ yields that

$$\lim_{k\to\infty}(\xi_k,y_k)_{L^2(\Omega)} = (\xi^\star,y^\star)_{L^2(\Omega)} \geq 0.$$

Since all other terms in the auxiliary objective $\tilde{J}_{\gamma_k,r_k}(y_k,u_k,\xi_k)$ are non-negative, we derive the uniform bound

$$\delta(\gamma_k)\,(\xi_k,y_k)_{L^2(\Omega)} \leq \tilde{J}_{\gamma_k,r_k}(y_k,u_k,\xi_k) \leq \frac{1}{2}\sum_{w\in I} y_w^2 + \frac{\nu}{2}\,\|f\|^2_{L^2(\Omega)}$$

from (35) and so, $\delta(\gamma_k) \to \infty$ implies that $\lim_{k\to\infty}(\xi_k,y_k)_{L^2(\Omega)} \leq 0$. In order to prove optimality of the limiting element, we assume that (y,u,ξ) is feasible for problem (25). Then, for every $k \in \mathbb{N}$, (y,u,ξ) is feasible for the respective penalized auxiliary problem (29). Utilizing Lemma 3.4 for the (constant) sequence $(y)_{k\in\mathbb{N}}$ we infer from the fact that the penalization terms vanish in (y,u,ξ) that

$$J(y,u) = \lim_{k\to\infty} \tilde{J}_{\gamma_k,r_k}(y,u,\xi) = \liminf\{\tilde{J}_{\gamma_k,r_k}(y,u,\xi)\,|\,k\in\mathbb{N}\}.$$

The optimality of (y_k,u_k,ξ_k) additionally yields that $\tilde{J}_{\gamma_k,r_k}(y,u,\xi) \geq \tilde{J}_{\gamma_k,r_k}(y_k,u_k,\xi_k)$ and we hence have

$$J(y,u) = \liminf\{\tilde{J}_{\gamma_k,r_k}(y,u,\xi)\,|\,k\in\mathbb{N}\} \geq \liminf\{\tilde{J}_{\gamma_k,r_k}(y_k,u_k,\xi_k)\,|\,k\in\mathbb{N}\}.$$

We finally utilize Lemma 3.7 to obtain

$$J(y,u) \geq \liminf\{\tilde{J}_{\gamma_k,r_k}(y_k,u_k,\xi_k)\,|\,k\in\mathbb{N}\} \geq J(y^\star,u^\star). \qquad \square$$

3.2. **First Order Stationarity Conditions.** In the same way as in Section 2 we use [33, Thm. 3.1] to derive a system of first order conditions for the auxiliary problem (29) and perform a limiting analysis to derive necessary first order conditions for the original problem.

For

$$x = (y,u,\xi) \in \mathcal{C} = H_0^1(\Omega)\times U_{ad}\times \Xi_{ad} \subset X = H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)$$

define $g : X \to Y = H^{-1}(\Omega)$ by $g(y,u,\xi) = Ay - u - \xi - f$. Then, the constraint set in problem (29), which does not depend on the parameters γ_k, ϵ_k, is described by $\{x\in\mathcal{C}\,|\,g(x)\in\{0\}\subset H^{-1}(\Omega)\}$. In order to show that $Y \subset g'(x_k)\mathcal{C}(x_k)$ (where $x_k = (y_k,u_k,\xi_k)$, and $\mathcal{C}(x_k)$ is the conical hull of $\mathcal{C}-x_k$) assume that $z\in Y$. Choose $c_u \in U_{ad}$, $c_\xi = 0$, and $c_y = A^{-1}(f + c_u + z)$ to obtain that

$$g'(y_k,u_k,\xi_k)(\beta(c_y-y_k),\beta(c_u-u_k),\beta(c_\xi-\xi_k)) = \beta(Ac_y - c_u - c_\xi - f) = z, \tag{36}$$

and thus, every feasible point $(y_k,u_k,\xi_k)\in H_0^1(\Omega)\times U_{ad}\times\Xi_{ad}$ of (29) is regular in the sense of [33, Eq. (1.4)]. This yields the following result.

Proposition 3.9. *If $(y,u,\xi) \in H_0^1(\Omega)\times U_{ad}\times\Xi_{ad}$ is optimal for the auxiliary problem (29) with Assumption 3.1 and parameters $(\gamma,r,\delta)\in(\mathbb{R}^{>0})^3$, then there exist $p\in H_0^1(\Omega)$ and $\tau = (\tau)_+ + (\tau)_- \in H_0^1(\Omega)$ such that the following first order conditions hold:*

$$A^\star p + \sum_{w\in I}(y-y_w)\bar{\chi}_{B_{r_k}(w)} + \gamma\,(y)_- + \delta\xi = 0 \qquad \text{in } H^{-1}(\Omega), \tag{37a}$$

$$u - \mathrm{Proj}_{U_{ad}}\left(\frac{1}{\nu}p\right) = 0 \qquad \text{in } L^2(\Omega), \tag{37b}$$

$$p + \tau - \delta y = 0 \qquad \text{in } H_0^1(\Omega), \tag{37c}$$

$$\langle \xi, (\tau)_+\rangle_{H^{-1}(\Omega)} = 0, \qquad \langle \xi-\phi, (\tau)_-\rangle_{H^{-1}(\Omega)} = 0. \tag{37d}$$

We now turn our attention to limiting first order conditions. Let $(r_k)_{k\in\mathbb{N}}$, $(\gamma_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ and $\delta : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ satisfy Assumption 3.6 and assume that for every $k \in \mathbb{N}$, $(y_k, u_k, \xi_k) \in H_0^1(\Omega) \times U_{ad} \times \Xi_{ad}$ is feasible for problem (29) with parameters $(\gamma_k, r_k, \delta(\gamma_k))$ and under Assumption 3.1. Moreover, let $p_k \in H_0^1(\Omega)$ and $\tau_k \in H_0^1(\Omega)$ be given such that the first order conditions (37) hold. We define

$$\lambda_k = -\gamma_k (y_k)_- - \delta(\gamma_k)\xi_k, \qquad \mu_k = (\tau_k)_+ - \delta(\gamma_k) y_k. \tag{38}$$

Proposition 3.11 below yields uniform bounds and hence the existence of accumulation points of the sequence $(y_k, u_k, \xi_k, p_k, \lambda_k, \mu_k)_{k\in\mathbb{N}}$. In its proof, we need the following assumption.

