

Cutting through form factors and cross sections of non-protected operators in $\mathcal{N} = 4$ SYM

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ABSTRACT: We study the form factors of the Konishi operator, the prime example of non-protected operators in $\mathcal{N} = 4$ SYM theory, via the on-shell unitarity methods. Since the Konishi operator is not protected by supersymmetry, its form factors share many features with those in QCD, such as the occurrence of rational terms and of UV divergences that require renormalization. A subtle point is that this operator depends on the spacetime dimension. This requires a modification when calculating its form factors via unitarity methods. We derive a rigorous prescription that implements this modification to all loop orders and obtain the two-point form factor up to two-loop order and the three-point form factor to one-loop order. From these form factors, we construct an IR-finite cross section type quantity, namely the inclusive decay rate of the (off-shell) Konishi operator to any final (on-shell) state. Via the optical theorem, it is connected to the imaginary part of the two-point correlation function. We extract the Konishi anomalous dimension up to two-loop order from it.

Contents

1	Introduction	2
2	Cross sections for two-point correlation functions in a nutshell	7
2.1	Renormalization of composite operators and their two-point functions	7
2.2	Two-point correlation functions and cross sections	8
3	Form factors for \mathcal{K}_6 via unitarity	11
3.1	Some BPS form factor results	12
3.2	Tree-level two- and three-point form factors	13
3.3	One-loop two-point form factor	14
3.4	Two-loop two-point form factor	17
3.5	One-loop three-point form factor	23
4	Konishi vs. \mathcal{K}_6	29
4.1	A subtlety in the dimension of intermediate states	29
4.2	Lifting intermediate states for form factors	31
4.3	Final Konishi form factors	33
5	BPS and Konishi cross sections	34
5.1	BPS cross section up to one-loop	34
5.2	Konishi cross section up to two-loop	36
5.2.1	One-loop result	37
5.2.2	Two-loop result	38
6	Conclusion and outlook	43
A	Fourier transformation of the propagator	46
B	Feynman integrals	46
C	Passarino-Veltman reductions	50
D	Checks of the three-point one-loop Konishi form factor	51
E	Phase-space parametrization	53
F	Anomalous dimensions via two-point form factors	54
G	Renormalization-scheme transformations	57
H	Feynman diagrams	59

1 Introduction

So far, the framework of quantum field theories (QFTs) is very successful in describing the high-energy processes measured at colliders such as LHC. However, theoretical predictions are usually restricted to the weak-coupling regime, which admits a perturbative expansion in terms of the small coupling constants. The individual contributions to the perturbation series can be calculated by employing Feynman diagrams. Thereby, a large proliferation of diagrams is in general encountered when one proceeds to higher-order corrections, and hence concrete calculations are mainly restricted to the first few orders.

The investigation of alternative techniques that bypass this limitation is thus of high importance. It might not only allow to push perturbation theory to higher orders, but could also deepen our understanding of fundamental principles and mechanisms encoded in QFTs. The so-called ‘on-shell’ techniques are such an alternative. They allow one to build amplitudes from other amplitudes with a lower number of external legs and loops via recursion relations [2, 3] and unitarity [4, 5]. They have been successfully used in supersymmetric gauge theories as well as in QCD, see [6–8] for pedagogical reviews and references therein.

In particular, the maximally supersymmetric Yang-Mills ($\mathcal{N} = 4$ SYM) theory with gauge group $SU(N_c)$ in four dimensions plays an important role in the aforementioned developments. According to the AdS/CFT correspondence [9–11] it has a dual description in terms of a string theory, allowing its study also at strong coupling. Moreover, in the planar limit [12], it shows signs of integrability at weak as well as at strong coupling, and it is believed to be present even at any coupling; based on the conjectured integrability, new predictions for the spectrum, i.e. for the anomalous scaling dimensions of gauge-invariant composite operators were made, see [13] for a review. This rises the hope that the theory is exactly solvable, and it is hence sometimes even referred to as the “harmonic oscillator of the 21st century”.

Given the success of the on-shell techniques for amplitudes, it is an intriguing question whether they can be applied for determining off-shell quantities such as correlation functions or the anomalous dimensions as well. A bridge between the purely on-shell amplitudes and the purely off-shell correlation functions is provided by form factors. They also contain the information necessary to determine the anomalous dimensions. An n -point form factor describes the overlap of an off-shell initial state, described by a composite operator, into an on-shell final state consisting of n elementary fields. It is given by

$$\mathcal{F}_{\mathcal{O}}(1, \dots, n) = \int d^D x e^{iq \cdot x} \langle 1 \cdots n | \mathcal{O}(x) | 0 \rangle = \delta^{(D)}\left(q - \sum_{i=1}^n p_i\right) \langle 1 \cdots n | \mathcal{O}(0) | 0 \rangle, \quad (1.1)$$

where the particles labeled by $i = 1, \dots, n$ carry individual on-shell momenta p_i and the operator \mathcal{O} carries momentum q . If the number n and type of the external fields exactly match those contained in \mathcal{O} , the form factor is called minimal form factor. Minimal form factors with $n = 2$ points are denoted as Sudakov form factors.

In the $\mathcal{N} = 4$ SYM theory, the most intensively studied form factors are the ones of

the half-BPS operator

$$\mathcal{O}_{\text{BPS}} = \text{tr}(\phi_{(I}\phi_{J)}) , \quad (1.2)$$

where the parenthesis denote traceless-symmetrization of the flavor indices $I, J = 1, \dots, N_\phi$ of the N_ϕ scalar field flavors. This operator belongs to the stress tensor supermultiplet. Its Sudakov form factor was first studied by van Neerven [14] and analyzed up to four loops [15, 16] in the recent past. The Sudakov form factor exhibits exponentiation [17–19], a feature which was seen to be the key for predicting the all-loop IR behavior of scattering amplitudes [20].

The form factors of the stress tensor multiplet with general n external legs can be analyzed in analogy to the scattering amplitudes with modern on-shell techniques. The n -point form factor with the bosonic operator (1.2) was first studied in [21, 22], and later generalized to the full stress tensor multiplet in [23, 24]. Up to one loop order, compact expressions for general n -point MHV as well as some NMHV form factors have been computed in [21, 23–26, 33]. The two-loop three-point form factor was computed in [27]. The form factors of half-BPS operators with k scalar fields (as well as the supermultiplet) have been studied in [22, 28, 29], where n -point tree and one-loop MHV results are presented in [28] and the minimal form factors (for $n = k$) were computed at two-loop [29]. Form factors have also been studied at strong coupling via the AdS/CFT correspondence [30], and a Y-system formulation was given in [31] for AdS₃ and in [32] for AdS₅.

The aforementioned studies have shown that form factors share very similar recursive and analytic properties with scattering amplitudes, at least for the protected operators. Moreover, the robust set of on-shell techniques for computing on-shell objects is also applicable here. This rises the hope that also fully off-shell quantities can be studied using on-shell methods, and that such an enhancement of the toolkit allows to detect new features of the theory. Indeed, it was found that certain correlation functions can be constructed via generalized unitarity from amplitudes, form factors and their generalizations involving several operator insertions [33]. In the recent parallel work [34], one of us has determined at tree level the minimal form factors of a generic operator and at one-loop order their cut-constructible parts. The one-loop results yield the complete one-loop dilatation operator of the theory.

Scattering amplitudes as well as form factors are themselves not physical observables, since they contain infrared (IR) divergences from the integration of loop momenta. Adding the so-called Bremsstrahlung contributions, their IR divergences from the real emissions of soft and collinear particles cancel the IR divergences coming from virtual loop corrections according to the Kinoshita-Lee-Nauenberg theorem [35, 36], and one obtains an observable. In particular, the cross sections are free of IR divergences and hence physical observables. They are, however, in general not well defined in a CFT such as the $\mathcal{N} = 4$ SYM theory, where asymptotic states are ill defined. Some cross-section-type quantities have been defined by using coherent states as asymptotic states [37]. Alternatively, we can consider the decay of an initial off-shell state described by an operator $\mathcal{O}(q)$, which is timelike ($q^2 > 0$) into any final on-shell multi-particle state. The probability of this inclusive decay is the total decay rate of $\mathcal{O}(q)$. This decay process may occur as part of a total cross section of

a scattering process in which $\mathcal{O}(q)$ is produced as an intermediate state.¹ The probability for the inclusive decay of $\mathcal{O}(q)$ into a final state X with total momentum $q = p_X$ is defined by

$$\sigma_{\mathcal{O}}(q) = \sum_X \delta^{(D)}(q - p_X) |\langle X | \mathcal{O}(0) | 0 \rangle|^2, \quad (1.3)$$

where the sum ensures that that the quantity is inclusive, i.e. all contributions, which are specified by the number and type of the particles in the final states are integrated over the respective phase space and are summed up. This cross-section type quantity depends on the matrix element $\langle X | \mathcal{O}(0) | 0 \rangle$, which is precisely the form factor of \mathcal{O} with final state X . Via the optical theorem, (1.3) is related to the imaginary part of the (time-ordered) two-point correlation function $\langle 0 | \bar{\mathcal{O}}(x) \mathcal{O}(0) | 0 \rangle$ after transforming to momentum space.

Finally, although not considered in this paper, we would like to mention that by modifying (1.3), ‘event shapes’ such as energy or charge correlation functions were studied in the $\mathcal{N} = 4$ SYM theory [33, 39–41]. Also, Wilson coefficients for the deep inelastic scattering were considered [42]. For simplicity, we will follow the terminology of [40] and denote the cross-section-type quantity defined in (1.3) as total cross section, or simply cross section.

In this paper, we will study the form factor (1.1) and the cross section (1.3) for the Konishi operator as a first example for an operator that is not protected by supersymmetry. Hence, UV divergences appear in addition to the aforementioned IR divergences that already emerge for protected operators. The Konishi primary operator is given by

$$\mathcal{K} = \text{tr}(\phi_I \phi_I), \quad (1.4)$$

where a sum over all $I = 1, \dots, N_\phi$ scalar field flavors is implicitly understood. In strictly $D = 4$ dimensions $N_\phi = 6$. The Konishi scaling dimension $\Delta_{\mathcal{K}} = \Delta_{\mathcal{K}}^{(0)} + \gamma_{\mathcal{K}}$ consists of the bare dimension $\Delta_{\mathcal{K}}^{(0)} = 2$ and an anomalous dimension $\gamma_{\mathcal{K}}$. It is a power series in the coupling constant

$$g^2 = \frac{g_{\text{YM}}^2 N_c}{(4\pi)^2} (4\pi e^{-\gamma_E})^\epsilon, \quad (1.5)$$

which depends on the Yang-Mills coupling constant g_{YM} , the number of colors N_c and is the loop-counting parameter in the modified Dimensional Reduction ($\overline{\text{DR}}$) scheme. In the planar limit, the Konishi anomalous dimension is given by

$$\begin{aligned} \gamma_{\mathcal{K}} = & 6[2g^2 - 8g^4 + 56g^6 - 16(26 - 6\zeta_3 + 15\zeta_5)g^8 \\ & + 16(158 + 72\zeta_3 - 54\zeta_3^2 - 90\zeta_5 + 315\zeta_7)g^{10}] + \mathcal{O}(g^{12}), \end{aligned} \quad (1.6)$$

¹ The operator may be of different physical origin. For example, it can be part of a vertex that couples to a massive particle in an effective Lagrangian. Then, (1.3) yields the decay rate of this particle. A concrete example from the Standard Model is an effective Higgs-gluons vertex $H \text{tr}(F_{\mu\nu} F^{\mu\nu})$ obtained by integrating out a heavy quark loop, see e.g. [38]. The operator may be also be a (conserved) current describing a two-particle scattering. Examples of this type are e^+e^- annihilation into a virtual photon or Drell-Yan scattering, where the two incoming particles are annihilated into a virtual photon or gluon, respectively, exciting the QCD vacuum and decaying into e.g. quarks, gluons or leptons.

where the one- and two-loop contributions, which we reproduce as a check in this paper, were obtained by explicit Feynman diagram calculations in [43, 44] and [45–47].²

The operator given in (1.4) is the so-called primary operator of the Konishi supermultiplet. Its anomalous dimension given in (1.6) was mainly obtained by considering certain descendent operators within the Konishi multiplet rather than the Konishi primary operator (1.4). This is possible, since all members of a supermultiplet have the same anomalous dimension.³ In fact, we will see that the Konishi primary defined in (1.4) and involving a sum over the N_ϕ scalar field flavors depends on the dimension D as $N_\phi = 10 - D$ is required to ensure supersymmetry. This becomes important when regulating the divergences by continuing the theory from $D = 4$ to $D = 4 - 2\epsilon$ dimensions.

We will apply four-dimensional unitarity in order to compute the form factors. Within this framework, all on-shell component fields can be conveniently combined into Nair's $\mathcal{N} = 4$ on-shell superfield [63]. The on-shell superfield reads

$$\Phi(p, \eta) = g_+(p) + \eta^A \psi_A(p) + \frac{\eta^A \eta^B}{2!} \phi_{AB}(p) + \frac{\varepsilon_{ABCD} \eta^A \eta^B \eta^C}{3!} \tilde{\psi}^D(p) + \eta^1 \eta^2 \eta^3 \eta^4 g_-(p), \quad (1.7)$$

where η^A are Grassmann variables that encode the flavor and helicity of the component fields, and $A = 1, \dots, 4$ is the $SU(4)$ R-symmetry index. In the above superfield, the six real on-shell scalars ϕ_I transforming in the fundamental representation of $SO(6)$ are represented as the anti-symmetric product representation of two fundamental $SU(4)$ representations, $\phi_{AB} = -\phi_{BA}$, employing the isomorphism of the Lie-algebras $\mathfrak{so}(6)$ and $\mathfrak{su}(4)$.

Using (1.7), each n -point scattering amplitude with fixed total helicity can be efficiently packed into a single superamplitude. In analogy, also the form factors for the BPS operator (1.2) can be packed into super form factors if the BPS operator is expressed in terms of the scalar fields ϕ_{AB}

$$\mathcal{O}_{\text{BPS}} = \text{tr}(\phi_{AB} \phi_{CD}) - \frac{1}{12} \varepsilon_{ABCD} \text{tr}(\phi^{EF} \phi_{EF}), \quad (1.8)$$

where the last term subtracts the trace in the space of scalar flavors. Without loss of generality we will focus in the rest of this paper on its particular component

$$\mathcal{O}_{\text{BPS}} = \text{tr}(\phi_{AB} \phi_{AB}), \quad (1.9)$$

where doubled indices are not summed. Expressing also the Konishi operator in terms of the scalar fields ϕ_{AB} yields

$$\mathcal{K}_6 = \frac{1}{8} \varepsilon^{ABCD} \text{tr}(\phi_{AB} \phi_{CD}) = \text{tr}(\phi_{12} \phi_{34}) - \text{tr}(\phi_{13} \phi_{24}) + \text{tr}(\phi_{14} \phi_{23}), \quad (1.10)$$

²The Konishi anomalous dimension $\gamma_{\mathcal{K}}$ is currently known up to five loops from field theory calculations and up to nine loops from the conjectured integrability. The three-loop result was conjectured in [48] and confirmed in [49, 50]. The four-loop result was determined by calculating the wrapping corrections to the integrability-based asymptotic dilatation operator in [51, 52] and by a computer-based direct calculation in [53]. The integrability-based four-loop expression of [54] matches this result. The five-loop result was predicted from integrability in [55–57], and confirmed in [58] from an OPE analysis of the four-point correlation function of stress-tensor multiplets. The results at six [59], seven [60], eight [61] and nine loops [62] are so far only based on the conjectured integrability.

³Working with certain descendants which are non-singlet states of the $SU(4)$ R-symmetry instead of the primary operator (1.4), which is an $SU(4)$ singlet, simplifies the calculations in both, the field theory and integrability-based approach.

where the subscript 6 reminds us that the operator is identical to the Konishi primary (1.4) only for $N_\phi = 6$, i.e. only in strictly $D = 4$ dimensions.

There is a subtlety originating from the fact that in $D \neq 4$ dimensions the Konishi operator \mathcal{K} in (1.4) cannot be identified with \mathcal{K}_6 in (1.10). The four-dimensional unitarity method applies to the operator \mathcal{K}_6 . In this formulation, the operator stays the same if the encountered IR- and UV-divergences are regularized by changing the spacetime dimension from $D = 4$ to $D = 4 - 2\epsilon$. But in $D = 4 - 2\epsilon$ dimensions the Konishi operator \mathcal{K} is *not* identical to the operator \mathcal{K}_6 . Hence, the unitarity-based results for \mathcal{K}_6 do not directly yield those for the Konishi operator \mathcal{K} . Instead, modifications have to be made which take into account that one should have used \mathcal{K} and not \mathcal{K}_6 in order to obtain the results for the Konishi operator regularized in $D = 4 - 2\epsilon$ dimensions.

In the main part of the paper we elaborate on the ideas mentioned above. In section 2 we discuss two-point correlation functions of gauge-invariant local operators, their renormalization and the transformation to momentum space. We identify the imaginary part of such a correlation function with the cross section defined in (1.3). Finally, we present the general strategy of computing the total cross section for a given operator using its form factors as the building blocks.

In section 3, we present our computation of the form factors for \mathcal{K}_6 at the one- and two-loop orders, which are based on the unitarity method and on-shell superspace. Since the Konishi operator is not protected, several new interesting features appear in the results, such as the UV divergences and rational terms.

In section 4, we discuss in detail the aforementioned subtleties arising from the fact that in $D = 4 - 2\epsilon$ dimensions the Konishi operator \mathcal{K} cannot be identified with \mathcal{K}_6 . We derive a rigorous prescription of how to implement the substitution of \mathcal{K}_6 by \mathcal{K} in the results of the previous section.

In section 5, we present the computation of the cross section starting with the BPS operator up to one-loop order as a simple example to make the reader become familiar with our strategy. We find the expected non-trivial cancelation of the IR divergences between real and virtual channels. Then, we compute the cross section for the Konishi operator up to two loops. We extract the renormalization constant and hence the anomalous dimension from the UV divergence of the bare result. They match the known results. We present the finite result for the renormalized cross section and discuss its dependence on the renormalization scheme.

Finally, in section 6 we summarize the main results of our paper and the interesting features associated with them. We also present some future directions and open questions.

In the appendices A–G, we elaborate on the various technical aspects of our computations as well as non-trivial features of our results which were not seen before in similar computations for BPS operators or on-shell objects like scattering amplitudes. In the final appendix H, we summarize direct Feynman-diagrammatic calculations of the one- and two-loop form factors for both, the BPS and the Konishi operator, which served as checks for our approach and guided us to the modifications discussed in section 4.

2 Cross sections for two-point correlation functions in a nutshell

In this section, we review some facts about the form of the two-point correlation function of a renormalized composite operator in spacetime and in momentum space. Via the optical theorem, its imaginary part yields a cross-section-type quantity. It can be directly obtained from the form factors of the respective operator.

2.1 Renormalization of composite operators and their two-point functions

Gauge-invariant local composite operators can be regarded as external states of the $\mathcal{N} = 4$ SYM theory, and they can occur in correlation functions in the same way as the elementary fields. Such correlation functions in general contain UV divergences, which are associated with the presence of these operators, requiring their renormalization in analogy to that of the elementary fields and vertices of the theory. In this paper, we only consider composite operators that are eigenstates under renormalization. Such a renormalized operator is given in terms of the bare operator as

$$\mathcal{O}_R = \mathcal{Z}_\mathcal{O}(g, \epsilon) \mathcal{O}_B, \quad (2.1)$$

where $\mathcal{Z}_\mathcal{O}$ is the renormalization constant. It depends on the coupling constant g and absorbs the UV divergences, which appear as poles in ϵ when the theory is regularized by changing the spacetime dimension from $D = 4$ to $D = 4 - 2\epsilon$. The renormalization constant determines the anomalous dimension

$$\gamma_\mathcal{O} = \sum_{\ell=1}^{\infty} g^{2\ell} \gamma_\mathcal{O}^{(\ell)} = \lim_{\epsilon \rightarrow 0} \epsilon g \frac{\partial}{\partial g} \log \mathcal{Z}_\mathcal{O}, \quad (2.2)$$

which is added to the bare scaling dimension $\Delta_\mathcal{O}^{(0)}$ in order to obtain the conformal dimension $\Delta_\mathcal{O}$. Since $\gamma_\mathcal{O}$ is finite when the limit $\epsilon \rightarrow 0$ is taken in the above equation, the form of $\mathcal{Z}_\mathcal{O}$ as a power series in g is fixed to

$$\mathcal{Z}_\mathcal{O} = \exp \left(\sum_{\ell=1}^{\infty} \frac{g^{2\ell}}{2\ell\epsilon} \gamma_\mathcal{O}^{(\ell)} \right) = 1 + g^2 \frac{\gamma_\mathcal{O}^{(1)}}{2\epsilon} + g^4 \left(\frac{(\gamma_\mathcal{O}^{(1)})^2}{8\epsilon^2} + \frac{\gamma_\mathcal{O}^{(2)}}{4\epsilon} \right) + \mathcal{O}(g^6). \quad (2.3)$$

Conformal symmetry also completely fixes the form of the two-point function of the operator \mathcal{O}_R . In Minkowski spacetime, it reads

$$G_{2\mathcal{O},R}(x) = \langle 0 | \bar{\mathcal{O}}_R(x) \mathcal{O}_R(0) | 0 \rangle = \frac{M}{(-x^2 + i0)^{\Delta_\mathcal{O}} \mu^{2\gamma_\mathcal{O}}}, \quad \Delta_\mathcal{O} = \Delta_\mathcal{O}^{(0)} + \gamma_\mathcal{O}, \quad (2.4)$$

where our conventions for the $i0$ description are given in appendix A. The parameter μ has the dimension of mass and is introduced in order to fix the mass dimension to $\Delta_\mathcal{O}^{(0)}$. The coupling-dependent dimensionless factor M has a perturbative expansion as

$$M = \sum_{\ell=0}^{\infty} g^{2\ell} M^{(\ell)}, \quad (2.5)$$

and it can be absorbed into the normalization of \mathcal{O}_R .

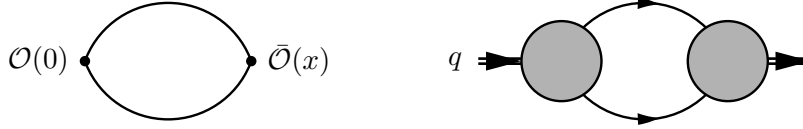


Figure 1: The two-point function in position space and momentum space.

We will work in momentum space, and hence need the Fourier transformation of (2.4). According to appendix A, it is given by

$$\tilde{G}_{2\mathcal{O},\text{R}}(q^2) = \int d^D x e^{iq \cdot x} G_{2\mathcal{O},\text{R}}(x) = (-i)2^{D-2\Delta_{\mathcal{O}}} \pi^{\frac{D}{2}} \frac{\Gamma(\frac{D}{2} - \Delta_{\mathcal{O}})}{\Gamma(\Delta_{\mathcal{O}})} \frac{M}{(-q^2 - i0)^{\frac{D}{2} - \Delta_{\mathcal{O}}} \mu^{2\gamma_{\mathcal{O}}}}. \quad (2.6)$$

When expanding the above expression first for small g and then for small ϵ , one obtains $\frac{1}{\epsilon^k}$ -poles for any $k \geq 1$, which for $k \geq 2$ are proportional to powers of $\gamma_{\mathcal{O}}$ [64]. Since $G_{2\mathcal{O},\text{R}}(x)$ is the finite (renormalized) Green function, these poles cannot come from UV divergences. In fact, they arise from integrating over the origin $x = 0$ of spacetime, where $G_{2\mathcal{O},\text{R}}(x)$ is singular. This can be most easily seen for the half-BPS operator \mathcal{O}_{BPS} defined in (1.2). Since this operator is protected, $\gamma_{\text{BPS}} = 0$, and all poles of order $k \geq 2$ disappear, but a simple $\frac{1}{\epsilon}$ -pole remains. In momentum space, this pole is associated with the one-loop bubble integral. It is obtained when inserting Fourier expressions for the two scalar propagators⁴ $\frac{1}{(-x^2 + i0)^{1-\epsilon}}$ connecting the two operators as depicted in figure 1 and performing the integration over x in (2.6), which yields a δ -function of momentum conservation. For the tree-level two-point function, the steps are as follows:

$$\frac{1}{(-x^2 + i0)^{2-2\epsilon}} \xrightarrow{\text{FT}} \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2(l-q)^2} \sim \frac{1}{\epsilon(-q^2 - i0)^\epsilon}. \quad (2.7)$$

This simple pole (for the BPS operator) and all the further $\frac{1}{\epsilon^k}$ -poles, $k \geq 2$, (for non-protected operators) are absent when taking the imaginary part of the momentum-space Green function (2.6).

As we will see in the next subsection, via the optical theorem this part yields a cross-section-type quantity: the probability of the inclusive decay of the renormalized operator \mathcal{O}_{R} with off-shell momentum. It has to be finite in the limit $\epsilon \rightarrow 0$, since it is free of IR divergences and — due to renormalization — also of UV divergences.

2.2 Two-point correlation functions and cross sections

Via the optical theorem, the imaginary part of a two-point correlation function is related to the inclusive decay width of the renormalized operator \mathcal{O}_{R} with off-shell momentum q , where $q^2 > 0$. As motivated in the introduction, we will simply denote this as cross section $\sigma_{\mathcal{O},\text{R}}$ in this paper. It is given by

$$\sigma_{\mathcal{O},\text{R}} = \text{Im}[2i \tilde{G}_{2\mathcal{O},\text{R}}(q^2)] = \sum_X \delta^D(q - p_X) |\langle X | \mathcal{O}_{\text{R}}(0) | 0 \rangle|^2, \quad (2.8)$$

⁴In D dimensions, the scaling dimension of a scalar field is given by $\Delta_\phi^{(0)} = \frac{D}{2} - 1 = 1 - \epsilon$

where one sums over all final on-shell states X , and the squared matrix element is given by the product of two form factors

$$\hat{\mathcal{F}}_{\mathcal{O},X} = \langle X | \mathcal{O}(0) | 0 \rangle . \quad (2.9)$$

The form factor is given by a perturbative expansion as

$$\hat{\mathcal{F}}_{\mathcal{O},X} = \sum_{\ell=0}^{\infty} g^{2\ell} \hat{\mathcal{F}}_{\mathcal{O},X}^{(\ell)} , \quad (2.10)$$

where g defined in (1.5) is the parameter of the loop expansion. Concretely, in $\mathcal{N} = 4$ SYM theory in the modified dimensional reduction ($\overline{\text{DR}}$) scheme,⁵ the coupling constant is given in (1.5). Moreover, the summation over all final states X in (2.8) involves in particular a summation over the number n of particles in the final state, i.e. of the n -point form factors $\hat{\mathcal{F}}_{\mathcal{O},n}^{(\ell)}$ over n . The number n is directly related to powers of the Yang-Mills coupling constant g_{YM} . In analogy to amplitudes (see e.g. [66]), the n -point form factors possess a decomposition in terms of the possible color structures as

$$\begin{aligned} \hat{\mathcal{F}}_{\mathcal{O},n}^{(\ell)}(\{a_i, p_i, \eta_i\}) &= g_{\text{YM}}^{n-2} \sum_{\sigma \in S_n/Z_n} \text{tr}(\mathbf{T}^{a_{\sigma(1)}} \dots \mathbf{T}^{a_{\sigma(n)}}) \mathcal{F}_{\mathcal{O},n}^{(\ell)}(\{p_{\sigma(i)}, \eta_{\sigma(i)}\}) \\ &+ \text{multi-trace terms} , \end{aligned} \quad (2.11)$$

where \mathbf{T}^a , $a = 1, \dots, N_c^2 - 1$, are the gauge-group generators of $SU(N_c)$ normalized as

$$\text{tr}(\mathbf{T}^a \mathbf{T}^b) = \delta^{ab} . \quad (2.12)$$

In (2.11), the i^{th} particle, $i = 1, \dots, n$, with momentum p_i carries the adjoint gauge-group index a_i . Via Nair's superfield (1.7), its flavor and helicity are encoded in terms of the Grassmann variables η_i , on which the color-ordered super form factors $\mathcal{F}_{\mathcal{O},n}^{(\ell)}$ on the rhs. also depend.

