Convergence of adaptive finite element methods for a nonconvex double-well minimisation problem

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The relaxation of the two-well model problem in the analysis of solid-solid phase transitions leads to a variational problem with a quasiconvex energy density which fails to be convex if the phases are not compatible. This paper presents an adaptive algorithm for the computation of minimizers for this functional in finite element spaces with Courant elements and with successive loops of the form SOLVE, ESTIMATE, MARK, and REFINE. Convergence of the total energy of the approximating deformations and strong convergence of all except one component of the corresponding deformation gradients is established. The proof relies on the decomposition of the energy density into a degenerate convex part and a null-Lagrangian, some convexity control of the degenerate convex part, and some refined estimator reduction compatible with the translation energy.

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1 Introduction

Mathematical models in the framework of nonlinear elasticity for phase transformations in solids lead to variational problems for which the existence of minimizers cannot be obtained by the direct method in the calculus of variations, see [1, 10, 2] and the literature quoted therein. In particular, infimizing sequences tend to develop oscillations on finer and finer scales and converge only weakly but not strongly. Typically the weak limit is not a minimizer of the problem and has to be replaced by a generalized minimizer, the gradient Young measure associated to the sequence of deformation gradients [16, 24].

The numerical simulation of problems of this kind is a challenging task and a direct minimization of the nonconvex energy in a finite element space leads to strongly mesh dependent effects [21, 9]. An alternative approach is based on a minimization of the

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associated relaxed variational problem [13]. It is obtained by replacing the energy density W by its quasiconvex relaxation W^{qc}, that is, one minimizes

$$I^{\mathrm{qc}}(v) := \int_\Omega W^{\mathrm{qc}}(\varepsilon(v))\,\mathrm{d}x - \int_\Omega f\cdot v\,\mathrm{d}x \quad \text{among all } v \text{ in } \mathcal{A} := u_D + H_0^1(\Omega;\mathbb{R}^2)\,. \tag{1.1}$$

Here the function $u_D \in H^1(\Omega;\mathbb{R}^2)$ defines the Dirichlet boundary conditions for the problem and $\varepsilon(v)$ denotes the symmetric part of the deformation gradient Du. In Section 5 we comment on the analogues of our results in the case that the energy density depends on the full gradient and not only on its symmetric part. The original and the relaxed variational problem are closely related. Since the energy density in the relaxed minimization problem satisfies the necessary convexity conditions in the vector-valued calculus of variations, it has a minimizer. Moreover, any minimizer u characterizes a macroscopic deformation of the original problem in the sense that there exists a sequence $(u_j)_{j\in\mathbb{N}}$ which infimizes the energy of the original variational problem and converges weakly to u. If this convergence is also strong in $H^1(\Omega;\mathbb{R}^2)$, then the minimum of the energy is attained and u is a classical minimizer of the original problem.

This approach is very appealing, in particular if an explicit formula for W^{qc} is known. In this case one can construct for a given deformation gradient F a corresponding gradient Young measure ν with center of mass F which realizes the relaxed energy, $W^{\mathrm{qc}}(F) = \langle W, \nu\rangle$, and provides at the same time a representation for the stress variable $\sigma(F) = DW^{\mathrm{qc}}(F) = \langle DW, \nu\rangle$; see [3] and [6] for a discussion of the regularity of the stress variable. In this way one obtains the associated stresses which are of fundamental importance in engineering applications. A successful example of this approach in the numerical analysis of a relaxed problem can be found in [7].

From the point of view of numerical analysis, one striking advantage of the relaxed minimization problem is that the macroscopic deformation u can, in principle, be computed with a strongly convergent sequence of minimizers in suitable finite element spaces. The reliability-efficiency gap [5] does not prevent the convergence proof of the associated stresses for a large class of variational problems with energy densities that fail to be strictly convex [4].

In this paper, we carry out the convergence analysis for the relaxation of the classical model energy, which we also refer to as two-well energy with linear kinematics,

$$W(E) = \min\Big\{\frac{1}{2}\,\langle \mathbb{C}(E-A_1), E-A_1\rangle + w_1\,,\frac{1}{2}\,\langle \mathbb{C}(E-A_2), E-A_2\rangle + w_2\Big\} \tag{1.2}$$

in a two-dimensional setting for which the relaxation was obtained in [19, 20, 25]; see (3.2) below for the precise formula with given symmetric matrices A_1 and A_2. It turns out that the quasiconvex relaxation is in fact the convex relaxation if and only if the two preferred strains A_1 and A_2 are compatible, i.e., if $A_1 - A_2 = c\otimes d + d\otimes c$ for two vectors c, $d \in \mathbb{R}^2$ [19, Lemma 4.1] for necessary and sufficient conditions for compatibility. The case of compatible wells was analyzed in [4] and therefore we focus on the incompatible case in this paper. Moreover, we assume that the matrix $A_1 - A_2$ is not proportional to the identity matrix since in this case the uniqueness of minimizers may be lost [27, Remark 2.2]. Hence we assume that the eigenvalues η_1 and η_2 of the matrix $A_1 - A_2$ satisfy

$$0 < \eta_1 < \eta_2 \,. \tag{1.3}$$

We refer to the problem as nonconvex since the relaxation is not convex but quasiconvex. See Remark 3.3 for a short discussion of uniqueness of minimizers and counterexamples to uniqueness in the case $\eta_1 = \eta_2$.

Our first main result shows strong convergence for three out of four components in the deformation gradient. The fact that the last component cannot be controlled is related to the degenerate convexity of the relaxed energy. We refer to Section 2 for the definition of the notation used below.

Theorem 1.1 *Suppose that W is given by (1.2) with assumptions (1.3), that $u \in \mathcal{A}$ is a minimizer of I^{qc}, and that u_h is a minimizer of*

$$I^{\mathrm{qc}}(v_h) = \int_\Omega W^{\mathrm{qc}}(\varepsilon(v_h))\,\mathrm{d}x - \int_\Omega f \cdot v_h \,\mathrm{d}x$$

in a finite element space $u_D + \mathbb{V}_{h,0}$ with $u_D \in \mathbb{V}_h$ and Courant finite element method with respect to some shape-regular triangulation $\mathcal{T}_h$. Then there exist constants C_1 and C_2 which depend on the triangulation only through the constant κ^* defined in (2.1) such that, in a suitable coordinate system with $A_1 - A_2 = \mathrm{diag}(\eta_1, \eta_2)$,

$$\begin{aligned} ||\partial_1(u-u_h)_1||_{H^{-1}(\Omega)} + \sum_{j,k=1,2;(j,k)\neq(1,1)} ||\partial_k(u-u_h)_j||_{L^2(\Omega)} \\ \leq C_1 \min_{v_h \in u_D + \mathbb{V}_{h,0}} \left(I^{\mathrm{qc}}(v_h) - I^{\mathrm{qc}}(u)\right). \end{aligned}$$

If $u \in H^2(\Omega;\mathbb{R}^2)$ then

$$\min_{v_h \in u_D + \mathbb{V}_{h,0}} \left(I^{\mathrm{qc}}(v_h) - I^{\mathrm{qc}}(u)\right) \leq C_2 h ||D^2 u||_{L^2(\Omega)}.$$

Our second main result concerns the design of an adaptive scheme of Section 4.2 which allows the computation of a sequence of triangulations $\mathcal{T}_\ell$ and minimizers $u_\ell \in u_D + \mathbb{V}_0^{(\ell)}$ and so generalises [4] to some nonconvex minimisation problem.

Theorem 1.2 *Suppose that the assumptions in Theorem 1.1 hold. Then the sequence $(u_\ell)_{\ell\in\mathbb{N}}$ with $u_\ell \in u_D + \mathbb{V}_0^{(\ell)}$, $\ell \in \mathbb{N}_0$, computed by the adaptive scheme converges with respect to the weak topology of $H^1(\Omega;\mathbb{R}^2)$ to the unique minimizer u of the variational integral I^{qc} in the class of admissible functions $\mathcal{A}$. Moreover, the energies $I^{\mathrm{qc}}(u_\ell)$ converge, i.e.,*

$$\lim_{\ell\to\infty} I^{\mathrm{qc}}(u_\ell) = I^{\mathrm{qc}}(u) = \min_{v \in u_D + H_0^1(\Omega;\mathbb{R}^2)} I^{\mathrm{qc}}(v)\,,$$

and, in a suitable coordinate system with $A_1 - A_2 = \mathrm{diag}(\eta_1, \eta_2)$, all components of the deformation gradient except the $(1,1)$-component converge strongly $L^2(\Omega)$, i.e.,

$$||\partial_1(u-u_\ell)_1||_{H^{-1}(\Omega)} + \sum_{j,k=1,2;(j,k)\neq(1,1)} ||\partial_k(u-u_\ell)_j||_{L^2(\Omega)} \to 0 \text{ as } \ell \to \infty.$$

One key ingredient in the proof is the observation [19] that the relaxation of the energy (1.2) can be written as the sum of a convex and a polyaffine function which in the case at hand is a multiple of the determinant. This special structure has, e.g., been used in [27, 28] to obtain uniqueness results and regularity of phase boundaries while our approach is in the spirit of the translation method which has been widely used in homogenization theory to separate nonconvex terms with special structure, usually polyaffine functions, from others terms, see the discussion in Section 5 in [19] for more details and references. The crucial observation in this paper is that the convex function Φ allows a convexity control in the sense of [7, 6, 17, 4], i.e., some suitable constant λ_1 satisfies

$$\lambda_1|D\Phi(A) - D\Phi(B)|^2 \leq \Phi(A) - \Phi(B) - \langle D\Phi(B), A - B\rangle \text{ for all } A,\, B \in \mathbb{M}^{2\times 2}. \quad (1.4)$$

The structure of the remaining parts of this paper is as follows. We introduce in Section 2 standard notation including our assumptions on shape regular triangulations. Section 3 reviews the necessary results on the relaxation of the two-well energy which are used in our proofs. The first key feature is the decomposition of the relaxed energy density in the form $W^{\mathrm{qc}} = \Phi + \gamma \det$. The convexity control (1.4) of the translated energy Φ is presented in Section 4.1. Section 4.2 states the adaptive algorithm, presents the error estimator, and introduces the refinement scheme. The proofs of Theorem 1.1 and Theorem 1.2 are presented in Section 4.3. The second key observation is a refined error estimator reduction introduced in the proof of Theorem 1.2 which allows one to relate errors in the approximation of the pseudostresses $D\Phi$ and the true stress DW^{qc}. The concluding Section 5 presents the analogous results in the case that the energy depends on the full deformation gradient.