Assumption 3.10. We assume the following uniform bounds:

$$\frac{\delta(\gamma_k)^3}{\gamma_k} \le C, \tag{39a}$$

$$\big(\delta(\gamma_k)\phi + \gamma_k (y_k)_-, (\tau_k)_-\big)_{L^2(\Omega)} \le C, \tag{39b}$$

$$\|u_k\|_{L^2(\Omega)} \le C. \tag{39c}$$

Note that since we are free to choose $\delta : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$, the first assumption (39a) can be guaranteed a priori. The second bound can be understood as an implication of a convergence rate of $\left\|(-y_k)_-\right\|_{L^\infty(\Omega)}$ and (39a) (see [21]). Or, if the upper constraint on ξ is chosen such that it is not active in the iterates (and in the solution), then $(\tau_k)_- = 0$ guarantees (39b). The third bound is satisfied if U_{ad} is bounded in $L^2(\Omega)$, and apart from that, a typical assumption that becomes important in the analysis of merely stationary points.

We observe that

$$\langle (y_k)_-, y_k\rangle_{H^{-1}(\Omega)} = \big((y_k)_-, y_k\big)_{L^2(\Omega)} = \left\|(y_k)_-\right\|^2_{L^2(\Omega)}. \tag{40}$$

Proposition 3.11. *With the notation from above, let Assumption 3.10 hold true. Then, we have the following uniform bounds:*

$$\|y_k\|_{W_0^{1,q}(\Omega)} \le C, \quad \|\xi_k\|_{L^2(\Omega)} \le C, \quad \|p_k\|_{H_0^1(\Omega)} \le C, \quad \|\lambda_k\|_{H^{-1}(\Omega)} \le C,$$

for some constant $C \ge 0$ which does not depend on k.

Proof. The admissible set Ξ_{ad} directly yields the uniform bound on $(\xi_k)_{k\in\mathbb{N}}$ in $L^2(\Omega)$. Then, the primal equation (29d) and the embedding of $L^2(\Omega)$ into $W^{-1,q}(\Omega)$ provides a bound for y_k in $W_0^{1,q}(\Omega)$. Utilizing the definition of λ_k in (38), the adjoint equation (37a) reads

$$A^\star p_k - \lambda_k = -\sum_{w\in I} (y_k - y_w)\bar{\chi}_{B_{r_k}(w)}.$$

Testing with p_k yields the estimate

$$c\,\|p_k\|^2_{H_0^1(\Omega)} \le \langle A^\star p_k, p_k\rangle_{H^{-1}(\Omega)} = \langle \lambda_k, p_k\rangle_{H^{-1}(\Omega)} - \sum_{w\in I} \fint_{B_{r_k}(w)} (y_k - y_w) p_k \, dx,$$

with $0 < c$. The uniform bound on y_k in $W_0^{1,q}(\Omega)$, which embeds continuously into $L^\infty(\Omega)$, then gives the estimate

$$c\,\|p_k\|^2_{H_0^1(\Omega)} - \langle \lambda_k, p_k\rangle_{H^{-1}(\Omega)} \le C\,\|p_k\|_{H_0^1(\Omega)}, \tag{41}$$

where $0 < c \leq C$. In order to obtain a bound on $\|p_k\|_{H_0^1(\Omega)}$, we provide an upper bound for the dual pairing $\langle \lambda_k, p_k \rangle_{H^{-1}(\Omega)}$. The definition of λ_k in (38) and (37c) yield that

$$\begin{aligned}\langle \lambda_k, p_k \rangle_{H^{-1}(\Omega)} =& \gamma_k \langle (y_k)_-, \tau_k \rangle_{H^{-1}(\Omega)} - \gamma_k \delta(\gamma_k) \langle (y_k)_-, y_k \rangle_{H^{-1}(\Omega)} \\ &+ \delta(\gamma_k) \langle \xi_k, \tau_k \rangle_{H^{-1}(\Omega)} - \delta(\gamma_k)^2 \langle \xi_k, y_k \rangle_{H^{-1}(\Omega)}.\end{aligned}$$

The second term on the right hand side can be simplified using (40). Furthermore we split $\tau_k = (\tau_k)_+ + (\tau_k)_-$ and use the complementarities in (37d) to obtain that

(42)
$$\begin{aligned}\langle \lambda_k, p_k \rangle_{H^{-1}(\Omega)} =& \gamma_k \langle (y_k)_-, (\tau_k)_+ \rangle_{H^{-1}(\Omega)} + \gamma_k \langle (y_k)_-, (\tau_k)_- \rangle_{H^{-1}(\Omega)} \\ &- \gamma_k \delta(\gamma_k) \left\|(y_k)_-\right\|^2_{L^2(\Omega)} + \delta(\gamma_k) \langle \phi, (\tau_k)_- \rangle_{H^{-1}(\Omega)} - \delta(\gamma_k)^2 \langle \xi_k, y_k \rangle_{H^{-1}(\Omega)}.\end{aligned}$$

We drop the first product as it is certainly not positive and estimate

$$\begin{aligned}\langle \lambda_k, p_k \rangle_{H^{-1}(\Omega)} \leq& \langle \gamma_k (y_k)_- + \delta(\gamma_k)\phi, (\tau_k)_- \rangle_{H^{-1}(\Omega)} \\ &- \gamma_k \delta(\gamma_k) \left\|(y_k)_-\right\|^2_{L^2(\Omega)} - \delta(\gamma_k)^2 \langle \xi_k, y_k \rangle_{H^{-1}(\Omega)}.\end{aligned} \tag{43}$$