The imaginary part of (2.6) can be obtained by taking the discontinuity, which for timelike ($q^2 > 0$) momentum reads⁶

$$2i \text{Im}(-q^2 - i0_+)^x = (-q^2 - i0)^x - (-q^2 + i0)^x = \frac{2\pi i}{\Gamma(x)\Gamma(1-x)} (q^2)^x . \quad (2.13)$$

Using this relation in order to determine the imaginary part of (2.6) and then inserting the result into (2.8) yields

$$\frac{\sigma_{\mathcal{O},\text{R}}}{\sigma_{\mathcal{O}}^{(0)}} = \frac{M(g)}{M^{(0)}} \frac{\Gamma(\Delta_{\mathcal{O}}^{(0)})\Gamma(\frac{D}{2} - \Delta_{\mathcal{O}})}{\Gamma(\Delta_{\mathcal{O}})\Gamma(\frac{D}{2} - \Delta_{\mathcal{O}}^{(0)})} \frac{\Gamma(\Delta_{\mathcal{O}}^{(0)} - \frac{D}{2})\Gamma(1 + \frac{D}{2} - \Delta_{\mathcal{O}}^{(0)})}{\Gamma(\Delta_{\mathcal{O}} - \frac{D}{2})\Gamma(1 + \frac{D}{2} - \Delta_{\mathcal{O}})} \left(\frac{q^2}{4\mu^2}\right)^{\gamma_{\mathcal{O}}} , \quad (2.14)$$

where we have divided $\sigma_{\mathcal{O},\text{R}}$ by its classical part $\sigma_{\mathcal{O}}^{(0)} = \text{Im}[2i \tilde{G}_{2\mathcal{O},\text{R}}^{(0)}(q^2)]$. Indeed, as mentioned at the end of the previous subsection, both $\sigma_{\mathcal{O}}^{(0)}$ and $\sigma_{\mathcal{O},\text{R}}$ are free of $\frac{1}{\epsilon}$ -poles,

⁵This scheme employs dimensional reduction as regularization and for the subtraction of the divergences a modified minimal subtraction which absorbs the same finite terms in addition to the UV-divergences into the renormalization constant as the famous $\overline{\text{MS}}$ scheme [65].

⁶Our conventions for the $i0$ description are given in appendix A.

since the poles are canceled by the extra Γ -functions introduced via (2.13). This can also directly be seen for the bubble integral in (2.7): its imaginary part is obtained by applying a double-cut, which just yields a finite constant.

By taking the logarithm of (2.14), we can expose the dependence on q^2 as follows

$$\log\left(\frac{\sigma_{\mathcal{O},R}}{\sigma_{\mathcal{O}}^{(0)}}\right) = \gamma_{\mathcal{O}} \log\frac{q^2}{\mu^2} + C + \mathcal{O}(\epsilon) , \quad (2.15)$$

where the constant C is scale-independent but depends on $\gamma_{\mathcal{O}}$ and the expansion coefficients of the normalization factor (2.5) as

$$\begin{aligned} C = & g^2 \left(\frac{M^{(1)}}{M^{(0)}} - (1 - 2\gamma_E)\gamma^{(1)} \right) \\ & + g^4 \left(\frac{M^{(2)}}{M^{(0)}} - \frac{1}{2} \left(\frac{M^{(1)}}{M^{(0)}} \right)^2 + \frac{3 - \pi^2}{6} (\gamma^{(1)})^2 - (1 - 2\gamma_E)\gamma^{(2)} \right) + \mathcal{O}(g^6) . \end{aligned} \quad (2.16)$$

It is also renormalization-scheme-dependent as discussed at the end of section 5. However, the $\log q^2$ term is universal and scheme-independent. The anomalous dimension is given by the coefficient of $\log\frac{q^2}{\mu^2}$. In this paper, we will verify this structure for the Konishi operator up to two loops.

Strategy of computing cross sections

The cross section is obtained from (2.8) in more detail as follows:

$$\sigma = \sum_n \int \text{dPS}_n \sum_{\text{colors}} \sum_{\substack{\text{spins} \\ \text{helicities}}} \left\{ \underbrace{\left(\begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \right) \hat{\mathcal{F}}_n \left(\begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \right) \hat{\mathcal{F}}_n \left(\begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \right)}_{\mathcal{M}_n} \right\} . \quad (2.17)$$

This relation holds for both, the bare and the renormalized cross section, if $\hat{\mathcal{F}}_n$ represents the bare and the renormalized form factors, respectively. The evaluation of (2.17) requires three main steps: (1) determining the form factors $\hat{\mathcal{F}}_n$, (2) taking the absolute square of $\hat{\mathcal{F}}_n$, and (3) performing the n -particle phase space integrals. More concretely, (2.17) is expanded in powers of g as follows:

$$\sigma = \sum_{\ell=0}^{\infty} g^{2\ell} \sigma^{(\ell)} , \quad \sigma^{(\ell)} = \sum_{n=2}^{\ell+2} g^{2(2-n)} \int \text{dPS}_n \mathcal{M}_n^{(\ell+2-n)} , \quad (2.18)$$

where the squared matrix elements are given by

$$\mathcal{M}_n^{(\ell)} = \frac{1}{n!} \sum_{a_i} \int \prod_{i=1}^n d^4 \eta_i \sum_{k=0}^m \sum_{l=0}^{\ell} \hat{\mathcal{F}}_{\mathcal{O},n}^{N^k \text{MHV},(l)}(\{a_i, p_i, \eta_i\}) \hat{\mathcal{F}}_{\mathcal{O},n}^{*,N^{m-k} \text{MHV},(\ell-l)}(\{a_i, p_i, \eta_i\}) , \quad (2.19)$$

in which $\hat{\mathcal{F}}_n^{(\ell)}(\{a_i, p_i, \eta_i\})$ is the ℓ -loop n -point non-color-ordered super form factor defined in (2.11), and $\hat{\mathcal{F}}_n^*(\{a_i, p_i, \eta_i\})$ is its complex conjugate.⁷ We have indicated the maximally helicity violating (MHV) degree, which depends on the degree of η . For the BPS and Konishi operator considered in this paper, the MHV form factors have degree 4 in η and $m = n - 2$ is fixed. The squared matrix element involves sums over all numbers n and types of external particles and their color degree of freedom. The sum over the types of particles is given in terms of integrations over the fermionic variables η_i^A , $A = 1, 2, 3, 4$, and a sum over the MHV degree k .

Given the squared matrix elements, as a next step, the integration over the phase space of the n particles in the final state has to be performed. The respective measure is given by

$$d\text{PS}_n = \prod_{\ell=1}^n \frac{d^D p_\ell}{(2\pi)^D} 2\pi \delta_+(p_\ell^2) \cdot (2\pi)^D \delta^D\left(q - \sum_{\ell=1}^n p_\ell\right), \quad (2.20)$$

where $\delta_+(p^2) = \delta(p^2)\theta(p_0)$ with $\theta(p_0)$ being the Heaviside step function that imposes the positivity condition on p_0 . In appendix E, we give explicit parametrizations of the two-particle and three-particle phase space integrals.

Finally, the sum over the different channels, i.e. over the different particle numbers n , has to be performed. This leads to a cancellation among the different soft and collinear IR divergences such that the final result is IR finite. If non-protected operators are involved, as in the Konishi case, their renormalization constants have to be taken into account.

The use of the super form factors encoding the particle content and their polarizations via the on-shell superfields (1.7) requires that the number of scalars $N_\phi = 6$, such that they fit in the antisymmetric product of two fundamental $SU(4)$ representations. Moreover, in order to preserve supersymmetry, this demands that the polarization vectors ε_i^μ of the gluons are kept four-dimensional when the spacetime dimension is continued to $D = 4 - 2\epsilon$ dimensions. The scheme imposing these descriptions is called the four-dimensional helicity (FDH) scheme [67, 68]. This is analogous to the procedure pursued in section 3 where we sum the internal particles via the four-dimensional on-shell superspace. There are, however, some subtleties when considering the case of the Konishi operator, since it has $N_\phi = 6 + 2\epsilon$ scalar flavors in $D = 4 - 2\epsilon$ dimensions rather than only 6 that are included in the on-shell superfield. This has to be considered by modifying the result. The corresponding prescription will be discussed in details in section 4.

3 Form factors for \mathcal{K}_6 via unitarity

In the previous section, we have defined the cross section for gauge-invariant operators \mathcal{O} in $\mathcal{N} = 4$ SYM theory in terms of its squared matrix elements. As discussed around (2.19), the building blocks of these squared matrix elements are the non-color ordered super form factors for the particular operator. In this section, we will present the building blocks

⁷Note that in (2.19) the complex conjugate of tree-level form factors is already encoded in changing \mathcal{O} to be its conjugate $\bar{\mathcal{O}}$ and also changing the MHV degree from k to $m - k$. Therefore, the “*” corresponds to taking the conjugate of the $\ell \geq 1$ contributions only. This will be explained in explicit examples in section 5. See the discussion around (5.6).

necessary for computing the cross section of the Konishi operator (1.4) up to two loops, which are the two-point form factor up to two-loop order, and the three-point form factor at one-loop order.⁸

We use the notation for the non-color-ordered super form factors $\hat{\mathcal{F}}_{\mathcal{O},n}^{(\ell)}(\{a_i, p_i, \eta_i\})$ and color-ordered super form factors $\mathcal{F}_{\mathcal{O},n}^{(\ell)}(\{p_i, \eta_i\})$ as introduced in (2.11). We denote the bosonic color-ordered form factors with fixed external states by $F_{\mathcal{O},n}^{(\ell)}(\{p_i\})$, and more specifically for a precise external state, e.g. of two scalars and one gluon as $F_{\mathcal{O}}^{(\ell)}(1_\phi, 2_\phi, 3_g)$ or simply $F_{\mathcal{O},(\phi,\phi,g)}^{(\ell)}$. These can be obtained from $\mathcal{F}_{\mathcal{O},n}^{(\ell)}(\{p_i, \eta_i\})$ by taking a specific term in the η_i expansion. We also introduce the normalized bosonic form factors $f_{\mathcal{O},n}^{(\ell)}$ as the ratio between the ℓ -loop and tree-level color-ordered bosonic form factors

$$f_{\mathcal{O},n}^{(\ell)}(\{p_i\}) = \frac{F_{\mathcal{O},n}^{(\ell)}(\{p_i\})}{F_{\mathcal{O},n}^{(0)}(\{p_i\})}. \quad (3.1)$$

Our computation will focus on the colored-ordered form factors, and it is straightforward to obtain the full non-color ordered super form factor from them.

The computation of form factors in this section are based on the on-shell superspace formulation (1.7). Therefore, the operator in the form factor is \mathcal{K}_6 defined in (1.10) and *not* the Konishi operator \mathcal{K} defined in (1.4). We denote the resulting form factors by $F_{\mathcal{K}_6,n}^{(\ell)}$. As we will see in the next section, we have to modify the results presented in this section to obtain the Konishi form factors. This will be discussed in detail in section 4.

3.1 Some BPS form factor results

We start by presenting some known BPS form factor results, which are also useful building blocks for the Konishi form factors. Unless otherwise specified, the BPS form factor in this paper will always refer to that of the half-BPS operator $\text{tr}(\phi_{AB}^2)$ defined in (1.9), and we use the abbreviation $\mathcal{F}_{\text{BPS},n}^{(\ell)} = \mathcal{F}_{\text{tr}(\phi_{AB}^2),n}^{(\ell)}$.

The n -point maximally helicity-violating (MHV) tree-level BPS super-form factor is given by [23]

$$\mathcal{F}_{\text{BPS},n}^{(0),\text{MHV}}(1, 2, \dots, n) = \frac{\delta^{(4)AB}(\sum_{i=1}^n \lambda_i \eta_i)}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}, \quad (3.2)$$

where $\delta^{(4)AB}(\sum_i \lambda_i \eta_i)$ is understood as taking η in the delta function with only A, B indices, or more explicitly

$$\delta^{(4)AB}(\sum_i \lambda_i \eta_i) = \left(\sum_{i < j} \langle ij \rangle \eta_i^A \eta_j^A \right) \left(\sum_{k < l} \langle kl \rangle \eta_k^B \eta_l^B \right). \quad (3.3)$$

Note that in this and all following expressions for form factors we do not explicitly write the momentum-conserving delta function $\delta^{(4)}(q - \sum_{i=1}^n p_i)$, where q is the four-momentum carried by the gauge invariant operator.

⁸The tree-level four-point Konishi form factor essentially agrees with the BPS result, as we will discuss in subsection 3.2.

We give the loop corrections to the BPS MHV form factor in terms of the normalized form factor defined in (3.1). For the purpose of this paper, we only need the following three results [14, 21, 22]:

$$f_{\text{BPS},2}^{(1)} = -2s_{12} \begin{array}{c} p_1 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ p_2 \end{array}, \quad (3.4)$$

$$f_{\text{BPS},2}^{(2)} = s_{12}^2 \left(4 \begin{array}{c} p_1 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ p_2 \end{array} + \begin{array}{c} p_1 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ p_2 \end{array} \right), \quad (3.5)$$

$$f_{\text{BPS},3}^{(1)} = -\frac{s_{12}s_{23}}{2} \begin{array}{c} p_1 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ p_2 \quad p_3 \end{array} - \frac{s_{13} + s_{23}}{2} \begin{array}{c} p_1 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ p_2 \quad p_3 \end{array} - \frac{s_{12} + s_{31}}{2} \begin{array}{c} p_1 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ p_2 \quad p_3 \end{array} \\ + \text{cyclic perm. of } \{p_1, p_2, p_3\}. \quad (3.6)$$

Each graph corresponds to a Feynman integral which is defined in appendix B. Throughout this paper, all external on-shell momenta labeled by p_i are understood as outgoing.

For the two-point case, only the MHV configuration exists, while at three-point there are the MHV and the next-to-MHV (NMHV) configuration. The NMHV tree-level form factor can be obtained from (3.2) by first taking the conjugation $\lambda \rightarrow \tilde{\lambda}$ and $\eta^A \rightarrow \tilde{\eta}_A$, and then applying a fermionic Fourier transformation as⁹

$$\mathcal{F}_{\text{BPS},3}^{(0),\text{NMHV}}(1,2,3) = \prod_{i=1}^3 \int d^4 \tilde{\eta}_i e^{\sum_C \eta_i^C \tilde{\eta}_{i,C}} \frac{\delta_{AB}^{(4)}(\sum_{i=1}^3 \tilde{\lambda}_i \tilde{\eta}_i)}{[12][23][31]}. \quad (3.7)$$

Both, the MHV and the NMHV three-point form factor, share the same loop correction (3.6).

3.2 Tree-level two- and three-point form factors

We now turn to the Konishi form factor. In this subsection, we consider the tree-level form factors for \mathcal{K}_6 . They are identical to those of the Konishi operator \mathcal{K} . The expression for \mathcal{K}_6 in (1.10) contains the individual fields $\phi_{AB}\phi_{CD}$ where A, B, C, D assume distinct values instead of ϕ_{AB}^2 as is the case for the BPS operator. For the tree-level bosonic form factor with specified external particles, however, the index structure of the external scalars and fermions do not play any role in the result, which is obvious from the Feynman diagram computation. Therefore, the tree-level bosonic form factors for the Konishi operator are identical to the corresponding BPS form factors.

The super form factors, on the other hand, take different forms. Taking into account all the components, the two-point super form factor reads¹⁰

$$\mathcal{F}_{\mathcal{K}_6}^{(0)}(1,2) = -\frac{1}{4} \frac{\langle 12 \rangle^2}{\langle 12 \rangle \langle 21 \rangle} \sum_{A,B,C,D=1}^4 \varepsilon_{ABCD} (\eta_1^A \eta_1^B) (\eta_2^C \eta_2^D), \quad (3.8)$$

⁹Note the operator also becomes the conjugate one, $\text{tr}((\bar{\phi}^{AB})^2)$, where $\bar{\phi}^{AB} = \frac{1}{2} \varepsilon^{ABCD} \phi_{CD}$.

¹⁰The normalization factor is fixed to be consistent with the definition of the operator \mathcal{K}_6 in (1.10).

where $\varepsilon_{1234} = 1$. The bosonic two-point form factor

$$F_{\mathcal{K}_6}^{(0)}(1_{\phi_{12}}, 2_{\phi_{34}}) = -\frac{\langle 12 \rangle^2}{\langle 12 \rangle \langle 21 \rangle} = 1, \quad (3.9)$$

can be obtained by taking the $(\eta_1^1 \eta_1^2)(\eta_2^3 \eta_2^4)$ component of the tree-level form factor $\mathcal{F}_{\mathcal{K}_6,2}^{(0)}$ in (3.8); it is identical to the BPS result as can be seen by taking the $(\eta_1^A \eta_1^B)(\eta_2^A \eta_2^B)$ component of (3.2) at $n = 2$. There are two other possible scalar field configurations at the external legs, namely $\{(\phi_{13}, \phi_{24}), (\phi_{14}, \phi_{23})\}$, and for both these cases we obtain the same bosonic form factor as above.

The three-point MHV super form factor is given by the following expression:

$$\begin{aligned} \mathcal{F}_{\mathcal{K}_6}^{(0)}(1, 2, 3) = \frac{-1}{4\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \sum_{A,B,C,D} \left(\langle 12 \rangle^2 \varepsilon_{ABCD} (\eta_1^A \eta_1^B) (\eta_2^C \eta_2^D) \right. \\ \left. + 2\langle 13 \rangle \langle 23 \rangle \varepsilon_{ABCD} \eta_1^A \eta_2^B (\eta_3^C \eta_3^D) + \text{cyclic perm.} \right). \end{aligned} \quad (3.10)$$

It has two distinct configurations of the external states: scalar-scalar-gluon and fermion-fermion-scalar. Taking the coefficients of $(\eta_1^1 \eta_1^2)(\eta_2^3 \eta_2^4)$ and $\eta_1^1 \eta_2^2 (\eta_3^3 \eta_3^4)$, we find

$$F_{\mathcal{K}_6}^{(0)}(1_{\phi_{12}}, 2_{\phi_{34}}, 3_{g^+}) = -\frac{\langle 12 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}, \quad F_{\mathcal{K}_6}^{(0)}(1_{\psi_1}, 2_{\psi_2}, 3_{\phi_{34}}) = -\frac{\langle 13 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}, \quad (3.11)$$

which are also identical to the corresponding BPS form factors. The NMHV form factor can be obtained from the MHV result in a similar way as in the BPS case (3.7).

3.3 One-loop two-point form factor

In this and the following subsection, we compute the form factor of \mathcal{K}_6 at one- and two-loop level by using four-dimensional unitarity [4, 5].

The general idea of unitarity in this context is to reconstruct loop corrections to the form factors at integrand level from their discontinuities, i.e. by applying cut. Here, a cut denotes setting a propagator on-shell according to

$$\frac{i}{l_i^2} \rightarrow 2\pi \delta_+(l_i^2), \quad (3.12)$$

where $\delta_+(l_i^2)$ was defined after (2.20). On the cut, the loop expression factorizes into a product of (known) tree-level or lower-loop form factors and amplitudes. These have to be summed over all possible particles exchanged in the cut channel, which can be done by taking the super form factors as well as the super amplitudes and integrating over the Grassmannian degrees of freedom in the cut legs. Then, one can apply the spinor algebra to write the result in a form that can be identified as a sum of cut integrals. In this way, an ansatz for the uncut integrals occurring in the loop correction is assembled. In general, not all integrals appear in a given cut, and additional cuts have to be taken to complement the ansatz. The complete ansatz has to be consistent with all possible cut. Finally, the cut integrals have to be lifted to the uncut integrals, as discussed in appendix B.

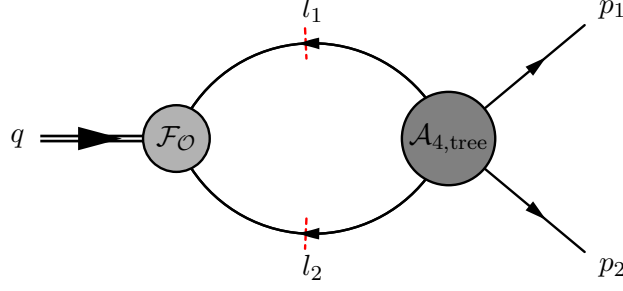


Figure 2: The simple $(p_1 + p_2)^2$ double cut.

In the following, we apply this technique to the form factor of \mathcal{K}_6 and start with the computation of the one-loop two-point form factor. For the sake of explicitness, we choose a fixed combination of external scalar states, namely $\{\phi_{12}, \phi_{34}\}$. As in the tree-level case, the other two choices of external scalars $\{\phi_{13}, \phi_{24}\}$ and $\{\phi_{14}, \phi_{23}\}$ lead to the same result. We abbreviate $F_{\mathcal{K}_6}^{(\ell)}(1_{\phi_{12}}, 2_{\phi_{34}})$ as $F_{\mathcal{K}_6, (\phi, \phi)}^{(\ell)}$.

Only one cut needs to be considered: the two-particle cut in the channel $(p_1 + p_2)^2 = q^2$.¹¹ It cuts the internal propagators carrying momenta l_1 and l_2 as shown in figure 2. The building blocks on the two sides of the cut are the color-ordered two-point form factor (3.8) and the color-ordered four-point MHV amplitude given in the standard MHV form [69] as¹²

$$\mathcal{A}_n^{(0)} = i \frac{\delta^{(8)}(\sum_{i=1}^n \lambda_i \eta_i)}{\langle 1 2 \rangle \langle 2 3 \rangle \dots \langle n 1 \rangle}. \quad (3.13)$$

The sum over all possible particles exchanged along the cut is considered by integrating over the fermionic coordinates of the exchanged particles as $\int d^8 \eta_{1,2} = \int d^4 \eta_1 d^4 \eta_2$ while keeping the external state fixed.

The q^2 -cut integral reads¹³

$$\begin{aligned} F_{\mathcal{K}_6, (\phi, \phi)}^{(1)} \Big|_{q^2} &= \int d\text{PS}_{2, \{l\}} d^8 \eta_{1,2} \mathcal{F}_{\mathcal{K}_6, 2}^{(0)}(-l_1, -l_2) \times \mathcal{A}_4^{(0)}(p_1, p_2, l_2, l_1) \Big|_{(\eta_1^1 \eta_1^2)(\eta_2^3 \eta_2^4)} \\ &= F_{\mathcal{K}_6, (\phi, \phi)}^{(0)} i \underbrace{\int d\text{PS}_{2, \{l\}} \frac{\langle l_1 2 \rangle^2 \langle l_2 1 \rangle^2 + 4 \langle l_1 1 \rangle \langle l_1 2 \rangle \langle l_2 1 \rangle \langle l_2 2 \rangle + \langle l_1 1 \rangle^2 \langle l_2 2 \rangle^2}{\langle l_1 1 \rangle \langle 1 2 \rangle \langle 2 l_2 \rangle \langle l_2 l_1 \rangle}}_{\mathcal{C}_{(\phi, \phi)}}. \end{aligned} \quad (3.14)$$

Since the external states are fixed to be $\{\phi_{12}, \phi_{34}\}$, we take the $(\eta_1^1 \eta_1^2)(\eta_2^3 \eta_2^4)$ component of the cut integrand. The phase-space integration measure, $d\text{PS}_{2, \{l\}}$, is defined according to (2.20), with the integration variables being the momenta of the cut propagators $\{l_1, l_2\}$; hence the subscript in the notation for $d\text{PS}_{2, \{l\}}$.

¹¹The other two two-particle cuts occur in the p_1^2 and p_2^2 channels. Since these legs have $p_1^2 = p_2^2 = 0$, massless bubble integrals in these channels vanish identically in dimensional regularization. Hence, all integrals can be detected by the q^2 cut.

¹²Recall that we are always suppressing the momentum-conserving delta function in the notation.

¹³For the reversing of momentum $l \rightarrow -l$, such as in $\mathcal{F}_{\mathcal{K}_6, 2}^{(0)}(-l_1, -l_2)$, we follow the convention $\lambda_l \rightarrow -\lambda_l$ and $\tilde{\lambda}_l \rightarrow \tilde{\lambda}_l$ in the spinor helicity formalism.

The cut integral can be simplified at the integrand level as¹⁴

$$\begin{aligned} \mathcal{C}_{(\phi,\phi)} &= \int d\text{PS}_{2,\{l\}} \left(\frac{\langle l_1 l_2 \rangle \langle 1 2 \rangle}{\langle l_1 1 \rangle \langle 2 l_2 \rangle} - 6 \frac{\langle l_1 2 \rangle \langle l_2 1 \rangle}{\langle 1 2 \rangle \langle l_1 l_2 \rangle} \right) = \int d\text{PS}_{2,\{l\}} \left(\frac{-s_{12}}{(l_1 + p_1)^2} + 6 \frac{(l_1 + p_2)^2}{s_{12}} \right) \\ &= -s_{12} \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \\ \text{---} \end{array} + 6 \frac{(l_1 + p_2)^2}{s_{12}} \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \\ \text{---} \end{array}, \end{aligned} \quad (3.15)$$

where the flow of the momenta is as specified in figure 2. In the above equation, the integral over the two-particle phase space is shown by the dashed cut line of the triangle and bubble graph. For the triangle graph, the denominator in the integrand is the uncut propagator $\frac{1}{(l_1 + p_1)^2}$ and the numerator coefficient is $-s_{12}$. The shown bubble graph has no uncut propagator, but it has a loop-momentum-dependent numerator factor, which is written in front of the graph.

As described in appendix B, the cut integrals (3.15) can be lifted to the full integrals. The full normalized form factor as defined in (3.1) then becomes¹⁵

$$f_{\mathcal{K}_6,(\phi,\phi)}^{(1)} = 2 \left(-s_{12} \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \\ \text{---} \end{array} + 6 \frac{s_{2l}}{s_{12}} \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right), \quad (3.16)$$

where the factor of 2 is due to the permutation of the two external legs, and we use the short notation $s_{il_j} = (p_i + l_j)^2$. Note that the prefactors that depend on the loop momentum are understood to appear in the integrand of the integral represented by the respective graph it multiplies.

The bubble integral with loop momentum in the numerator can be reduced to the scalar bubble integral via Passarino-Veltman (PV) reduction, see appendix C for details. Thus, we obtain the form factor¹⁶

$$f_{\mathcal{K}_6,(\phi,\phi)}^{(1)} = \underbrace{-2s_{12} \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \\ \text{---} \end{array}}_{f_{\text{BPS},2}^{(1)}} - 6 \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \\ \text{---} \end{array}. \quad (3.17)$$

Note that the contribution to the form factor involving the triangle integral is the same as the BPS form factor $f_{\text{BPS},2}^{(1)}$ in (3.4). The integrals corresponding to the graphs are given in appendix B. An independent computation of this result via Feynman diagrams is shown in appendix H.

From the above calculation at one-loop, we see that the IR-divergent part of the form factor of \mathcal{K}_6 is the same as the one of the BPS operator. The extra contribution coming from the UV divergent bubble integral yields a non-vanishing anomalous dimension unlike in the BPS case. We will equally organize all subsequent results for the form factor in terms of part identical to the BPS form factor and an additional contribution that is unique to the form factor of \mathcal{K}_6 .

¹⁴The first line can be obtained via the Schouten identity $\langle ab \rangle \langle cd \rangle = \langle ac \rangle \langle bd \rangle + \langle ad \rangle \langle cb \rangle$ for $\langle l_1 2 \rangle \langle l_2 1 \rangle$ in (3.14).

¹⁵The coupling dependence can be recovered as shown in appendix B.

¹⁶For convenience, we will from now on refer to the normalized form factor as form factor too.