The results presented in this paper are the first affirmative convergence results for a non-convex minimization problem in the spirit of [15, 8, 29, 4].

2 Notation

Throughout this paper, $\Omega \subset \mathbb{R}^2$ denotes an open and bounded domain with polygonal boundary and $u_D \in H^1(\Omega; \mathbb{R}^2)$ is piecewise affine and belongs to all finite element spaces $\mathbb{V}^{(\ell)}$ and $\mathbb{V}_h$.

Standard notation for respective Lebesgue and Sobolev spaces applies to the norms like $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{p;\Omega} = \|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}(\Omega)} = \|\cdot\|_{k,p;\Omega} = \|\cdot\|_{k,p}$. The domain is neglected if it is clear from the context. The space of real 2×2 matrices is denoted by $\mathbb{M}^{2\times 2}$ and the symmetric part of a given matrix $F \in \mathbb{M}^{2\times 2}$ by $E = \widehat{F} = (F + F^T)/2 \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$. The inner product between two vectors a and b reads $a \cdot b$ while that of the two matrices A and B reads $A : B$; the symbol $\langle \cdot, \cdot \rangle$ abbreviates the inner product in any dimension.

Generic constants may change from line to line. Unless indicated otherwise, all constants are independent of the underlying triangulation.

A triangulation $\mathcal{T}$ of a domain $\Omega \subset \mathbb{R}^2$ is a finite set of closed triangles which partitions Ω in the sense that

$$\bigcup_{T\in\mathcal{T}} T = \overline{\Omega}\,.$$

Moreover, if $T_1, T_2 \in \mathcal{T}$, $T_1 \neq T_2$, are two triangles, then $\mathring{T}_1 \cap \mathring{T}_2 = \emptyset$ and if the intersection of two triangles $T_1, T_2 \in \mathcal{T}$, $T_1 \neq T_2$, is not empty, then it is either a common edge, called interior edge, or a common vertex, also called node. The set of all nodes (resp. edges) reads $\mathcal{N}$ (resp. $\mathcal{E}$) and the set of all interior nodes (resp. interior edges) by $\mathring{\mathcal{N}}$ (resp. $\mathring{\mathcal{E}}$). A family of triangulations $\mathcal{T}_\ell$, $\ell \in \mathbb{N}$, is said to be shape regular in the sense of [11] if there exists a universal constant κ^* with $0 < \kappa^* < 1/2$ which is independent of the level $\ell \in \mathbb{N}$ such that the area $|T|$ of each triangle $T \in \mathcal{T}_\ell$ satisfies a two-sided bound in terms of the diameter $h_T = \mathrm{diam}(T)$ in the sense of

$$\kappa^* \, h_T^2 \leq |T| \leq h_T^2/\kappa^* \,. \tag{2.1}$$

We write $\mathcal{T}_h$ if h_T is bounded by h for all $T \in \mathcal{T}_h$. Throughout this paper, we use Courant elements at each fixed refinement level $\ell \in \mathbb{N}_0$. Let $P_k(T)$ denote the set of all real-valued polynomials of total degree at most k on the triangle $T \in \mathcal{T}_\ell$ and let

$$P_k(\mathcal{T}_\ell) = \left\{ v_\ell \in L^2(\Omega) : \forall T \in \mathcal{T}_\ell,\, v_\ell|_T \in P_k(T) \right\}.$$

Finally we define the corresponding vector-valued functions $P_k(\mathcal{T}_\ell; \mathbb{R}^2) = P_k(\mathcal{T}_\ell) \times P_k(\mathcal{T}_\ell)$ and introduce the finite element spaces (plus the analogous definitions for triangulations $\mathcal{T}_h$)

$$\mathbb{V}(\mathcal{T}_\ell) = P_1(\mathcal{T}_\ell; \mathbb{R}^2) \cap H^1(\Omega; \mathbb{R}^2) \quad \text{and} \quad \mathbb{V}_0(\mathcal{T}_\ell) = P_1(\mathcal{T}_\ell; \mathbb{R}^2) \cap H^1_0(\Omega; \mathbb{R}^2)\,.$$

3 Review of fundamental properties of the relaxation of the double well problem

The starting point is the nonconvex energy density W for a two-dimensional model in linear elasticity with linear kinematics for a phase transforming material with two preferred elastic strains A_1 and $A_2 \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ and elasticity tensor $\mathbb{C}$ for which

$$W(E) := \min\left\{W_1(E),\, W_2(E)\right\} \quad \text{for all } E \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$$

with suitable constants $w_j \in \mathbb{R}$ and

$$W_j(E) := \frac{1}{2}\, \langle \mathbb{C}(E - A_j), E - A_j \rangle + w_j \quad \text{for } j = 1,\, 2\,.$$

The attention in this contribution lies on the classical case of an isotropic Hooke's law with bulk modulus $\kappa > 0$ and shear modulus $\mu > 0$, i.e.,

$$\mathbb{C}E = \kappa (\operatorname{tr} E)\, \mathbb{I} + 2\mu \big(E - \frac{1}{2}\, (\operatorname{tr} E)\, \mathbb{I}\big)\,.$$

See Section 5 for comments on the situation with nonlinear kinematics $W = W(F)$ instead of $W = W(E)$ in an isotropic model. Since A_1 and A_2 are symmetric matrices, we may relabel the matrices in such a way that the eigenvalues η_1 and η_2 of $A_1 - A_2$ satisfy $\eta_1 \geq |\eta_2|$ and after a suitable change of coordinates we may suppose that the eigenvectors are parallel to the coordinate axes, i.e., $A_1 - A_2 = \operatorname{diag}(\eta_1, \eta_2)$. It is well-established (see, e.g., Lemma 4.1 in [19]) that A_1 and A_2 are incompatible as linear elastic strains, if and only if $\eta_2 > 0$. The relaxed energy density $W^{\mathrm{qc}}(E)$ was computed by Kohn [19], Lurie and Cherkaev [20] and Pipkin [25]. As mentioned, e.g., in [19], Section 4, the relaxation is piecewise quadratic and globally C^1, and in the notation of this reference given by the expression below. In order to simplify the formulas, set [19]

$$\begin{aligned}
\mathcal{P}_1 &= \left\{E \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \colon W_1(E) - W_2(E) + \frac{g}{2} \le 0\right\}, \\
\mathcal{P}_2 &= \left\{E \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \colon W_1(E) - W_2(E) - \frac{g}{2} \ge 0\right\}, \\
\mathcal{P}_{\mathrm{rel}} &= \left\{E \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \colon \left|W_1(E) - W_2(E)\right| \le \frac{g}{2}\right\},
\end{aligned}$$

as well as, for $j = 1, 2$,

$$\gamma_j = (\kappa - \mu)\operatorname{tr}(A_1 - A_2) + 2\mu\eta_j\,, \qquad g = \frac{\gamma_1^2}{\kappa + \mu} = \frac{\gamma_1^2}{\mu(\nu + 2)}\,, \qquad \nu = \frac{\kappa - \mu}{\mu}\,. \tag{3.1}$$

With this notation the quasiconvex envelope of the two-well energy is given by

$$W^{\mathrm{qc}}(E) = \begin{cases} W_1(E) & \text{if } E \in \mathcal{P}_1\,, \\ W_2(E) & \text{if } E \in \mathcal{P}_2\,, \\ W_2(E) - \dfrac{1}{2g}\left(W_2(E) - W_1(E) + \dfrac{1}{2}\,g\right)^2 & \text{if } E \in \mathcal{P}_{\mathrm{rel}}\,. \end{cases} \tag{3.2}$$

For future reference we note that in the case $\eta_2 > 0$ of incompatible tensors,

$$-1 < \zeta := (\nu + 1) - (\nu + 2)\frac{\gamma_2}{\gamma_1} < 1. \tag{3.3}$$

Moreover, $\zeta = -1$ if and only if $\gamma_1 = \gamma_2$ and $\zeta = 1$ if and only if $\eta_2 = 0$. In order to verify the upper bound, one uses that for $\eta_2 = 0$ the expression simplifies to

$$\zeta = \frac{\kappa}{\mu} - \frac{\kappa + \mu}{\mu}\,\frac{(\kappa - \mu)\eta_1}{(\kappa - \mu)\eta_1 + 2\mu\eta_1} = 1$$

and that the derivative $\partial\zeta/\partial\eta_1$ is less than or equal to zero on $[0, \eta_2]$.

Following [27] we define

$$H(E) := \frac{1}{2}\,\langle \mathbb{C}E, E\rangle - \frac{1}{2g}\langle E, \mathbb{C}(A_1 - A_2)\rangle^2 \quad \text{ for } E \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}\,. \tag{3.4}$$

Note that H is the quadratic part of the energy in the relaxed phase where the relaxation does not coincide with one of the two functions W_1 and W_2. The relaxed energy is non-convex due to a term proportional to the determinant in the relaxed phase. The key observation is, that this energy is given by a nonnegative quadratic form after a suitable translation with a term proportional to the determinant.

The next lemma is a crucial ingredient in the proof of the convexity control.