The first term is bounded by (39b) in Assumption 3.10. Note that although the term $-\gamma_k \delta(\gamma_k) \left\|(y_k)_-\right\|^2_{L^2(\Omega)}$ is clearly not positive, we rather keep it in the estimate and further analyze the sum

$$\gamma_k \delta(\gamma_k) \left\|(y_k)_-\right\|^2_{L^2(\Omega)} + \delta(\gamma_k)^2 \left(\xi_k, y_k\right)_{L^2(\Omega)},$$

which we need to bound from below. Similarly as in the proof of Proposition 3.2, we estimate

$$\delta(\gamma_k)^2 \left(\xi_k, y_k\right)_{L^2(\Omega)} \geq -\delta(\gamma_k)^2 \|\phi\|_{L^2(\Omega)} \left\|(y_k)_-\right\|_{L^2(\Omega)}$$

and study a quadratic function $\bar{h}_{\gamma_k}(v) = \gamma_k \delta(\gamma_k) v^2 - \delta(\gamma_k)^2 \|\phi\|_{L^2(\Omega)} v$. We hence find that $\bar{h}_{\gamma_k}(v) \geq \bar{h}_{\gamma_k}\left(\frac{\delta(\gamma_k)}{2\gamma_k} \|\phi\|_{L^2(\Omega)}\right)$ and so

$$\gamma_k \delta(\gamma_k) \left\|(y_k)_-\right\|^2_{L^2(\Omega)} + \delta(\gamma_k)^2 \left(\xi_k, y_k\right)_{L^2(\Omega)} \geq -\frac{\delta(\gamma_k)^3}{4\gamma_k} \|\phi\|^2_{L^2(\Omega)}. \tag{44}$$

The last term is bounded from below by (39a) in Assumption 3.10, and so, the dual pairing $\langle \lambda_k, p_k \rangle_{H^{-1}(\Omega)}$ from (43) is bounded from above. Plugging this into (41), we have

$$c \|p_k\|^2_{H_0^1(\Omega)} - \tilde{C} \leq C \|p_k\|_{H_0^1(\Omega)},$$

i.e., a uniform bound on $\|p_k\|_{H_0^1(\Omega)}$. One can then derive the boundedness of the sequence $(\|\lambda_k\|_{H^{-1}(\Omega)})_{k \in \mathbb{N}}$ from the adjoint equation (37a). □

The next lemma prepares the limiting analysis of a sequence of first order points.

Lemma 3.12. *Under the conditions of Proposition 3.11 it holds that*

$$\delta(\gamma_k) \left| \gamma_k \left\|(y_k)_-\right\|^2_{L^2(\Omega)} + \delta(\gamma_k) \left(\xi_k, y_k\right)_{L^2(\Omega)} \right| \leq C, \tag{45}$$

$$\lim_{k \to \infty} \delta(\gamma_k) \left\|(y_k)_-\right\|_{L^2(\Omega)} = 0, \tag{46}$$

$$\lim_{k \to \infty} \delta(\gamma_k) \left(\xi_k, y_k\right)_{L^2(\Omega)} = 0. \tag{47}$$

Proof. The estimates (44) and (39a) guarantee that

$$-\delta(\gamma_k)\left(\gamma_k \left\|(y_k)_-\right\|_{L^2(\Omega)}^2 + \delta(\gamma_k)\,(\xi_k, y_k)_{L^2(\Omega)}\right) \le \frac{\delta(\gamma_k)^3}{4\gamma_k}\,\|\phi\|_{L^2(\Omega)}^2 \le C. \tag{48}$$

Using the expression (42) for $\langle \lambda_k, p_k\rangle_{H^{-1}(\Omega)}$ in (41) and the uniform boundedness of $\|p_k\|_{H_0^1(\Omega)}$ one derives that

$$\begin{aligned}-\gamma_k\langle (y_k)_-\,,(\tau_k)_+\rangle_{H^{-1}(\Omega)} + \delta(\gamma_k)\left(\gamma_k \left\|(y_k)_-\right\|_{L^2(\Omega)}^2 + \delta(\gamma_k)\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)}\right)\\ \le C + \langle \gamma_k\,(y_k)_- + \delta(\gamma_k)\phi, (\tau_k)_-\rangle_{H^{-1}(\Omega)}.\end{aligned}$$

The non-negativity of the first term on the left hand side, and the uniform bound on the last term on the right hand side from Assumption 3.10 thus yield that

$$\delta(\gamma_k)\left(\gamma_k \left\|(y_k)_-\right\|_{L^2(\Omega)}^2 + \delta(\gamma_k)\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)}\right) \le C.$$

Combined with (48) this proves (45). Estimating

$$\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)} \ge -\|\phi\|_{L^2(\Omega)}\left\|(y_k)_-\right\|_{L^2(\Omega)}$$

we obtain that

$$\delta(\gamma_k)\left(\gamma_k \left\|(y_k)_-\right\|_{L^2(\Omega)}^2 - \delta(\gamma_k)\,\|\phi\|_{L^2(\Omega)}\left\|(y_k)_-\right\|_{L^2(\Omega)}\right) \le C.$$

Then eq. (34) for $s = \gamma_k$, $t = \delta(\gamma_k)\,\|\phi\|_{L^2(\Omega)}$, $\kappa = \frac{C}{\delta(\gamma_k)}$ and $v = \left\|(y_k)_-\right\|_{L^2(\Omega)}$ gives the bound

$$\left\|(y_k)_-\right\|_{L^2(\Omega)} \le \sqrt{\frac{C}{\delta(\gamma_k)\gamma_k} + \frac{\delta(\gamma_k)^2}{4\gamma_k^2}\,\|\phi\|_{L^2(\Omega)}^2} + \frac{\delta(\gamma_k)}{\gamma_k}\,\|\phi\|_{L^2(\Omega)}\,.$$

Therefore, $\left\|(y_k)_-\right\|_{L^2(\Omega)} \to 0$ for $k \to \infty$, and, owing to the convergence $\frac{\delta(\gamma_k)^2}{\gamma_k} \to 0$, we even have $\delta(\gamma_k)\left\|(y_k)_-\right\|_{L^2(\Omega)} \to 0$. We utilize this convergence to see that from

$$\delta(\gamma_k)\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)} \ge -\delta(\gamma_k)\,\|\phi\|_{L^2(\Omega)}\left\|(y_k)_-\right\|_{L^2(\Omega)}$$

it follows that

$$\liminf_{k\to\infty} \delta(\gamma_k)\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)} \ge \liminf_{k\to\infty} -\delta(\gamma_k)\,\|\phi\|_{L^2(\Omega)}\left\|(y_k)_-\right\|_{L^2(\Omega)} = 0. \tag{49}$$