Vanishing one-loop form factors

Before proceeding to two-loops, we briefly discuss two other possible form factors with gluon or fermion external states, namely the form factors $F_{\mathcal{K}_6}^{(1)}(1_{g-}, 2_{g+})$ and $F_{\mathcal{K}_6}^{(1)}(1_{\psi_1}, 2_{\psi_{234}})$. At tree level, they are zero since no Feynman diagram for this configuration exists. At higher loops this is not obvious. Here, we use unitarity to show explicitly that they are zero at least at one-loop order. Consider the q^2 -cut as in (3.14), but to obtain $F_{\mathcal{K}_6}^{(1)}(1_{g-}, 2_{g+})$ and $F_{\mathcal{K}_6}^{(1)}(1_{\psi_1}, 2_{\psi_{234}})$ take the components $\eta_1^1 \eta_1^2 \eta_1^3 \eta_1^4$ and $\eta_1^1 (\eta_2^2 \eta_2^3 \eta_2^4)$ of the cut integrand, respectively. This yields for the two cases

$$F_{\mathcal{K}_6, (g-, g+)}^{(1)} \Big|_{q^2} = i \int \text{dPS}_{2, \{l\}} \frac{6 \langle l_1 1 \rangle \langle l_2 1 \rangle^2}{\langle 1 2 \rangle \langle l_1 l_2 \rangle \langle l_2 2 \rangle} = -i 6 \frac{\langle 1 | l_1 | 2 \rangle^2}{s_{12}} = \text{triangle diagram} \quad (3.18)$$

$$F_{\mathcal{K}_6, (\psi_1, \psi_{234})}^{(1)} \Big|_{q^2} = i \int \text{dPS}_{2, \{l\}} \frac{3 \langle l_1 1 \rangle \langle l_1 2 \rangle \langle l_2 1 \rangle^2 + 3 \langle l_1 1 \rangle^2 \langle l_2 1 \rangle \langle l_2 2 \rangle}{\langle 1 2 \rangle \langle l_1 l_2 \rangle \langle l_1 1 \rangle \langle l_2 2 \rangle} \\ = i 3 \langle 1 | l_1 | 2 \rangle = \text{triangle diagram} + i 6 \frac{\langle 1 | l_1 | 2 \rangle}{s_{12}} = \text{bubble diagram} \quad (3.19)$$

When we lift these expressions to the full triangle and bubble integrals and perform the PV reduction, we obtain zero. Since we use four-dimensional unitarity, we also have to check that there is no contribution from potential rational terms. A similar (but simpler) study as in appendix D shows that rational terms are indeed absent.

Finally, there is an easy way to see that $F_{\mathcal{K}_6}(1_{g-}, 2_{g+}) = 0$ to all loop orders. Using the gauge freedom, we can choose the polarization vectors of the outgoing gluons as $\varepsilon_1^- = \varepsilon_2^+ \propto \lambda_1 \tilde{\lambda}_2$. It is then obvious that the form factor must be zero, since it is proportional to $\varepsilon_i \cdot p_j$ or $\varepsilon_1 \cdot \varepsilon_2$.

One can also compute $F_{\mathcal{K}_6, (g, g)}^{(1)}$ directly by using Feynman diagrams. A simple computation gives

$$F_{\mathcal{K}_6, (g, g)}^{(1)} = \left[2(\varepsilon_1 \cdot \varepsilon_2) - \frac{(\varepsilon_1 \cdot p_2)(\varepsilon_2 \cdot p_1)}{s_{12}} \right] I_3^D[\ell_\epsilon^2], \quad (3.20)$$

where the integral $I_3^D[\ell_\epsilon^2] = \frac{1}{2} + \mathcal{O}(\epsilon)$ is given in (B.7) for $p_2 \rightarrow 0$ and the relabeling $p_3 \rightarrow p_2$. This result applies for the polarization vectors $\varepsilon_{1,2}^\pm$ taken to be in general $4 - 2\epsilon$ dimensions. Since $I_3[\ell_\epsilon^2]$ is finite and its prefactor is of order $\mathcal{O}(\epsilon)$ (as it vanishes when $D = 4$), the form factor itself $F_{\mathcal{K}_6, (g, g)}^{(1)}$ is of order $\mathcal{O}(\epsilon)$. This is consistent with the unitarity-based calculation.

3.4 Two-loop two-point form factor

Next, we compute the two-loop two-point form factor of \mathcal{K}_6 . As in the one-loop case, we specify the external states to be $\{\phi_{12}, \phi_{34}\}$.

Two-particle cut

We first study the two-particle cut in the q^2 -channel. We follow a similar procedure as the one being used in computing the BPS form factor [27]. We first quote the q^2 -cut integral

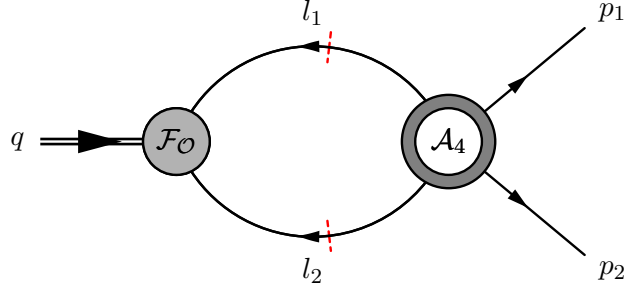


Figure 3: The $(p_1 + p_2)^2$ double cut at two loops that contributes to the planar ladder integral. The building blocks are the color-ordered one-loop amplitude and the color-ordered tree-level form factor.

given by the equation (2.6) of [27]¹⁷

$$F_{\mathcal{O},2}^{(2)} \Big|_{q^2} = \int d\text{PS}_{2,\{l\}} d^8\eta_{l_{1,2}} \mathcal{F}_{\mathcal{O},2}^{(0)}(-l_1, -l_2) \left(4\mathcal{A}_4^{(1)}(p_1, p_2, l_2, l_1) + \mathcal{A}_4^{(1)}(p_1, l_1, p_2, l_2) \right), \quad (3.21)$$

where the building blocks are the two-point tree-level form factor (3.8) and the one-loop color-ordered four-point amplitudes [70]

$$\mathcal{A}_4^{(1)}(p_1, p_2, p_3, p_4) = \mathcal{A}_4^{(0)}(p_1, p_2, p_3, p_4) (-s_{12}s_{23}) I_4^{(1)}(p_1, p_2, p_3, p_4). \quad (3.22)$$

The tree-level super amplitude $\mathcal{A}_4^{(0)}(p_1, p_2, p_3, p_4)$ in (3.22) contains all the dependence on the fermionic coordinates, and the term multiplying it is a massless scalar box integral $I_4^{(1)}$ defined in (B.6).¹⁸

Let us briefly explain (3.21); see [27] for a derivation in full details. The above cut integral is obtained by taking the product of the *non-color ordered* four-point amplitude and the two-point form factor. The one-loop four-point amplitude contains a single-trace contribution, as well as a double-trace contribution which is sub-leading in color. However, after the contraction of the color factors with the two-point form factor, both contribute to the cut integral with the single-trace color factor $\delta^{ab} = \text{tr}(\mathbf{T}^a \mathbf{T}^b)$.¹⁹ The final building blocks in the cut integral are the *color-ordered* form factor and amplitude as given in (3.21). The two contributions in the parentheses of (3.21) are depicted in figure 3 and 4 respectively.²⁰ We consider them one by one below.

We first study the contribution from the first term in the parentheses of (3.21), which is shown in figure 3. Using the one-loop result (3.22) for the amplitude and taking the

¹⁷Note that (3.21) applies to any composite operator with two elementary fields, in particular to the form factor of \mathcal{K}_6 .

¹⁸The minus sign in (3.22) is related to the convention of the box integral we use in (B.6).

¹⁹The enhancement of the power in N_c of the apparently suppressed double-trace term in the amplitude is the wrapping effect analysed earlier for the spectral problem [71].

²⁰The factor 4 in the first term comes from the different contributions of the color factor contractions, two of them come from the single-trace four-point amplitudes, the other two from the double-trace four-point amplitude, as explained in [27]. A different way to understand the factor 4 is to look at the two-particle cut with the one-loop form factor on the left hand side and the tree-level amplitude on the right hand side. It then arises from twice applying the reasoning that gave us the factor 2 at one loop.

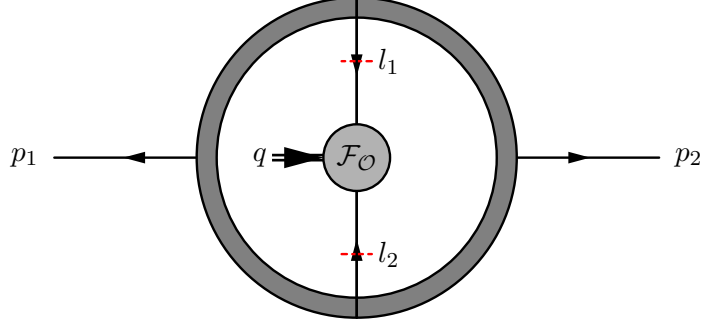


Figure 4: The q^2 -cut at two loops that contributes to the crossed ladder integral. The building blocks are the the color-ordered one-loop amplitude and the tree-level form factor.

external states to be $\{\phi_{12}, \phi_{34}\}$, the corresponding cut integral can be written as

$$F_{\mathcal{K}_6, (\phi, \phi)}^{(2)} \Big|_{q^2}^{\text{I}} = \int \text{dPS}_{2, \{l\}} \text{d}^8 \eta_{l_{1,2}} \mathcal{F}_{\mathcal{K}_6, 2}^{(0)}(-l_1, -l_2) \mathcal{A}_4^{(0)}(p_1, p_2, l_2, l_1) \Big|_{(\eta_1^1 \eta_1^2)(\eta_2^3 \eta_2^4)} \quad (3.23)$$

$$\times (-) s_{12} s_{1l_1} I_4^{(1)}(p_1, p_2, l_2, l_1) .$$

The first line in (3.23) given by the product of tree factors is the same as the cut integrand of the previously studied one-loop case in (3.14). Therefore, we can perform exactly the same calculation as in (3.15) to obtain

$$F_{\mathcal{K}_6, (\phi, \phi)}^{(2)} \Big|_{q^2}^{\text{I}} = -F_{\mathcal{K}_6, (\phi, \phi)}^{(0)} \int \text{dPS}_{2, \{l\}} \left(\frac{-s_{12}}{s_{1l_1}} + 6 \frac{s_{2l_1}}{s_{12}} \right) s_{12} s_{1l_1} I_4^{(1)}(p_1, p_2, l_2, l_1) \quad (3.24)$$

$$= F_{\mathcal{K}_6, (\phi, \phi)}^{(0)} \left(s_{12}^2 - 6 s_{1l_1} s_{2l_1} \right) = \text{Diagram} .$$

The diagram shows a triangle with vertices at the top and bottom. The top vertex is labeled p_1 and the bottom vertex is labeled p_2 . A vertical line segment is drawn from the top vertex to the bottom vertex. A red dashed line segment is drawn from the top vertex to the bottom vertex, crossing the vertical line segment. The red dashed line is labeled l_1 at the top and l_2 at the bottom.

The above cut integral can be lifted to the two-loop planar ladder integral. This integral can be drawn in two different ways, namely

$$= \text{Diagram 1} , \text{Diagram 2} . \quad (3.25)$$

Diagram 1: A triangle with vertices at the top and bottom. The top vertex is labeled p_1 and the bottom vertex is labeled p_2 . A vertical line segment is drawn from the top vertex to the bottom vertex. A horizontal line segment is drawn from the left side to the right side, crossing the vertical line segment. Diagram 2: A triangle with vertices at the top and bottom. The top vertex is labeled p_1 and the bottom vertex is labeled p_2 . A vertical line segment is drawn from the top vertex to the bottom vertex. A horizontal line segment is drawn from the left side to the right side, crossing the vertical line segment. The horizontal line segment is labeled with an equals sign.

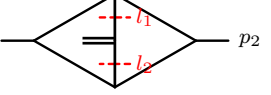
Furthermore, there are two other planar graphs by permuting the external legs $p_1 \leftrightarrow p_2$. So, all together we have 4 diagrams drawn in four different ways, which all give equivalent planar ladder integrals. This provides a diagrammatic interpretation of the factor 4 in the first term of (3.21). As we will see later in the triple cut, it is also very important to separately draw the ladder graphs in different ways according to (3.25) in order to compute the cut integrand correctly.

Next we consider the second term inside the parentheses in (3.21), which is depicted in figure 4. The corresponding cut integral is given by

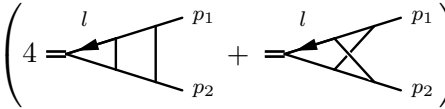
$$F_{\mathcal{K}_6, (\phi, \phi)}^{(2)} \Big|_{q^2}^{\text{II}} = - \int \text{dPS}_{2, \{l\}} \text{d}^8 \eta_{l_{1,2}} \mathcal{F}_{\mathcal{K}_6, 2}^{(0)}(-l_1, -l_2) \mathcal{A}_4^{(0)}(p_1, l_1, p_2, l_2) \quad (3.26)$$

$$\times s_{12} s_{1l_1} I_4^{(1)}(p_1, l_1, p_2, l_2) \Big|_{(\eta_1^1 \eta_1^2)(\eta_2^3 \eta_2^4)} .$$

Following similar steps as in the previous case, the cut integral is expressed as a two-particle cut of the two-loop crossed ladder integral as shown below

$$F_{\mathcal{K}_6,(\phi,\phi)}^{(2)} \Big|_{q^2}^{\text{II}} = F_{\mathcal{K}_6,(\phi,\phi)}^{(0)} \left(s_{12}^2 - 6s_{1l_1}s_{2l_1} \right) p_1 \text{ --- } \text{---} p_2 \text{ .} \quad (3.27)$$


Now, lifting the cut integral (3.27) together with the previous planar contribution in (3.24), as described in appendix B, we find the following contribution to the two-loop form factor of \mathcal{K}_6

$$4f_{\mathcal{K}_6,(\phi,\phi)}^{(2),\text{I}} + f_{\mathcal{K}_6,(\phi,\phi)}^{(2),\text{II}} = \left(s_{12}^2 - 6s_{1l}s_{2l} \right) \left(4 \text{---} p_1 \text{---} p_2 + \text{---} p_1 \text{---} p_2 \right) \text{ ,} \quad (3.28)$$


where, as explained around (3.25), the factor 4 is included for $f_{\mathcal{K}_6,(\phi,\phi)}^{(2),\text{I}}$.

There is another q^2 -cut which is similar to the one in figure 3. It has the one-loop two-point form factor on the left hand side and the tree-level four-point amplitude on the right hand side. This case is a bit subtle. Naively, one would assume that only the one-loop two-point form factor with scalar external states $F_{\mathcal{K}_6,(\phi,\phi)}^{(1)}$ can occur on the left hand side, since the other possibly contributing form factors with gluon and fermion external states $F_{\mathcal{K}_6,(g-,g+)}^{(1)}$ and $F_{\mathcal{K}_6,(\psi_1,\psi_{234})}^{(1)}$, respectively, vanish. However, this turns out to be incorrect. There are non-vanishing contributions from these two cases: only the integrated one-loop form factors are zero, but the integrands are not, as we can see from (3.18) and (3.19). One should take their integrands into account in the unitarity cuts, and then obtains a result which is consistent with the one found from the q^2 -cut of figure 3.²¹

The above result obtained by using only the two-particle cuts is not guaranteed to give the full form factor. One problem is that the numerator coefficient of the crossed ladder integral may have an ambiguity up to terms proportional to l^2 . Due to the on-shell condition of the cut propagators, such terms are not detected by the double cuts. Moreover, there may be other basis integrals which cannot be detected by the double cuts. Both these issues can be fixed by studying the three particle cuts, which we do next.

Three-particle cut

The three-particle cut, or triple cut (TC), across the q^2 -channel is shown in figure 5. Unlike for the BPS form factor, the triple cut will indeed give some new contribution to the form factor of \mathcal{K}_6 , which is not detectable by the previous double cut.

²¹Since the non-planar ladder does not contribute to this cut, one only obtains the contribution coming from the planar ladder integral in (3.28).

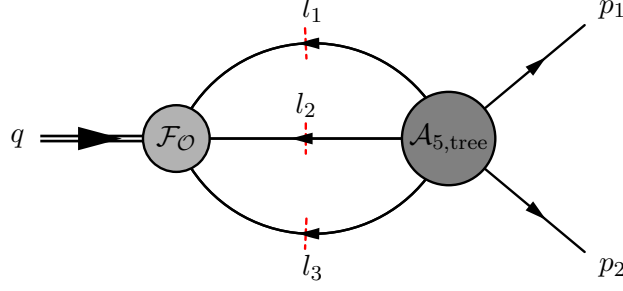


Figure 5: The two loop $(p_1 + p_2)^2 = q^2$ triple cut.

The cut integral is given as

$$\begin{aligned}
F_{\mathcal{K}_{6,(\phi,\phi)}}^{(2)} \Big|_{\text{TC}} &= \int \text{dPS}_{3,\{\ell\}} \prod_{i=1}^3 \text{d}^4 \eta_i \left(\mathcal{F}_{\mathcal{K}_{6,3}}^{(0),\text{MHV}}(-l_1, -l_2, -l_3) \mathcal{A}_5^{(0),\text{NMHV}}(p_1, p_2, l_3, l_2, l_1) \right. \\
&\quad \left. + \mathcal{F}_{\mathcal{K}_{6,3}}^{(0),\text{NMHV}}(-l_1, -l_2, -l_3) \mathcal{A}_5^{(0),\text{MHV}}(p_1, p_2, l_3, l_2, l_1) \right) \Big|_{(\eta_1^1 \eta_1^2)(\eta_2^3 \eta_2^4)} \\
&= F_{\mathcal{K}_{6,(\phi,\phi)}}^{(0)} i \mathcal{C}_{\text{TC}} .
\end{aligned} \tag{3.29}$$

Note that besides the MHV form factors and amplitudes also the NMHV form factors and amplitudes appear as building blocks. The two terms in the above sum are in fact conjugate to each other.

After performing the fermionic integrations²² and some spinor algebra, the cut integral can be simplified at the integrand level into the following form:²³

$$\begin{aligned}
\mathcal{C}_{\text{TC}} &= \int \text{dPS}_{3,\{\ell\}} \left(\frac{s_{12}^2 - 6s_{1l_1} s_{2l_1}}{s_{2l_3} s_{l_1 l_2} s_{l_2 l_3}} + \frac{s_{12}^2 - 6s_{1l_3} s_{2l_3}}{s_{1l_1} s_{l_1 l_2} s_{l_2 l_3}} + \frac{s_{12}^2 - 6s_{1l_2} s_{2l_2}}{s_{2l_3} s_{l_1 l_2} s_{l_1 l_3}} \right. \\
&\quad \left. + \frac{s_{12}^2 - 6s_{1l_2} s_{2l_2}}{s_{1l_1} s_{l_1 l_3} s_{l_2 l_3}} + \frac{s_{12}^2 - 6s_{1l_2} s_{2l_2}}{s_{1l_1} s_{2l_3} s_{l_1 l_3}} + \frac{18}{s_{12}} - \frac{18s_{1l_3}}{s_{12} s_{l_1 l_2}} - \frac{18s_{2l_1}}{s_{12} s_{l_2 l_3}} \right) .
\end{aligned} \tag{3.30}$$

Note that the first five terms in (3.30) can be obtained directly from the result determined by the two-particle cut in the previous paragraph, namely the planar ladder contribution in (3.24) and the crossed ladder in (3.26). Let us first look at the first term in (3.28), the contribution from the planar ladder, which contains the numerical prefactor 4. As mentioned earlier, this factor 4 stems from the four different ways of drawing the planar ladder graph. The two configurations shown in (3.25) contribute to the above triple cut. In order to account for all the possible triple cuts on these two diagrams, we cut each

²²To obtain the cut integrand in a compact form, it is convenient to take the product of the bosonic form factor and amplitude expressions and sum over all helicity configurations, since the NMHV result of both three-point form factor and five-point amplitudes take simple $\overline{\text{MHV}}$ form. We have checked that the expression obtained in this way is equivalent to the expression by using super form factor and amplitudes and doing the fermionic integration directly.

²³In practice this form can be obtained easily as follows. First, one can write down immediately the contribution of the first five terms by using the result (3.28) obtained from the double cuts, as explained below. Then, subtracting them from the cut integrand, the remaining terms take a very simple form which can be easily simplified into the last three terms.

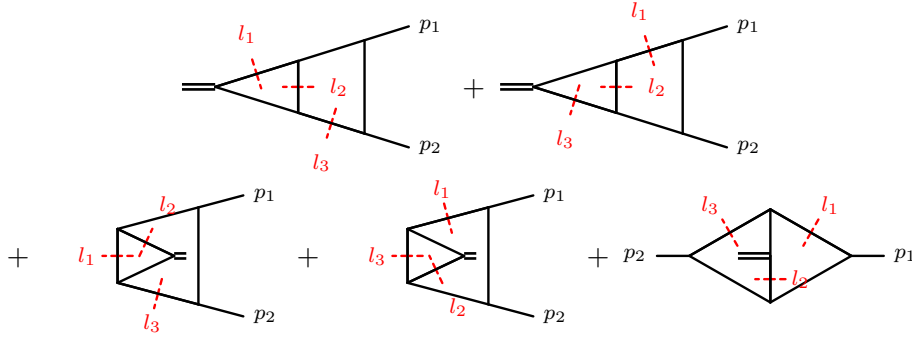


Figure 6: Triple cut of the integrals that correspond to the first five terms in (3.30). The flow of the momenta is as specified in figure 5.

in two ways as shown in figure 6. Thus, the first four terms in (3.30) correspond to the first four diagrams in figure 6, which are just the planar ladder integrals. The remaining fifth term in (3.30) correspond to the last diagram in figure 6, which is the crossed ladder integral with only one possible triple cut. Hence, the first five terms in (3.30) do not result in any new contribution but reproduce the double-cut result in (3.28).

The remaining three terms in (3.30), however, are new contributions to the two-loop ansatz detected by the three-particle cut. They can be expressed as the three-particle cut of the following three integrals:

$$f_{\mathcal{K}_{6,(\phi,\phi)}}^{(2)} \Big|_{\text{CT}}^{\text{III}} = i 18 \left(\frac{1}{s_{12}} \text{Diagram 1} - \frac{s_{1l_3}}{s_{12}} \text{Diagram 2} - \frac{s_{2l_1}}{s_{12}} \text{Diagram 3} \right). \quad (3.31)$$

These above three cut integrals can be lifted to full integrals, which can be simplified further at the integral level to give a single scalar integral:

$$f_{\mathcal{K}_{6,(\phi,\phi)}}^{(2),\text{III}} = \frac{18}{s_{12}} \text{Diagram 4}. \quad (3.32)$$

Complete two-loop result

Now, we combine the results from all the cuts, (3.28) and (3.32), and obtain the two-loop two-point form factor,²⁴

$$\begin{aligned}
 f_{\mathcal{K}_6,(\phi,\phi)}^{(2)} &= 4f_{\mathcal{K}_6,(\phi,\phi)}^{(2),\text{I}} + f_{\mathcal{K}_6,(\phi,\phi)}^{(2),\text{II}} + 2f_{\mathcal{K}_6,(\phi,\phi)}^{(2),\text{III}} \\
 &= -6(l+p_1)^2(l+p_2)^2 \left(4 \text{ [Diagram 1]} + \text{ [Diagram 2]} \right) \\
 &\quad + \frac{36}{s_{12}} \text{ [Diagram 3]} + \underbrace{s_{12}^2 \left(4 \text{ [Diagram 4]} + \text{ [Diagram 5]} \right)}_{f_{\text{BPS},2}^{(2)}}, \tag{3.33}
 \end{aligned}$$

where the integrals corresponding to the graphs are given in appendix B. Note that we have multiplied $f_{\mathcal{K}_6,(\phi,\phi)}^{(2),\text{III}}$ by 2 to include the contribution from the permutation of the external legs $p_1 \leftrightarrow p_2$. As in the one-loop case, we have presented the result by separating a part that is identical to the BPS form factor $f_{\text{BPS},2}^{(2)}$ given in (3.5).

The double and triple cuts we have considered should be able to detect all possible basis integrals up to potential rational terms that might be missing when using four dimensional unitarity. Comparing our result (3.33) with the one which we obtained from the Feynman diagrams of appendix H, we have confirmed that such rational terms are absent.

As will be explained in section 4, the result given by (3.33) is, however, only for the operator \mathcal{K}_6 defined in (1.10), but not for the Konishi operator \mathcal{K} defined in (1.4). This subtlety will be discussed in details in section 4. We will see that by a rigorous prescription we can modify the above result in order to obtain the Konishi form factor.

3.5 One-loop three-point form factor

In this subsection, we compute the one-loop three-point form factor of \mathcal{K}_6 . The computation is similar to what we have done for the previous two-point case. We need to consider cuts in all possible kinematic channels, which, apart from the q^2 -cut employed earlier for the two-point form factors, contain also the s_{ab} -cut, as shown in figure 7. Combining the results from both cuts ensures that no contribution to the ansatz is missed.

Unlike for the BPS form factor, the loop corrections to the tree-level form factor of \mathcal{K}_6 are in general different for different configurations of external particles. Therefore, we need to consider the form factors with specific configurations of the external states individually. We consider the scalar-scalar-gluon and fermion-fermion-scalar cases.²⁵ We will discuss the scalar-scalar-gluon case in some detail. The fermion-fermion-scalar result can be obtained in the same way and we only present the final result.

²⁴The above result matches the one in the unpublished notes of Boucher-Veronneau, Dixon and Pennington [72].

²⁵There could also be other external states with combinations of different fields, such as $F_{\mathcal{K}_6}^{(1)}(1_{g_-}, 2_{g_+}, 3_{g_+})$ and $F_{\mathcal{K}_6}^{(1)}(1_{\psi_1}, 2_{\psi_{234}}, 3_{g_+})$. Such form factors, however, do not contribute to the two-loop cross section studied in section 5 as the corresponding tree-level results are zero, and we will not consider them in this paper.

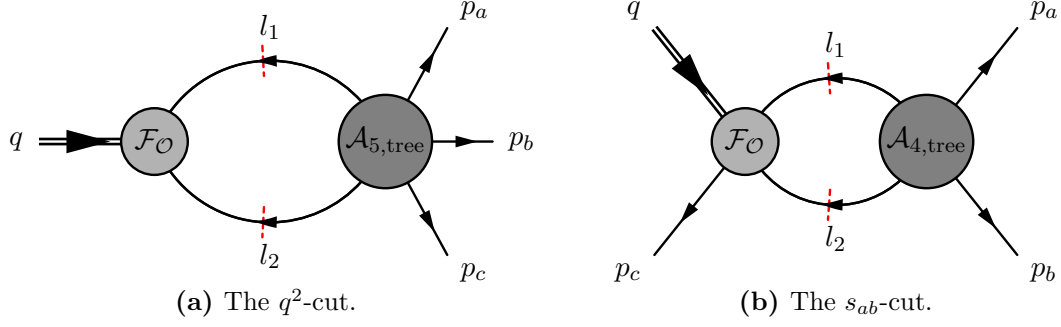


Figure 7: The cuts needed to compute the one-loop three-point form factor of \mathcal{K}_6 .

$$F_{\mathcal{K}_6}^{(1)}(\mathbf{1}_{\phi_{AB}}, \mathbf{2}_{\phi_{CD}}, \mathbf{3}_{g_+})$$

We first consider the form factor of \mathcal{K}_6 with scalar-scalar-gluon external states. For the sake of explicitness, we focus on $F_{\mathcal{K}_6}^{(1)}(1_{\phi_{12}}, 2_{\phi_{34}}, 3_{g_+})$, which we abbreviate as $F_{\mathcal{K}_6, (\phi, \phi, g)}^{(1)}$. As shown in figure 7, we need to consider both the q^2 -cut and the s_{ab} -cut. Furthermore, since the operator is a color singlet, we need to consider all possible cyclic permutations of external on-shell legs in the cuts, as they contribute to the same color-ordered form factor. Explicitly, we need to consider three cases for each channel in figure 7:

$$\{a, b, c\} \rightarrow \text{(I)} \{1_{\phi_{12}}, 2_{\phi_{34}}, 3_{g_+}\}, \quad \text{(II)} \{2_{\phi_{34}}, 3_{g_+}, 1_{\phi_{12}}\}, \quad \text{(III)} \{3_{g_+}, 1_{\phi_{12}}, 2_{\phi_{34}}\}. \quad (3.34)$$

In total, there are six cut channels to consider: (a-I), (a-II), (a-III) and (b-I), (b-II), (b-III), such that (a-I)-(a-III) are the q^2 -cut while (b-I)-(b-III) are the s_{ab} -cut. Note the (I) and (III) cases are actually related to each other by a flipping symmetry.

