Six-loop anomalous dimension of twist-two operators
in planar $\mathcal{N} = 4$ SYM theory

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Abstract: We compute the general form of the six-loop anomalous dimension of twist-two operators in planar $\mathcal{N} = 4$ SYM theory. First we find the contribution from the asymptotic Bethe ansatz. Then we reconstruct the wrapping terms from the first 35 even spin values of the full six-loop anomalous dimension computed using the quantum spectral curve of the theory. The obtained anomalous dimension satisfies all known constraints coming from the BFKL equation and the generalized double-logarithmic equation.

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1 Introduction

The anomalous dimension of composite gauge-invariant operators in $\mathcal{N} = 4$ SYM theory can be calculated with the help of integrability. Integrability in the context of AdS/CFT-correspondence [1–3] was found from the study of the single-trace operators [4] in the leading order of perturbative theory in ref. [5]. Generalization to higher orders together with the studies of the integrable structures from the superstring theory side, started in ref. [10], allowed formulating all-loop asymptotic Bethe equations [11–22, 25–30]. For the operators of finite length the asymptotic Bethe equations give a non-complete result due to appearance of wrapping effects [31, 32]. Again, the computations of wrapping corrections can be performed using integrability [33–46]. Independent tests of the obtained results were performed with the perturbative field theory computations in refs. [47–50]. Moreover, the results for twist-2 and spin $M$ operators passed very non-trivial tests coming from the Balitsky-Fadin-Kuraev-Lipatov (BFKL) [51–53] and generalized double-logarithmic [54–60] equations which impose all-loop constrains on the structure of the result analytically continued to negative values of $M$.

In principle, we can do the inverse: we can use constraints from the BFKL and the generalized double-logarithmic equations to reconstruct the wrapping corrections if the part of the full anomalous dimension related to the asymptotic Bethe equations is already

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*Earlier, similar integrability was discovered in quantum chromodynamics in the Regge limit [6–8] and for some operators [9].*
known. This was done at five loops by one of the authors in ref. [61]. However, one could not obtain any reasonable results at six loops in this way.

Recently, an advanced method for computing the anomalous dimension of operators with arbitrary but fixed twist and spin was developed by two of the authors [62]. This approach is based on a perturbative solution of the quantum spectral curve (QSC) equations [63, 64] and can in principle produce results to any loop order. In this paper we reconstruct a general form of the six-loop anomalous dimension for twist-two operators and use all available constraints to check its correctness.

In section 2 we compute the part of the six-loop anomalous dimension for twist-2 operators coming from the ABA. In section 3 we briefly describe the method of computation of the six-loop anomalous dimension for twist-2 operators with fixed spin. In section 4 we reconstruct the general form of the six-loop anomalous dimension from the first 35 even values, obtained with the method described in section 3. In section 5 we provide the constraints which will be used to verify the obtained result, together with the description of their origin. In Appendices we give the results for the most complicated parts of the six-loop anomalous dimension and its analytic continuation at $M = -2 + \omega$.

2 The six-loop anomalous dimension from Bethe ansatz

Twist-two operators belong to the $\mathfrak{sl}(2)$ sub-sector of the full theory. In this sector, the highest-weight (primary) states consist of two scalar fields $Z$ and $M^\dagger$ covariant derivatives $D$

$$\text{Tr} \left( Z D^M Z \right),$$

(2.1)

where $M$ should be even. In the spin chain picture which is valid at one loop at weak coupling, such single-trace operators are identified with the states of the non-compact $\mathfrak{sl}(2)$ spin $=-1/2$ length-two Heisenberg magnet with $M$ excitations.

We denote the total scaling dimension of these states as

$$\Delta = 2 + M + \gamma(g), \quad \text{with} \quad \gamma(g) = \sum_{\ell=1}^{\infty} \gamma_{2\ell} g^{2\ell}. \tag{2.2}$$

Here, $\gamma(g)$ is the anomalous part of the dimension depending on the coupling constant

$$g^2 = \frac{\lambda}{16 \pi^2}, \tag{2.3}$$

and $\lambda = N g_{\mathfrak{sl}(2)}^2$ is the 't Hooft coupling constant. The anomalous dimension $\gamma(g)$ may be determined up to the three-loop order $O(g^6)$ with help of the asymptotic Bethe ansatz [19].

The asymptotic Bethe equations for the $\mathfrak{sl}(2)$ operators can be found in [20, 30]

$$\left( \frac{x_j^+}{x_k^-} \right)^L = \prod_{\substack{j=1 \atop j \neq k}}^{M} \frac{x_k^- - x_j^+}{1 - g^2/x_k^- x_j^+} \frac{1 - g^2/x_k^- x_j^+}{1 - g^2/x_k^- x_j^+} \exp \left( 2i \theta(u_k, u_j) \right), \quad \prod_{k=1}^{M} x_k^+ = 1, \tag{2.4}$$

\footnote{$M$ is the value of the Lorentz spin. Hence the notation $S$ is often used instead of $M$. We chose to use $M$ to not interfere with the notation for harmonic sums.}
where the variables $x_k^\pm$ are related to $u_k$ through
\[
x_k^\pm = x(u_k^\pm), \quad u_k^\pm = u \pm \frac{i}{2}, \quad x(u) = \frac{u}{2} \left(1 + \sqrt{1 - 4 \frac{g^2}{u^2}}\right).
\]