Assume on the other hand that $\limsup\{\delta(\gamma_k)\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)} \,|\, k \in \mathbb{N}\} = 2\varepsilon > 0$. Then we have a subsequence denoted the same and a natural number $K \in \mathbb{N}$ such that for all $k \ge K$ it holds that $\delta(\gamma_k)\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)} > \varepsilon$. This implies

$$\delta(\gamma_k)\left(\gamma_k \left\|(y_k)_-\right\|_{L^2(\Omega)}^2 + \delta(\gamma_k)\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)}\right) > \delta(\gamma_k)\gamma_k\left\|(y_k)_-\right\|_{L^2(\Omega)}^2 + \delta(\gamma_k)\varepsilon$$

which is a contradiction to the boundedness of the term on the left hand side. We thus have $\limsup\{\delta(\gamma_k)\langle \xi_k, y_k\rangle_{H^{-1}(\Omega)} \,|\, k \in \mathbb{N}\} \le 0$, which yields, together with (49), the convergence in (47). □

Theorem 3.13. *With the notation of Assumptions 3.1, 3.6 and 3.10 let* $(y_k, u_k, \xi_k) \in H_0^1(\Omega)\times U_{ad}\times \Xi_{ad}$ *be a first order point for problem* (29) *with parameters* $(\gamma_k, r_k, \delta(\gamma_k))$ *and let* $p_k \in H_0^1(\Omega)$, $\tau_k \in H_0^1(\Omega)$ *be the respective multipliers for every* $k \in \mathbb{N}$.

Moreover, let λ_k, μ_k *be given by* (38)*. Then there exists a subsequence of first order points (denoted the same) with*

$$y_k \to y \text{ in } W_0^{1,q}(\Omega), \qquad u_k \to u \text{ in } L^2(\Omega), \qquad \xi_k \rightharpoonup \xi \text{ in } L^2(\Omega),$$
$$p_k \rightharpoonup p \text{ in } H_0^1(\Omega), \qquad \lambda_k \rightharpoonup \lambda \text{ in } W^{-1,q'}(\Omega).$$

The limit point $(y, u, \xi, p, \lambda, \mu)$ *satisfies the following conditions:*

$$Ay - u - \xi = f, \tag{50a}$$
$$u \in U_{ad}, \qquad \xi \in \Xi_{ad}, \tag{50b}$$
$$y \geq 0, \qquad \langle \xi, y \rangle_{H^{-1}(\Omega)} = 0, \tag{50c}$$
$$A^* p + \sum_{w \in I} (y(w) - y_w)\delta_w - \lambda = 0, \tag{50d}$$
$$\langle \lambda, y \rangle_{W^{-1,q'}(\Omega)} = 0. \tag{50e}$$

If $(\xi_k)_{k\in\mathbb{N}}$ *is a (sub-)sequence such that* $\xi_k < \phi$ *a.e. on* Ω *for all* $k \in \mathbb{N}$ *and* $\lim_{k\to\infty} \frac{\delta(\gamma)^3}{\gamma_k} = 0$*, then it holds additionally that*

$$\mu_k \rightharpoonup \mu \text{ in } H_0^1(\Omega), \quad (\xi, \mu)_{L^2(\Omega)} = 0, \quad \liminf\{\langle \lambda_k, \mu_k \rangle_{H^{-1}(\Omega)} \,|\, k \in \mathbb{N}\} \geq 0. \tag{50f}$$

Proof. We start with similar arguments as in the proof of Proposition 3.8 on consistency of the penalty scheme. The bound on $(\|p_k\|_{H_0^1(\Omega)})_{k\in\mathbb{N}}$ from Proposition 3.11 yields a weak limit $p \in H_0^1(\Omega)$ of a subsequence of $(p_k)_{k\in\mathbb{N}}$, and the strong convergence of this subsequence in $L^2(\Omega)$ guarantees that

$$u_k = \mathrm{Proj}_{U_{ad}}\left(\frac{1}{\nu} p_k\right) \to \mathrm{Proj}_{U_{ad}}\left(\frac{1}{\nu} p\right) = u \in U_{ad}.$$

The slack constraint set Ξ_{ad} is bounded and weakly closed, which means that $(\xi_k)_{k\in\mathbb{N}}$ contains a subsequence with weak limit $\xi \in \Xi_{ad}$ and

$$y_k = A^{-1}(u_k + \xi_k + f) \to A^{-1}(u + \xi + f) =: y \qquad \text{in } W_0^{1,q}(\Omega)$$

for $k \to \infty$. We employ the adjoint equation (37a) and the definition of λ_k in (38) to derive the convergence of $(\lambda_k)_{k\in\mathbb{N}}$ as follows. Firstly, $A^* : W_0^{1,q'}(\Omega) \to W^{-1,q'}(\Omega)$ is a bounded linear operator and as such weakly continuous such that $A^* p_k \rightharpoonup A^* p$ in $W^{-1,q'}(\Omega)$. Owing to the convergence of $(y_k)_{k\in\mathbb{N}}$ to y in $W_0^{1,q}(\Omega)$ we can apply Lemma 3.4 to obtain that

$$\lambda_k = A^* p_k + \sum_{w\in I} (y_k - y_w)\bar{\chi}_{B_{r_k}(w)} \rightharpoonup A^* p + \sum_{w \in I}(y(w) - y_w)\delta_w = \lambda \quad \text{ in } W^{-1,q'}(\Omega).$$

We hence showed (50a), (50b) and (50d). The non-negativity of the limit state y in (50c) follows from (46) in Lemma 3.12: Since $y_k \to y$ for $k \to \infty$ we have

$$\left\|(-y)_+\right\|_{L^2(\Omega)} = \lim_{k\to\infty} \left\|(-y_k)_+\right\|_{L^2(\Omega)} = 0.$$