(a-I)-cut:

This is the q^2 -cut in figure 7 with the choice of external legs $\{p_a, p_b, p_c\}$ corresponding to the particles $\{1_{\phi_{12}}, 2_{\phi_{34}}, 3_{g_+}\}$. As in the previous subsections, the cut integral is given by the following equation,

$$\begin{aligned} F_{\mathcal{K}_6, (\phi, \phi, g)}^{(1)} \Big|_{(a-I)} &= \int d\text{PS}_{2, \{l\}} d^8 \eta_{l_{1,2}} \mathcal{F}_{\mathcal{K}_6, 2}^{(0)}(-l_1, -l_2) \mathcal{A}_5^{(0), \text{MHV}}(p_1, p_2, p_3, l_2, l_1) \Big|_{(\eta_1^1 \eta_1^2)(\eta_2^3 \eta_2^4)} \\ &= F_{\mathcal{K}_6, (\phi, \phi, g)}^{(0)} i \underbrace{\int d\text{PS}_{2, \{l\}} \frac{\langle l_1 2 \rangle^2 \langle l_2 1 \rangle^2 + 4 \langle l_1 1 \rangle \langle l_1 2 \rangle \langle l_2 1 \rangle \langle l_2 2 \rangle + \langle l_1 1 \rangle^2 \langle l_2 2 \rangle^2 \langle 1 3 \rangle}{\langle l_1 1 \rangle \langle 3 l_2 \rangle \langle l_2 l_1 \rangle}}_{\mathcal{C}_{(a-I)}} \frac{\langle 1 2 \rangle^2}{\langle 1 2 \rangle^2}, \end{aligned} \quad (3.35)$$

where the tree-level form factor $F_{\mathcal{K}_6, (\phi, \phi, g)}^{(0)} = F_{\mathcal{K}_6}^{(0)}(1_{\phi_{12}}, 2_{\phi_{34}}, 3_{g_+})$ is given in (3.11).

The above result can be reduced to an appropriate cut of integrals by using some

spinor algebra. Without going through the detail, we present the result below:

$$\begin{aligned}
\mathcal{C}_{(a-I)} = & -\frac{s_{12}s_{23}}{2} \begin{array}{c} p_1 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ p_2 \quad p_3 \end{array} \begin{array}{c} l_1 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ l_2 \end{array} - \frac{s_{12} + s_{13}}{2} \begin{array}{c} p_1 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ p_2 \quad p_3 \end{array} \begin{array}{c} l_1 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ l_2 \end{array} \\
& - \left[\frac{s_{13} + s_{23}}{2} - \frac{3s_{2l_1}((s_{13} + s_{23})s_{1l_1} + s_{12}s_{23})}{s_{12}^2} \right] \begin{array}{c} p_1 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ p_2 \quad p_3 \end{array} \begin{array}{c} l_1 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ l_2 \end{array} \\
& + 3 \left[\frac{(q^2 - 2s_{13})(s_{13} + s_{1l_2})}{s_{12}^2} - \frac{s_{13}(s_{13}s_{3l_2} - s_{12}s_{2l_2})}{s_{12}^2 s_{123}} \right] \begin{array}{c} p_1 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ p_2 \quad p_3 \end{array} \begin{array}{c} l_1 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ l_2 \end{array} .
\end{aligned} \tag{3.36}$$

(a-II)-cut:

This is the q^2 cut in figure 7 with the choice of external legs $\{p_a, p_b, p_c\}$ corresponding to a different order of particles, namely $\{2_{\phi_{34}}, 3_{g_+}, 1_{\phi_{12}}\}$. The cut integral can be computed as

$$\begin{aligned}
F_{\mathcal{K}_{6,(\phi,\phi,g)}}^{(1)} \Big|_{(a-II)} &= \int d\text{PS}_{2,\{l\}} d^8\eta_{l_{1,2}} \mathcal{F}_{\mathcal{K}_{6,2}}^{(0)}(-l_1, -l_2) \mathcal{A}_5^{(0),\text{MHV}}(p_2, p_3, p_1, l_2, l_1) \Big|_{(\eta_1^1 \eta_1^2)(\eta_2^3 \eta_2^4)} \\
&= F_{\mathcal{K}_{6,(\phi,\phi,g)}}^{(0)} i \underbrace{\int d\text{PS}_{2,\{l\}} \frac{\langle l_1 1 \rangle^2 \langle l_2 2 \rangle^2 + 4\langle l_1 1 \rangle \langle l_1 2 \rangle \langle l_2 1 \rangle \langle l_2 2 \rangle + \langle l_1 2 \rangle^2 \langle l_2 1 \rangle^2}{\langle l_1 2 \rangle \langle 2 1 \rangle \langle 1 l_2 \rangle \langle l_2 l_1 \rangle}}_{\mathcal{C}_{(a-II)}} .
\end{aligned} \tag{3.37}$$

After some spinor algebra, the above result can be expressed as cut of the following integrals,

$$\begin{aligned}
\mathcal{C}_{(a-II)} = & -\frac{s_{23}s_{31}}{2} \begin{array}{c} p_2 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ p_3 \quad p_1 \end{array} \begin{array}{c} l_1 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ l_2 \end{array} - \frac{s_{12} + s_{13}}{2} \begin{array}{c} p_2 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ p_3 \quad p_1 \end{array} \begin{array}{c} l_1 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ l_2 \end{array} - \frac{s_{12} + s_{23}}{2} \begin{array}{c} p_2 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ p_3 \quad p_1 \end{array} \begin{array}{c} l_1 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ l_2 \end{array} \\
& - 3 \left(1 + \frac{s_{1l_1}s_{2l_2} - s_{2l_1}s_{1l_2}}{s_{12}s_{123}} \right) \begin{array}{c} p_2 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ p_3 \quad p_1 \end{array} \begin{array}{c} l_1 \\ / \quad \backslash \\ \text{---} \quad \text{---} \\ \backslash \quad / \\ l_2 \end{array} .
\end{aligned} \tag{3.38}$$

(b-I) cut:

This is the $(q-p_3)^2$ cut in figure 7 with the choice of external legs $\{p_a, p_b, p_c\}$ corresponding to the particles $\{1_{\phi_{12}}, 2_{\phi_{34}}, 3_{g_+}\}$. In this case, one of the building blocks is the tree-level three-point form factor in (3.10). The cut integral is given by,

$$\begin{aligned}
F_{\mathcal{K}_{6,(\phi,\phi,g)}}^{(1)} \Big|_{(b-I)} &= \int d\text{PS}_{2,\{l\}} d^8\eta_{l_{1,2}} \mathcal{F}_{\mathcal{K}_{6,3}}^{(0),\text{MHV}}(-l_1, -l_2, p_3) \mathcal{A}_4^{(0)}(p_1, p_2, l_2, l_1) \Big|_{(\eta_1^1 \eta_1^2)(\eta_2^3 \eta_2^4)} \\
&= F_{\mathcal{K}_{6,(\phi,\phi,g)}}^{(0)} i \underbrace{\int d\text{PS}_{2,\{l\}} \frac{\langle l_1 2 \rangle^2 \langle l_2 1 \rangle^2 + 4\langle l_1 1 \rangle \langle l_1 2 \rangle \langle l_2 1 \rangle \langle l_2 2 \rangle + \langle l_1 1 \rangle^2 \langle l_2 2 \rangle^2 \langle 1 3 \rangle \langle 2 3 \rangle}{\langle l_1 1 \rangle \langle 2 l_2 \rangle \langle 3 l_2 \rangle \langle l_1 3 \rangle} \frac{\langle 1 3 \rangle \langle 2 3 \rangle}{\langle 1 2 \rangle^2}}_{\mathcal{C}_{(b-I)}} .
\end{aligned} \tag{3.39}$$

It can be written as cut of integrals as

$$\begin{aligned}
\mathcal{C}_{(b-I)} = & -\frac{s_{12}s_{13}}{2} \text{diag}_1 - \frac{s_{12}s_{23}}{2} \text{diag}_2 \\
& + \left[-\frac{s_{13} + s_{23}}{2} + \frac{3s_{1l_2}((s_{13} + s_{23})s_{2l_2} + s_{12}s_{13})}{s_{12}^2} \right] \text{diag}_3 \\
& + \left[-\frac{s_{13} + s_{23}}{2} + \frac{3s_{2l_1}((s_{13} + s_{23})s_{1l_1} + s_{12}s_{23})}{s_{12}^2} \right] \text{diag}_4 .
\end{aligned} \tag{3.40}$$

(b-II) cut:

This is the case of the $(q - p_1)^2$ cut, the last of the independent cut channels, in figure 7 with the choice of external legs $\{p_a, p_b, p_c\}$ corresponding to the particles in the order of $\{2_{\phi_{34}}, 3_{g^+}, 1_{\phi_{12}}\}$. The cut integral is given by

$$\begin{aligned}
F_{\mathcal{K}_{6,(\phi,\phi,g)}}^{(1)}|_{(b-II)} &= \int d\text{PS}_{2,\{l\}} d^8\eta_{l_{1,2}} \mathcal{F}_{\mathcal{K}_{6,3}}^{(0),\text{MHV}}(-l_1, -l_2, p_1) \mathcal{A}_4^{(0)}(p_2, p_3, l_2, l_1) \Big|_{(\eta_1^1 \eta_1^2)(\eta_2^3 \eta_2^4)} \\
&= F_{\mathcal{K}_{6,(\phi,\phi,g)}}^{(0)} i \underbrace{\int d\text{PS}_{2,\{l\}} \frac{(\langle l_1 1 \rangle \langle l_2 2 \rangle - \langle l_1 2 \rangle \langle l_2 1 \rangle)^2 \langle 3 1 \rangle}{\langle l_1 2 \rangle \langle 3 l_2 \rangle \langle 1 l_2 \rangle \langle l_1 1 \rangle \langle 2 1 \rangle}}_{\mathcal{C}_{(b-II)}} ,
\end{aligned} \tag{3.41}$$

which leads to

$$\begin{aligned}
\mathcal{C}_{(b-II)} = & -\frac{s_{12}s_{23}}{2} \text{diag}_1 - \frac{s_{23}s_{31}}{2} \text{diag}_2 \\
& - \frac{s_{12} + s_{13}}{2} \text{diag}_3 - \frac{s_{12} + s_{13}}{2} \text{diag}_4 .
\end{aligned} \tag{3.42}$$

As previously mentioned, the cuts (a-III) and (b-III) give similar results to (a-I) and (b-I), and can be obtained by a flipping symmetry, namely $\{p_1 \leftrightarrow p_2\}$. Hence, we will not give them explicitly.

From the cuts to the full form factor

We find that all the above cut results are consistent with each other, i.e. the prefactors of the graphs are identical when the same graph appears in different cut channels apart from terms that vanish due to the on-shell condition for the cut momenta. Given all these cut results, it is straightforward to lift the cut integrals, as described in appendix B, to obtain

the full form factor²⁶

$$\begin{aligned}
f_{\mathcal{K}_6,(\phi,\phi,g)}^{(1)} = & \left\{ 3 \left[\frac{(s_{123} - 2s_{13})(s_{13} + s_{1l})}{s_{12}^2} - \frac{s_{13}(s_{13}s_{3l} - s_{12}s_{2l})}{s_{12}^2 s_{123}} - \frac{1}{2} \right] \right. \\
& + \left. \frac{3s_{2l}(s_{1l}(s_{13} + s_{23}) + s_{12}s_{23})}{s_{12}^2} \right. \\
& \left. = \right. \text{[Diagram: Triangle with loop } l \text{ and external lines } p_1, p_2, p_3 \text{]} \\
& \left. + \{p_1 \leftrightarrow p_2\} \right\} + f_{\text{BPS},3}^{(1)}, \tag{3.43}
\end{aligned}$$

where $f_{\text{BPS},3}^{(1)}$ denotes the BPS part that is given in (3.5). We point out that the above result applies to form factors with any other non-zero scalar-scalar-gluon configuration $F_{\mathcal{K}_6}^{(1)}(1_{\phi_{AB}}, 2_{\phi_{CD}}, 3_{g_{\pm}})$.

$$F_{\mathcal{K}_6}^{(1)}(\mathbf{1}_{\psi_1}, \mathbf{2}_{\psi_2}, \mathbf{3}_{\phi_{34}})$$

For the form factor with fermion-fermion-scalar external states like $F_{\mathcal{K}_6}^{(1)}(\mathbf{1}_{\psi_1}, \mathbf{2}_{\psi_2}, \mathbf{3}_{\phi_{34}})$, one can proceed in the above steps for computing the cut integrand in all possible channels and lifting the cut result to the full answer. Without giving details, we present the final result, denoted by $f_{\mathcal{K}_6,(\psi,\psi,\phi)}^{(1)}$,

$$\begin{aligned}
f_{\mathcal{K}_6,(\psi,\psi,\phi)}^{(1)} = & \left\{ -\frac{3s_{23}}{2} \right. \text{[Diagram: Triangle with external lines } p_1, p_2, p_3 \text{]} \\
& - 3 \left(1 - \frac{s_{12} - s_{13}}{s_{23}} \right) \text{[Diagram: Triangle with loop } l \text{ and external lines } p_1, p_2, p_3 \text{]} \\
& + 3 \left(\frac{s_{12}s_{2l} - s_{13}s_{3l}}{s_{23}s_{123}} - \frac{s_{2l} - s_{3l}}{s_{23}} \right) \text{[Diagram: Triangle with loop } l \text{ and external lines } p_1, p_2, p_3 \text{]} \\
& + 3 \left(\frac{s_{12} + s_{13}}{2} + \frac{s_{12}s_{3l} - s_{13}s_{2l}}{s_{23}} \right) \text{[Diagram: Triangle with loop } l \text{ and external lines } p_1, p_2, p_3 \text{]} \\
& \left. + \frac{3s_{23}s_{31}}{2} \right. \text{[Diagram: Triangle with external lines } p_1, p_2, p_3 \text{]} \\
& \left. + f_{\text{BPS},3}^{(1)} \right\}. \tag{3.44}
\end{aligned}$$

Note that in the above result not all the contribution from box graphs are incorporated in the analogous BPS part given in (3.5), unlike in the expression for the scalar-scalar-gluon form factor (3.43). Also, in the first line of the equation there are new one-mass triangles which do not exist in (3.43).

PV reduction and some interesting features of the results

We have obtained the full integral expressions for the form factors (3.43) and (3.44) of \mathcal{K}_6 . The results are obtained by using unitarity method fully at the integrand level. As a result, the integrals still contain loop-momentum-dependent numerators. Such integrals can be reduced further via PV reduction, see appendix C for details.

²⁶ Recall that the prefactors that depend on the loop momenta are understood to appear in the integrand of the integral represented by the respective graph each prefactor multiplies.

After PV reduction, the results (3.43) and (3.44) are simplified to

$$\begin{aligned}
f_{\mathcal{K}_6,(\phi,\phi,g)}^{(1)} &= -3 \left[1 + \frac{s_{13}^2 + s_{23}^2}{(s_{13} + s_{23})^2} \right] \text{Bubble}(p_1, p_2, p_3) - \frac{6s_{13}s_{23}}{(s_{13} + s_{23})^2} \text{Cross}(p_1, p_2, p_3) \\
&\quad + \frac{12s_{13}s_{23}}{s_{12}(s_{13} + s_{23})} I_3^D[\ell_\epsilon^2] + f_{\text{BPS},3}^{(1)}, \tag{3.45}
\end{aligned}$$

$$\begin{aligned}
f_{\mathcal{K}_6,(\psi,\psi,\phi)}^{(1)} &= 3 \left(\frac{s_{12} - s_{13}}{s_{12} + s_{13}} + \frac{s_{12} - s_{23}}{s_{12} + s_{23}} \right) \text{Bubble}(p_1, p_2, p_3) \\
&\quad - 3 \left(1 + \frac{s_{12} - s_{13}}{s_{12} + s_{13}} \right) \text{Cross}(p_1, p_2, p_3) - 3 \left(1 + \frac{s_{12} - s_{23}}{s_{12} + s_{23}} \right) \text{Cross}(p_2, p_3, p_1) \\
&\quad + \frac{3s_{23}s_{31}}{2} \text{Fin} \left(\text{Triangle}(p_1, p_2, p_3) \right) + f_{\text{BPS},3}^{(1)}, \tag{3.46}
\end{aligned}$$

where all relevant integrals are given in appendix B, and the finite box function is defined in (4.8).

There are several interesting features in the above results we would like to comment on.

- Going back to the expressions (3.43) and (3.44), we notice that besides a part identical to the BPS form factor $f_{\text{BPS},3}^{(1)}$, there are still triangle or box integrals left, which separately contain IR divergences. This might cause a net IR divergences in addition to the one contained in $f_{\text{BPS},3}^{(1)}$, i.e. it would spoil the universality of the IR divergence. However, as evident from the results (3.45) and (3.46), these additional IR divergences cancel after PV reduction and hence the only the universal IR divergence of the BPS part remains.
- There are remaining divergences given by bubble integrals. These are the UV divergences, which have to be canceled by renormalizing the composite operator. See section 5 for a further discussion.
- Besides the common BPS part, the two results (3.45) and (3.46) are quite different from each other. This directly shows that the form factors of \mathcal{K}_6 with different external legs (even with the same MHV degree) have very different structure and need to be studied case by case.
- There a term in (3.43) involving the integral $I_3^D[\ell_\epsilon^2]$ given in (B.7). It evaluates to a rational term. Interestingly, we have found this contribution by applying four-dimensional unitarity. This may not be surprising, since we apply the four-dimensional unitarity to compute the integrand expressed in terms of a tensor-integral basis. The rational term only appears after the PV reduction when the basis is reduced to scalar integrals. In the usual one-loop (generalized) unitarity computation [73, 74], one computes the coefficients of the scalar integrals directly and hence one would miss this rational term. In appendix D, we have checked that the rational term in (3.43)

matches with an independent Feynman diagrammatic computation, and hence the final result for the form factor is complete.

- The integral coefficients of the form factors in (3.45) and (3.46) appear to contain unphysical poles, such as $\frac{1}{s_{13}+s_{23}} = \frac{1}{s_{123}-s_{12}}$. These are, however, just spurious poles which cancel when all contributions are taken into account. We demonstrate this in details in appendix D.

Finally, we would like to mention that most of the above features (except the IR divergence) do not occur for the one-loop scattering amplitudes and BPS form factors of the $\mathcal{N} = 4$ SYM theory. In QCD they are, however, common as e.g. for the one-loop amplitudes [6].

Last but not least, recall that the above results for \mathcal{K}_6 still need to be modified to get the correct ones for the Konishi form factor, as will be described in the next section. This does not affect any of the above listed properties.

4 Konishi vs. \mathcal{K}_6

In this section, we discuss some important subtleties that arise when regulating the theory by continuing the spacetime dimension from $D = 4$ to $D = 4 - 2\epsilon$. Our unitarity-based calculation made use of the on-shell superfield (1.7) that only captures all degrees of freedom in strictly $D = 4$ dimensions. Hence, we had to keep the external states in the intermediate steps in strictly $D = 4$. The on-shell superspace formulation has been so far successfully used in computing scattering amplitudes and form factors of BPS operators. However, in general this is not enough and the final result has to be lifted to $D = 4 - 2\epsilon$ dimensions. We explain this in details below, taking the Konishi form factor as a concrete (counter)example. In this section, the dimension always refers to the dimension of physical states rather than the loop momenta. The loop momenta are lifted to $D = 4 - 2\epsilon$ dimensions.

4.1 A subtlety in the dimension of intermediate states

When regulating the theory by continuing the spacetime dimension to $D = 4 - 2\epsilon$, one has to also specify how the various fields are continued. In conventional dimensional regularization (CDR) [75] and the 't Hooft Veltman (HV) scheme [76], the number of fermion flavors N_ψ and also the number of scalar flavors N_ϕ remain as in four dimensions and are hence kept as $N_\psi = 4$ and $N_\phi = 6$, respectively. This does, however, break supersymmetry, since the polarization vector ϵ^μ is taken in $D = 4 - \epsilon$ dimension.

A scheme that preserves supersymmetry is dimensional reduction (DR) [77].²⁷ In this scheme, the number of scalar fields is changed to $N_\phi = 6 + 2\epsilon$, such that $D + N_\phi = 10$ is independent of ϵ . It exploits the fact that four-dimensional $\mathcal{N} = 4$ SYM theory can be obtained by dimensional reduction of ten-dimensional $\mathcal{N} = 1$ SYM theory. Performing the dimensional reduction to $D = 4 - 2\epsilon$ dimensions instead, one obtains a regulated theory that preserves $\mathcal{N} = 4$ supersymmetry. The ten-dimensional gauge field A_M , $M = 1, \dots, 10$,

²⁷Supersymmetry is also preserved in the four-dimensional helicity (FDH) scheme [67, 68], although in a different way.

then reduces to the D -dimensional gauge field A_μ and to $N_\phi = 10 - D = 6 + 2\epsilon$ scalar fields ϕ_I . Similarly, the ten-dimensional metric g^{MN} reduces to the D -dimensional metric $g^{\mu\nu}$ and δ^{IJ} .

The above fact forces us to revisit the previous calculations based on strictly four-dimensional unitarity, since we have lifted the loop momenta in the integrals to $D = 4 - 2\epsilon$ but kept the degrees of freedom of the external states in four dimensions. In particular, the DR scheme is incompatible with the use of the $\mathcal{N} = 4$ on-shell superfield (1.7), since the $N_\phi = 6 + 2\epsilon$ scalar fields do not fit into the six-dimensional antisymmetric representation of the $SU(4)$ R-symmetry group. In order to detect the differences between working with external states in strictly four dimensions and the DR scheme, we examine the underlying Feynman diagrams.

In Feynman diagrams, explicit factors of $D = g^\mu{}_\mu$ and $N_\phi = \delta^I{}_I$ arise whenever a gauge or scalar field runs in a loop in which the respective Lorentz or flavor index also forms a closed loop. These factors occur even if the index loop is apparently interrupted when the gauge or scalar field splits into a pair of fermions that themselves build a loop. This follows from the relations of the spacetime and scalar Clifford algebra into which the ten-dimensional Clifford algebra splits; these are the algebras governing the interactions of the bosons with the fermions. The corresponding splitting of the ten-dimensional vector-fermion coupling is illustrated in figure 8. In the following, we understand closed loops to

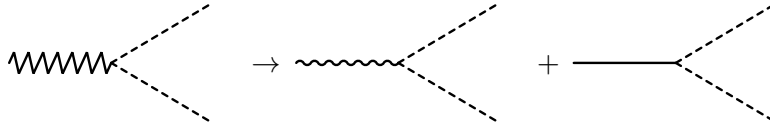


Figure 8: In dimensional reduction, the interaction of the ten-dimensional vector (zigzag line) with fermions (dashed) decomposes to one of a D -dimensional vector (wiggly line) with fermions and one of a $10 - D$ -dimensional scalar (solid line) with fermions.

mean closed index loops. We call an index loop *externally closed* if fields of the composite operator are involved in the index loop and *internally closed* if the operator is not involved in the index loop.

We look at internally closed index loops first. From the dimensional reduction from ten dimensions, we know that an internally closed vector index loop always occurs together with an internally closed scalar index loop. This is illustrated in figure 9. Hence, each

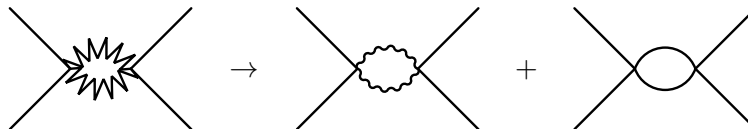


Figure 9: In dimensional reduction, a Feynman diagram with a closed ten-dimensional vector loop (zigzag) decomposes to one with a closed $4(-2\epsilon)$ -dimensional vector loop (wiggly) and one with a $6(+2\epsilon)$ -dimensional scalar loop (plain).

factor of D for a closed Lorentz index loop is accompanied by a factor of N_ϕ for a closed

scalar flavor index loop. This can also be seen in the concrete Feynman diagrams in appendix H, e.g. by comparing the first two lines in table 2. The sum of both contributions is proportional to $D + N_\phi = 10$, both in the DR scheme and in strictly four dimensions. Hence, as far as internally closed index loops are concerned, one is free to work with strictly four-dimensional external fields.

The situation changes for externally closed index loops. Generically, the fields of a composite operator involved in such a loop are only a subset of the fields in the theory, e.g. only the scalar fields. In this case, a diagram in which the externally closed scalar loop generates a factor N_ϕ is not paired with a diagram in which a vector field can circulate in the loop and generate a factor D . Hence, the result is sensitive to working with external states in strictly four dimensions or the DR scheme. On the other hand, in scattering amplitudes and form factors of the BPS operators such as $\text{tr}(\phi_{12}^k)$, no externally closed index loops can occur. This is why the FDH scheme works so well for these quantities.

Let us consider the particular example of the Konishi primary operator (1.4); it is defined as the trace over all scalars and can hence be part of an externally closed scalar index loop. The Konishi primary is the highest-weight state of a so-called long supermultiplet of $\mathfrak{psu}(2, 2|4)$. Supersymmetry guarantees that all members of this supermultiplet have the same anomalous dimension (1.6) – unless it is broken by the regularization scheme. In the supersymmetry-preserving DR scheme, the Konishi primary is defined as the trace over all $N_\phi = 10 - D = 6 + 2\epsilon$ scalars. While the expression (1.4) can easily be modified to sum over this regularization-dependent number of scalars, the expression (1.10) is only valid for $N_\phi = 6$ and hence it is not the highest-weight state of the superconformal multiplet. In particular, its anomalous dimension does not equal (1.6), which is usually calculated using a descendent of the Konishi operator in the $SU(2)$ or $SL(2)$ sector.²⁸

A priori, this discrepancy requires the use of methods beyond those based on the on-shell superspace. Fortunately, this is not the case. In the following, we will argue that – at least for the cases at hand – the strictly four-dimensional result can be lifted to the D -dimensional result with a simple prescription.

4.2 Lifting intermediate states for form factors

Consider a generic multi-loop diagram contributing to the two-point form factor of the operator $\text{tr}(\phi_I \phi_J)$ with outgoing scalar fields ϕ_K and ϕ_L . It can only have one of the three types of R -charge flow depicted in figure 10 together with the respective tensor structures. Only in the case (c), an externally closed scalar index loop exists. The BPS operator $\text{tr}(\phi_I \phi_J)$ defined in (1.2) obtains contributions from the cases (a) and (b) but not from case (c). The Konishi operator $\text{tr}(\phi_I \phi_I)$ defined in (1.4) obtains contributions from all three cases. Thus, we can isolate the contribution from case (c) by subtracting the result for the BPS operator from the result for the Konishi operator. In D dimensions, the single externally closed scalar index loop present in the case (c) should not generate a factor 6 but instead a factor $N_\phi = 10 - D = 6 + 2\epsilon$. Hence, in order to lift the 4-dimensional result

²⁸The additional 2ϵ scalar components in the D -dimensional continuation of the Konishi operator are an example of so-called evanescent operators, which also appear in the context of QCD [78]. See [75] for a textbook treatment.