Lemma 3.1 (see [27]) *Let $\gamma := \mu\big(\nu - (\nu+2)\frac{\gamma_2}{\gamma_1}\big)$ and $F \in \mathbb{M}^{2\times 2}$ with symmetric part $E := \widehat{F}$. If $E \in \mathcal{P}_{rel}$, then the quadratic part T of the translation of the relaxed energy $W^{\mathrm{qc}}(E) - \gamma \det F$ satisfies*

$$T(F) = H(E) - \gamma \det F = \frac{1}{2}\,\langle \mathbb{C}E, E\rangle - \frac{1}{2g}\,\langle \mathbb{C}E, A_1 - A_2\rangle^2 - \gamma \det F \geq 0\,. \tag{3.5}$$

Proof. We include a sketch of the proof for future reference since we will need the nonnegativity of certain terms. The explicit expression for H follows immediately from the definition of the relaxed energy in (3.2). Since $\mathbb{C}(A_1 - A_2) = \mathrm{diag}(\gamma_1, \gamma_2)$ we can evaluate the quadratic form H and find that

$$\begin{aligned} H(E) &= \frac{\mu}{2}\,(2+\nu)\big(1 - \frac{\gamma_2^2}{\gamma_1^2}\big)F_{22}^2 + \mu\big(\nu - (\nu+2)\frac{\gamma_2}{\gamma_1}\big)F_{11}F_{22} + 2\mu E_{12}^2 \\ &= \frac{\mu}{2}\,(2+\nu)\big(1 - \frac{\gamma_2^2}{\gamma_1^2}\big)F_{22}^2 + \widetilde{Q}(F) + \mu\big(\nu - (\nu+2)\frac{\gamma_2}{\gamma_1}\big)\det F \end{aligned} \tag{3.6}$$

with $c_0 = c_0(\gamma_1, \gamma_2) \geq 0$ and

$$\begin{aligned} \widetilde{Q}(F) &= \frac{\mu}{2}\,F_{12}^2 + \frac{\mu}{2}\,F_{21}^2 + \mu\big(\nu + 1 - (\nu+2)\frac{\gamma_2}{\gamma_1}\big)F_{12}F_{21} \\ &= \frac{\mu}{2}\begin{pmatrix} F_{12} \\ F_{21} \end{pmatrix}^T \begin{pmatrix} 1 & \nu + 1 - (\nu+2)\frac{\gamma_2}{\gamma_1} \\ \nu + 1 - (\nu+2)\frac{\gamma_2}{\gamma_1} & 1 \end{pmatrix} \begin{pmatrix} F_{12} \\ F_{21} \end{pmatrix} \\ &\geq c_0\big(F_{12}^2 + F_{21}^2\big). \end{aligned} \tag{3.7}$$

Indeed, up to a factor of $\mu^2/4$, the determinant of the matrix in the last formula equals

$$(\nu+2)\big(1 - \frac{\gamma_2}{\gamma_1}\big)\,\frac{(2+\nu)\gamma_2 - \nu\gamma_1}{\gamma_1} = 4\mu\eta_2(\nu+1)(\nu+2)\frac{(\gamma_1 - \gamma_2)^2}{\gamma_1^2}\,.$$

In fact, $c_0 > 0$ if $\eta_2 > 0$ and $\gamma_2 < \gamma_1$; the former inequality holds if the two linear strains are not compatible and the latter if $A_1 - A_2$ is not isotropic, that is, not proportional to the identity matrix. □

Under the foregoing assumptions, a minimizer of the relaxed functional exists in the class of admissible functions $\mathcal{A}$ and is unique. We include a short proof of the theorem for the convenience of the reader and in order to emphasize that existence and uniqueness follows in the finite element space $\mathbb{V}_h$ as well. Moreover, we will follow the same outline in the case of models with nonlinear kinematics.

Theorem 3.2 ([27], Theorem 2.1) *Suppose that W is given by (1.2) with assumptions (1.3) and that $u_D \in H^1(\Omega; \mathbb{R}^2)$ and $f \in L^2(\Omega)$. Then there exists a unique minimizer of the variational problem: Minimize I^{qc} with*

$$I^{\mathrm{qc}}(v) = \int_\Omega W^{\mathrm{qc}}(\varepsilon(v))\mathrm{d}x - \int_\Omega f \cdot v \mathrm{d}x \quad \textit{among all } v \in \mathcal{A}.$$

Moreover, any solution of the Euler-Lagrange equations coincides with the minimizer. Finally, if $u \in \mathcal{A}$ is a minimizer of I^{qc} and $v \in \mathcal{A}$, then the difference $e = u - v$ satisfies

$$\frac{\mu}{2} \int_\Omega \Big(\alpha e_{2,2}^2 + e_{1,2}^2 + e_{2,1}^2 + \beta e_{1,2} e_{2,1} \Big) \mathrm{d}x = \int_\Omega H(e(x))\, \mathrm{d}x \le I^{\mathrm{qc}}(v) - I^{\mathrm{qc}}(u)$$

with $\alpha := (2+\nu)\big(1 - \frac{\gamma_2^2}{\gamma_1^2}\big) > 0$ *and* $-2 < \beta := 2\zeta = 2\big((1+\nu) - (2+\nu)\frac{\gamma_2}{\gamma_1}\big) < 2.$

Note that $\beta = -2$ if $A_1 - A_2$ is isotropic and $\beta \ge 2$ if A_1 and A_2 are compatible as linear strains.

Proof. The existence of a minimizer follows from the direct method in the calculus of variations. To prove the remaining assertions, we follow the arguments in [27, Section 3]. Since the relaxed energy is globally C^1 (see Section 4 in [19]), any critical point u satisfies the Euler-Lagrange equations

$$\int_\Omega DW^{\mathrm{qc}}(\varepsilon(u))\colon \varepsilon(v)\mathrm{d}x - \int_\Omega f \cdot v \mathrm{d}x = 0 \qquad \text{for all } v \in H_0^1(\Omega;\mathbb{R}^2)\,. \tag{3.8}$$

For all $A, B \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$, the Taylor expansion about A implies

$$\begin{aligned} &W^{\mathrm{qc}}(B) - W^{\mathrm{qc}}(A) - DW^{\mathrm{qc}}(A)\colon (B-A) \\ &\quad = \frac{1}{2}\int_0^1 D^2 W^{\mathrm{qc}}(A + s(B-A))[B-A, B-A]\, \mathrm{d}s\,. \end{aligned} \tag{3.9}$$

Note that (3.4) implies

$$H(E) \le \frac{1}{2}\, D^2 W^{\mathrm{qc}}(C)[E,E] \qquad \text{for all } C, E \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}\,. \tag{3.10}$$

We set $A = \varepsilon(u)$ and $B = \varepsilon(v)$, and use this estimate with $C = A + s(B-A)$ and $E = A - B$ to obtain a lower bound for the right-hand side in the Taylor expansion. After integration over Ω one obtains in view of (3.8)

$$\int_\Omega H\big(\varepsilon(v-u)\big)\mathrm{d}x \le I^{\mathrm{qc}}(v) - I^{\mathrm{qc}}(u)\,.$$

We deduce from (3.6) and the fact that the determinant is a Null-Lagrangian that

$$\begin{aligned} I^{\mathrm{qc}}(v) - I^{\mathrm{qc}}(u) \ge \frac{\mu}{2} \int_\Omega \Big(&(2+\nu)\big(1 - \frac{\gamma_2^2}{\gamma_1^2}\big)\big(\partial_2(v_2 - u_2)\big)^2 \\ &+ \big(\partial_1(v_2-u_2)\big)^2 + \big(\partial_2(v_1-u_1)\big)^2 \\ &+ 2\big[(\nu+1) - (\nu+2)\frac{\gamma_2}{\gamma_1}\big]\partial_1(v_2-u_2)\partial_2(v_1-u_1)\Big)\mathrm{d}x \ge 0\,, \end{aligned} \tag{3.11}$$

as asserted. Finally suppose that v is a minimizer and that u is a critical point. Then (3.11) implies $\partial_2(v_2 - u_2) = 0$ and, by Poincaré's inequality, that $v_2 - u_2$ vanishes identically. We then conclude that $\partial_2(v_1 - u_1) = 0$ and hence $v_1 - u_1 = 0$ as well. This establishes the proof of the theorem. □

We conclude this section with an example which demonstrates the loss of uniqueness in the isotropic case [25, 27]. The question of uniqueness is an open problem for general boundary conditions, [18] contains some affirmative results in the case of strict quasiconvexity and affine boundary conditions.

Remark 3.3 Suppose that the material is isotropic, i.e., that $\eta_1 = \eta_2$ and that

$$A_1 - A_2 = \eta_1 I\,, \quad A_1 = A_2 + \eta_1 I, \quad \overline{A} = \frac{1}{2}\,(A_1 + A_2) = A_2 + \frac{\eta_1}{2}\,I$$

with diagonal matrices A_1 and A_2. For simplicity we assume that $\kappa = \mu = 1/2$. In this case, $\mathbb{C}$ is the identity tensor and the relevant constants are given by $\gamma_1 = \gamma_2 = \eta_1$, $\nu = 0$ and $g = \eta_1^2 > 0$, see (3.1). Moreover,

$$W_1(\overline{A}) = \frac{1}{2}\,\big|\overline{A} - A_1\big|^2 = \frac{1}{2}\,\big|\frac{\eta_1}{2}\,I\big|^2 \quad \text{and} \quad W_2(\overline{A}) = \frac{1}{2}\,\big|\overline{A} - A_2\big|^2 = \frac{1}{2}\,\big|\frac{\eta_1}{2}\,I\big|^2.$$

Hence $\overline{A}$ is a matrix in the interior of the relaxed phase $\mathcal{P}_{\mathrm{rel}}$ which is an open set in the space of all deformation gradients. On this subset, the quadratic part of the relaxed energy reads

$$H(E) = \frac{1}{4}\,(F_{12} - F_{21})^2 - \det F\,.$$

Fix any $\phi \in C_c^\infty(\Omega)$ with compact support in Ω. For δ small enough, the deformation gradient of the deformation $u_\delta(x) = \overline{A}x + \delta D\phi(x)$ is symmetric and lies in the open set $\mathcal{P}_{\mathrm{rel}}$. Thus the total elastic energy of the affine function $u_D(x) = \overline{A}x$ and the functions u_δ (which satisfy the same boundary conditions) are equal. This establishes nonuniqueness for constant applied forces f. To obtain forces which are not constant one can choose $f = \operatorname{curl}\psi$ with $\psi \in C^\infty(\Omega)$. In particular, the stress fields of the deformations are different. Thus our results cannot be extended to the case of isotropic materials.

4 Kinematically linear models

We begin our analysis with the case of linear kinematics.