The convergence (47) in Lemma 3.12 implies that

$$\langle \xi, y \rangle_{H^{-1}(\Omega)} = \lim_{k\to\infty} \langle \xi_k, y_k \rangle_{H^{-1}(\Omega)} = 0.$$

This proves (50c). We next analyze the dual pairing

$$\langle \lambda, y \rangle_{W^{-1,q'}(\Omega)} = \lim_{k\to\infty} \langle \lambda_k, y_k \rangle_{W^{-1,q'}(\Omega)}.$$

The definition of λ_k in resolves to

$$|\langle \lambda_k, y_k \rangle_{W^{-1,q'}(\Omega)}| = |\gamma_k \left\|(-y_k)_+\right\|^2_{L^2(\Omega)} + \delta(\gamma_k)\left(\xi_k, y_k\right)_{L^2(\Omega)}|.$$

The term on the right hand side converges to zero by (45) in Lemma 3.12 and we thus proved (50e). In the second part of the assertion, it holds for all $k \in \mathbb{N}$ that $\xi_k - \phi < 0$ and $\left(\xi_k - \phi, (\tau_k)_-\right)_{L^2(\Omega)} = 0$, which indicates that $(\tau_k)_- = 0$. This implies that

$$\mu_k = (\tau_k)_+ - \delta(\gamma_k) y_k = \tau_k - \delta(\gamma_k) y_k = -p_k$$

and we infer the weak convergence of μ_k to $\mu = -p$ in $H^1_0(\Omega)$ from the respective convergence of the adjoint state variables p_k. We write $(\xi, \mu)_{L^2(\Omega)} = \lim_{k\to\infty} (\xi_k, \mu_k)_{L^2(\Omega)}$ and compute

$$(\xi_k, \mu_k)_{L^2(\Omega)} = \left(\xi_k, (\tau_k)_+ - \delta(\gamma_k) y_k\right)_{L^2(\Omega)} = -\delta(\gamma_k)\left(\xi_k, y_k\right)_{L^2(\Omega)}.$$

Using (47) in Lemma 3.12 we directly obtain that $(\xi, \mu)_{L^2(\Omega)} = 0$. Finally, for all $k \in \mathbb{N}$, the definition of λ_k and μ_k in (38) yields

$$\begin{aligned}
\langle \lambda_k, \mu_k \rangle_{H^{-1}(\Omega)} &= \left(-\gamma_k\left(y_k\right)_- - \delta(\gamma_k)\xi_k, (\tau_k)_+ - \delta(\gamma_k) y_k\right)_{L^2(\Omega)} \\
&= -\gamma_k\left((y_k)_-, (\tau_k)_+\right)_{L^2(\Omega)} + \gamma_k \delta(\gamma_k)\left\|(y_k)_-\right\|^2_{L^2(\Omega)} + \delta(\gamma_k)^2\left(\xi_k, y_k\right)_{L^2(\Omega)} \\
&\geq \delta(\gamma_k)\left(\gamma_k\left\|(y_k)_-\right\|^2_{L^2(\Omega)} + \delta(\gamma_k)\left(\xi_k, y_k\right)_{L^2(\Omega)}\right).
\end{aligned}$$

The bound from Lemma 3.12 resolves to

$$\liminf\{\langle \lambda_k, \mu_k \rangle_{H^{-1}(\Omega)} \,|\, k \in \mathbb{N}\} \geq -C,$$

but one can consider (48) to refine this estimate and derive (50f) by the assumption that $\lim_{k\to\infty} \frac{\delta(\gamma)^3}{\gamma_k} = 0$. □

4. Numerical Tests

In Section 4.1, we set up a function space algorithm according to the penalty schemes discussed above.

4.1. **Function Space Method.**

We provide a path-following method for the solution of optimal control problems of the type (1) in function space, cf. Algorithm 1. In this connection, the so-called *path* is a sequence of first order points $x_k = (y_k, u_k, \xi_k)$ of the auxiliary problem (3) or (25), with respective multiplier vectors Λ_k for a sequence of penalty parameters $(\gamma_k)_{k\in\mathbb{N}} \subset \mathbb{R}^{>0}$ with $\gamma_k \to \infty$ for $k \to \infty$ and, in the point tracking case, $\delta : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ and $(r_k)_{k\in\mathbb{N}} \subset \mathbb{R}^{>0}$. We choose $(\gamma_k)_{k\in\mathbb{N}} = (\gamma \cdot (\delta\gamma)^k)_{k\in\mathbb{N}}$ for an initial penalty parameter $\gamma > 0$ and a factor $\delta\gamma > 1$. The sequence $(x_k, \Lambda_k)_{k\in\mathbb{N}}$ is computed in steps 6 or 10 of the outer loop (Algorithm 1) by the subroutine `solvePenMPEC` which will be referred to as the inner loop. In each iteration, the multipliers λ_k, μ_k that occur in the stationarity systems (14) and (50) are reconstructed from (y_k, u_k, ξ_k, p_k), see steps 7, 8 / 11, 12. Then, the norm of the residual pertinent to the conditions in (14) and (50) is computed in step 14.