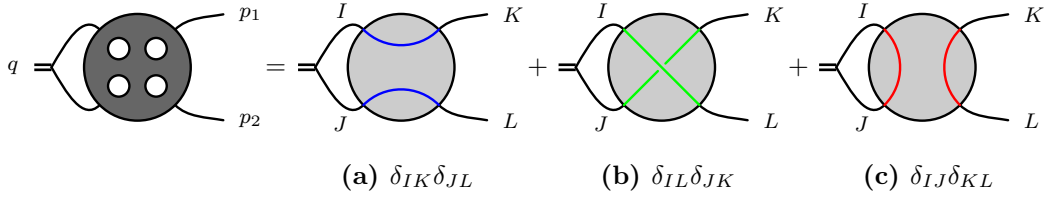


Figure 10: According to R-charge conservation, only three different contractions of the scalar flavors can exist in a generic multi-loop diagram with incoming operator $\text{tr}(\phi_I \phi_J)$ and outgoing scalar fields ϕ_K and ϕ_L : (a) $\delta_{IK} \delta_{JL}$ (blue), (b) $\delta_{IL} \delta_{JK}$ (green) and (c) $\delta_{IJ} \delta_{KL}$ (red).

to the $4 - 2\epsilon$ -dimensional result, we simply have to multiply the contributions of case (c), i.e. the difference of the Konishi and BPS case, by the ratio

$$r_\phi = \frac{N_\phi}{6} = \frac{6 + 2\epsilon}{6}. \quad (4.1)$$

Similar arguments are valid for the three-point form factor of the Konishi operator. In our calculation, only its components with either two scalar legs and one gluon leg or two fermion legs and one scalar legs appear. While in the former case the previous arguments directly apply, in the latter case a slight modification is necessary. A generic multi-loop diagram of the latter type, which has incoming operator $\text{tr}(\phi_I \phi_J)$ and outgoing fields ϕ_K , ψ_A and ψ_B , can only have one of the three possible R-charge flows show in figure 11. An

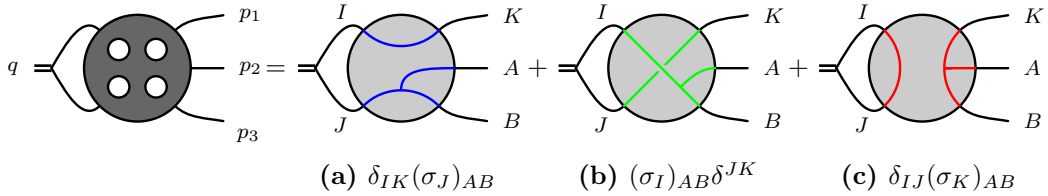


Figure 11: According to R-charge conservation, only three different contractions of the scalar flavors can exist in a generic multi-loop diagram with incoming operator $\text{tr}(\phi_I \phi_J)$, outgoing scalar fields ϕ_K and outgoing fermion fields ψ_A and ψ_B : a) $\delta_{IK} (\sigma_J)_{AB}$ (blue), b) $(\sigma_I)_{AB} \delta^{JK}$ (green) and c) $\delta_{IJ} (\sigma_K)_{AB}$ (red).

externally closed scalar index loop exists only in the case (c). In analogy to the case of the two-point form factor, we can isolate this case by subtracting the result for the BPS operator from the result for the Konishi operator. Then, we correct the number of scalars by multiplying this difference by r_ϕ .

In fact, these arguments can be generalized to any number n of points. Note that we have performed the analysis for the Konishi form factor using real scalars ϕ_I . The results of our analysis, however, are formulated in terms of a part identical to the form factor of

the BPS operator and a part unique for the Konishi operator, where the former stems from the contribution of (a)–(b) and the latter from the contribution of (c). In particular, this formulation of the results of our analysis makes no reference to the kind of scalars we are using to express these operators. This allows us to perform the calculations using scalars transforming in the antisymmetric representation of $SU(4)$, and hence $\mathcal{N} = 4$ on-shell super space – as done in the previous section.

In all form factor ratios in the previous section, we have to introduce r_ϕ by replacing

$$f_{\mathcal{K}_6,n}^{(\ell)} = f_{\text{BPS},n}^{(\ell)} + \tilde{f}_{\mathcal{K}_6,n}^{(\ell)} \xrightarrow{r_\phi} f_{\text{BPS},n}^{(\ell)} + r_\phi \tilde{f}_{\mathcal{K}_6,n}^{(\ell)} = f_{\text{BPS},n}^{(\ell)} + \tilde{f}_{\mathcal{K},n}^{(\ell)} = f_{\mathcal{K},n}^{(\ell)}, \quad (4.2)$$

where $f_{\text{BPS},n}$ is the part identical to the BPS form factor and

$$\tilde{f}_{\mathcal{K}_6,n}^{(\ell)} = f_{\mathcal{K}_6,n}^{(\ell)} - f_{\text{BPS},n}^{(\ell)}, \quad \tilde{f}_{\mathcal{K},n}^{(\ell)} = f_{\mathcal{K},n}^{(\ell)} - f_{\text{BPS},n}^{(\ell)}, \quad (4.3)$$

are the parts unique for the operators (1.10) and (1.4), respectively. More explicitly, the replacement rule reads

$$\tilde{f}_{\mathcal{K}_6,n}^{(\ell)} \xrightarrow{r_\phi} r_\phi \tilde{f}_{\mathcal{K}_6,n}^{(\ell)} = \tilde{f}_{\mathcal{K},n}^{(\ell)}. \quad (4.4)$$

We have focused on the form factor of the Konishi operator. A similar discussion should also be applicable to other operators containing a contraction of flavor or vector indices. As in the Konishi case, it is essential to be able to formulate the results in terms of two parts, one that contains an externally closed index loop and the other that does not. The part without externally closed index loop should be able to be computed independently, such as the BPS part in the Konishi form factor. Given such a decomposition, one can then use the efficient on-shell techniques, together with a simple modification rule as (4.2). Another example of an operator with contracted flavor indices is $\text{tr}(\phi_I \phi_I \phi_K)$, which has one-loop anomalous dimension 8. An example with contracted vector indices is $\text{tr}(D^\mu \phi_{12} D_\mu \phi_{12})$, which has one-loop anomalous dimension 12. In this case, the differences in the one-loop two-point form factor between intermediate states in $D = 4$ and $D = 4 - 2\epsilon$ dimensions are precisely given by the rational terms in the PV reduction formula (C.5).

4.3 Final Konishi form factors

Finally, we list the non-vanishing results for the Konishi form factor, which will be used as the input in the next section to calculate the cross section. They can be obtained by using the form factors computed in section 3 and integral results in appendix B. Note that the $\tilde{f}_{\mathcal{K}}$ parts are corrected by the r_ϕ factor according to the prescription (4.4). The full form factors can be obtained as $f_{\mathcal{K},n}^{(\ell)} = f_{\text{BPS},n}^{(\ell)} + \tilde{f}_{\mathcal{K},n}^{(\ell)}$.

Two-point one-loop

$$\begin{aligned} f_{\text{BPS},2}^{(1)} &= \left(\frac{\mu^2}{-q^2}\right)^\epsilon \left[-\frac{2}{\epsilon^2} + \frac{\pi^2}{6} + \frac{14}{3}\zeta_3\epsilon + \frac{47}{720}\pi^4\epsilon^2 \right] + \mathcal{O}(\epsilon^3), \\ \tilde{f}_{\mathcal{K},(\phi,\phi)}^{(1)} &= \left(\frac{\mu^2}{-q^2}\right)^\epsilon \left[-\frac{6}{\epsilon} - 14 - \left(28 - \frac{\pi^2}{2}\right)\epsilon - \left(56 - \frac{7\pi^2}{6} - 14\zeta_3\right)\epsilon^2 \right] + \mathcal{O}(\epsilon^3). \end{aligned} \quad (4.5)$$

Two-point two-loop

$$\begin{aligned}
f_{\text{BPS},2}^{(2)} &= \left(\frac{\mu^2}{-q^2}\right)^{2\epsilon} \left[\frac{2}{\epsilon^4} - \frac{\pi^2}{6\epsilon^2} - \frac{25\zeta_3}{3\epsilon} - \frac{7\pi^4}{60} \right] + \mathcal{O}(\epsilon) , \\
\tilde{f}_{\mathcal{K},(\phi,\phi)}^{(2)} &= \left(\frac{\mu^2}{-q^2}\right)^{2\epsilon} \left[\frac{12}{\epsilon^3} + \frac{46}{\epsilon^2} + \frac{152 - 2\pi^2}{\epsilon} + \left(484 - \frac{35\pi^2}{3} - 56\zeta_3\right) \right] + \mathcal{O}(\epsilon) .
\end{aligned} \tag{4.6}$$

Three-point one-loop

$$\begin{aligned}
f_{\text{BPS},3}^{(1)} &= -\frac{c_\Gamma}{\epsilon^2} \left[\left(\frac{\mu^2}{-s_{12}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{23}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{31}}\right)^\epsilon \right] \\
&\quad + \text{FB}(p_1, p_2, p_3, -q) + \text{FB}(p_2, p_3, p_1, -q) + \text{FB}(p_3, p_1, p_2, -q) , \\
\tilde{f}_{\mathcal{K},(\phi,\phi,g)}^{(1)} &= -\frac{r_\phi c_\Gamma}{\epsilon(1-2\epsilon)} \left\{ 3 \left[1 + \frac{s_{13}^2 + s_{23}^2}{(s_{13} + s_{23})^2} \right] \left(\frac{\mu^2}{-q^2}\right)^\epsilon + \frac{6s_{13}s_{23}}{(s_{13} + s_{23})^2} \left(\frac{\mu^2}{-s_{12}}\right)^\epsilon \right. \\
&\quad \left. + \frac{12s_{13}s_{23}}{s_{12}(s_{13} + s_{23})} \frac{\epsilon}{(2-2\epsilon)} \frac{1}{s_{13} + s_{23}} \left[s_{12} \left(\frac{\mu^2}{-s_{12}}\right)^\epsilon - q^2 \left(\frac{\mu^2}{-q^2}\right)^\epsilon \right] \right\} , \\
\tilde{f}_{\mathcal{K},(\psi,\psi,\phi)}^{(1)} &= -\frac{r_\phi c_\Gamma}{\epsilon(1-2\epsilon)} \left\{ 3 \left(1 + \frac{s_{12} - s_{13}}{s_{12} + s_{13}} \right) \left(\frac{\mu^2}{-s_{23}}\right)^\epsilon + 3 \left(1 + \frac{s_{12} - s_{23}}{s_{12} + s_{23}} \right) \left(\frac{\mu^2}{-s_{13}}\right)^\epsilon \right. \\
&\quad \left. - 3 \left(\frac{s_{12} - s_{13}}{s_{12} + s_{13}} + \frac{s_{12} - s_{23}}{s_{12} + s_{23}} \right) \left(\frac{\mu^2}{-q^2}\right)^\epsilon \right\} + \frac{3r_\phi}{2} \text{FB}(p_2, p_3, p_1, -q) ,
\end{aligned} \tag{4.7}$$

in which we have defined the finite part of the one-mass box integral as

$$\text{FB}(p_1, p_2, p_3, -q) = -\frac{c_\Gamma}{\epsilon^2} \left[\left(\frac{\mu^2}{-s_{12}}\right)^\epsilon h\left(-\frac{s_{31}}{s_{23}}\right) + \left(\frac{\mu^2}{-s_{23}}\right)^\epsilon h\left(-\frac{s_{31}}{s_{12}}\right) - \left(\frac{\mu^2}{-q^2}\right)^\epsilon h\left(-\frac{s_{31}q^2}{s_{12}s_{23}}\right) \right] , \tag{4.8}$$

where $h(x) = {}_2F_1(1, -\epsilon, 1 - \epsilon, x) - 1$ and $q^2 = s_{12} + s_{23} + s_{31}$.

5 BPS and Konishi cross sections

In this section, we compute the cross section discussed in section 2. We first discuss in detail the case of the BPS operator (1.2) as a warmup example. Then, we compute one of our main results: the Konishi cross section to two-loop order. We will use the Konishi form factors given in subsection 4.3 that were obtained from the form factors of \mathcal{K}_6 computed in section 3 by applying the prescription of section 4. All form factors $f_{\mathcal{K},n}^{(\ell)}$ and $\tilde{f}_{\mathcal{K},n}^{(\ell)}$ appearing in this section are hence those for the Konishi operator (1.4) which are explicitly given in subsection 4.3.

5.1 BPS cross section up to one-loop

As a warmup, we first consider in details the cross section corresponding to the imaginary part of the two-point correlation function of BPS operators, $\langle 0 | \text{tr}(\phi_{12}^2)(x) \text{tr}(\phi_{34}^2)(0) | 0 \rangle$. Since the operators are protected, the cross section has no loop corrections, i.e.

$$\sigma_{\text{BPS}} = \sigma_{\text{BPS}}^{(0)} + \mathcal{O}(\epsilon) . \tag{5.1}$$

We check this explicitly up to one-loop level.

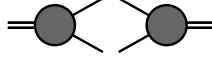


Figure 12: Tree-level squared matrix element.

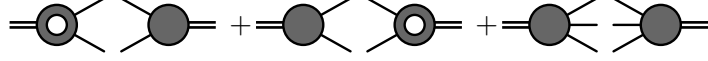


Figure 13: One-loop squared matrix elements.

Tree level

Let us start with the tree-level cross section. The squared matrix element, as shown in figure 12, is the product of two two-point tree-level BPS form factors, one for $\text{tr}(\phi_{12}^2)$ and one for its conjugate $\text{tr}(\phi_{34}^2)$. The tree-level BPS non color-ordered super form factor can be obtained from (2.11) and (3.2). It is easy to perform the color factor summation and the fermionic integration. This yields the squared matrix element

$$\mathcal{M}_{\text{BPS},2}^{(0)} = \frac{1}{2!} \sum_{a_1, a_2} \int d^4\eta_1 d^4\eta_2 \hat{\mathcal{F}}_{\text{BPS}}^{(0)}(1, 2) \hat{\mathcal{F}}_{\text{BPS}}^{*(0)}(1, 2) = \frac{N_c^2 - 1}{2}. \quad (5.2)$$

The tree-level cross section is given by the integral (E.2) of $\mathcal{M}_{\text{BPS},2}^{(0)}$ over the two-particle phase space in $D = 4 - 2\epsilon$ dimensions. This yields

$$\sigma_{\text{BPS}}^{(0)} = \int d\text{PS}_2 \mathcal{M}_{\text{BPS},2}^{(0)} = \left(\frac{\mu^2}{q^2}\right)^\epsilon \frac{1}{4(16\pi)^{\frac{1}{2}-\epsilon} \Gamma(\frac{3}{2}-\epsilon)} \frac{N_c^2 - 1}{2}. \quad (5.3)$$

One loop

The one-loop cross section is given by the sum of a two-particle and a three-particle channel, as shown in figure 13:

$$\sigma_{\text{BPS}}^{(1)} = \int d\text{PS}_2 \mathcal{M}_{\text{BPS},2}^{(1)} + \frac{1}{g^2} \int d\text{PS}_3 \mathcal{M}_{\text{BPS},3}^{(0)}. \quad (5.4)$$

Two-particle channel

The squared matrix element of the two-particle channel corresponds to the first two graphs of figure 13. As an equation, it reads

$$\mathcal{M}_{\text{BPS},2}^{(1)} = \frac{1}{2!} \sum_{a_1, a_2} \int d^4\eta_1 d^4\eta_2 \left[\hat{\mathcal{F}}_{\text{BPS},2}^{(1)} \hat{\mathcal{F}}_{\text{BPS},2}^{*(0)} + \hat{\mathcal{F}}_{\text{BPS},2}^{(0)} \hat{\mathcal{F}}_{\text{BPS},2}^{*(1)} \right] = 2\mathcal{M}_{\text{BPS},2}^{(0)} \Re(f_{\text{BPS},2}^{(1)}), \quad (5.5)$$

where \Re denotes the real part, and $f_{\mathcal{O},n}^{(\ell)}$ is the ratio between the ℓ -loop and tree-level n -point form factor of the operator \mathcal{O} as defined in (3.1). The tree-level form factor is absorbed into $\mathcal{M}_{\text{BPS},2}^{(0)}$. For short notation, we denote $\hat{\mathcal{F}}_{\mathcal{O}}(1, \dots, n)$ as $\hat{\mathcal{F}}_{\mathcal{O},n}$.

There is an important point related to the $i0$ -prescription to be explained here. Two-point form factors acquire a factor of $(-q^2 \pm i0)^{-\epsilon}$ for each loop. The function $f_{\text{BPS},2}^{*(1)}$ is the complex conjugate of $f_{\text{BPS},2}^{(1)}$ and can be obtained from the latter by replacing $(-q^2 - i0)^{-\epsilon}$ with $(-q^2 + i0)^{-\epsilon}$. The sum of them amounts to taking the real part of $f_{\text{BPS},2}^{(1)}$. Hence, we need the real part of $(-q^2 \pm i0)^{-\epsilon}$, which is given by (see e.g. [79])

$$\Re(-q^2 \pm i0)^x = \frac{\Gamma(1+x)\Gamma(1-x)}{\Gamma(1+2x)\Gamma(1-2x)}(q^2)^x. \quad (5.6)$$

Using this result to determine the real part of the form factor (4.5) and then inserting it into (5.5) together with the tree-level result (5.2) and performing the two-particle phase space integral (E.2), we obtain for the first term in (5.4)

$$\sigma_{\text{BPS},2}^{(1)} = \int \text{dPS}_2 \mathcal{M}_{\text{BPS},2}^{(1)} = \sigma_{\text{BPS}}^{(0)} \left(\frac{\mu^2}{q^2}\right)^\epsilon \left(-\frac{4}{\epsilon^2} + \frac{7\pi^2}{3}\right) + \mathcal{O}(\epsilon). \quad (5.7)$$

Three-particle channel

The squared matrix element of the three-particle channel is given by the last graph of figure 13. The MHV and NMHV non-color-ordered three-point form factor (2.11) can be obtained using (3.2) and (3.7). Performing the color summation and fermionic integration, we find the squared matrix element

$$\begin{aligned} \mathcal{M}_{\text{BPS},3}^{(0)} &= \frac{1}{3!} \sum_{a_1, a_2, a_3} \int \text{d}^4\eta_1 \text{d}^4\eta_2 \text{d}^4\eta_3 \left[\hat{\mathcal{F}}_{\text{BPS},3}^{\text{MHV},(0)} \hat{\mathcal{F}}_{\text{BPS},3}^{*\text{NMHV},(0)} + \hat{\mathcal{F}}_{\text{BPS},3}^{\text{NMHV},(0)} \hat{\mathcal{F}}_{\text{BPS},3}^{*\text{MHV},(0)} \right] \\ &= \frac{2}{3} g_{\text{YM}}^2 N_c (N_c^2 - 1) \frac{(q^2)^2}{s_{12}s_{23}s_{31}}. \end{aligned} \quad (5.8)$$

Performing the three-particle phase space integral by using (E.3), we obtain for the second term in (5.4)

$$\sigma_{\text{BPS},3}^{(1)} = \frac{1}{g^2} \int \text{dPS}_3 \mathcal{M}_{\text{BPS},3}^{(0)} = \sigma_{\text{BPS}}^{(0)} \left(\frac{\mu^2}{q^2}\right)^\epsilon \left(\frac{4}{\epsilon^2} - \frac{7\pi^2}{3}\right) + \mathcal{O}(\epsilon). \quad (5.9)$$

Summing (5.7) and (5.9) together as prescribed by (5.4), we see that both contributions cancel and hence that (5.1) exactly holds at one-loop level.

5.2 Konishi cross section up to two-loop

Next, we compute the Konishi cross section. We start with the discussion of an important simplification for the computation, which exploits the fact that BPS cross section (5.1) is protected and occurs as a contribution in the Konishi cross section.

First, at tree level, the squared matrix elements of the Konishi and the BPS cross section satisfy the following simple relation²⁹

$$\mathcal{M}_{\mathcal{K},n}^{(0)} = \sum_{\text{colors}} \sum_{\substack{\text{spins} \\ \text{helicities}}} \hat{\mathcal{F}}_{\mathcal{K},n}^{(0)} \hat{\mathcal{F}}_{\mathcal{K},n}^{*(0)} = 6 \sum_{\text{colors}} \sum_{\substack{\text{spins} \\ \text{helicities}}} \hat{\mathcal{F}}_{\text{BPS},n}^{(0)} \hat{\mathcal{F}}_{\text{BPS},n}^{*(0)} = 6 \mathcal{M}_{\text{BPS},n}^{(0)}, \quad (5.10)$$

²⁹Note that the Konishi tree-level form factors with specified external legs are identical to the corresponding BPS form factors.

where the factor 6 originates from the contribution of all scalar flavor degrees of freedom in the two-point function of the Konishi operator (1.4) that does not occur for the BPS operator.³⁰

Furthermore, the loop correction of the Konishi form factor can be written as linear combination of two contributions: one that is identical to the BPS form factor and the other one that is defined in (4.3). We can introduce a corresponding squared matrix element that includes a subtraction of the BPS part as

$$\tilde{\mathcal{M}}_{\mathcal{K},n}^{(\ell)} = \mathcal{M}_{\mathcal{K},n}^{(\ell)} - 6 \mathcal{M}_{\text{BPS},n}^{(\ell)} , \quad (5.11)$$

where the factor 6 takes into account that at any loop order ℓ the contribution $\mathcal{M}_{\text{BPS},n}^{(\ell)}$ built from two BPS-type components of the Konishi form factor receives a factor 6 as in (5.10).

Since the BPS cross section (5.1) receives no loop corrections, this means that at loop level, we only need to consider the squared matrix elements $\tilde{\mathcal{M}}_{\mathcal{K}}^{(\ell)}$ defined in (5.11), namely

$$\sigma_{\mathcal{K}}^{(\ell)} = \sum_{n=2}^{\ell+1} g^{2(2-n)} \int \text{dPS}_n \tilde{\mathcal{M}}_{\mathcal{K},n}^{(\ell+2-n)} , \quad \ell \geq 1 , \quad (5.12)$$

which does not contain a pure BPS contribution compared to (2.18). Note in particular that $\tilde{\mathcal{M}}_{\mathcal{K},n}^{(0)} = 0$, therefore, the sum of n can be terminated already at $\ell + 1$. As we will see, this simplifies the computation dramatically.

From (5.10), it immediately follows that the tree-level cross section for the Konishi operator also contains an extra factor 6 compared to the one of the BPS-operator. Hence,

$$\sigma_{\mathcal{K}}^{(0)} = \int \text{dPS}_2 \mathcal{M}_{\mathcal{K},2}^{(0)} = 6 \sigma_{\text{BPS}}^{(0)} . \quad (5.13)$$

At loop-level, it is convenient to factorize out $\sigma_{\mathcal{K}}^{(0)}$.

5.2.1 One-loop result

The bare one-loop Konishi cross section receives contributions from products of tree-level and one-loop two-point form factors and of tree-level three-point form factors as shown in figure 13. The squared matrix element of the two-particle channel is given by

$$\mathcal{M}_{\mathcal{K},2}^{(1)} = \frac{1}{2!} \sum_{a_1, a_2} \int \text{d}^4 \eta_1 \text{d}^4 \eta_2 \left(\hat{\mathcal{F}}_{\mathcal{K},2}^{(1)} \hat{\mathcal{F}}_{\mathcal{K},2}^{*(0)} + \hat{\mathcal{F}}_{\mathcal{K},2}^{(0)} \hat{\mathcal{F}}_{\mathcal{K},2}^{*(1)} \right) = 2 \mathcal{M}_{\mathcal{K},2}^{(0)} \Re(f_{\mathcal{K},(\phi,\phi)}^{(1)}) , \quad (5.14)$$

where we use the abbreviation $f_{\mathcal{K},(\phi,\phi)}^{(1)} = f_{\mathcal{K},2}^{(1)}(1_{\phi_{12}}, 2_{\phi_{34}})$.³¹

³⁰Since we use the FDH scheme, we get an integer factor 6 compared to the BPS result which holds for (1.9) where no summation over the scalar flavors occurs. In another prescription, one might have to replace 6 by $N_\phi = 6 + 2\epsilon$. In any case, this factor cancels out when the cross section is divided by the tree-level cross section as e.g. in (2.14).

³¹The non-zero two-point tree-level form factor must contain two external scalar legs. Therefore, it is not necessary to consider other external states such as $f_{\mathcal{K},2}^{(1)}(1_{g_+}, 2_{g_-})$, which are anyway zero as shown in section 3. A similar argument applies also to the following two-loop computation.

$$\left(\mathcal{Z}_{\mathcal{K}}^{(1)} \right) = \text{diagram 1} + \text{diagram 2}$$

Figure 14: One-loop correction from one-loop renormalization constant.

As discussed above, the result for the three-particle channel cancels with the BPS part in the two-particle channel. Therefore, we can subtract the BPS part from (5.14), as in (5.11) This gives

$$\tilde{\mathcal{M}}_{\mathcal{K},2}^{(1)} = 2\mathcal{M}_{\mathcal{K},2}^{(0)} \Re(\tilde{f}_{\mathcal{K},(\phi,\phi)}^{(1)}) , \quad (5.15)$$

where $\tilde{f}_{\mathcal{K},(\phi,\phi)}^{(1)}$ is given in (4.5). Performing the two-particle phase space integral (E.2), the one-loop bare cross section reads

$$\sigma_{\mathcal{K}}^{(1)} = \int \text{dPS}_2 \tilde{\mathcal{M}}_{\mathcal{K},2}^{(1)} = \sigma_{\mathcal{K}}^{(0)} \left(\frac{\mu^2}{q^2} \right)^\epsilon \left(-\frac{12}{\epsilon} - 28 \right) + \mathcal{O}(\epsilon) . \quad (5.16)$$

The divergence in (5.16) should be canceled by the one-loop correction of the Konishi operator obtained from the one-loop term $\mathcal{Z}_{\mathcal{K}}^{(1)}$ in the operator renormalization constant $\mathcal{Z}_{\mathcal{K}}$, as shown in figure 14, which is imply given by

$$\sigma_{\mathcal{Z}^{(1)}\mathcal{K}}^{(1)} = 2\mathcal{Z}^{(1)}\sigma_{\mathcal{K}}^{(0)} . \quad (5.17)$$

Since this contribution has to cancel the overall UV divergence of (5.16), we immediately find

$$\mathcal{Z}_{\mathcal{K}}^{(1)} = \frac{6}{\epsilon} . \quad (5.18)$$

Comparing this result with the one-loop term of the expansion (2.3) reproduces the one-loop Konishi anomalous dimension $\gamma^{(1)} = 12$ first obtained in [43, 44].

The renormalized one-loop cross section is hence given by

$$\sigma_{\mathcal{K},\text{R}}^{(1)} = \sigma_{\mathcal{K}}^{(1)} + \sigma_{\mathcal{Z}^{(1)}\mathcal{K}}^{(1)} = \sigma_{\mathcal{K}}^{(0)} \left(12 \log \frac{q^2}{\mu^2} - 28 \right) + \mathcal{O}(\epsilon) . \quad (5.19)$$

As predicted in (2.15), the coefficient of $\log \frac{q^2}{\mu^2}$ also reproduces the correct one-loop anomalous dimension.

5.2.2 Two-loop result

The two-loop cross section is obtained from the contributions to the squared matrix elements that are depicted in figure 15. As discussed at the beginning of this section, we can neglect the contribution that is proportional to the BPS cross section. In particular, it is not necessary to consider the four-particle channel contribution in figure 15c, which involves the complicated four-particle phase space integral. This simplifies the computation significantly. In the following, we compute the contributions from the two-particle and three-particle channel separately.

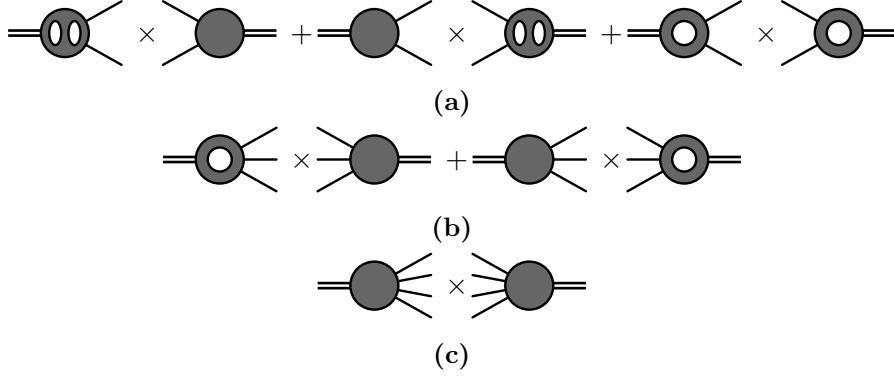


Figure 15: The two-loop bare squared matrix element.