4.1 Convexity control of the translated energy

One key observation is that the translated energy allows for convexity control in the sense of [4, 17].

Theorem 4.1 *Let $\gamma := \mu\big(\nu - (\nu+2)\frac{\gamma_2}{\gamma_1}\big)$ and define the translation of the energy W^{qc} as $\Phi : \mathbb{M}^{2\times 2} \to \mathbb{R}$ for all $X \in \mathbb{M}^{2\times 2}$ by*

$$\Phi(X) = W^{\mathrm{qc}}(\widehat{X}) - \gamma \det X \quad \textit{for} \quad \widehat{X} := \frac{1}{2}\,(X + X^T). \tag{4.1}$$

Then Φ allows convexity control in the sense that there exists a constant λ_1 with $0 < \lambda_1 < \infty$ such that

$$\lambda_1 |D\Phi(A) - D\Phi(B)|^2 \le \Phi(A) - \Phi(B) - \langle D\Phi(B), A - B\rangle \textit{ for all } A,\, B \in \mathbb{M}^{2\times 2}. \tag{4.2}$$

Note that the energy Φ depends on the full deformation gradient and not only on its symmetric part.

The proof requires a useful observation on nonnegative quadratic forms.

Lemma 4.2 *Given any nonnegative quadratic form* $Q : \mathbb{M}^{m\times n} \to \mathbb{R}$ *there exists a constant* $\lambda_0 > 0$ *such that, for all* $A, B, X \in \mathbb{M}^{m\times n}$,

$$\lambda_0 |DQ(X)|^2 \le Q(X)$$

and

$$\lambda_0 |DQ(A) - DQ(B)|^2 \le Q(B) - Q(A) - \langle DQ(A), B - A\rangle = Q(B - A)\,. \tag{4.3}$$

Proof. The identification of $\mathbb{M}^{m\times n}$ with $\mathbb{R}^{mn}$ shows that one needs to prove the lemma for $m = 1$ and any $n \in \mathbb{N}$ and may identify the bilinear form with some matrix $M \in \mathbb{R}^{n\times n}$, i.e., $Q(X) = X \cdot MX$ for all $X \in \mathbb{R}^n$. Without loss of generality we may and will suppose that M is symmetric.

The terms $Q(X) = X \cdot MX$ and $|DQ(X)|^2 = 4\,|MX|^2$ are invariant under orthogonal transformations and the spectral theorem shows that it is sufficient to prove the assertion for any diagonal matrix M. The latter follows immediately form the scalar case with $\lambda_0 = 1/(4\lambda_{\max})$ for the maximal positive eigenvalue of M (when $M \neq 0$ and else for any λ_0). This concludes the proof of the first assertion.

Since Q is quadratic, the Taylor series expansion of Q at A in terms of $X = B - A$ up to the quadratic term equals $Q(B)$. Furthermore, the second derivative $\frac{1}{2}\,D^2Q(A)[X, X]$ equals $Q(X)$. Hence, the Taylor series expansion proves the equality in (4.3). That equality plus the first assertion imply the claimed inequality in (4.3). □

Proof of Theorem 4.1. By definition, Φ is the translation of W^{qc} by a multiple of the determinant which is (in two dimensions) a quadratic form. In particular, if we collect all terms involving the translation on the right-hand side in (4.2), we obtain, for all $A, B \in \mathbb{M}^{2\times 2}$, that

$$-\gamma\big(\det(A) - \det(B) - D\det(B) \colon (A - B)\big) = -\gamma \det(A - B)\,. \tag{4.4}$$

Recall that the energy W^{qc} in (3.2) is given by three distinct expressions in the three domains $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_{\mathrm{rel}}$ with $W^{\mathrm{qc}} = W_j$ on P_j for $j = 1, 2$. To simplify the notation we set $W_{\mathrm{rel}} = W^{\mathrm{qc}}$ on $\mathcal{P}_{\mathrm{rel}}$. It follows from the chain rule that

$$\frac{\partial}{\partial F_{jk}}\, W(E) = \frac{1}{2}\,\frac{\partial}{\partial E_{jk}}\, W(E) + \frac{1}{2}\,\frac{\partial}{\partial E_{kj}}\, W(E)$$

and hence $\partial_F W$ is the symmetric part of $\partial_E W$. However, since the derivative of W with respect to E is symmetric, we may write DW without indicating whether the derivative is with respect to E or F. The same applies to W^{qc} and the three distinct parts in the formula for W and W^{qc}.

We need to check all combinations of arguments A and B lying in each of the three domains $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_{\mathrm{rel}}$. Suppose thus that $\widehat{A} \in \mathcal{P}_j$ and $\widehat{B} \in \mathcal{P}_k$ with $j,\, k \in \{1, 2, \mathrm{rel}\}$. Then

$$\begin{aligned}\mathrm{RHS} &= \Phi(A) - \Phi(B) - D\Phi(B) \colon (A - B)\\ &= W_j(\widehat{A}) - W_k(\widehat{A}) + W_k(\widehat{A}) - W_k(\widehat{B}) - DW_k(\widehat{B}) \colon (\widehat{A} - \widehat{B}) - \gamma \det(A - B)\,.\end{aligned}$$

Since W_k is a quadratic polynomial we obtain for all symmetric arguments X that

$$D^2W_k(B)[X, X] = D^2W_k(\widehat{B})[X, X] = \mathbb{C}[X, X] \geq 2H(X)$$

for $\widehat{B} \in \mathcal{P}_k$ and $k = 1, 2$ while

$$D^2W_{\mathrm{rel}}(B)[X, X] = D^2W_{\mathrm{rel}}(\widehat{B})[X, X] = 2H(X)$$

for $\widehat{B} \in \mathcal{P}_{\mathrm{rel}}$, see (3.4). Hence the right-hand side is equal to

$$\mathrm{RHS} = W_j(\widehat{A}) - W_k(\widehat{A}) + \frac{1}{2}\, D^2W_k(\widehat{B})[\widehat{A} - \widehat{B}, \widehat{A} - \widehat{B}] - \gamma \det(A - B)\,. \tag{4.5}$$

The assertion follows for $j = k$ from (4.3) and Lemma 3.1 since

$$\frac{1}{2}\, D^2W_k(\widehat{B})[\widehat{A} - \widehat{B}, \widehat{A} - \widehat{B}] - \gamma \det(A - B) \geq H(\widehat{A} - \widehat{B}) - \gamma \det(A - B) \geq 0\,.$$

The strategy in the remaining cases is to rearrange the terms in such a way that they are equal to $T(A-B)$ plus some nonnegative terms where T was defined in (3.5). Then the expression $D\Phi(A)-D\Phi(B)$ on the left-hand side is transformed to $DT(A-B)$ plus error terms. Finally one notes that in all cases the squares of the error terms are bounded by the additional nonnegative terms.

We include a sketch of the calculations for the four relevant cases and omit the remaining two (symmetric cases) for brevity.

Case 1: $\widehat{B} \in \mathcal{P}_{rel}$, $\widehat{A} \in \mathcal{P}_1$. The right-hand side is given by (4.5), i.e.,

$$\begin{aligned}\mathrm{RHS} &= W_1(\widehat{A}) - W_{\mathrm{rel}}(\widehat{A}) + \frac{1}{2}\, D^2W_{\mathrm{rel}}(\widehat{B})[\widehat{A} - \widehat{B}, \widehat{A} - \widehat{B}] - \gamma \det(A - B)\\ &= \frac{1}{2g}\, \big(W_2(\widehat{A}) - W_1(\widehat{A}) - \frac{g}{2}\big)^2 + T(A - B)\,.\end{aligned} \tag{4.6}$$

The relevant expression on the left-hand side (under the square) is equal to

$$\begin{aligned}&DW_1(\widehat{A}) - DW_{\mathrm{rel}}(\widehat{B}) - \gamma D \det(A - B)\\ &\qquad = DT(A - B) + \frac{1}{g}\big(W_2(\widehat{A}) - W_1(\widehat{A}) - \frac{g}{2}\big)\mathbb{C}(A_1 - A_2)\,.\end{aligned}$$

We conclude that

$$\begin{aligned}&\big|DW_1(\widehat{A}) - DW_{\mathrm{rel}}(\widehat{B}) - \gamma D \det(A - B)\big|^2\\ &\qquad \leq \frac{2}{g}\, (\gamma_1^2 + \gamma_2^2)\big(W_2(\widehat{A}) - W_1(\widehat{A}) - \frac{g}{2}\big)^2 + 2|DT(A - B)|^2\,.\end{aligned} \tag{4.7}$$

The first term in (4.7) is estimated by the first term in (4.6) (up to constants, recall that $T \geq 0$) and the second term in this expression is estimated by (4.3). Thus there exists a constant λ_1 with the asserted properties.

Case 2: $\widehat{B} \in \mathcal{P}_{rel}$, $\widehat{A} \in \mathcal{P}_2$. In this case the right-hand side is given by

$$\begin{aligned} &W_2(\widehat{A}) - W_{\mathrm{rel}}(\widehat{A}) + \frac{1}{2} D^2 W_{\mathrm{rel}}(\widehat{B})[\widehat{A} - \widehat{B}, \widehat{A} - \widehat{B}] - \gamma \det(A - B) \\ &\quad = \frac{1}{2g} \big(W_2(\widehat{A}) - W_1(\widehat{A}) + \frac{g}{2}\big)^2 + T(A - B) \end{aligned}$$

while the left-hand side is equal to

$$\begin{aligned} &DW_2(\widehat{A}) - DW_{\mathrm{rel}}(\widehat{B}) - \gamma D \det(A - B) \\ &\quad = DT(A - B) + \frac{1}{g}\big(W_2(\widehat{A}) - W_1(\widehat{A}) + \frac{g}{2}\big)\mathbb{C}(A_1 - A_2)\,. \end{aligned}$$

The assertion follows as in the previous case.