The break criterion for the outer loop relies on a sufficient decrease of the residual by means of a prescribed $\bar{r} > 0$. In case that the break criterion is not satisfied, the penalty parameter is increased and the corresponding auxiliary problem is solved. Since we proved the convergence only on a subsequence, it can happen that the

Algorithm 1 solveMPEC

Input: Data for problem (2) or, if PT, for problem (25), initial values for y, u, ξ

1: Choose $0 < \bar{r} \ll 1$, $\gamma = \gamma_0 > 0$, $\delta\gamma > 1$ and $M \in \mathbb{N}$, set $i := 1$.
2: If PT, set $\delta = \gamma^{1/3}$, $r = \gamma^{-1}$.
3: **loop**
4:     **while** $i \leq M$ **do**
5:         **if** PT **then**
6:             $(y, u, \xi, p, \tau) =$ `solvePenPT`$(\mathrm{DATA}, \gamma, \delta, r, y, u, \xi)$
7:             Set $\lambda := A^*p + \sum_{w \in I}(y(w) - y_w)\delta_w$ according to (50d).
8:             Set $\mu := (\tau)_+ - \delta y$ according to (38).
9:         **else**
10:             $(y, u, \xi, p) =$ `solvePen`$(\mathrm{DATA}, \gamma, y, u, \xi)$
11:             Set $\lambda := A^*p + j'(y)$ according to (14c).
12:             Set $\mu := -p$ according to (14e).
13:         **end if**
14:         Compute $r_\gamma =$`residual`$(y, u, \xi, p, \lambda(, \mu, \tau))$ due to (14) or (50)
15:         **if** $r_\gamma \leq \bar{r}$ **then**
16:             **return** y, u, ξ.
17:         **end if**
18:         Set $\gamma = \delta\gamma \cdot \gamma$, $i = i + 1$, and, if PT, set $\delta = \gamma^{1/3}$, $r = \gamma^{-1}$.
19:     **end while**
20:     Reset $\gamma = \gamma_0$ and if PT, set $\delta\gamma = \frac{8\delta\gamma+1}{9}$, $\delta = \gamma^{1/3}$, $r = \gamma^{-1}$.
21: **end loop**

algorithm does not converge. We indicate this by defining a maximum number of iterations M, and act on the assumption that the sequence does not converge if M is reached and the residual pertinent to C-stationarity of the suggested solution is not satisfactory small. In this case we reset γ to a fixed value (here γ_0) and decrease $\delta\gamma$ in step 20.

The auxiliary problems (3) and (29) can be solved with standard optimization tools for problems with smooth objective and linear constraints. In our numerical test computations, we discretize the auxiliary stationarity systems and solve the resulting finite dimensional complementarity system by a damped semi-smooth Newton method (cf. [14, 8, 31]).

4.2. Numerical Results for an L^2-tracking Problem.

We consider an example from [15, Example 6.1].

Example 4.1. We set $A = -\Delta$ on the square domain $\Omega = (0,1) \times (0,1)$ and

$$\begin{aligned}
z_1(x_1) &= -4096x_1^6 + 6144x_1^5 - 3072x_1^4 + 512x_1^3, \\
z_2(x_2) &= -244.140625x_2^6 + 585.9375x_2^5 - 468.75x_2^4 + 125x_2^3, \\
y^*(x_1, x_2) &= \begin{cases} z_1(x_1)z_2(x_2) & \text{in } (0, 0.5) \times (0, 0.8), \\ 0 & \text{else,} \end{cases} \\
u^*(x_1, x_2) &= y^*(x_1, x_2), \\
\xi^*(x_1, x_2) &= 2\max\{0, -|x_1 - 0.8| - |(x_2 - 0.2)x_1 - 0.3| + 0.35\}.
\end{aligned}$$

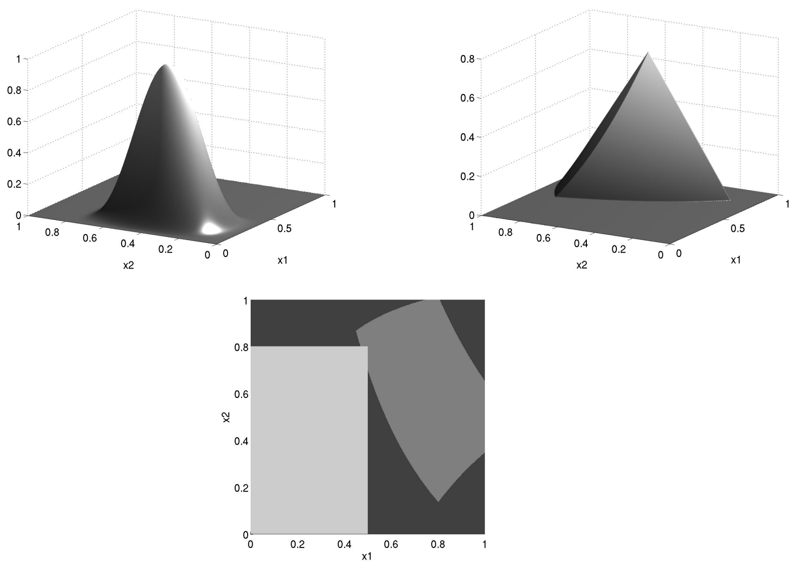

FIGURE 1. Solution graphs for Example 4.1, showing y ($= u$, upper left), ξ ($= -\lambda$, upper right) and the active sets (lower plot). The inactive set is depicted in light grey, the strongly active set in medium gray, and the biactive set in dark gray.

The data f, y_d is set to

$$f = -\Delta y^* - u^* - \xi^*, \qquad\qquad y_d = y^* + \xi^* - \nu \Delta u^*.$$

The parameter for the cost of the control is $\nu = 1$, there are no constraints on the control, and the objective functional is defined by

$$J(y,u) = \frac{1}{2}\left\|y - y_d\right\|^2_{L^2(\Omega)} + \frac{\nu}{2}\left\|u\right\|_{L^2(\Omega)}.$$

The solution $y = u = p$, the multiplier $\xi = -\lambda$ and the strongly active and biactive sets are shown in Figure 1.

We discretize the state as well as the control space by use of P^1-finite elements on a (regular) triangulation of the domain Ω. The MPEC solver can be run on a fixed mesh, or, in order to improve the efficiency of the method, it can be run on gradually refined meshes, optionally also with an a posteriori error estimation procedure from [13] which adaptively adjusts the discretization to the solution of the concrete problem. On each refinement level, the optimization routine (Algorithm 1) is employed with $\bar{r} = 10^{-6}$, $\gamma = 10^{-3}$, $\delta\gamma = 1.5$ and $M = 500$.