Two-particle channel

The full contribution of the two-particle channel consists of three terms

$$\int d\text{PS}_2 \tilde{\mathcal{M}}_{\mathcal{K},2}^{(2)} + \int d\text{PS}_2 \mathcal{M}_{\mathcal{Z}^{(1)}\mathcal{K},2}^{(2)} + \int d\text{PS}_2 \mathcal{M}_{\mathcal{Z}^{(2)}\mathcal{K},2}^{(2)}, \quad (5.20)$$

where the first term is the bare contribution, and the second and the third term involve the one- and two-loop contributions of the renormalized Konishi operator that are given as multiplication with the respective term of the renormalization constant $\mathcal{Z}_{\mathcal{K}}$. We compute the first two terms and then determine $\mathcal{Z}_{\mathcal{K}}^{(2)}$ from the condition that all divergences are canceled.

The squared matrix element obtained from the bare form factors is shown in figure 15a. In analogy to (5.15), the first two graphs yield

$$\tilde{\mathcal{M}}_{\mathcal{K},2}^{(2),\text{I}} = 2\mathcal{M}_{\mathcal{K},2}^{(0)} \Re(\tilde{f}_{\mathcal{K},(\phi,\phi)}^{(2)}), \quad (5.21)$$

where $\tilde{f}_{\mathcal{K},(\phi,\phi)}^{(2)}$ is given in (4.6).

The third graph of figure 15a contributing to the squared matrix element has no lower-loop counterpart and needs to be discussed in detail. It is the product of two one-loop Konishi form factors, and each of them is a linear combination of the BPS part $f_{\text{BPS},2}^{*(1)}$ and the $\tilde{f}_{\mathcal{K},2}^{(1)}$ part. After subtracting the product of two BPS parts, we obtain

$$\tilde{\mathcal{M}}_{\mathcal{K},2}^{(2),\text{II}} = \mathcal{M}_{\mathcal{K},2}^{(0)} \left[2 \Re\left(\tilde{f}_{\mathcal{K},(\phi,\phi)}^{(1)} f_{\text{BPS},2}^{*(1)}\right) + \tilde{f}_{\mathcal{K},(\phi,\phi)}^{(1)} \tilde{f}_{\mathcal{K},(\phi,\phi)}^{*(1)} \right], \quad (5.22)$$

where the form factors are given in (4.5).³²

Integrating the sum of the two previous contributions over the two-particle phase space (E.2) yields the bare cross section of the two-particle channel. It explicitly reads

$$\begin{aligned} \sigma_{\mathcal{K},2}^{(2)} &= \int d\text{PS}_2 (\tilde{\mathcal{M}}_{\mathcal{K},2}^{(2),\text{I}} + \tilde{\mathcal{M}}_{\mathcal{K},2}^{(2),\text{II}}) \\ &= \sigma_{\mathcal{K}}^{(0)} \left(\frac{\mu^2}{q^2}\right)^{2\epsilon} \left[\frac{48}{\epsilon^3} + \frac{184}{\epsilon^2} + \frac{584 - 56\pi^2}{\epsilon} + 1724 - \frac{668}{3}\pi^2 - 224\zeta_3 \right] + \mathcal{O}(\epsilon). \end{aligned} \quad (5.23)$$

Figure 16: The contribution from the one-loop renormalization constant to the two-particle channel.

Next, we consider the contribution involving the one-loop renormalization constant. It is shown in figure 16 and cancels some one-loop subdivergences contained in the bare cross section (5.23). The squared matrix element is given by the expression

$$\mathcal{M}_{\mathcal{Z}^{(1)}\mathcal{K},2}^{(2)} = \mathcal{M}_{\mathcal{K},2}^{(0)} \left[4\Re(f_{\mathcal{K},(\phi,\phi)}^{(1)})\mathcal{Z}_{\mathcal{K}}^{(1)} + (\mathcal{Z}_{\mathcal{K}}^{(1)})^2 \right]. \quad (5.24)$$

Inserting the explicit expressions (4.5) into (5.24) and performing two-point phase space integration (E.2), we obtain

$$\begin{aligned} \sigma_{\mathcal{Z}^{(1)}\mathcal{K},2}^{(2)} &= \int \text{dPS}_2 \mathcal{M}_{\mathcal{Z}^{(1)}\mathcal{K},2}^{(2)} \\ &= \sigma_{\mathcal{K}}^{(0)} \left[\left(\frac{\mu^2}{q^2} \right)^\epsilon \left(-\frac{48}{\epsilon^3} - \frac{144}{\epsilon^2} - \frac{336 - 28\pi^2}{\epsilon} - 672 + 84\pi^2 + 112\zeta_3 \right) \right. \\ &\quad \left. + \frac{36}{\epsilon^2} - \frac{12}{\epsilon} + 4 \right] + \mathcal{O}(\epsilon). \end{aligned} \quad (5.25)$$

Three-particle channel

There are two contributions to the two-loop cross section in the three-particle channel:

$$\frac{1}{g^2} \left[\int \text{dPS}_3 \tilde{\mathcal{M}}_{\mathcal{K},3}^{(1)} + \int \text{dPS}_3 \mathcal{M}_{\mathcal{Z}^{(1)}\mathcal{K},3}^{(1)} \right]. \quad (5.26)$$

The contribution involving the bare form factor is determined from the diagrams of figure 15b. The expression reads

$$\begin{aligned} \mathcal{M}_{\mathcal{K},3}^{(1)} &= \frac{1}{3!} \sum_{a_i} \int \prod_{i=1}^3 \text{d}^4\eta_i \sum_{\ell=0}^1 \left[\hat{\mathcal{F}}_{\mathcal{K},3}^{(\ell),\text{MHV}} \hat{\mathcal{F}}_{\mathcal{K},3}^{*(1-\ell),\text{NMHV}} + \hat{\mathcal{F}}_{\mathcal{K},3}^{(\ell),\text{NMHV}} \hat{\mathcal{F}}_{\mathcal{K},3}^{*(1-\ell),\text{MHV}} \right] \\ &= 6\mathcal{M}_{\mathcal{K},3}^{(0)} \left[2\Re\left(f_{\mathcal{K},(\phi,\phi,g)}^{(1)}\right) \frac{s_{12}^2}{(q^2)^2} + 2\Re\left(f_{\mathcal{K},(\psi,\psi,\phi)}^{(1)}\right) \frac{s_{13}s_{23}}{(q^2)^2} \right]. \end{aligned} \quad (5.27)$$

In the second line, we have not indicated the MHV degree, since the loop correction is the same for the MHV and the NMHV amplitude. This allows us to use the abbreviation $f_{\mathcal{K},3}^{(1)} = f_{\mathcal{K},3}^{(1),\text{MHV}} = f_{\mathcal{K},3}^{(1),\text{NMHV}}$ for any fixed three-particle final state. Moreover, we have abbreviated the form factors of the two different final states as $f_{\mathcal{K},(\phi,\phi,g)}^{(1)} = f_{\mathcal{K},3}^{(1)}(1_{\phi_{12}}, 2_{\phi_{34}}, 3_{g_+})$ and $f_{\mathcal{K},(\psi,\psi,\phi)}^{(1)} = f_{\mathcal{K},3}^{(1)}(1_{\psi_1}, 2_{\psi_2}, 3_{\phi_{34}})$.

³²Note that for the Konishi form factors we have specified the external legs, while for BPS we do not.

$$\left(\mathcal{Z}_{\mathcal{K}}^{(1)} \right) + \left(\mathcal{Z}_{\mathcal{K}}^{(1)} \right)$$

Figure 17: The contribution from the one-loop renormalization constant to the three-particle cross section.

Since the one-loop corrections differ from each other, as given in (4.7), we need to treat the contribution of these two form factors separately. Note the factors $\frac{s_{12}^2}{(q^2)^2}$ and $\frac{s_{13}s_{23}}{(q^2)^2}$ come from the square of tree-level form factors divided by the tree-level matrix element $\mathcal{M}_{\mathcal{K},3}^{(0)}$.

After subtracting the BPS part, we find

$$\tilde{\mathcal{M}}_{\mathcal{K},3}^{(1)} = 6\mathcal{M}_{\mathcal{K},3}^{(0)} \left[2\Re\left(\tilde{f}_{\mathcal{K},(\phi,\phi,g)}^{(1)}\right) \frac{s_{12}^2}{(q^2)^2} + 2\Re\left(\tilde{f}_{\mathcal{K},(\psi,\psi,\phi)}^{(1)}\right) \frac{s_{13}s_{23}}{(q^2)^2} \right]. \quad (5.28)$$

Inserting the explicit results (4.7) and performing the three-particle phase space integration (E.3), we find that the contribution to the cross section is given by

$$\sigma_{\mathcal{K},3}^{(2)} = \sigma_{\mathcal{K}}^{(0)} \left(\frac{\mu^2}{q^2} \right)^{2\epsilon} \left[-\frac{48}{\epsilon^3} - \frac{112}{\epsilon^2} - \frac{224 - 56\pi^2}{\epsilon} - 544 + \frac{632}{3}\pi^2 + 848\zeta_3 \right] + \mathcal{O}(\epsilon). \quad (5.29)$$

The one-loop renormalization constant (5.18) contributes as shown in figure 17. Together with (5.24) these terms cancel all one-loop subdivergences contained in the bare cross section (5.23). The squared matrix element reads

$$\mathcal{M}_{\mathcal{Z}^{(1)}\mathcal{K},3}^{(1)} = 2\mathcal{M}_{\mathcal{K},3}^{(0)} \mathcal{Z}_{\mathcal{K}}^{(1)}. \quad (5.30)$$

The corresponding cross section can be computed as in (5.9), and is given by

$$\sigma_{\mathcal{Z}^{(1)}\mathcal{K},3}^{(2)} = \frac{1}{g^2} \int \text{dPS}_3 \mathcal{M}_{\mathcal{Z}^{(1)}\mathcal{K},3}^{(1)} = \sigma_{\mathcal{K}}^{(0)} \left(\frac{\mu^2}{q^2} \right)^{\epsilon} \left[\frac{48}{\epsilon^3} - \frac{28\pi^2}{\epsilon} - 400\zeta_3 \right] + \mathcal{O}(\epsilon). \quad (5.31)$$

Summing (5.23), (5.25), (5.29) and (5.31), we find the cross section, which only contains the two-loop overall UV divergence after the subdivergences have been cancelled.³³ It reads

$$\begin{aligned} & \sigma_{\mathcal{K},2}^{(2)} + \sigma_{\mathcal{Z}^{(1)}\mathcal{K},2}^{(2)} + \sigma_{\mathcal{K},3}^{(2)} + \sigma_{\mathcal{Z}^{(1)}\mathcal{K},3}^{(2)} \\ &= \sigma_{\mathcal{K}}^{(0)} \left[-\frac{36}{\epsilon^2} + \frac{24}{\epsilon} + 72 \log^2 \frac{q^2}{\mu^2} - 384 \log \frac{q^2}{\mu^2} + 508 + 72\pi^2 + 336\zeta_3 \right] + \mathcal{O}(\epsilon). \end{aligned} \quad (5.32)$$

Two-loop renormalization constant

So far we have not included the third contribution of (5.20) involving the two-loop renormalization constant. In analogy to (5.17), it reads

$$\sigma_{\mathcal{Z}^{(2)}\mathcal{K}}^{(2)} = 2\sigma_{\mathcal{K}}^{(0)} \mathcal{Z}_{\mathcal{K}}^{(2)}. \quad (5.33)$$

³³All infrared divergences should be already canceled between the different channels.

Since this contribution has to cancel the overall UV divergence of (5.32), we immediately find

$$\mathcal{Z}_{\mathcal{K}}^{(2)} = \frac{18}{\epsilon^2} - \frac{12}{\epsilon} . \quad (5.34)$$

Comparing this with the expansion of (2.3) in terms of the anomalous dimension to two-loop order yields the known one- and two-loop Konishi anomalous dimension first obtained in [43, 44]

$$\gamma_{\mathcal{K}}^{(1)} = 12 , \quad \gamma_{\mathcal{K}}^{(2)} = -48 . \quad (5.35)$$

Adding (5.33) to (5.32) yields the renormalized two-loop cross section

$$\sigma_{\mathcal{K},\text{R}}^{(2)} = \sigma_{\mathcal{K}}^{(0)} \left[72 \log^2 \frac{q^2}{\mu^2} - 384 \log \frac{q^2}{\mu^2} + 508 + 72\pi^2 + 336\zeta_3 \right] + \mathcal{O}(\epsilon) , \quad (5.36)$$

Finally, we compute the second order term in the expansion of the logarithm

$$\left[\log \left(\frac{\sigma_{\mathcal{K},\text{R}}}{\sigma_{\mathcal{K}}^{(0)}} \right) \right]^{(2)} = \frac{\sigma_{\mathcal{K},\text{R}}^{(2)}}{\sigma_{\mathcal{K}}^{(0)}} - \frac{1}{2} \left(\frac{\sigma_{\mathcal{K},\text{R}}^{(1)}}{\sigma_{\mathcal{K}}^{(0)}} \right)^2 = -48 \log \frac{q^2}{\mu^2} + 116 + 72\pi^2 + 336\zeta_3 + \mathcal{O}(\epsilon) , \quad (5.37)$$

We find that the coefficient of $\log \frac{q^2}{\mu^2}$ gives the correct two-loop anomalous dimension, as expected from (2.15).

Including also the one-loop result (5.19), the Konishi cross section is given by

$$\log \left(\frac{\sigma_{\mathcal{K},\text{R}}}{\sigma_{\mathcal{K}}^{(0)}} \right) = g^2 \left(12 \log \frac{q^2}{\mu^2} - 28 \right) + g^4 \left(-48 \log \frac{q^2}{\mu^2} + 116 + 72\pi^2 + 336\zeta_3 \right) + \mathcal{O}(g^6, \epsilon) . \quad (5.38)$$

Indeed, this result is in accord with (2.15) since the prefactor of the logarithm is the Konishi-anomalous dimension to two-loop order given e.g. in (5.34). Moreover, the remaining finite terms yield the constant C , and by a comparison with (2.16) they determine the one- and two-loop terms of the constant M in (2.4). It would be nice to have an independent check of the two-loop contribution to C .

Some discussion

There are different routes one can pursue to compute the renormalized cross section. In the above presentation, we have treated the bare contribution and the terms involving the renormalization constant separately at the cross section level. One may also perform the renormalization of the form factors first, as described in appendix F, and then compute the renormalized cross section directly with them. Furthermore, the terms involving the renormalization constant can be obtained directly by expanding the relation (G.1). For example, the sum of (5.25) and (5.31) that involve the one-loop renormalization constant can be obtained as³⁴

$$\sigma_{\mathcal{Z}^{(1)}\mathcal{K},2}^{(2)} + \sigma_{\mathcal{Z}^{(1)}\mathcal{K},3}^{(2)} = 2\mathcal{Z}^{(1)}\sigma_{\mathcal{K}}^{(1)} + (\mathcal{Z}^{(1)})^2 . \quad (5.39)$$

³⁴Note in this case, one needs the result of one-loop cross section up to $\mathcal{O}(\epsilon)$ order. The result (5.16) based on (5.12) is not enough, since σ_{BPS} is not zero at $\mathcal{O}(\epsilon)$ order.

We have checked that these different ways give the same result.

As discussed in appendix G, the above result depends on the renormalization scheme. One can define the new coupling at which the subtraction is performed as $g_\rho = g e^{\rho\epsilon}$ and then expand the expressions in terms of the original coupling g . This scheme change can be implemented by simply replacing in all the above equations $\mathcal{Z}_K^{(\ell)} \rightarrow \mathcal{Z}_K^{(\ell)} e^{2\rho\epsilon}$. With such a modification in the above computation, one finds that the renormalized cross section (5.38) acquires a finite additive contribution $2\gamma_{K\rho}$, demonstrating that M in (2.4) is scheme-dependent. This is exactly as expected from (G.7), since the scheme change can be understood as a change of the 't Hooft mass $\mu \rightarrow \mu e^{-\rho}$.

Finally, let us briefly comment on the FDH scheme we have chosen in the computation. In the FDH scheme, we set the number of external scalars to 6 and use polarization vectors in $D = 4$ dimensions for the form factors. As discussed in section 2, this corresponds to the prescription given in (2.19), where the sum of the degrees of freedom for the external legs is performed by the η -integration based on the $SU(4)$ representation. One can also perform a detailed analysis at the diagrammatic level, as for the form factors in section 4, which leads to an alternative prescription for obtaining the cross section in $D = 4 - 2\epsilon$ dimensions. We will not present the details in the paper, but we have checked that both prescriptions give identical results at least up to the two-loop order.

6 Conclusion and outlook

In this paper, we have studied form factors of non-protected operators in $\mathcal{N} = 4$ SYM theory, specifically the Konishi operator, using on-shell unitarity techniques. We have obtained explicit new results of the three-point form factor at one-loop and two-point form factors up to two-loop order, given in (4.5)–(4.7). The application of on-shell methods to determine such form factors, which are partial off-shell quantities involving both, composite operators and on-shell states, provides a step to deepen our understanding of the connection between modern on-shell techniques and the off-shell world of correlation functions.

Another important aspect of the paper is to provide a physical observable within $\mathcal{N} = 4$ SYM, given by a cross-section-type quantity: the inclusive decay rate of a state, described by a composite operator carrying off-shell momentum q , into any final on-shell multi-particle state. We gave a formulation of how to compute this observable. Using the Konishi form factor results mentioned above, we performed an explicit computation of the total cross section up to two-loop order, given in (5.38).

The UV divergences appearing in the Konishi form factors together with the IR divergences require the renormalization of the operator. This is carried out explicitly in the computation of the total cross section in which the IR divergences cancel. We reproduced the known Konishi anomalous dimension up to two-loop order from the renormalization constant and also identified it as the coefficient of the $\log \frac{q^2}{\mu^2}$ term in the renormalized cross section (5.38).

Since the Konishi operator is not protected by supersymmetry, interesting subtleties and new features appeared, which we now summarize.

First, an important subtlety occurs in the unitarity-based computation of the Konishi form factors. The Konishi primary is a trace of all scalars. In order to preserve supersymmetry, the Konishi operator has to be continued to $D = 4 - 2\epsilon$ dimensions, i.e. the number of scalar field flavors that is summed over has to be $N_\phi = 10 - D$. However, working with 4-dimensional unitarity based on Nair's on-shell superspace means that one computes the form factor for the different operator \mathcal{K}_6 , which in $D = 4 - 2\epsilon$ dimensions has $N_\phi = 6$ scalars rather than $N_\phi = 6 + 2\epsilon$. In order to find the Konishi form factor, the results based on four-dimensional unitarity have to be modified when they are lifted to $D = 4 - 2\epsilon$ dimensions where the occurring divergences are regularized. We provide a rigorous prescription (4.2) for this lift that yields the form factors of the Konishi operator.

Second, some interesting features are not present for other on-shell quantities in $\mathcal{N} = 4$ SYM theory studied so far, such as scattering amplitudes or the BPS form factors as partial off-shell quantities. The bare Konishi form factor contains bubble integrals and bubble subintegrals which contain the UV divergences. Moreover, the one-loop three-point result contains a rational term.³⁵ Also, the coefficients of the individual integrals occurring in the form factor results involve spurious poles, which only disappear after multiplication with the basis integrals and summation over all contributions. Last but not least, the loop corrections of the Konishi form factors that have different external states turn out to have quite different structures, even if they are in the same MHV sector. The emergence of these features that are familiar from QCD may be traced back to the fact that a non-protected operator has been inserted into the action, formally breaking its supersymmetry.

Finally, let us briefly mention some further directions one can pursue following this work.

First, it should be straightforward to generalize the computation of the one-loop Konishi form factor to the higher-point cases. Since the Konishi form factors share similar features as amplitudes in QCD, it would be interesting to study possible connection between these quantities. It is also interesting to proceed to higher loop orders. In particular, using the known IR exponentiation property of the Sudakov form factor, the knowledge of the two-point Konishi form factor alone should be enough to extract the Konishi anomalous dimension. Turning the logic around, we give in appendix F a prediction for the three-loop two-point Konishi form factor apart from finite terms, only using the known three-loop anomalous dimension and the IR exponentiation.

Second, our detailed example of how to apply four-dimensional unitarity to compute the Konishi form factor by understanding an encountered subtlety and providing a solution is a solid stepping stone for further studies of other non-protected operators, based on generalizing the prescription we give in section 4. Combining our insights with those from the recent one-loop calculation in [34], it should be possible to compute the minimal form factors for general operators at two-loop order via on-shell methods. This would allow us to determine the complete two-loop dilatation operator of $\mathcal{N} = 4$ SYM theory which yields all two-loop anomalous dimensions as eigenvalues. Besides the anomalous dimensions, the other important CFT data is given by the structure constants, which can be computed

³⁵A similar rational term also occurs for the minimal form factor of operators in the $SL(2)$ subsector [80].

from three-point functions. It would be very interesting to use similar unitarity-based techniques to compute them.

Furthermore, as given in [33, 39–41], the so-called energy energy correlation function can be computed as a weighted cross section which is very similar to the total cross section studied in this paper. Although in [40, 41] different techniques have been used to evaluate them, it would be interesting to perform a cross section computation directly. The interpretation of the cross-section-type quantities at strong coupling via the AdS/CFT correspondence is also an open problem. In particular, it would be interesting to consider the phase space integration with strong coupling form factors in the framework of string theory.

Finally, as a cousin of $\mathcal{N} = 4$ SYM, the so-called ABJM theory [81] has been intensively studied in recent years. In particular, the form factors for half-BPS operators have been determined in this theory as well [82–84]. It would be interesting to pursue a similar study as in this paper for the ABJM theory, especially for the form factors of non-protected operators.

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A Fourier transformation of the propagator

In Euclidean space, we have the following relation from the Fourier transformation:

$$\frac{1}{(x_{\text{E}}^2)^\Delta} = 2^{D-2\Delta} \pi^{\frac{D}{2}} \frac{\Gamma(\frac{D}{2} - \Delta)}{\Gamma(\Delta)} \int \frac{d^D q_{\text{E}}}{(2\pi)^D} \frac{e^{iq_{\text{E}} \cdot x_{\text{E}}}}{(q_{\text{E}}^2)^{\frac{D}{2} - \Delta}}, \quad (\text{A.1})$$

where $q_{\text{E}} \cdot x_{\text{E}} = q_{\text{E},0} x_{\text{E},0} + \sum_{i=1}^{D-1} q_i x_i$.

In Minkowski space, where the integrand has poles at $q_0 = \pm|\vec{q}|$, we want positive energies $q_0 > 0$ to propagate into the future $x_0 > 0$. Hence, for a mostly-minus-signature metric where the exponent from the Fourier transformation is given by $-iq \cdot x$, the pole at $q_0 = |\vec{q}|$ has to be picked when for $x_0 > 0$ the integral over q_0 is closed in the negative imaginary half plane such that the exponential factor vanishes for $q_0 \rightarrow -i\infty$. This is guaranteed if we replace $q_{\text{E}}^2 \rightarrow -q^2 - i0$ in the denominator of the above expression. It fixes the Wick-rotation to be counterclockwise in momentum space, i.e. $q_0 = iq_{\text{E},0}$, and clockwise in configuration space, i.e. $x_0 = -ix_{\text{E},0}$. This leaves the exponential invariant, and it can be transformed to Minkowski signature by flipping the sign of the spatial momenta q_i . We hence find

$$\frac{1}{(-x^2 + i0)^\Delta} = (-i) 2^{D-2\Delta} \pi^{\frac{D}{2}} \frac{\Gamma(\frac{D}{2} - \Delta)}{\Gamma(\Delta)} \int \frac{d^D q}{(2\pi)^D} \frac{e^{-iq \cdot x}}{(-q^2 - i0)^{\frac{D}{2} - \Delta}}, \quad (\text{A.2})$$

where $q \cdot x = q_0 x_0 - \sum_{i=1}^{D-1} q_i x_i$.

B Feynman integrals

In this appendix, we present all integrals that enter the form factor results in section 3, as well as our conventions. Moreover, we show how the cut integrals are lifted to full integrals.

As a regularization procedure, the four-dimensional $\mathcal{N} = 4$ SYM theory can be continued to $D = 4 - 2\epsilon$ dimensions. Both IR and UV divergences are then captured as poles in ϵ . Moreover, the Yang-Mills coupling constant g_{YM} has to be replaced by $g_{\text{YM}} \mu^\epsilon$, where μ is the 't Hooft mass that is introduced in order to keep g_{YM} dimensionless [85]. Hence, in the planar limit, the combination $g\mu^\epsilon$ with g given in (1.5) is the effective loop expansion parameter of the perturbation series.

From Feynman diagrams, the following combination of the coupling constant, 't Hooft mass μ and loop integral occurs at ℓ loops

$$(g_{\text{YM}} \mu^\epsilon)^{2\ell} N_c^\ell (-i)^\ell \int \frac{d^D l_1}{(2\pi)^D} \cdots \frac{d^D l_\ell}{(2\pi)^D} \frac{f(l_1, \dots, l_\ell)}{\prod_j D_j} = g^{2\ell} I^{(\ell)}[f(l_1, \dots, l_\ell)], \quad (\text{B.1})$$

where, with the definition of the effective planar coupling constant given in (1.5), the integral $I^{(\ell)}$ is of the following form

$$I^{(\ell)}[f(l_1, \dots, l_\ell)] = (e^{\gamma_E} \mu^2)^{\ell\epsilon} \int \frac{d^D l_1}{i\pi^{\frac{D}{2}}} \cdots \frac{d^D l_\ell}{i\pi^{\frac{D}{2}}} \frac{f(l_1, \dots, l_\ell)}{\prod_j D_j}. \quad (\text{B.2})$$

In these formulae, the D_j 's are the propagators, i.e. $D_j = k_j^2 + i0$ for k_j being the combination of external momenta and loop momenta that flows through the propagator.

Lifting the cut integral

Let us explain our procedure and convention for lifting the cut integrals to the full integral.

Consider the triangle term in (3.15) as an explicit example. We have

$$g_{\text{YM}}^2 N_c \int d\text{PS}_{2,\{l\}} \frac{s_{12}}{(l_1 + p_1)^2} = g_{\text{YM}}^2 N_c s_{12} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} p_1 \\ p_2 \end{array} \xrightarrow{\text{lifting}} -i g^2 s_{12} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} p_1 \\ p_2 \end{array}, \quad (\text{B.3})$$

where the phase space integration measure $d\text{PS}_{2,\{l\}}$ is defined according to (2.20) with measure factor $\frac{d^D l}{(2\pi)^D}$. To lift the cut integral to the full integral, two steps have to be performed.

One is to replace the cut propagator as

$$2\pi\delta_+(l_j^2) \longrightarrow \frac{i}{l_j^2}. \quad (\text{B.4})$$

The other is to change the measure factor and coupling constant as in (B.1), such that the integrals of *uncut* graphs are defined in terms of (B.2). This prescription is employed throughout section 3.

List of integrals

We define $q = \sum_i p_i$ and introduce the factor

$$c_\Gamma = e^{\gamma_E \epsilon} \frac{\Gamma(1 - \epsilon)^2 \Gamma(1 + \epsilon)}{\Gamma(1 - 2\epsilon)}. \quad (\text{B.5})$$

In the results for the integrals, all $(-q^2)^{\ell\epsilon}$ should be understood as $(-q^2 - i0)^{\ell\epsilon}$ and similarly for $(-s_{ij})^{\ell\epsilon}$.