Case 3: $\widehat{B} \in \mathcal{P}_1$, $\widehat{A} \in \mathcal{P}_{rel}$. The right-hand side is equal to

$$\begin{aligned} &W_{\mathrm{rel}}(\widehat{A}) - W_1(\widehat{A}) + \frac{1}{2} D^2 W_1(\widehat{B})[\widehat{A} - \widehat{B}, \widehat{A} - \widehat{B}] - \gamma \det(A - B) \\ &\quad = -\frac{1}{2g} \big(W_2(\widehat{A}) - W_1(\widehat{A}) - \frac{g}{2}\big)^2 + \frac{1}{2g} \langle \mathbb{C}(\widehat{A} - \widehat{B}), A_1 - A_2 \rangle^2 + T(A - B)\,. \end{aligned}$$

Note that in the situation at hand

$$|W_2(\widehat{A}) - W_1(\widehat{A})| \leq \frac{g}{2}\,, \quad W_1(\widehat{B}) - W_2(\widehat{B}) + \frac{g}{2} \leq 0$$

and that the first two terms can be rearranged to

$$\begin{aligned} &-\frac{1}{2g} \big(W_2(\widehat{A}) - W_1(\widehat{A}) - \frac{g}{2}\big)^2 + \frac{1}{2g} \langle \mathbb{C}(\widehat{A} - \widehat{B}), A_1 - A_2 \rangle^2 \\ &= -\frac{1}{g} \big(W_2(\widehat{A}) - W_1(\widehat{A}) - \frac{g}{2}\big)\big(W_2(\widehat{B}) - W_1(\widehat{B}) - \frac{g}{2}\big) + \frac{1}{2g} \big(W_2(\widehat{B}) - W_1(\widehat{B}) - \frac{g}{2}\big)^2\,. \end{aligned}$$

In particular, the first term is nonnegative and the right-hand side is bounded from below by

$$\frac{1}{2g} \big(W_2(\widehat{B}) - W_1(\widehat{B}) - \frac{g}{2}\big)^2 + T(A - B)\,.$$

On the left-hand side we obtain

$$\begin{aligned} &DW_{\mathrm{rel}}(\widehat{A}) - DW_1(\widehat{B}) - \gamma D \det(A - B) \\ &\quad = -\frac{1}{g} \big(W_2(\widehat{B}) - W_1(\widehat{B}) - \frac{g}{2}\big)\mathbb{C}(A_1 - A_2) + DT(A - B) \end{aligned}$$

and the assertion follows as before.

Case 4: $\widehat{B} \in \mathcal{P}_1$, $\widehat{A} \in \mathcal{P}_2$. In this case,

$$W_1(\widehat{B}) - W_2(\widehat{B}) + \frac{g}{2} \leq 0\,, \quad W_1(\widehat{A}) - W_2(\widehat{A}) - \frac{g}{2} \geq 0$$

and the right-hand side is equal to

$$\begin{aligned} &W_2(\widehat{A}) - W_1(\widehat{A}) + \frac{1}{2}\, D^2 W_1(\widehat{B})[\widehat{A} - \widehat{B}, \widehat{A} - \widehat{B}] - \gamma \det(A - B) \\ &\quad = W_2(\widehat{A}) - W_1(\widehat{A}) + \frac{1}{2g}\, \langle \mathbb{C}(\widehat{A} - \widehat{B}), A_1 - A_2 \rangle^2 + T(A - B)\,. \end{aligned}$$

We focus on the first three terms which we rewrite as

$$\begin{aligned} &W_2(\widehat{A}) - W_1(\widehat{A}) + \frac{1}{2g}\, \langle \mathbb{C}(\widehat{A} - \widehat{B}), A_1 - A_2 \rangle^2 \\ &= \frac{1}{2g}\, \big(W_2(\widehat{A}) - W_1(\widehat{A}) + \frac{g}{2}\big)^2 - \frac{1}{g}\, \big(W_2(\widehat{A}) - W_1(\widehat{A}) - \frac{g}{2}\big)\big(W_2(\widehat{B}) - W_1(\widehat{B}) - \frac{g}{2}\big) \\ &\qquad + \frac{1}{2g}\, \big(W_2(\widehat{B}) - W_1(\widehat{B}) - \frac{g}{2}\big)^2\,. \end{aligned}$$

Note that the middle term is by assumption nonnegative. The terms on the left-hand side are

$$\begin{aligned} &DW_2(\widehat{A}) - DW_1(\widehat{B}) - \gamma D \det(A - B) \\ &= \frac{1}{g}\, \big[\big(W_2(\widehat{A}) - W_1(\widehat{A}) + \frac{g}{2}\big) - \big(W_2(\widehat{B}) - W_1(\widehat{B}) - \frac{g}{2}\big)\big]\mathbb{C}(A_1 - A_2) + DT(A - B)\,. \end{aligned}$$

If we square the right-hand side we obtain three squares which are all balanced on the left-hand side. The proof is complete. □

4.2 Adaptive Algorithm

This section describes the adaptive algorithm. Given an initial shape-regular triangulation $\mathcal{T}_0$, this scheme generates a sequence of triangulations $\mathcal{T}_\ell$ and corresponding finite element spaces $\mathbb{V}^{(\ell)}$ which all satisfy the estimate (2.1) with a constant κ^* determined form the initial configuration. In particular, all constants are independent of ℓ.

4.2.1 `INPUT`

The input required by the numerical scheme is a shape-regular triangulation $\mathcal{T}_0$ of the bounded domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\partial\Omega$ into closed triangles, the associated finite element space $\mathbb{V}^{(0)} = \mathbb{V}(\mathcal{T}_0)$ of continuous functions which are on all elements affine polynomials with values in $\mathbb{R}^2$, and a fixed parameter Θ with $0 < \Theta < 1$ for the marking strategy. Moreover, we assume that the Dirichlet condition u_D is contained in $\mathbb{V}^{(0)}$.

4.2.2 SOLVE and the discrete minimization problems

Given the triangulation $\mathcal{T}_\ell$, $\ell \in \mathbb{N}_0$, with the corresponding discrete spaces $\mathbb{V}^{(\ell)} = \mathbb{V}(\mathcal{T}_\ell)$ and $\mathbb{V}_0^{(\ell)} = \mathbb{V}_0(\mathcal{T}_\ell)$ on the level ℓ, compute the discrete solution $u_\ell \in u_D + \mathbb{V}_0^{(\ell)}$ as the unique minimizer of the energy functional I^{qc} on $u_D + \mathbb{V}_0^{(\ell)}$, see Corollary 4.3 below. For simplicity, we suppose that the discrete solution is computed exactly. Then, the discrete stress is given by

$$\sigma_\ell = DW^{\mathrm{qc}}(\varepsilon(u_\ell)) \in L^2(\mathcal{T}_\ell; \mathbb{M}^{2\times 2}_{\mathrm{sym}}).$$

Note that DW^{qc} is piecewise affine and globally continuous and hence globally Lipschitz continuous. Since $\varepsilon(u_\ell) \in P_0(\mathcal{T}_\ell; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ is piecewise constant, so is $\sigma_\ell \in L^2(\mathcal{T}_\ell; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$.

4.2.3 ESTIMATE

Suppose that T_+ and T_- are two distinct triangles in $\mathcal{T}_\ell$ with a common edge $E = \partial T_+ \cap \partial T_- \in \mathcal{E}_\ell(\Omega)$ of length $|E|$. The unit normal vector

$$\nu_E = \nu_{T_+}|_E = -\nu_{T_-}|_E \quad \text{along } E$$

is defined up to the orientation which we fix as the orientation of the outer normal ν_{T_+} of T_+ along E. Given the discrete stress $\sigma_\ell = DW^{\mathrm{qc}}(\varepsilon(u_\ell)) \in L^2(\mathcal{T}_\ell; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ of the previous subsection, the jump of σ_ℓ across the edge is defined as

$$[\sigma_\ell]_E \nu_E = \sigma_\ell|_{T_+}\nu_{T_+} + \sigma_\ell|_{T_-}\nu_{T_-} = \left(\sigma_\ell|_{T_+} - \sigma_\ell|_{T_-}\right)\nu_E \quad \text{along } E.$$

Let $\mathcal{E}(T)$ denote the set of the three edges of a triangle $T \in \mathcal{T}_\ell$ and $\mathring{\mathcal{E}}(T) = \mathcal{E}(T) \setminus \mathcal{E}_\ell(\partial\Omega)$ the subset of interior edges. To each triangle $T \in \mathcal{T}_\ell$ with area $|T|$ we associate the error estimator contribution $\eta_\ell(T)$ given by

$$\eta_\ell^2(T) = |T|\, ||f + \operatorname{div} \sigma_\ell||^2_{L^2(T)} + |T|^{1/2} \sum_{E \in \mathring{\mathcal{E}}(T)} ||[\sigma_\ell]_E \nu_E||^2_{L^2(E)}. \tag{4.8}$$

The sum

$$\eta_\ell^2 = \sum_{T \in \mathcal{T}_\ell} \eta_\ell^2(T)$$

is indeed an error estimator for the accompanying pseudostress approximations from the translated energy minimization problem, see the proof of Theorem 1.2. However, the upper bound η_ℓ of the pseudostress error is not sharp, the reliable error estimator η_ℓ is not efficient. This dramatic difficulty in the a posteriori error control is called reliability-efficiency gap in [5] and is caused by the degenerate convexity which is frequently encountered in relaxed variational problems in the modelling of microstructures.

4.2.4 MARK and REFINE

Suppose that all element contributions $(\eta_\ell^2(T) : T \in \mathcal{T}_\ell)$ defined in the previous subsection are known on the current level ℓ with triangulation $\mathcal{T}_\ell$. Given the input parameter

$\Theta \in (0,1)$ select a subset $\mathcal{M}_\ell$ of $\mathcal{T}_\ell$ (of minimal cardinality) with

$$\Theta \eta_\ell^2 \le \sum_{T \in \mathcal{M}_\ell} \eta_\ell^2(T) =: \eta_\ell^2(\mathcal{M}_\ell). \tag{4.9}$$

This selection condition is also called *bulk criterion* or Dörfler marking [15, 22] and is easily arranged with some greedy algorithm.

Any marked element is bisected according to the rules in Figure 1 and further mesh refinements may be necessary (e.g., via newest vertex bisection) such that $\mathcal{T}_{\ell+1}$ is a refinement of $\mathcal{T}_\ell$ with $\mathcal{M}_\ell \subset \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$.