Figure 2 shows the convergence history of the ℓ_1-penalty scheme for Example 4.1 on two different meshes. The L^2-error of the control is plotted in black, and the residual is plotted in gray in a logarithmic scale against the value of the penalty parameter γ. The errors and residuals of the respective last steps of the algorithm are not plotted, because the extremely small residual (around 10^{-15}) spoils the scale. The three plots correspond to mesh sizes of 2^{-3} (49 free nodes), 2^{-4} (225 free nodes) and 2^{-5} (961 free nodes) on the square domain. Note that the initial value for the penalty parameter γ is very small. All graphs in Figure 2 start with

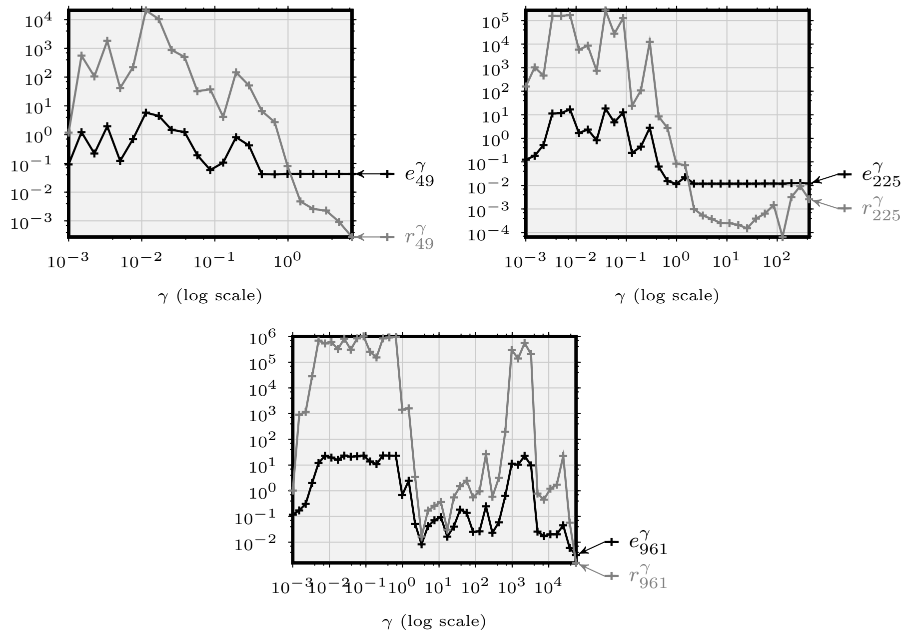

FIGURE 2. Convergence history for the elastic mode algorithm 1 without multigrid / adaptive refinement. The convergence of the error (e^{γ}, black) and the residual (r^{γ}, gray) are plotted against γ for Example 4.1 on three different meshes.

a sequence of more or less unreasonable solutions until a penalty parameter of around 10^{-1} is reached. When using this penalty scheme on fixed meshes without an adaptive or uniform refinement loop, the initial value should thus be increased. Especially in the first row, the discretization error can be seen: At some point, the L^2-error of u (i.e. the distance of the discrete control to the exact control in $L^2(\Omega)$), denoted by e_{γ}, does not decrease anymore while the solution of the discrete problem is not yet found, while the residual r_{γ} still decreases. This means that if one aims to compute an approximation of the function space solution, then the break criterion for the penalty method should be linked to the mesh size.

Figure 3 shows the convergence history of the algorithm including successive mesh refinement. The two plots show the data pertinent to uniformly refined meshes (left) and adaptive refinement (right). Every plot shows the number of nodes in the mesh in black and outer iterations on the horizontal axes. On a fixed mesh (i.e., for a constant number of nodes), the outer loop increases the penalty parameter γ until the residual that belongs to the C-stationarity system $r_{\gamma}= \texttt{residual}(x, \Lambda)$ in an iterate (x, Λ) (gray graph) is below a level $\bar{r}$ (dotted in gray). Then, the mesh is refined by bisection of every triangle or due to the estimator and thus the number of nodes increases. We cut each plot after 100 outer iterations. The L^2-norm of the error in the control variable $e_u= \|u^{\star} - u_k\|_{L^2(\Omega)}$ in an iterate u_k is plotted in black. Its value combines the discretization error with the error in optimality: It decreases when γ is increased until the discretization error is reached. When the mesh is refined and the iterate u_k is prolongated to the larger finite element space,

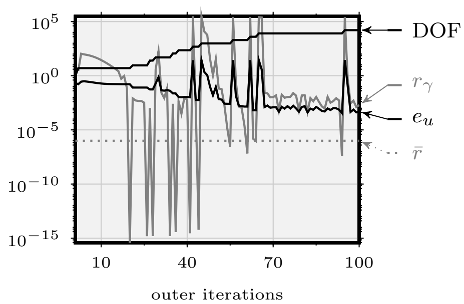

Figure 3. Convergence history for the elastic mode algorithm 1 with multigrid / adaptive refinement applied to Example 4.1. The left plot shows the convergence of the error (e^γ, black) and the residual (r^γ, gray) plotted against the outer iterations for the uniform method, the right plot refers to the respective data in the adaptive method.

$e_u$increases. For finer meshes, the discretization error becomes smaller. In this plot we did not cut the last steps and see the sudden decrease of the residual on each mesh. This indicates that the algorithm in fact finds an exact solution to the discrete problem, cf. Section 2.3.

4.3. **Numerical Results for a Point Tracking Problem.**

Example 4.2. We now consider the L-shaped domain $\Omega = (-1,0) \times (-1,1) \cup (-1,1) \times (0,1)$. The set of tracking points is

$$I = \{(-0.5,-0.5),(-0.5,0),(-0.5,0.5),(0,0.5),(0.5,0.5)\}$$

and we want the state variable to take the values $y_w = 1$ for $w \in \{(-0.5,-0.5),(-0.5,0)\}$, $y_w = -0.1$ for $w \in \{(-0.5,0.5),(0,0.5)\}$ and $y_w = 0$ for $w = (0.5,0.5)$. The parameter belonging to the control costs is set to $\nu = 0.01$, the force acting on the state is defined by

$$f(x_1,x_2) = 0.5 + 0.5(x_1 - x_2).$$

There are no control constraints. The solutions calculated by our algorithm are depicted in Figure 4. The example again admits a biactive set, and the solutions have a low regularity (consider, for instance, λ on the right hand side, middle row). For the penalty method 1 we choose $\bar{r} = 10^{-5}$, $\gamma_0 = 0.01$, $\delta\gamma = 1.5$ and $M = 300$.