In the convention introduced in (B.2), the one-loop integrals that are required to

compute the one-loop form factors read

$$\begin{aligned}
\text{Diagram 1} &= (e^{\gamma_E} \mu^2)^\epsilon \int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{1}{l^2(l+q)^2} = \frac{c_\Gamma}{\epsilon(1-2\epsilon)} \left(\frac{-q^2}{\mu^2}\right)^{-\epsilon}, \\
\text{Diagram 2} &= (e^{\gamma_E} \mu^2)^\epsilon \int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{1}{(l+p_1)^2 l^2 (l-p_2)^2} = -\frac{c_\Gamma}{\epsilon^2} \frac{1}{(-q^2)} \left(\frac{-q^2}{\mu^2}\right)^{-\epsilon}, \\
\text{Diagram 3} &= (e^{\gamma_E} \mu^2)^\epsilon \int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{1}{(l+p_1+p_2)^2 l^2 (l-p_3)^2} \\
&= -\frac{c_\Gamma}{\epsilon^2} \frac{1}{s_{13}+s_{23}} \left[\left(\frac{-s_{12}}{\mu^2}\right)^{-\epsilon} - \left(\frac{-q^2}{\mu^2}\right)^{-\epsilon} \right], \\
\text{Diagram 4} &= I_4^{1m}(p_1, p_2, p_3, -q) = (e^{\gamma_E} \mu^2)^\epsilon \int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{1}{l^2(l^2+p_1)^2(l+p_1+p_2)^2(l-q)^2} \\
&= \frac{c_\Gamma}{\epsilon^2} \frac{2}{s_{12}s_{23}} \left[-\left(\frac{-q^2}{\mu^2}\right)^{-\epsilon} {}_2F_1\left(1, -\epsilon, 1-\epsilon, -\frac{q^2 s_{13}}{s_{12}s_{23}}\right) \right. \\
&\quad \left. + \left(\frac{-s_{12}}{\mu^2}\right)^{-\epsilon} {}_2F_1\left(1, -\epsilon, 1-\epsilon, -\frac{s_{13}}{s_{23}}\right) \right. \\
&\quad \left. + \left(\frac{-s_{23}}{\mu^2}\right)^{-\epsilon} {}_2F_1\left(1, -\epsilon, 1-\epsilon, -\frac{s_{13}}{s_{12}}\right) \right], \tag{B.6}
\end{aligned}$$

where ${}_2F_1$ denote the Gaussian hypergeometric function. The box integral result can be found, for example, in [86].

Furthermore, we need the following integral, which evaluates to a rational term:

$$\begin{aligned}
I_3^D[l_\epsilon^2] &= (e^{\gamma_E} \mu^2)^\epsilon \int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{l_\epsilon^2}{(l+p_1+p_2)^2 l^2 (l-p_3)^2} \\
&= -\frac{c_\Gamma}{(1-2\epsilon)(2-2\epsilon)} \frac{1}{s_{13}+s_{23}} \left[s_{12} \left(\frac{-s_{12}}{\mu^2}\right)^{-\epsilon} - q^2 \left(\frac{-q^2}{\mu^2}\right)^{-\epsilon} \right], \tag{B.7}
\end{aligned}$$

where the notation of l_ϵ^2 is introduced in (C.1) in appendix C.

To compute the two-loop two-point form factor, we need the following two-loop integrals. Using IBP identities, e.g. via LiteRed [87], these can be reduced to master integrals as³⁶

³⁶Recall that the loop-momentum-dependent prefactors are understood to appear in the numerators of the depicted integrals.

$$\begin{aligned}
& \text{Diagram 1} = \frac{2-3\epsilon}{\epsilon(-q^2)} \text{Diagram 2} \\
(q^2)^2 & \text{Diagram 3} = -\frac{3(1-2\epsilon)(1-3\epsilon)(2-3\epsilon)}{\epsilon^3(-q^2)} \text{Diagram 4} \\
& + \frac{3(1-2\epsilon)(1-3\epsilon)}{2\epsilon^2} \text{Diagram 5} + \frac{(1-2\epsilon)^2}{\epsilon^2} \left(\text{Diagram 6} \right)^2 \\
s_{1l}s_{2l} & \text{Diagram 7} = \frac{(2-3\epsilon)(2-9\epsilon+10\epsilon^2-4\epsilon^3)}{(1-\epsilon)(1-2\epsilon)\epsilon^2(-q^2)} \text{Diagram 8} \\
& - \frac{1-4\epsilon+2\epsilon^2}{(1-\epsilon)\epsilon} \text{Diagram 9} \\
& - \frac{2-3\epsilon+2\epsilon^2}{2(1-\epsilon)\epsilon} \left(\text{Diagram 10} \right)^2 \\
s_{1l}s_{2l} & \text{Diagram 11} = \frac{(1-2\epsilon)(2-3\epsilon)(3-5\epsilon)}{\epsilon^2(1-4\epsilon)(-q^2)} \text{Diagram 12} \\
& - \frac{(1+\epsilon)(1-2\epsilon)}{\epsilon(1-4\epsilon)} \text{Diagram 13} - \frac{\epsilon(-q^2)^2}{(1-4\epsilon)} \text{Diagram 14},
\end{aligned} \tag{B.8}$$

where the master integrals are [88]

$$\begin{aligned}
& \text{Diagram 1} = e^{2\gamma_E\epsilon} \frac{\Gamma(1-\epsilon)^3\Gamma(1+2\epsilon)}{2\epsilon(1-2\epsilon)\Gamma(3-3\epsilon)} (-q^2) \left(-\frac{q^2}{\mu^2} \right)^{-2\epsilon}, \\
& \text{Diagram 2} = e^{2\gamma_E\epsilon} \frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}{2\epsilon^2(1-2\epsilon)\Gamma(2-3\epsilon)} \left(-\frac{q^2}{\mu^2} \right)^{-2\epsilon}, \\
& \text{Diagram 3} = e^{2\gamma_E\epsilon} \frac{1}{(-q^2)^2} \left(-\frac{q^2}{\mu^2} \right)^{-2\epsilon} \left[\frac{\Gamma(1-2\epsilon)^4\Gamma(1+2\epsilon)^3\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{\epsilon^4(1-4\epsilon)^2\Gamma(1+4\epsilon)} \right. \\
& + \frac{4\Gamma(1-\epsilon)^2\Gamma(1-2\epsilon)\Gamma(1+2\epsilon)}{\epsilon^2(1+\epsilon)(1+2\epsilon)\Gamma(1-4\epsilon)} {}_3F_2(1, 1, 1+2\epsilon; 2+\epsilon, 2+2\epsilon; 1) \\
& + \frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)\Gamma(1-2\epsilon)\Gamma(1+2\epsilon)}{2\epsilon^4\Gamma(1-3\epsilon)} {}_3F_2(1, -4\epsilon, -2\epsilon; 1-3\epsilon, 1-2\epsilon; 1) \\
& \left. + \frac{\Gamma(1-\epsilon)^3\Gamma(1+2\epsilon)}{2\epsilon^4\Gamma(1-3\epsilon)} {}_4F_3(1, 1-\epsilon, -4\epsilon, -2\epsilon; 1-3\epsilon, 1-2\epsilon, 1-2\epsilon; 1) \right],
\end{aligned} \tag{B.9}$$

and the one-loop bubble integral given in (B.6).

C Passarino-Veltman reductions

In this appendix, we summarize some results on Passarino-Veltman (PV) reduction [89], which we need in section 3.

We use the four-dimensional helicity (FDH) scheme of [67, 68], and decompose the D -dimensional loop momentum l into a four-dimensional part $l_{(4)}$ and a $(D - 4) = -2\epsilon$ dimensional part l_ϵ , where we assume that $\epsilon < 0$. This yields for the decomposition of the scalar product

$$\eta_{\mu\nu} l^\mu l^\nu = l_{(4)}^2 = l^2 + l_\epsilon^2, \quad (\text{C.1})$$

where $\eta_{\mu\nu}$ is the four-dimensional metric.³⁷ Arbitrary four-dimensional external reference momenta are denoted as k_i .

Bubble. The D -dimensional bubble integral with external momentum q defined in (B.6) may include a non-trivial polynomial $f(l)$ of the loop momentum l and the reference momenta k_i in its numerator. Denoting this as $I_2^D[f(l)](q^2)$, we find the relations

$$I_2^D[(l \cdot k_1)](q^2) = -\frac{(q \cdot k_1)}{2} I_2^D(q^2), \quad (\text{C.2})$$

$$I_2^D[(l \cdot k_1)(l \cdot k_2)](q^2) = \left(\frac{(q \cdot k_1)(q \cdot k_2)}{3} - \frac{q^2(k_1 \cdot k_2)}{12} \right) I_2^D(q^2) \\ - \left(\frac{(q \cdot k_1)(q \cdot k_2)}{3q^2} - \frac{(k_1 \cdot k_2)}{3} \right) I_2^D[l_\epsilon^2](q^2). \quad (\text{C.3})$$

Triangle. Next, we consider the D -dimensional triangle integral with numerator $f(l)$, which depends on two arbitrary momenta q_1 and q_2 . It is defined as

$$I_3^D[f(l)](q_1, q_2) = (e^{\gamma_E} \mu^2)^\epsilon \int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{f(l)}{l^2(l+q_1)^2(l+q_2)^2}. \quad (\text{C.4})$$

We find

$$I_3^D[(l \cdot k_1)](q_1, q_2) = \sum_{i=0}^2 \frac{a_i}{2} I_2^{D,(i)} - \sum_{i=1}^2 \frac{a_i q_i^2}{2} I_3^D, \quad (\text{C.5}) \\ I_3^D[(l \cdot k_1)(l \cdot k_2)](q_1, q_2) = \sum_{i=0}^2 C_2^{(i)} I_2^{D,(i)} + C_{3,0} I_3^D + C_{3,\epsilon} I_3^D[l_\epsilon^2](q_1, q_2),$$

where $I_2^{D,(0)} = I_2^D((q_1 - q_2)^2)$, $I_2^{D,(1)} = I_2^D(q_2^2)$, $I_2^{D,(2)} = I_2^D(q_1^2)$,

³⁷The sign in front of l_ϵ results from the mostly-minus metric.

$$C_2^{(1)} = -\frac{1}{4} \left(\sum_{i=1}^2 a_{1i} q_i^2 \right) a_{21} - \frac{1}{4} a_{11} (q_2 \cdot k_2) + \frac{1}{8} [(k_1 \cdot k_2) - b] q_2^2 \sum_{i=1}^2 (A^{-1})_{2i} , \quad (\text{C.6})$$

$$C_2^{(2)} = -\frac{1}{4} \left(\sum_{i=1}^2 a_{1i} q_i^2 \right) a_{22} - \frac{1}{4} a_{12} (q_1 \cdot k_2) + \frac{1}{8} [(k_1 \cdot k_2) - b] q_1^2 \sum_{i=1}^2 (A^{-1})_{1i} , \quad (\text{C.7})$$

$$C_2^{(0)} = -\sum_{i=1}^2 C_2^{(i)} + \frac{1}{4} (k_1 \cdot k_2) , \quad (\text{C.8})$$

$$C_{3,0} = \frac{1}{4} \left(\sum_{i=1}^2 a_{1i} q_i^2 \right) \left(\sum_{j=1}^2 a_{2j} q_j^2 \right) + \frac{1}{8} [b - (k_1 \cdot k_2)] q_1^2 q_2^2 \tilde{A} , \quad (\text{C.9})$$

$$C_{3,\epsilon} = \frac{(k_1 \cdot k_2) - b}{2} , \quad (\text{C.10})$$

and

$$a_{ij} = \sum_{m=1}^2 (k_i \cdot q_m) (A^{-1})_{mj} , \quad A_{ij} = q_i \cdot q_j , \quad (\text{C.11})$$

$$a_0 = -\sum_{i=1}^2 a_i , \quad a_i = a_{1i} , \quad i, j = 1, 2 , \quad (\text{C.12})$$

$$b = \sum_{i,j=1}^2 (k_1 \cdot q_i) (A^{-1})_{ij} (q_j \cdot k_2) , \quad \tilde{A} = \sum_{i,j=1}^2 (A^{-1})_{ij} . \quad (\text{C.13})$$

The integrals involving l_ϵ^2 give rational terms [90]. The rational term for the tensor-two triangle integral is given in (B.7).

D Checks of the three-point one-loop Konishi form factor

Rational term in $F_{\mathcal{K}}^{(1)}(1_{\phi_{12}}, 2_{\phi_{34}}, 3_{g^+})$

An interesting feature of the Konishi form factor is the occurrence of rational term at one loop.

For the form factor $F_{\mathcal{K}}^{(1)}(1_{\phi_{12}}, 2_{\phi_{34}}, 3_{g^+})$, denoted as $F_{\mathcal{K},(\phi,\phi,g)}^{(1)}$, this corresponds to the triangle integral containing the l_ϵ^2 -term, see (3.45).

Using the formula (B.7), the rational term, denoted by $\mathcal{R}[\bullet]$, can be computed as

$$\mathcal{R}[F_{\mathcal{K},(\phi,\phi,g)}^{(1)}] = F_{\mathcal{K},(\phi,\phi,g)}^{(0)} \frac{N_\phi s_{13} s_{23}}{s_{12}(s_{13} + s_{23})} . \quad (\text{D.1})$$

Since the computation in section 3 is based on the four-dimensional unitarity method, one might be concerned whether additional rational terms are missing. In the following, we show that the result is actually complete by comparing with a Feynman diagram computation following the strategy of [91].

$$\begin{aligned}
& \text{Diagram 1} = \text{Diagram 2} = \frac{-iN_\phi g_{\text{YM}}^3}{2\sqrt{2}s_{12}} \int \frac{d^D l}{(2\pi)^D} \frac{(2l + p_1 + p_2) \cdot (p_1 - p_2) (2l - p_3) \cdot \epsilon_3^+}{l^2 (l + p_1 + p_2)^2 (l - p_3)^2} .
\end{aligned}
\tag{D.3}$$

Figure 18: Feynman diagrams of the one-loop three-point Konishi form factor that contribute to the rational term.

First, from the power counting criterion given in [5], a one-loop integral has rational-term contributions only when it has high enough power of loop momentum l in the numerator of the loop integrand, which is given by

$$\begin{aligned}
m > n - 2 & \quad \text{for} \quad I_n^D [(l)^m] \quad \text{with} \quad n > 2 , \\
m > 1 & \quad \text{for} \quad I_2^D [(l)^m] .
\end{aligned}
\tag{D.2}$$

Furthermore, we can safely neglect Feynman diagrams that appear in the computation of the BPS form factor, since the sum of them is known to be free of rational terms. It turns out that only two diagrams need to be considered, which are shown in figure 18.

Using standard color-ordered Feynman rules (see e.g. [66]), these two graphs give

$$F_{\mathcal{K},(\phi,\phi,g)}^{(0)} \left(-\frac{4N_\phi}{\sqrt{2}s_{12}} \right) \frac{\langle 23 \rangle \langle 31 \rangle}{\langle 12 \rangle} (e^{\gamma_E} \mu^2)^\epsilon \int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{(l \cdot \epsilon_3^+) [l \cdot (p_1 - p_2)]}{l^2 (l - p_3)^2 (l + p_{12})^2} , \tag{D.4}$$

where the polarization vector is given by $\epsilon_3^+ = \frac{\sqrt{2}\xi\tilde{\lambda}_3}{\langle \xi\lambda_3 \rangle}$ and ξ is an arbitrary reference spinor. Then, applying the formula³⁸

$$\mathcal{R} \left[\int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{(l \cdot k_1)(l \cdot k_2)}{l^2 (l + q_1)^2 (l + q_2)^2} \right] = \frac{k_1 \cdot k_2}{2} - \frac{(k_1 \cdot k_2)(k_2 \cdot q_1)}{2q_1 \cdot q_2} , \quad k_1 \cdot q_1 = q_1^2 = 0 , \tag{D.5}$$

we can extract the rational term of (D.4) immediately, which, after some simple spinor algebra, turns out to be identical to that given in (D.1). Thus, we have proven that the unitarity method gives the complete rational terms.

Spurious poles

The coefficients of the integrals in (3.45) and (3.46) contain unphysical poles, such as the pole $\frac{1}{s_{13}+s_{23}} = \frac{1}{s_{123}-s_{12}}$. The physical consistency requires that such poles must cancel when multiplying the coefficients with the respective integrals and summing all contributions.³⁹ Here we check that this is indeed true. We focus on the pole $\frac{1}{s_{13}+s_{23}}$, the others are similar.

Let us first consider the case of $F_{\mathcal{K}}^{(1)}(1_\psi, 2_\psi, 3_\phi)$. Only the coefficients of the bubble integrals contain spurious poles. Summing over all bubble integrals, the $\frac{1}{\epsilon}$ term is free of

³⁸This identity can be obtained using PV reduction and (B.7). Formulae for more general cases can be found in [91].

³⁹This is a common feature for one-loop QCD amplitudes, see e.g [6].

the pole, and at finite order we find

$$-3 \frac{s_{12}s_{13}}{s_{12} + s_{13}} \log \left(\frac{s_{123}}{s_{23}} \right). \quad (\text{D.6})$$

This is indeed finite when $s_{12} + s_{13} \rightarrow 0$, as can be seen from the expansion

$$\frac{\log x}{x-1} = 1 - \frac{x-1}{2} + \frac{(x-1)^2}{3} + \mathcal{O}((x-1)^3). \quad (\text{D.7})$$

The $F_{\mathcal{K}}^{(1)}(1_\phi, 2_\phi, 3_g)$ case is a little more complicated. In this case, both bubble and triangle integral contain the pole $\frac{1}{s_{13}+s_{23}}$ in its coefficients. Expanding to finite order and extracting the terms that contain this pole, we find

$$-6 \frac{s_{13}s_{23}}{(s_{13} + s_{23})^2} \log \left(\frac{s_{123}}{s_{12}} \right) + 6 \frac{s_{13}s_{23}}{s_{12}(s_{13} + s_{23})}, \quad (\text{D.8})$$

where the first term stems from the sum of bubble integrals and the second term is the rational term. Each term itself is divergent when taking the limit $s_{13} + s_{23} \rightarrow 0$; however, the sum of the two terms is finite in the limit.

E Phase-space parametrization

In this appendix, we provide formulae for the parametrization of the phase-space integrals. Furthermore, we give details on a non-trivial three-particle phase-space integration encountered in section 5.

The n -particle phase space integral is defined as

$$\int \text{dPS}_n(\bullet) = \int \left(\prod_{j=1}^n \frac{\text{d}^D p_j}{(2\pi)^D} 2\pi \delta_+(p_j^2) \right) (2\pi)^D \delta^D \left(q - \sum_{j=1}^n p_j \right) (\bullet), \quad (\text{E.1})$$

where (\bullet) denotes the integrand, i.e. the squared matrix element.

When $n = 2$, the squared matrix element depends only on q^2 , and we can evaluate the two-particle phase space integral independently:

$$\int \text{dPS}_2(\bullet) = f_{\text{PS}2}(\bullet), \quad f_{\text{PS}2} = \frac{(q^2)^{-\epsilon}}{4(16\pi)^{\frac{1}{2}-\epsilon} \Gamma(\frac{3}{2}-\epsilon)}. \quad (\text{E.2})$$

The three-particle phase space can be parametrized as

$$\int \text{dPS}_3(\bullet) = f_{\text{PS}3} \int_0^1 dx x^{1-2\epsilon} (1-x)^{-\epsilon} \int_0^1 dy [y(1-y)]^{-\epsilon} (\bullet), \quad (\text{E.3})$$

with

$$f_{\text{PS}3} = \frac{(q^2)^{1-2\epsilon}}{2(4\pi)^{3-2\epsilon} \Gamma(2-2\epsilon)}. \quad (\text{E.4})$$

The ratios of Mandelstam variables occurring in the squared matrix element are parametrized as

$$\left\{ \frac{s_{ij}}{q^2}, \frac{s_{jk}}{q^2}, \frac{s_{ki}}{q^2} \right\} = \{x(1-y), 1-x, xy\}, \quad (\text{E.5})$$

in which (i, j, k) can be any permutation of $(1, 2, 3)$, since the phase space measure is totally symmetric for p_1, p_2, p_3 .

Some details about the three-particle phase space integral

The phase-space integration becomes non-trivial for the squared matrix element involving the three-point one-loop form factor. It contains the finite part of the box integral, which involves the hypergeometric functions ${}_2F_1$, as given in (4.8).

The corresponding phase space integrals are

$$\begin{aligned}
& \int d\text{PS}_3 \frac{1}{s_{13}} \left(\frac{\mu^2}{-s_{23}} \right)^\epsilon \frac{1}{\epsilon^2} {}_2F_1 \left(1, -\epsilon, 1 - \epsilon, -\frac{s_{13}}{s_{12}} \right) \\
&= \frac{f_{\text{PS3}}}{q^2} \left(\frac{\mu^2}{-q^2} \right)^\epsilon \frac{1}{\epsilon^2} \int_0^1 dx x^{-2\epsilon} (1-x)^{-2\epsilon} \int_0^1 dy y^{-1-\epsilon} (1-y)^{-\epsilon} {}_2F_1 \left(1, -\epsilon, 1 - \epsilon, -\frac{y}{1-y} \right) \\
&= \frac{f_{\text{PS3}}}{q^2} \left(\frac{\mu^2}{-q^2} \right)^\epsilon \frac{\Gamma(1-2\epsilon)\Gamma(-\epsilon)^2}{4\epsilon^2\Gamma(2-4\epsilon)} \left[(1+6\epsilon) {}_2F_1 \left(1, 1, 1-2\epsilon, 1 \right) \right. \\
&\quad \left. - (8\epsilon) {}_3F_2 \left(1, 1, -\epsilon; 1-\epsilon, -2\epsilon; 1 \right) \right]
\end{aligned} \tag{E.6}$$

and

$$\begin{aligned}
& \int d\text{PS}_3 \frac{1}{s_{13}} \left(\frac{\mu^2}{-q^2} \right)^\epsilon \frac{1}{\epsilon^2} {}_2F_1 \left(1, -\epsilon, 1 - \epsilon, -\frac{q^2}{s_{12}s_{23}} \right) \\
&= \frac{f_{\text{PS3}}}{q^2} \left(\frac{\mu^2}{-q^2} \right)^\epsilon \frac{1}{\epsilon^2} \int_0^1 dx x^{-2\epsilon} (1-x)^{-\epsilon} \int_0^1 dy y^{-1-\epsilon} (1-y)^{-\epsilon} \\
&\quad {}_2F_1 \left(1, -\epsilon, 1 - \epsilon, -\frac{1}{1-x} \frac{y}{1-y} \right) \\
&= \frac{f_{\text{PS3}}}{q^2} \left(\frac{\mu^2}{-q^2} \right)^\epsilon \left[-\frac{1}{\epsilon^3} - \frac{3}{\epsilon^2} - \frac{9 - \frac{5\pi^2}{6}}{\epsilon} + \left(-27 + \frac{17\pi^2}{6} + 21\zeta_3 \right) + \mathcal{O}(\epsilon) \right]
\end{aligned} \tag{E.7}$$

F Anomalous dimensions via two-point form factors

In the main part of this paper, we have determined the anomalous dimension of the Konishi operator from its cross section, i.e. from the imaginary part of its two-point function. It is also possible to determine the anomalous dimension of the Konishi operator from its two-point form factor alone. As seen throughout this paper, form factors of non-protected operators contain both UV and IR divergences. To extract the UV divergences, one needs to subtract the IR divergences. The computation of the cross section, as done in section 5, is one of the safest ways to do so. On the other hand, the IR divergences, in particular for Sudakov form factors, have an universal structure [17–19]. This allows one to subtract the IR divergences directly from the form factors. The remaining divergences are purely UV divergences, from which one can read off the anomalous dimension of the operator.⁴⁰

⁴⁰This route was also taken in the unpublished notes of Boucher-Veronneau, Dixon, and Pennington [72].

In terms of the effective planar coupling constant (1.5), the logarithm of the Sudakov form factor in $\mathcal{N} = 4$ SYM theory has the following structure [20]:⁴¹

$$\begin{aligned} \log f_{\mathcal{O},R} &= \sum_{\ell=1}^{\infty} g^{2\ell} (\log f_{\mathcal{O},R})^{(\ell)} \\ &= \sum_{\ell=1}^{\infty} g^{2\ell} \left(\frac{\mu^2}{-q^2} \right)^{\ell\epsilon} \left(-\frac{\gamma_{\text{cusp}}^{(\ell)}}{(2\ell\epsilon)^2} - \frac{\mathcal{G}_0^{(\ell)}}{2\ell\epsilon} + \text{Fin}^{(\ell)} \right) + \mathcal{O}(\epsilon), \end{aligned} \quad (\text{F.1})$$

where the divergent terms are determined by the universal cusp and collinear anomalous dimensions

$$\begin{aligned} \gamma_{\text{cusp}}(g) &= \sum_{\ell=1}^{\infty} \gamma_{\text{cusp}}^{(\ell)} g^{2\ell} = 8g^2 - 16\zeta_2 g^4 + 176\zeta_4 g^6 + \mathcal{O}(g^8), \\ \mathcal{G}_0(g) &= \sum_{\ell=1}^{\infty} \mathcal{G}_0^{(\ell)} g^{2\ell} = -4\zeta_3 g^4 + 16 \left(2\zeta_5 + \frac{5}{3}\zeta_2\zeta_3 \right) g^6 + \mathcal{O}(g^8). \end{aligned} \quad (\text{F.2})$$

The finite terms of the logarithm of the form factor depend on the specific properties of the form factor such as the choice of the operator. In particular, they contain a remainder function, which was studied in [27, 29].

For non-protected operators, renormalization is required. The renormalized form factor is given as

$$f_{\mathcal{O},R}^{(L)} = \sum_{\ell=1}^L \mathcal{Z}^{(\ell)} f_{\mathcal{O},B}^{(L-\ell)}, \quad (\text{F.3})$$

where the renormalization constant is related to the anomalous dimension as shown in (2.3).

The universal structure of IR divergence, together with the bare Konishi form factor, allow us to determine the renormalization constant and therefore the anomalous dimension. In the following, we employ the two-loop Konishi form factor to reproduce the Konishi anomalous dimension (1.6) up to two-loop order. Reversing the logic, we then give a prediction for the bare three-loop two-point Konishi form factor up to $\mathcal{O}(\epsilon^{-1})$ order by using the known three-loop anomalous dimension.