Theorem 1.2 does not need the refinement with five bisections to obtain the interior node property and may focus on green-blue-red or green-blue refinement strategies.

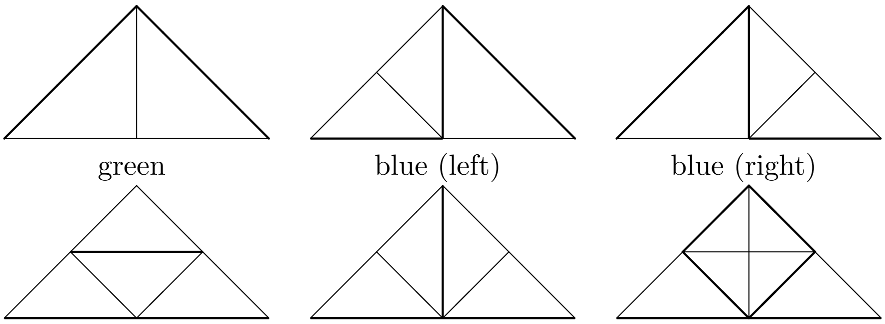

Figure 1: Possible refinements of a triangle (up to rotations).

4.2.5 `OUTPUT` and convergence result

For a given triangulation $\mathcal{T}_\ell$ the adaptive scheme generates the triangulation at the next level $\mathcal{T}_{\ell+1}$ by a successive completion of the subroutines

$$\texttt{SOLVE} \rightarrow \texttt{ESTIMATE} \rightarrow \texttt{MARK} \rightarrow \texttt{REFINE} \tag{4.10}$$

Based on the input triangulation $\mathcal{T}_0$, this scheme defines a sequence of meshes $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ and associated discrete subspaces

$$\mathbb{V}^{(0)} \subsetneqq \mathbb{V}^{(1)} \subsetneqq \cdots \subsetneqq \mathbb{V}^{(\ell)} \subsetneqq \mathbb{V}^{(\ell+1)} \subsetneqq \cdots \subsetneqq V = H^1(\Omega;\mathbb{R}^2) \tag{4.11}$$

with discrete minimizers $u_\ell \in u_D + \mathbb{V}_0^{(\ell)}$, $\ell \in \mathbb{N}_0$. The main properties of this sequence of solutions are formulated in Theorem 1.1 and Theorem 1.2, see also the steps in the proof in Section 4.3 for approximation of the pseudostresses associated to Φ.

4.3 Proofs of Theorem 1.1 and Theorem 1.2

This section presents the proofs for our main results which involve additional approximation estimates for the pseudostress τ.

4.3.1 Preliminary Remarks

Theorem 3.2 implies the existence and uniqueness of minimizers in our finite element spaces.

Corollary 4.3 *Suppose that $u_D \in H^1(\Omega;\mathbb{R}^2)$, that $\mathcal{T}_h$ is a shape regular triangulation with associated finite element space $\mathbb{V}_h$ with Courant elements and that $u_D \in \mathbb{V}_h$. Let $f \in L^2(\Omega)$. Then there exists a unique solution $u_h \in \mathbb{V}_h$ with $u_h = u_D$ on $\partial\Omega$ of the variational problem: Minimize $I^{\mathrm{qc}}(v_h)$ among all admissible functions $v_h \in u_D + \mathbb{V}_{h,0}$.*

We begin with a brief discussion of the relations between the original and the translated energy density $\Phi(X) = W^{\mathrm{qc}}(\widehat{X}) - \gamma \det X$. The first observation is that the determinant is a Null-Lagrangian, that is, for all $u \in \mathcal{A}$ defined in (1.1) the identity

$$\int_\Omega \det Du \mathrm{d}x = \int_\Omega \det Du_D \mathrm{d}x$$

holds, see, e.g., [23, Theorem 2.3]. This implies that the relaxed functional I^{qc} and the energy functional with the translated energy E^{qc} differ on $\mathcal{A}$ by a constant,

$$E^{\mathrm{qc}}(v) := \int_\Omega \Phi(Dv)\mathrm{d}x - \int_\Omega f \cdot v \,\mathrm{d}x = I^{\mathrm{qc}}(v) - \gamma \int_\Omega \det Du_D \mathrm{d}x \quad \text{for all } v \in \mathcal{A}\,.$$

Moreover, u is a minimizer for I^{qc} if and only if u is a minimizer for E^{qc}. Note that Φ depends on the full deformation gradient while W^{qc} depends only on its symmetric part.

An important consequence is that any minimizer of E or I^{qc} is a weak solution of the corresponding Euler-Lagrange systems,

$$\int_\Omega DW^{\mathrm{qc}}(\varepsilon(u)) : \varepsilon(v)\mathrm{d}x - \int_\Omega f \cdot v\mathrm{d}x = \int_\Omega \sigma : \varepsilon(v)\mathrm{d}x - \int_\Omega f \cdot v\mathrm{d}x = 0 \tag{4.12}$$

for all $v \in H_0^1(\Omega;\mathbb{R}^2)$ as well as

$$\int_\Omega D\Phi(Du) : Dv\mathrm{d}x - \int_\Omega f \cdot v\mathrm{d}x = \int_\Omega \tau : Dv\mathrm{d}x - \int_\Omega f \cdot v\mathrm{d}x = 0. \tag{4.13}$$

Here and throughout the paper, $\sigma := DW^{\mathrm{qc}}(\varepsilon(u))$ denotes the true stresses, $\tau := D\Phi(Du)$ denotes the pseudostress, i.e. the stress associated to the translated variational problem.

4.3.2 Proof of Theorem 1.1

The bound in terms of the energy difference follows from the algebraic estimates in Theorem 3.2, since for all $v_h \in \mathbb{V}_h$ the estimate

$$\frac{\mu}{2} \int_\Omega \Big(\alpha e_{2,2}^2 + \frac{2-\beta}{2}\left(e_{1,2}^2 + e_{2,1}^2\right)\Big) \le I^{\mathrm{qc}}(u_h) - I^{\mathrm{qc}}(u) \le I^{\mathrm{qc}}(v_h) - I^{\mathrm{qc}}(u)$$

holds. The weaker estimate for $\partial_1(u-u_h)$ follows with Poincaré's inequality from

$$\begin{aligned} ||\partial_1(u-u_h)_1||_{H^{-1}(\Omega)} &= \sup_{w\in H_0^1(\Omega),\, \|w\|_{1,2}\le 1} \int_\Omega (u-u_h)_1 \partial_1 w \mathrm{d}x \\ &\lesssim \sup_{w\in H_0^1(\Omega),\, \|w\|_{1,2}\le 1} \|\partial_2(u-u_h)_1\|_{L^2(\Omega)} \|Dw\|_{L^2(\Omega)} \\ &\lesssim \|\partial_2(u-u_h)_1\|_{L^2(\Omega)} . \end{aligned}$$

Moreover, the fact that W^{qc} is piecewise quadratic implies in view of the Taylor expansion (3.9) and the Euler-Lagrange system (4.12) for the minimizer u that

$$\begin{aligned} 0 \le I^{\mathrm{qc}}(v_h) - I^{\mathrm{qc}}(u) &= \int_\Omega \big(W^{\mathrm{qc}}(\varepsilon(v_h)) - W^{\mathrm{qc}}(\varepsilon(u))\big)\mathrm{d}x - \int_\Omega f\cdot(v_h-u)\mathrm{d}x \\ &= \int_\Omega \big(W^{\mathrm{qc}}(\varepsilon(v_h)) - W^{\mathrm{qc}}(\varepsilon(u)) - DW^{\mathrm{qc}}(\varepsilon(u))\colon (\varepsilon(v_h)-\varepsilon(u))\big)\mathrm{d}x \\ &\le C\int_\Omega \big|Dv_h - Du\big|^2 \mathrm{d}x . \end{aligned}$$

If $u\in H^2(\Omega;\mathbb{R}^2)$ then the error estimate follows if one chooses for v_h the usual nodal interpolation operator of u and uses the standard error estimates.

4.3.3 Proof of Theorem 1.2

We divide the proof in several steps. Let u_ℓ be the finite element minimizer in $\mathbb{V}^{(\ell)}$. Since the discrete spaces are nested, see the inclusions in (4.11), it follows that the sequence $(I^{\mathrm{qc}}(u_\ell))_{\ell\in\mathbb{N}}$ is monotone decreasing and bounded from below by $I^{\mathrm{qc}}(u)$, hence convergent. In the following $\mathcal{H}(\mathrm{div}=0)$ denotes the subspace of all matrix fields in $L^2(\Omega;\mathbb{M}^{2\times 2})$ for which the divergence of the rows vanishes in the sense of distributions,

$$\mathcal{H}(\mathrm{div}=0) := \big\{\tau\in L^2(\Omega;\mathbb{M}^{2\times 2}) : \mathrm{div}\,\tau = 0 \text{ in } \mathcal{D}'(\Omega;\mathbb{R}^2)\big\}.$$

Step 1: True stresses and pseudostresses. The key to the proof is the analysis of the convergence of the pseudostress $\tau_\ell = D\Phi(Du_\ell)$ which is piecewise constant. Since the derivative of the determinant as a map from $\mathbb{M}^{2\times 2}$ to $\mathbb{R}$ is the cofactor matrix, and since $\mathrm{div}\,\mathrm{cof}\,Du = 0$ in the sense of distributions, i.e., $\mathrm{cof}\,Du_\ell\in\mathcal{H}(\mathrm{div}=0)$, the true stress $\sigma_\ell = \sigma(Du_\ell)$ and the pseudostress τ_ℓ are related by

$$\sigma_\ell = D\Phi(Du_\ell) + \gamma\,\mathrm{cof}\,Du_\ell \in \tau_\ell + H(\mathrm{div}=0)\,.$$

Step 2: Error estimator reduction. There exist two constants $0<\rho<1$ and $0<\Lambda<\infty$ (which only depend on Θ and $\mathcal{T}_0$) such that, for any two consecutive levels ℓ and $\ell+1$ with corresponding finite element solutions u_ℓ and $u_{\ell+1}$ and discrete stress (resp. pseudo-stress) approximations σ_ℓ and $\sigma_{\ell+1}$ (resp. τ_ℓ and $\tau_{\ell+1}$),

$$\eta_{\ell+1}^2 \le \rho\,\eta_\ell^2 + \Lambda||\tau_{\ell+1}-\tau_\ell||^2_{L^2(\Omega)}. \tag{4.14}$$

When $||\tau_{\ell+1}-\tau_\ell||^2_{L^2(\Omega)}$ is replaced by $||\sigma_{\ell+1}-\sigma_\ell||^2_{L^2(\Omega)}$, the error reduction property (4.14) is a well-established tool in the convergence analysis of adaptive finite element methods and can be found in [4, 8] for elliptic problems in a very general setting. The proof employs the triangle and trace inequalities but no particular property of the piecewise polynomial approximations. Since the true stress and the pseudo-stress approximations differ merely by some piecewise constant divergence free cofactor matrix, the error estimator η_ℓ in terms of σ_ℓ is identical to the one with σ_ℓ replaced by τ_ℓ. This establishes (4.14).