We again start with a test of the algorithm on fixed meshes with different complexity. Figure 5 shows the convergence of the residuals for Example 4.2 on a mesh with 833 nodes (bright gray graph, $h = 2^{-4}$ on the L-shaped domain), with 3201 nodes (dark gray graph, $h = 2^{-5}$) and with 12545 nodes (black graph, $h = 2^{-6}$). The algorithm shows a clearly mesh independent convergence. Note that since we do not know the exact solution, we do not plot errors here.

Figure 6 shows the convergence history of the overall algorithm for both uniform (left) and adaptive mesh refinement (right). The structure here is the same as in Figure 3: We plot the outer iterations of Algorithm 1 and the mesh refinement steps on the horizontal axis. The black line shows the number of free nodes and

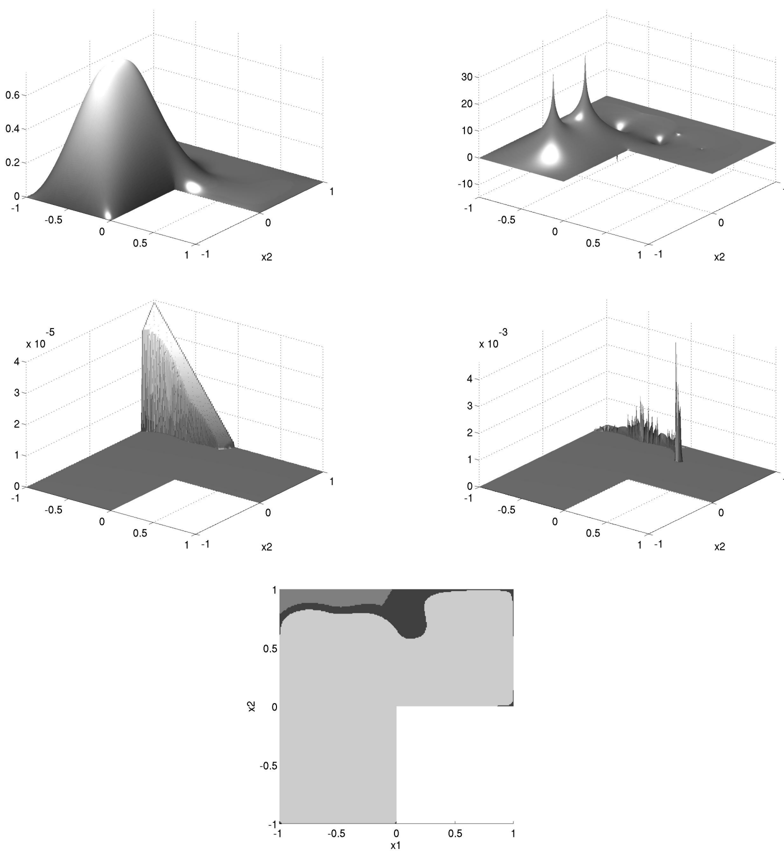

FIGURE 4. Solution graphs for Example 4.2, showing y (upper left), u (upper right), ξ (middle left), λ (middle right) and the active sets (lower plot). The inactive set is depicted in light grey, the strongly active set in medium gray, and the biactive set in dark gray.

thus indicates the refinement steps that are performed as soon as the residual is below $\bar{r}$.

The residual shows a similar trend as in Figure 3 on the L^2-tracking case. It decreases until, at a certain value of γ, the iterate 'falls' into the solution and the outer loop break in fact with a residual that is far below its bound $\bar{r}$. The adaptive method (right plot) has an advantage because of its capability of a rather accurate detection of the active sets. The sudden increase of the residual in the left plot comes from a reset of γ to γ_0: The upper bound M in Algorithm 1 is reached, and the solution pertinent to the new (small) penalty parameter is of course a bad candidate for a solution of the C-stationarity system.

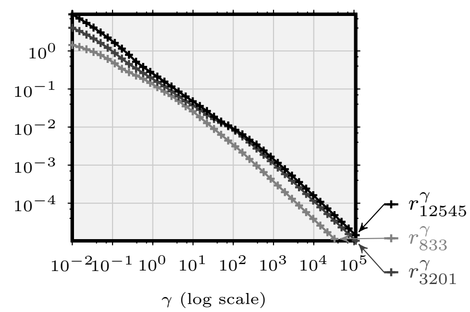

FIGURE 5. Convergence history of the residuals of the C-stationarity system for the elastic mode algorithm 1 without multigrid / adaptive refinement plotted against the value of the penalty parameter γ. The subscript values in the labels of the graphs give the complexity of the problem which they belong to.

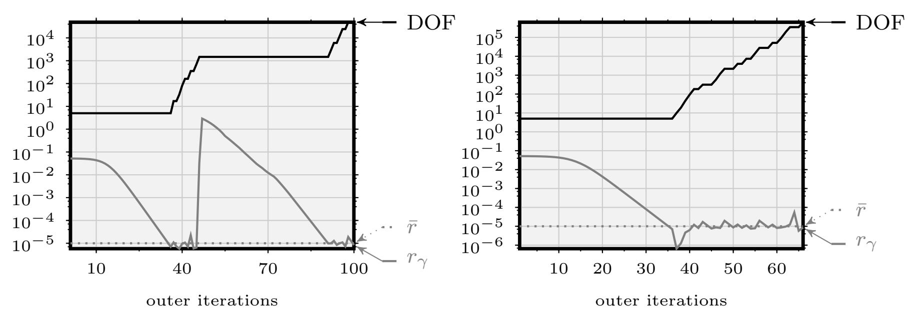

FIGURE 6. Convergence history for the elastic mode algorithm 1 with multigrid / adaptive refinement applied to Example 4.2. The left column shows the convergence of the residual r^γ (gray) and DOF (black) plotted against the outer iterations for the uniform method, the right column refers to the residual in the adaptive method.

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