One-loop form factor

The one-loop bare form factor is given in (4.5). From the universal IR structure, we know that

$$(\log f_{\mathcal{K},R})^{(1)} = f_{\mathcal{K},R}^{(1)} = f_{\mathcal{K},B}^{(1)} + \mathcal{Z}_{\mathcal{K}}^{(1)} = \left(\frac{\mu^2}{-q^2} \right)^{\epsilon} \left(-\frac{\gamma_{\text{cusp}}^{(1)}}{4\epsilon^2} - \frac{\mathcal{G}_0^{(1)}}{2\epsilon} \right) + \mathcal{O}(\epsilon^0), \quad (\text{F.4})$$

⁴¹This form was checked for the minimal form factor of the BPS operator $\text{tr}(\phi_{12}^2)$ up to the third loop order [15], for the minimal form factor of the BPS operator $\text{tr}(\phi_{12}^k)$ up to the second loop order [29], for the n -point MHV form factor of the BPS operator $\text{tr}(\phi_{12}^2)$ up to the first loop order [21] and for the 3-point MHV form factor of the BPS operator $\text{tr}(\phi_{12}^2)$ up to the second loop order [27].

where the one-loop cusp and collinear anomalous dimensions are given in (F.2). The simple pole in $f_{\mathcal{K},B}^{(1)}$ has to be canceled by the one-loop term in the operator renormalization constant, which yields

$$\mathcal{Z}_{\mathcal{K}}^{(1)} = \frac{6}{\epsilon}, \quad (\text{F.5})$$

in agreement with (5.18) and the known one-loop anomalous dimension. Thus, the one-loop renormalized form factor is

$$f_{\mathcal{K},R}^{(1)} = \left(\frac{\mu^2}{-q^2}\right)^\epsilon \frac{2(1+\epsilon+\epsilon^2)e^{\epsilon\gamma_E}\Gamma(-\epsilon)^2\Gamma(1+\epsilon)}{(-1+2\epsilon)\Gamma(1-2\epsilon)} + \frac{6}{\epsilon}. \quad (\text{F.6})$$

Two-loop form factor

The two-loop bare Konishi form factor is given in (4.6). From the universal IR structure, we know that

$$\begin{aligned} (\log f_{\mathcal{K},R})^{(2)} &= f_{\mathcal{K},R}^{(2)} - \frac{1}{2} \left(f_{\mathcal{K},R}^{(1)}\right)^2 = \left(f_{\mathcal{K},B}^{(2)} + \mathcal{Z}_{\mathcal{K}}^{(1)} f_{\mathcal{K},B}^{(1)} + \mathcal{Z}_{\mathcal{K}}^{(2)}\right) - \frac{1}{2} \left(f_{\mathcal{K}}^{(1)}\right)^2 \\ &= \left(\frac{\mu^2}{-q^2}\right)^{2\epsilon} \left(-\frac{\gamma_{\text{cusp}}^{(2)}}{16\epsilon^2} - \frac{\mathcal{G}_0^{(2)}}{4\epsilon}\right) + \mathcal{O}(\epsilon^0), \end{aligned} \quad (\text{F.7})$$

where the two-loop cusp and collinear anomalous dimensions are given in (F.2). This determines the two-loop term of the renormalization constant as

$$\mathcal{Z}_{\mathcal{K}}^{(2)} = \frac{18}{\epsilon^2} - \frac{12}{\epsilon}, \quad (\text{F.8})$$

which perfectly agrees with (5.34) and the known two-loop anomalous dimension. Hence, the two-loop renormalized form factor is

$$\begin{aligned} f_{\mathcal{K},R}^{(2)} &= \left(\frac{\mu^2}{-q^2}\right)^{2\epsilon} \left[\frac{2}{\epsilon^4} + \frac{28 - \frac{\pi^2}{6}}{\epsilon^2} + \frac{56 - \pi^2 - \frac{25\zeta_3}{3}}{\epsilon} + \left(316 - \frac{26\pi^2}{3} - 28\zeta_3 - \frac{7\pi^4}{60}\right) \right. \\ &\quad \left. + \left(1172 - \frac{131\pi^2}{3} - \frac{572\zeta_3}{3} - \frac{53\pi^4}{120} + \frac{23\pi^2\zeta_3}{18} + \frac{71\zeta_5}{5}\right)\epsilon \right] + \mathcal{O}(\epsilon^2). \end{aligned} \quad (\text{F.9})$$

Prediction for the three-loop bare Konishi form factor

Now, we reverse the logic. From the universal IR structure, we know that

$$\begin{aligned} (\log f_{\mathcal{K},R})^{(3)} &= f_{\mathcal{K},R}^{(3)} - f_{\mathcal{K},R}^{(2)} f_{\mathcal{K},R}^{(1)} + \frac{1}{3} \left(f_{\mathcal{K},R}^{(1)}\right)^3 \\ &= \left(f_{\mathcal{K},B}^{(3)} + \mathcal{Z}_{\mathcal{K}}^{(1)} f_{\mathcal{K},B}^{(2)} + \mathcal{Z}_{\mathcal{K}}^{(2)} f_{\mathcal{K},B}^{(1)} + \mathcal{Z}_{\mathcal{K}}^{(3)}\right) - f_{\mathcal{K},R}^{(2)} f_{\mathcal{K},R}^{(1)} + \frac{1}{3} \left(f_{\mathcal{K},R}^{(1)}\right)^3 \\ &= \left(\frac{\mu^2}{-q^2}\right)^{3\epsilon} \left(-\frac{\gamma_{\text{cusp}}^{(3)}}{36\epsilon^2} - \frac{\mathcal{G}_0^{(3)}}{6\epsilon}\right) + \mathcal{O}(\epsilon^0), \end{aligned} \quad (\text{F.10})$$

where the three-loop cusp and collinear anomalous dimensions are given in (F.2). Using the known one- and two-loop form factors, and together with the counter term up to three-loop,

we can predict the three-loop bare Konishi form factor as:

$$f_{\mathcal{K},\text{B}}^{(3)} = \left(\frac{\mu^2}{-q^2}\right)^{3\epsilon} \left[-\frac{4}{3\epsilon^6} - \frac{12}{\epsilon^5} - \frac{64}{\epsilon^4} - \frac{284 - 2\pi^2 - \frac{22\zeta_3}{3}}{\epsilon^3} - \frac{1180 - \frac{65\pi^2}{3} - 78\zeta_3 - \frac{247\pi^4}{3240}}{\epsilon^2} - \frac{4744 - 141\pi^2 - 554\zeta_3 - \frac{51\pi^4}{40} + \frac{85\pi^2\zeta_3}{54} + \frac{878\zeta_5}{15}}{\epsilon} \right] + \mathcal{O}(\epsilon^0). \quad (\text{F.11})$$

This should be compared with a direct computation.

G Renormalization-scheme transformations

In this appendix, we review transformations between different mass-independent renormalization schemes and derive the behavior of the cross section (5) under such transformations.

A renormalization scheme specifies a regularization procedure for the UV divergences encountered in perturbation theory beyond tree-level and a prescription for the subtraction of these divergences into renormalized fields, coupling constants and composite operators. The subtraction prescription specifies how the UV divergences are removed from the perturbation series. In particular, it has to be indicated which finite parts are absorbed together with the UV divergences into the counter terms or — equivalently — the renormalization constants determining the relations between bare and renormalized quantities.

A modified renormalization scheme, which contains a different prescription for subtracting the UV divergences from the perturbation series in g , can be described by applying the subtraction of the original scheme but to the perturbation series in a modified coupling constant $g_\varrho = g e^{\varrho\epsilon}$. Thereby, the parameter ϱ specifies the finite terms that are subtracted together with the UV divergences. Since the combination $g\mu^\epsilon$ of the coupling constant g and 't Hooft mass μ is the expansion parameter of the perturbation series, the change between schemes, i.e. between g and g_ϱ , can easily be implemented by changing μ . If we demand $g_\varrho\mu_\varrho^\epsilon = g\mu^\epsilon$, the transformation of the perturbation series to the scheme ϱ , but written in terms of the original coupling constant g , is given by replacing $\mu \rightarrow \mu_\varrho = \mu e^{-\varrho}$.

A widely used renormalization scheme is the dimensional reduction (DR) scheme, which combines regularization by dimensional reduction with a minimal subtraction of the divergences into counter terms or —equivalently— renormalization constants. Minimal subtraction means that no finite terms are subtracted. In the DR scheme, minimal subtraction is applied to the perturbation series in the coupling constant $\frac{\sqrt{\lambda}}{4\pi}$, $\lambda = g_{\text{YM}}^2 N_c$.

In this paper, we work in the modified dimensional reduction ($\overline{\text{DR}}$) scheme. It employs dimensional reduction as regularization procedure, but the subtraction is non-minimal in terms of the coupling constant $\frac{\sqrt{\lambda}}{4\pi}$, $\lambda = g_{\text{YM}}^2 N_c$. It is, however, minimal in terms of the coupling constant g defined in (1.5). Hence, the subtraction procedures of the DR and $\overline{\text{DR}}$ scheme are related in the same way as those of the famous MS and $\overline{\text{MS}}$ schemes defined in [85] and [65], respectively, that employ dimensional regularization as regularization procedure. The expressions in this scheme are obtained from the ones in the DR scheme by replacing $\mu \rightarrow \mu e^{-\varrho}$, where $\varrho = \frac{1}{2}(\log 4\pi - \gamma_E)$.

Consider the renormalized cross section σ_{R} which when inserting (2.1) into (2.8) is given as a product of the squared operator renormalization constant $\mathcal{Z}_{\mathcal{O}}$ introduced in

(2.1) and the bare cross section σ_B

$$\sigma_R = \mathcal{Z}_O(g, \epsilon)^2 \sigma_B . \quad (\text{G.1})$$

The logarithm of the ratio of the bare and the tree-level cross section $\sigma^{(0)}$ then has the following expansion up to two-loop order

$$\log \frac{\sigma_B}{\sigma^{(0)}} = g^2 \left(\frac{\mu^2}{q^2} \right)^\epsilon \left(-\frac{\gamma_O^{(1)}}{\epsilon} + s_0^{(1)} \right) + g^4 \left(\frac{\mu^2}{q^2} \right)^{2\epsilon} \left(-\frac{\gamma_O^{(2)}}{2\epsilon} + s_0^{(2)} - \frac{(s_0^{(1)})^2}{2} \right) + \mathcal{O}(g^6, \epsilon) . \quad (\text{G.2})$$

In the $\overline{\text{DR}}$ scheme where the coupling constant is (1.5), the subtraction of only the $\frac{1}{\epsilon}$ -poles into \mathcal{Z}_O then shows that the finite terms $s_0^{(\ell)}$ become the coefficients of the perturbative expansion of the ratio of the renormalized and the tree-level cross section

$$\frac{\sigma_R}{\sigma^{(0)}} = \left(\frac{q^2}{4\mu^2} \right)^{\gamma_O} \left[1 + g^2 (s_0^{(1)} + s_1^{(1)} \epsilon) + g^4 (s_0^{(2)} + \mathcal{O}(\epsilon)) + \mathcal{O}(g^6, \epsilon^2) \right] . \quad (\text{G.3})$$

The condition $g_\rho \mu_\rho^\epsilon = g \mu^\epsilon$ implies that the expression (G.2) is the same in all schemes. However, only the subtraction prescription of the $\overline{\text{DR}}$ scheme leads to the above expression for the renormalized cross section.

The renormalization constant $\mathcal{Z}_O(g, \epsilon)$ of the $\overline{\text{DR}}$ scheme obtained by performing minimal subtraction at the coupling constant g can be expressed as the renormalization constant $\mathcal{Z}_{O,\rho} = \mathcal{Z}_O(g_\rho, \epsilon)$ in the scheme ρ obtained by performing minimal subtraction at the coupling constant g_ρ times a factor without poles in ϵ . Hence, the difference of the logarithms of these constants is finite and given by

$$\log \mathcal{Z}_O(g, \epsilon) = \log \mathcal{Z}_{O,\rho} + g_\rho^2 \Delta \mathcal{Z}_O^{(1)} + g_\rho^4 \Delta \mathcal{Z}_O^{(2)} + \mathcal{O}(g_\rho^6) , \quad (\text{G.4})$$

where

$$\begin{aligned} \Delta \mathcal{Z}_O^{(1)} &= \left(\left(\frac{g}{g_\rho} \right)^2 - 1 \right) \mathcal{Z}_O^{(1)} = -\gamma_O^{(1)} \rho (1 - \rho \epsilon) + \mathcal{O}(\epsilon^2) , \\ \Delta \mathcal{Z}_O^{(2)} &= \left(\left(\frac{g}{g_\rho} \right)^4 - 1 \right) \mathcal{Z}_O^{(2)} - \frac{(\Delta \mathcal{Z}_O^{(1)})^2}{2} - \mathcal{Z}_O^{(1)} \Delta \mathcal{Z}_O^{(1)} = -\gamma_O^{(2)} \rho + \mathcal{O}(\epsilon) . \end{aligned} \quad (\text{G.5})$$

We have used the expansion given in (2.3). In the expression $\Delta \mathcal{Z}_O^{(1)}$ we have kept the term linear in ϵ , since it leads to a finite term in ϵ in the expression $\Delta \mathcal{Z}_O^{(2)}$, when it is multiplied by $\mathcal{Z}_O^{(1)}$.

Adding (G.2) and (G.5) leads to the following relation of the renormalized cross sections in both schemes

$$\log \frac{\sigma_{R,\rho}}{\sigma^{(0)}} = \log \frac{\sigma_R}{\sigma^{(0)}} - 2(g^2 \Delta \mathcal{Z}_O^{(1)} + g^4 \Delta \mathcal{Z}_O^{(2)}) = \log \frac{\sigma_R}{\sigma^{(0)}} + 2\gamma_O \rho + \mathcal{O}(g^6, \epsilon) + \mathcal{O}(g^6, \epsilon) , \quad (\text{G.6})$$

where we have inserted $g_\rho = g e^{\rho \epsilon}$ and neglected terms that vanish when $\epsilon \rightarrow 0$. This relation can be interpreted in two ways, as follows.

First, one can insert the expansion (G.3) for σ_R and the same expression for $\sigma_{R,\rho}$ but with μ replaced by μ_ρ . Then, one obtains the relation $\mu_\rho = \mu e^{-\rho}$ mentioned already at

the beginning of this appendix. This shows that a scheme change can be performed by changing μ .

Second, one can insert the expansion (G.3) for σ_R and a similar expression for $\sigma_{R,\varrho}$ but with the finite expansion coefficients $s_0^{(\ell)}$ replaced by $s_{0,\varrho}^{(\ell)}$. Then, one obtains the behavior of the finite terms under a scheme change, given by the relations

$$s_{0,\varrho}^{(\ell)} = s_0^{(\ell)} + 2\gamma_{\mathcal{O}}^{(\ell)} \varrho . \quad (\text{G.7})$$

H Feynman diagrams

In this appendix, we compute the unrenormalized form factors of section 3 to two-loop order via Feynman diagrams. See e.g. [92] for the Feynman rules of the $\mathcal{N} = 4$ SYM theory in our conventions. In particular, we demonstrate how the analysis of section 4 works for the concrete diagrams and that we did not miss any rational terms in section 3.

One-loop self energies

For the calculation of the unrenormalized two-loop form factors, we need the one-loop self-energies of the gauge and scalar fields. They occur as subdiagrams in certain two-loop diagrams.

The one-loop self-energy of the gauge field is determined from diagrams in which the scalar fields, the fermion fields, the gauge field itself or the ghost field propagates in the loop. They evaluate to

$$\begin{aligned} \text{---} \circ \text{---} &= \frac{g^2}{2} N_\phi \delta^{ab} I_{s\mu\nu} , & \text{---} \text{---} \circ \text{---} &= g^2 N_\psi \delta^{ab} I_{f\mu\nu} , \\ \text{---} \text{---} \text{---} &= \frac{g^2}{2} \delta^{ab} (D I_{s\mu\nu} + I_{\text{ph}\mu\nu} + 2I_{\text{gh}\mu\nu}) , & \text{---} \text{---} \text{---} &= -g^2 \delta^{ab} I_{\text{gh}\mu\nu} , \end{aligned} \quad (\text{H.1})$$

where g is the coupling in the $\overline{\text{DR}}$ scheme defined in (1.5), and besides the number of scalar flavors $N_\phi = 6 + 2\epsilon$ we have also introduced the number of fermion flavors $N_\psi = 4$ of the $\mathcal{N} = 4$ SYM theory. Moreover, we have split the contribution from the gauge loop into the tensor integrals $I_{s\mu\nu}$, $I_{\text{gh}\mu\nu}$ occurring in case of the scalar- and ghost-loop contribution, respectively, and into $I_{\text{ph}\mu\nu}$, which is associated with the remaining physical degrees of freedom of the gauge-field polarizations. The occurring integrals are expressed in terms of the simple bubble integral in the first line of (B.6) as

$$\begin{aligned}
I_{s\mu\nu} &= (e^{\gamma_E} \mu^2)^\epsilon \int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{(q-2l)_\mu (q-2l)_\nu}{l^2 (q-l)^2} = \frac{1}{D-1} (q_\mu q_\nu - q^2 g_{\mu\nu}) = \text{Diagram} , \\
I_{f\mu\nu} &= (e^{\gamma_E} \mu^2)^\epsilon \int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{\text{tr} \tilde{\sigma}_\mu (q-l) \tilde{\sigma}_\nu l}{l^2 (q-l)^2} = \frac{D-2}{D-1} (q_\mu q_\nu - q^2 g_{\mu\nu}) = \text{Diagram} , \\
I_{\text{ph}\mu\nu} &= (e^{\gamma_E} \mu^2)^\epsilon \int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{1}{l^2 (q-l)^2} (-6q_\mu q_\nu + 4q_\mu (q-l)_\nu + 4(q-l)_\mu q_\nu - 8(q-l)_\mu (q-l)_\nu) \\
&\quad + (5q^2 - 2q \cdot (q-l) + 2(q-l)^2) g_{\mu\nu} \\
&= -\frac{4D-2}{D-1} (q_\mu q_\nu - q^2 g_{\mu\nu}) = \text{Diagram} , \\
I_{\text{gh}\mu\nu} &= (e^{\gamma_E} \mu^2)^\epsilon \int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{(l-q)_\mu l_\nu}{l^2 (q-l)^2} = \frac{1}{4(D-1)} (-(D-2)q_\mu q_\nu - q^2 g_{\mu\nu}) = \text{Diagram} .
\end{aligned} \tag{H.2}$$

Inserting the results for the tensor integrals into (H.1) and summing all contributions, we obtain

$$\begin{aligned}
\text{Diagram} &= \frac{g^2}{2} \delta^{ab} ((N_\phi + D) I_{s\mu\nu} + 2N_\psi I_{f\mu\nu} + I_{\text{ph}\mu\nu}) , \\
&= \frac{g^2}{2} \delta^{ab} \frac{N_\phi + D + 2(D-2)N_\psi - (4D-2)}{D-1} (q_\mu q_\nu - q^2 g_{\mu\nu}) = \text{Diagram} \tag{H.3} \\
&= 2g^2 \delta^{ab} (q_\mu q_\nu - q^2 g_{\mu\nu}) = \text{Diagram} .
\end{aligned}$$

The first line shows that our decomposition of the gauge loop contribution in (H.1) is advantageous: D and N_ϕ only appear in the combination $D + N_\phi$ which is insensitive to the simultaneous continuation of D and N_ϕ as prescribed by the $\overline{\text{DR}}$ scheme, cf. the discussion in section 4. We note that when inserting the appropriate numbers flavors in the second line, the dependence on D originating from the tensor integrals is also canceled.

The remaining one-loop self energies for the scalar and fermion fields read

$$\begin{aligned}
\text{Diagram} &= -2g^2 \delta^{ab} \delta_I^J q^2 = \text{Diagram} , \\
\text{Diagram} &= -4g^2 \delta^{ab} \delta_I^J q_{\dot{\alpha}}^\alpha = \text{Diagram} .
\end{aligned} \tag{H.4}$$

One-loop form factors

In the Feynman diagram approach, the one-loop form factors for the BPS operator (1.2) and the Konishi operator (1.4) are obtained from the two diagrams given in table 1. Completing the numerator of the second integral in table 1 to squared momenta occurring in the

diagram	$g^2(A\mathbb{1} + B\mathbb{T})$		$f(l)$	
	A	B		
	1	-1	$(l + p_2)^2$	} $f(l)$
	1	0	$(l + 2q - p_2) \cdot (l - p_2)$	

Table 1: Diagrams for the unrenormalized one-loop form factors. The prefactors $g^2(A\mathbb{1} + B\mathbb{T})$ of each diagram consist of the identity and trace operator in flavor space $\mathbb{1}$ and \mathbb{T} , respectively. For the BPS operator (1.2) and for the Konishi operator (1.4) the prefactors reduce to g^2A and $g^2(A + BN_\phi)$, respectively. They multiply the triangle integral which contains the numerator factor $f(l)$.

denominator, it can be transformed to the expression

$$(l + 2q - p_2) \cdot (l - p_2) \triangle = - \text{bubble} + \text{triangle}_1 + \text{triangle}_2 + (p_1^2 + p_2^2 - 2q^2) \triangle. \quad (\text{H.5})$$

Only the first three integrals are UV divergent. Moreover, they develop IR divergences if the corresponding external momentum square q^2 , p_1^2 or p_2^2 vanishes. In this case, the respective integral vanishes identically in dimensional reduction since its IR pole and its UV pole cancel. The fourth integral is UV finite, but it becomes IR divergent if at least one of the three momentum squares vanishes. In case that p_1^2, p_2^2 are not zero, also the self-energy corrections of the scalar fields contribute to the form factor. Using the expression for the one-loop scalar self energy given in (H.4), the respective contribution can be written as

$$\frac{1}{2} \left[\text{triangle}_1^{\text{self}} + \text{triangle}_2^{\text{self}} \right] = -g^2 \left[\text{triangle}_1 + \text{triangle}_2 \right], \quad (\text{H.6})$$

where the factor $\frac{1}{2}$ originates from the fact that the squareroot of the renormalization constant determined from the self energy contribution renormalizes the corresponding elementary field. When added to the sum of the two diagrams given in table 1, this contribution exactly cancels the second and third term in the expansion of the second integral given in (H.5), irrespective of the vanishing or non-vanishing of p_1^2, p_2^2 . In the case of the BPS operator, where both diagrams of table 1 only contribute with the coefficient A , the remaining UV divergence contained in the bubble integral cancels among the two diagrams given in table 1. Hence, in the BPS case, there is only a contribution from the triangle integral of (H.5). In the case of the Konishi operator, the contributions of the bubble integral do not cancel for the flavor-trace contribution, which comes with the coefficient B . The one-loop

form factors for the BPS operator and the Konishi operator hence read

$$\begin{aligned}
 f_{\text{BPS},2}^{(1)} &= (p_1^2 + p_2^2 - 2q^2) \text{ (triangle diagram) } , \\
 f_{\mathcal{K},(\phi,\phi)}^{(1)} &= -N_\phi \text{ (circle diagram) } + (p_1^2 + p_2^2 - 2q^2) \text{ (triangle diagram) } .
 \end{aligned}
 \tag{H.7}$$

We have calculated the above form factors for generic off-shell momenta $p_1^2 \neq 0$ and $p_2^2 \neq 0$. Hence, they are generalizations of the respective expressions with $p_1^2 = p_2^2$, given for the BPS operator in (3.4) and for the operator \mathcal{K}_6 in (3.16) into which the factor r_ϕ has to be introduced as prescribed in (4.2) in order to obtain the Konishi form factor.

The difference of the two form factors defined in (4.3) is free of any contribution from the triangle integral, and it is in particular independent of p_1^2 and p_2^2 . This explicitly confirms that the IR divergence is universal, i.e. the same for the BPS and the Konishi operator. Moreover, the UV divergence of the Konishi operator can be extracted from the final ϵ -expansions of the Konishi and the BPS form factor given in (4.5) in the on-shell case $p_1^2 = p_2^2 = 0$ where the $\frac{1}{\epsilon}$ -poles originate from both, the UV and the IR divergences.

Two-loop form factors

The one-particle-irreducible (1PI) diagrams for the two-loop form factors of the BPS operator (1.2) and the Konishi operator (1.4) are displayed in table 2.

diagram	$\lambda^2(A\mathbf{1} + B\mathbf{T})$		$f(k, l)$
	A	B	
	1	$N_\phi - 2$	$(k - l)^2(l + p_2)^2$
	0	d	$(k - l)^2(l + p_2)^2$
	1	-1	$(l + p_2)^2(k + l) \cdot (k + l + 2q)$
	0	0	$(l + q)^2(l - 2k) \cdot (l + 2p_2)$
	1	-1	$(k - l)^2(l - p_2) \cdot (l + 2q - p_2)$
	0	1	$(k - l)^2(l + 2p_2) \cdot (l + 2p_2 - q)$
	0	-1	$(l + p_2)^2(l - 2k) \cdot (l - 2k - q)$
	-4	4	$(l + q)^2(l + p_2)^2$
	$\frac{3}{2}(N_\phi - 3)$	3	$l^2(l + q)^2$
	$\frac{3}{2}d$	0	$l^2(l + q)^2$
	$-\frac{3}{2}$	0	$(l + q)^2(k + l) \cdot (l - p_2)$
	0	-8	$2(l - k) \cdot l(l + p_2) \cdot (l + q) - 2(l - k) \cdot (l + p_2)l \cdot (l + q) + 2(l - k) \cdot (l + q)l \cdot (l + p_2)$
	1	0	$(k + l) \cdot (k + l + 2q)(l - p_2) \cdot (l + 2q - p_2)$
	0	1	$(s - 2k - q) \cdot (l + p_2 - q)(l - 2k) \cdot (l + 2p_2)$

I_{tb}

Table 2 – continued on next page

Continued from previous page

diagram	$\lambda^2(A\mathbb{1} + B\mathbb{T})$		$f(k, l)$	
	A	B		
	$-\frac{3}{2}$	0	$(l + p_2)^2(k + l) \cdot (k + 2q - p_2)$	} I_{bt}
	$\frac{3}{2}$	0	$(k - l)^2(l - p_2) \cdot (k + 2q - p_2)$	
	$\frac{1}{2}$	0	$(l - 2k) \cdot (l + 2p_2)(k - 2l - p_2) \cdot (k + 2q - p_2)$	
	$-\frac{1}{2}$	0	$(2k - l + p_2) \cdot (l - p_2)(k + l) \cdot (k + 2q - p_2)$ $- (k + l + 2p_2) \cdot (k + l)(k + 2q - p_2) \cdot (l - p_2)$ $+ (2l - k + p_2) \cdot (k + 2q - p_2)(k + l) \cdot (l - p_2)$	
	-4	0	$2l \cdot (-l - p_2)(k + 2q - p_2) \cdot (k - l)$ $- 2l \cdot (k + 2q - p_2)(-l - p_2) \cdot (k - l)$ $+ 2l \cdot (k - l)(k + 2q - p_2) \cdot (-l - p_2)$	
	-2	0	$l^2(k - p_2) \cdot (k + 2q - p_2)$ $-\frac{l^2}{(k + p_2)^2}(k + p_2) \cdot (k - p_2)(k + p_2) \cdot (k + 2q - p_2)$	
	-2	0	$(l + p_2)^2(k - p_2) \cdot (k + 2q - p_2)$	
	-4	2	$2l \cdot (l + p_2)(k + q - p_2) \cdot k - 2l \cdot (k + q - p_2)k \cdot (l + p_2)$ $+ 2l \cdot k(l + p_2) \cdot (k + q - p_2)$	} I_{bb}
	$\frac{1}{2}$	0	$(2k + l + 2q - p_2) \cdot (l - p_2)(k + 2l) \cdot (k + 2q - 2p_2)$	

Table 2: Diagrams for the unrenormalized two-loop form factors. The prefactors $g^2(A\mathbb{1} + B\mathbb{T})$ of each diagram consist of the identity and the trace operator in flavor space, $\mathbb{1}$ and \mathbb{T} , respectively. For the BPS operator (1.2) and for the Konishi operator (1.4), the prefactors reduce to g^4A and $g^4(A + BN_\phi)$, respectively. They multiply the corresponding integral t_b , t_{bt} or t_{bb} , which are given in (H.8) and contain the numerator factors $f(k, l)$. For all diagrams which are not symmetric under a reflection at the horizontal axis, also the corresponding reflected version has to be considered.

The occurring integrals are given by

$$\begin{aligned}
I_{\text{tb}} = f(k, l) &= \text{[diagram]} = (e^{\gamma_E} \mu^2)^{2\epsilon} \int \frac{d^D k}{i\pi^{\frac{D}{2}}} \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{f(k, l)}{k^2(k+q)^2(k-l)^2 l^2(l+q)^2(l+p_1)^2}, \\
I_{\text{bt}} = f(k, l) &= \text{[diagram]} = (e^{\gamma_E} \mu^2)^{2\epsilon} \int \frac{d^D k}{i\pi^{\frac{D}{2}}} \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{f(k, l)}{k^2(k+q)^2(k+p_1)^2(k-l)^2 l^2(l+p_1)^2}, \\
I_{\text{bb}} = f(k, l) &= \text{[diagram]} = (e^{\gamma_E} \mu^2)^{2\epsilon} \int \frac{d^D k}{i\pi^{\frac{D}{2}}} \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{f(k, l)}{(k+l)^2(k+l+q)^2 k^2(k+p_1)^2 l^2(l+p_2)^2}.
\end{aligned}
\tag{H.8}$$

For $p_1^2 \neq 0$, $p_2^2 \neq 0$, contributions from diagrams involving the two-loop self-energy of the scalar fields have to be considered in addition to the 1PI diagrams shown in table 2. Also, the second diagram coming with the integral I_{bt} yields a non-vanishing contribution, while it vanishes otherwise. All graphs are then IR finite, and the UV-divergence can easily be extracted by setting e.g. one external momentum to zero and the other one to q^2 such that no new IR divergences are accidentally created. Moreover, since all integrals are superficially logarithmically divergent, one can neglect external momenta in the numerators as convenient for maximal simplifications. We have checked that this produces the known result for the two-loop overall UV-divergence of the Konishi operator when subdivergences are subtracted by considering also the corresponding counter-term diagrams. This also produces a vanishing result for the BPS operator.

For $p_1^2 = p_2^2 = 0$, where the 1PI diagrams shown in table 2 are the only contributions to the form factors, it is advantageous to express the scalar products in the numerators in terms of squares of momenta found in the denominator from the propagators. Then, one can use IBP reduction as e.g. implemented in `LiteRed` [87] in order to further reduce the integrals to a set of master integrals. The results exactly match the ones given in (4.6). This confirms the absence of further rational terms that might not have been detected in the unitarity-based approach.

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