Step 3: Bounds on the difference of successive pseudostresses. For any $\ell \in \mathbb{N}$ the L^2-norm of the difference of stresses at successive levels is estimated by

$$\lambda_1 \|\tau_{\ell+1} - \tau_\ell\|^2_{L^2(\Omega)} \leq E^{\mathrm{qc}}(u_\ell) - E^{\mathrm{qc}}(u_{\ell+1})\,.$$

To prove this estimate, we evaluate the convexity control estimate in (4.2) for x in the interior of an element in $\mathcal{T}_{\ell+1}$ in $A = Du_\ell(x)$ and $B = Du_{\ell+1}(x)$ and integrate on Ω to obtain

$$\lambda_1 \|\tau_{\ell+1} - \tau_\ell\|^2_{L^2(\Omega)} \leq \int_\Omega (\Phi(Du_\ell) - \Phi(Du_{\ell+1}))\,\mathrm{d}x - \int_\Omega \tau_{\ell+1} : D(u_\ell - u_{\ell+1})\mathrm{d}x\,. \quad (4.15)$$

Since $u_{\ell+1}$ minimizes E in $u_D + \mathbb{V}_0^{(\ell+1)}$ we may use the discrete Euler-Lagrange equations which are analogous to (4.13); that is,

$$\int_\Omega \tau_{\ell+1} : Dv_{\ell+1}\,\mathrm{d}x = \int_\Omega f \cdot v_{\ell+1}\,\mathrm{d}x \quad \text{for all } v_{\ell+1} \in \mathbb{V}_0^{(\ell+1)}\,.$$

Since $\mathbb{V}^{(\ell)} \subseteq \mathbb{V}^{(\ell+1)}$, $v_{\ell+1} = u_\ell - u_{\ell+1} \in \mathbb{V}_0^{(\ell+1)}$ is an admissible test function and hence

$$\int_\Omega \tau_{\ell+1} : D(u_\ell - u_{\ell+1})\mathrm{d}x = \int_\Omega f \cdot (u_\ell - u_{\ell+1})\mathrm{d}x.$$

We substitute this identity in (4.15) and obtain the assertion.

Step 4: Convergence of the error estimator. The error estimators η_ℓ, $\ell \in \mathbb{N}$, converge to zero, that is, $\lim_{\ell\to\infty} \eta_\ell = 0$. In fact, the error estimator reduction (4.14) and the discrete stress control of Step 3 imply

$$\eta^2_{\ell+1} \leq \rho\,\eta^2_\ell + \Lambda/\lambda_1\,(E^{\mathrm{qc}}(u_\ell) - E^{\mathrm{qc}}(u_{\ell+1})) \quad \text{for all } \ell \in \mathbb{N}\,.$$

Mathematical induction shows, for m, $n \in \mathbb{N}$, that

$$\begin{aligned} \eta^2_{m+n} &\leq \rho^n \eta^2_m + \frac{\Lambda}{\lambda_1} \sum_{k=0}^{n-1} \rho^{n-k-1} \big(E^{\mathrm{qc}}(u_{m+k}) - E^{\mathrm{qc}}(u_{m+k+1})\big) \\ &\leq \rho^n \eta^2_m + \frac{\Lambda}{\lambda_1} \big(E^{\mathrm{qc}}(u_m) - E^{\mathrm{qc}}(u_{m+n})\big)\,. \end{aligned}$$

For $m = 0$ we obtain uniform boundedness of the sequence $(\eta_n)_{n\in\mathbb{N}}$, and since $(E^{\mathrm{qc}}(u_\ell))_{\ell\in\mathbb{N}}$ is a Cauchy sequence and $0 < \rho < 1$ we conclude $\lim_{\ell\to\infty} \eta_\ell = 0$.

Step 5: Error estimates for the pseudo-stress. Let $\ell \in \mathbb{N}$, then

$$\lambda_1 \|\tau - \tau_\ell\|^2_{L^2(\Omega)} \leq E^{\mathrm{qc}}(u_\ell) - E^{\mathrm{qc}}(u)$$

and

$$\lambda_1 \|\tau - \tau_\ell\|^2_{L^2(\Omega)} \leq E^{\mathrm{qc}}(u) - E^{\mathrm{qc}}(u_\ell) + \int_\Omega (\tau - \tau_\ell) : D(u - u_\ell)\mathrm{d}x \,. \tag{4.16}$$

The first assertion follows as in Step 3 by replacing $u_{\ell+1}$ with u. To prove (4.16), let x be a Lebesgue point of Du which lies in the interior of an element in $\mathcal{T}_\ell$. For such an x we evaluate the convexity control estimate (4.2) in $A = Du(x)$ and $B = Du_\ell(x)$. Since almost all points are Lebesgue points, we may integrate on Ω and obtain

$$\lambda_1 \|\tau - \tau_\ell\|^2_{L^2(\Omega)} \leq \int_\Omega \left(\Phi(Du) - \Phi(Du_\ell)\right) \mathrm{d}x - \int_\Omega \tau_\ell : D(u - u_\ell)\mathrm{d}x \,.$$

The pseudo-stress τ satisfies the Euler-Lagrange equations (4.13) and the assertion follows in view of the definition of the energy.

Step 6: Explicit residual-based reliable error control I. There exists a constant C_{rel} such that, for all $\ell \in \mathbb{N}_0$,

$$\lambda_1 \|\tau - \tau_\ell\|^2_{L^2(\Omega)} + E^{\mathrm{qc}}(u_\ell) - E^{\mathrm{qc}}(u) \leq C_{rel}\, \eta_\ell ||D(u - u_\ell)||_{L^2(\Omega)} \,.$$

To prove this, let $e_\ell := u - u_\ell \in H^1_0(\Omega; \mathbb{R}^2)$ denote the error on the ℓth level of the scheme and let J_ℓ be a quasi-interpolation of H^1_0 onto $\mathbb{V}^{(\ell)}_0$ in the sense of [12, 26]. We denote by h_ℓ the mesh-size function of $\mathcal{T}_\ell$ which is constant on the elements in $\mathcal{T}_\ell$. Then there exists a constant C_{apx} which depends only on $\mathcal{T}_0$ such that [30]

$$||h_\ell^{-1}(e - J_\ell e)||^2_{L^2(\Omega)} + \sum_{E\in\mathcal{E}(\mathcal{T}_\ell)} |E|^{-1} \|e - J_\ell e\|^2_{L^2(E)} \leq C_{apx} ||De||^2_{L^2(\Omega)} \,. \tag{4.17}$$

We use (4.16) and the Euler-Lagrange equations for the solutions u and u_ℓ to obtain for all $v_\ell \in u_D + \mathbb{V}^{(\ell)}_0$,

$$\lambda_1 \|\tau - \tau_\ell\|^2_{L^2(\Omega)} + E^{\mathrm{qc}}(u_\ell) - E^{\mathrm{qc}}(u) \leq \int_\Omega f \cdot (u - v_\ell)\mathrm{d}x - \int_\Omega \tau_\ell : D(u - v_\ell)\mathrm{d}x \,.$$

Let $v_\ell = u_\ell + J_\ell(u - u_\ell) \in u_D + \mathbb{V}^{(\ell)}_0$ so that $u - v_\ell = e_\ell - J_\ell e_\ell$. In the second integral we use integration by parts on the individual triangles. In order to simplify the notation we do not replace integrals over Ω by a sum over all triangles. Instead we denote by div_ℓ the local divergence on all elements in $\mathcal{T}_\ell$. A careful rearrangement of the boundary terms shows that

$$\begin{aligned} &\int_\Omega f \cdot (u - v_\ell)\mathrm{d}x - \int_\Omega \tau_\ell : D(u - v_\ell)\mathrm{d}x \\ = &\int_\Omega (f + \mathrm{div}_\ell\, \tau) \cdot (e_\ell - J_\ell e_\ell)\mathrm{d}x - \sum_{E\in\mathring{\mathcal{E}}(\mathcal{T}_\ell)} \int_E (e_\ell - J_\ell e_\ell) \cdot [\tau_\ell]_E \nu_E \,\mathrm{d}s \,. \end{aligned}$$

Cauchy's inequality and the approximation error estimate (4.17) lead to the upper bound

$$\Big(\|h_\ell(f+\operatorname{div}_\ell \tau_\ell)\|^2_{L^2(\Omega)} + \sum_{E\in\mathring{\mathcal{E}}(\mathcal{T}_\ell)} |E|\,\|[\tau_\ell]_E\nu_E\|^2_{L^2(E)}\Big)^{1/2} C_{apx}^{1/2}||De_\ell||_{L^2(\Omega)}.$$

The equivalence of local mesh-size and the square root of the area of the elements (which follows from the shape-regularity) implies the existence of a reliability constant C_{rel} and the corresponding upper bound $\eta_\ell C_{rel}||De_\ell||_{L^2(\Omega)}$. This verifies the asserted estimate.

Step 7: Explicit residual-based reliable error control II. Let u_ℓ be the sequence of functions computed by the adaptive finite element scheme. Then

$$\lim_{\ell\to\infty} E^{\mathrm{qc}}(u_\ell) = E^{\mathrm{qc}}(u) \quad \text{and} \quad \lim_{\ell\to\infty} \|\tau-\tau_\ell\|_{L^2(\Omega)} = 0\,.$$

Note that the energy density W^{qc} satisfies two-sided growth conditions of the from

$$c_1|E|^2 - c_2 \le W^{\mathrm{qc}}(E) \le c_3(|E|^2+1) \quad \text{ for all } E\in\mathbb{M}^{2\times2}_{\mathrm{sym}}$$

with positive constants c_1, c_2, c_3. Thus the symmetric parts of the deformation gradients of the minimizers u and u_ℓ are uniformly bounded in L^2 and since $u-u_\ell\in H^1_0(\Omega)$ we obtain from Korn's inequality that $\|Du-Du_\ell\|_{L^2}$ is uniformly bounded. Step 4 shows $\eta_\ell\to 0$ as $\ell\to\infty$ and Step 6 implies that

$$\lim_{\ell\to\infty}\Big(\lambda_1\|\tau-\tau_\ell\|^2_{L^2(\Omega)} + E^{\mathrm{qc}}(u_\ell) - E^{\mathrm{qc}}(u)\Big) = 0.$$

This estimate implies the assertion of the Theorem 1.2.

Step 8: Convergence of the deformation gradient. This follows from Theorem 3.2 and the convergence of the energy from Step 7. □

5 Kinematically nonlinear models

In this section we extend the foregoing results to the case of energies which depend on the full deformation gradient and not only its symmetric part. The analysis for the relaxation of the double-well energy in the kinematically linear case can be performed in the nonlinear case as well and leads to the same formula (3.2), see Section 7 in [19].

5.1 Results

In the special case of an isotropic material with

$$W(F) = \frac{1}{2}\min\big\{|\alpha F - A_1|^2 + w_1,\, |\alpha F - A_2|^2 + w_2\big\}\,, \quad \alpha>0,\, w_1,\, w_2\in\mathbb{R}$$

the constant g reads (see Formula (7.1) in [19])

$$\alpha\lambda_{\max}\big((A_1-A_2)^T(A_1-A_2)\big).$$

By a change of coordinates we may assume that $A_2 = -A_1 = \Lambda = \mathrm{diag}(\alpha_1, \alpha_2)$ with $\alpha_1 > |\alpha_2| > 0$ and $\alpha = 1$. In particular, the two matrices are not compatible in the sense that the rank of the matrix $A_1 - A_2$ is bigger than one and that the matrix $A^T A$ is not proportional to the identity matrix. These assumptions lead to

$$W(F) = \frac{1}{2} \min\{|F - \Lambda|^2 + w_1, |F + \Lambda|^2 + w_2\} \quad \text{with} \quad g = \lambda_{\max}(4\Lambda^2) = 4\alpha_1^2 . \tag{5.1}$$

In this situation we have the following uniqueness result which parallels [27, Theorem 2.1] or Theorem 3.2 from this paper; the case $w_1 = w_2$ was already noted in [14].

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded and open domain with Lipschitz boundary, let $A \in \mathbb{M}^{m\times n}$ be a matrix with* $\mathrm{rank}(A) > 1$ *and let W be given by (5.1). Given $f \in L^2(\Omega;\mathbb{R}^N)$ and $u_D \in H^1(\Omega;\mathbb{R}^m)$, consider the variational integral*

$$I^{\mathrm{qc}}[v] = \int_\Omega W^{\mathrm{qc}}(Dv)\mathrm{d}x - \int_\Omega f \cdot v\mathrm{d}x \quad \textit{for all } v \in H^1(\Omega;\mathbb{R}^m) \tag{5.2}$$

in the class of admissible functions

$$\mathcal{A} = \{u \in H^1(\Omega;\mathbb{R}^N) |\, u = u_D \textit{ on } \partial\Omega\} .$$

Then, I^{qc} has a unique minimizer u in $\mathcal{A}$.

The analysis in Section 4 can be performed for the nonlinear setting as well. The corresponding results are summarized in the next theorem.

Theorem 5.2 *Let I^{qc} be the functional given in (5.2) with the energy density given in (5.1).*

1. *A priori estimates: Suppose that $u_D \in \mathbb{V}_h$ for some $h > 0$, that u is a minimizer of the functional I^{qc} in the class of admissible functions $\mathcal{A}$, and that u_h is a minimizer of I^{qc} in the finite element space $u_D + \mathbb{V}_{h,0}$ based on Courant finite elements on an underlying shape-regular triangulation $\mathcal{T}_h$. Then there exists a constant C_1 such that, in a suitable coordinate system with $A_1 - A_2 = \mathrm{diag}(\eta_1, \eta_2)$,*

$$\sum_{j,k=1,2;(j,k)\neq(1,1)} ||\partial_k(u - u_h)_j||_{L^2(\Omega)} \le C_1 \min_{v_h \in u_D + \mathbb{V}_{h,0}} \big(I^{\mathrm{qc}}(v_h) - I^{\mathrm{qc}}(u)\big) .$$

   Moreover, the $(1,1)$-component satisfies the weaker estimate

$$||\partial_1(u - u_h)_1||_{H^{-1}(\Omega)} \le C_1 \min_{v_h \in u_D + \mathbb{V}_{h,0}} \big(I^{\mathrm{qc}}(v_h) - I^{\mathrm{qc}}(u)\big) .$$

   If $u \in H^2(\Omega;\mathbb{R}^2)$ then there exists a constant C_2 such that

$$\min_{v_h \in u_D + \mathbb{V}_{h,0}} \big(I^{\mathrm{qc}}(v_h) - I^{\mathrm{qc}}(u)\big) \le C_2 h ||D^2 u||_{L^2(\Omega)} .$$

2. *Convergence of the adaptive scheme: Let* $(u_\ell)_{\ell\in\mathbb{N}}$ *with* $u_\ell \in u_D + \mathbb{V}_0^{(\ell)}$, $\ell \in \mathbb{N}_0$, *be the sequence computed by the adaptive scheme described in Section 4.2. Then this sequence converges with respect to the weak topology of* $H^1(\Omega;\mathbb{R}^2)$ *to the unique minimizer* u *of the variational integral* I^{qc} *in the class of admissible functions* $\mathcal{A}$. *Moreover, the energies* $I^{\mathrm{qc}}(u_\ell)$ *converge, i.e.,*

$$\lim_{\ell\to\infty} I^{\mathrm{qc}}(u_\ell) = I^{\mathrm{qc}}(u) = \min_{v\in u_D+H_0^1(\Omega;\mathbb{R}^2)} I^{\mathrm{qc}}(v)\,,$$

and all components of the deformation gradient except the $(1,1)$*-component converge strongly* $L^2(\Omega)$, *i.e.,*

$$\|\partial_1(u-u_\ell)_1\|_{H^{-1}(\Omega)} + \sum_{j,k=1,2;(j,k)\neq(1,1)} \|\partial_k(u-u_\ell)_j\|_{L^2(\Omega)} \to 0 \text{ as } \ell\to\infty.$$

In the following sections we sketch the proof of this theorem.

5.2 Convexity control of the translated energy

The key idea in Section 4.1 was the definition of the quadratic form H which was the quadratic part of the energy in the phase $\mathcal{P}_{\mathrm{rel}}$. This motivates to define

$$H(F) = \frac{1}{2}\,|F|^2 - \frac{2}{g}\,\langle F, A\rangle^2 \le \frac{1}{2}\,D^2W^{\mathrm{qc}}(G)[F,F] \quad \text{for all } F,\, G \in \mathbb{M}^{2\times 2}\,. \tag{5.3}$$

We define $\gamma = -\alpha_2/\alpha_1 \in (-1,1)$ and note that

$$T(F) = H(F) - \gamma \det F \ge \frac{1}{2}\,(1-\gamma^2)F_{22}^2 + \frac{1}{2}\,(1-\gamma^2)\,(F_{12}^2 + F_{21}^2)$$

defines a nonnegative quadratic form. In analogy to Theorem 4.1 one obtains convexity control for the translated energy.

Theorem 5.3 *Suppose that* W *is given by (5.1). Let* $\Phi : \mathbb{M}^{2\times 2} \to \mathbb{R}$ *be given by*

$$\Phi(X) = W^{\mathrm{qc}}(X) - \gamma \det X,\ X \in \mathbb{M}^{2\times 2}\,. \tag{5.4}$$

Then the translated energy Φ *allows convexity control in the sense that there exists a constant* λ_1 *with* $0 < \lambda_1 < \infty$ *such that*

$$\lambda_1 |D\Phi(A) - D\Phi(B)|^2 \le \Phi(A) - \Phi(B) - \langle D\Phi(B), A-B\rangle \text{ for all } A,\, B \in \mathbb{M}^{2\times 2}\,. \tag{5.5}$$

The proof is identical to the proof of Theorem 4.1.

5.3 Convergence analysis and proofs of Theorem 5.1 and Theorem 5.2

The proof of Theorem 5.1 is with minor changes identical to the proof of Theorem 3.2. We now obtain for any stationary point $u \in \mathcal{A}$ of I^{qc} and any $v \in \mathcal{A}$ the estimate

$$\begin{aligned} I^{\mathrm{qc}}(v) - I^{\mathrm{qc}}(u) &\geq \int_\Omega H\big(D(u-v)\big)\mathrm{d}x \\ &\geq \frac{1}{2}\,(1-\gamma^2) \int_\Omega \big[\big(\partial_2(v_1-u_1)\big)^2 + \big(\partial_1(v_2-u_2)\big)^2 + \big(\partial_2(v_2-u_2)\big)^2\big]\mathrm{d}x\,. \end{aligned}$$

In particular, all stationary states are minimizers and uniquely defined. This estimate implies immediately the a priori estimates in Theorem 5.2. The convergence analysis follows the lines of the one for the case of linear kinematics in Section 4.3 with the true stresses $\sigma_\ell = DW^{\mathrm{qc}}(Du_\ell)$ and the pseudostresses $\tau = D\Phi(Du_\ell)$.

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