The dressing phase $\theta(u, v)$ has been conjectured in [30] and shown to be solution of the crossing equation [22] in [23, 24]. To the sixth order in perturbation theory it has the following form
\[
\theta(u_k, u_j) = \left(4 \zeta_3 g^6 - 40 \zeta_5 g^8 + 420 \zeta_7 g^{10}\right) (q_2(u_k) q_3(u_j) - q_3(u_k) q_2(u_j))
\]
\[
- 8 \zeta_5 g^{10} (q_2(u_k) q_5(u_j) - q_5(u_k) q_2(u_j))
\]
\[
+ 24 \zeta_5 g^{10} (q_3(u_k) q_4(u_j) - q_4(u_k) q_3(u_j)) + \mathcal{O}(g^{12}),
\]

where $q_r(u)$ are the eigenvalues of the conserved magnon charges, see [20]. The anomalous dimension is given by
\[
\gamma_{ABA}^{\text{ABA}}(g) = 2 g^2 \sum_{k=1}^M \left(\frac{i}{x_k} - \frac{i}{x_{-k}}\right) = \sum_{l=1}^\infty g^{2l} \gamma_{2l}^{\text{ABA}}(M)
\]

and it is decomposed at six loops as
\[
\gamma_{12}^{\text{ABA}}(M) = \hat{\gamma}_{12} = \hat{\gamma}_{rational}^{\text{rational}} + \hat{\gamma}_{12}^{\zeta_3} \zeta_3 + \hat{\gamma}_{12}^{\zeta_5} \zeta_5 + \hat{\gamma}_{12}^{\zeta_7} \zeta_7.
\]

At one loop the Bethe roots $u_k$ are given by the zeros of the Hahn polynomial [28, 65]
\[
P_M(u) = 3 F_2 \left(-M, M + 1, \frac{1}{2} + i u \right| 1, 1\).
\]

In order to obtain a general expression for the anomalous dimension of twist-two operator for arbitrary $M$ we solve eq. (2.4) perturbatively for fixed values of the spin $M$ and match the coefficients in an appropriate ansatz which assumes the maximal transcendentality principle [67]. The basis for the ansatz is formed from the harmonic sums defined by the following recurrent procedure (see [72])
\[
S_a(M) = \sum_{j=1}^M \frac{(\text{sgn}(a))^j}{j^{|a|}},
\]
\[
S_{a_1,\ldots,a_n}(M) = \sum_{j=1}^M \frac{(\text{sgn}(a_1))^j}{j^{|a_1|}} S_{a_2,\ldots,a_n}(j).
\]

The transcendentality $k$ of each sum $S_{a_1,\ldots,a_n}$ is given by the sum of the absolute values of its indices
\[
k = |a_1| + \ldots |a_n|,
\]

\[\text{‡\ The hypothesis about maximal transcendentality principle [67] was confirmed by direct perturbative calculations at the two-loop order [68] and then successfully applied at the three-loop order [69], when corresponding results were obtained in QCD [70, 71].}\]
and the transcendentality of a product of harmonic sums is equal to the sum of the transcendentality of its constituents. According to the maximal transcendentality principle [32, 67–69] the anomalous dimension of twist-two operators can contain only harmonic sums with maximal transcendentality at a given order of perturbative theory. The number of harmonic sums entering into this basis at the $\ell$-loop order for the transcendentality $k = 2\ell - 1$ is equal to $((1 - \sqrt{2})^k + (1 + \sqrt{2})^k)/2$, and at six loops we will need more than 8800 sums, see Table 1.

Fortunately, the generalized Gribov-Lipatov reciprocity [73, 74] enters the game and allows us to significantly reduce the dimension of the basis. Define the reciprocity-respecting function $P_{ABA}(M)$ [73–75] by the relation

$$\gamma_{ABA}(M) = P_{ABA}(M + \frac{1}{2}\gamma_{ABA}(M)).$$

(2.13)

The reciprocity-respecting splitting function $P(x)$ [73, 74], related to $P(M)$ through a Mellin transformation, should satisfy the Gribov-Lipatov relation [76]

$$P(x) = -xP\left(\frac{1}{x}\right)$$

(2.14)

at all orders of the perturbation theory.

The practical output which we will use is that the reciprocity function $P_{ABA}(M)$ should be expressed only in terms of the binomial sums (see [72])

$$S_{i_1,...,i_k}(N) = (-1)^N \sum_{j=1}^{N} (-1)^j \binom{N}{j} \binom{N+j}{j} S_{i_1,...,i_k}(j),$$

(2.15)

which is the necessary and sufficient condition to satisfy (2.14) [36].

Table 1. The number of harmonic sums in the basis for different contributions.

<table>
<thead>
<tr>
<th>Contribution</th>
<th>Rational</th>
<th>$\zeta_3$</th>
<th>$\zeta_5$</th>
<th>$\zeta_3^2$</th>
<th>$\zeta_7$</th>
<th>$\zeta_5\zeta_3$</th>
<th>$\zeta_3^3$</th>
<th>$\zeta_9$</th>
<th>Total</th>
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<tr>
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<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>11</td>
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<td>17</td>
<td></td>
<td></td>
<td></td>
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<td>8812</td>
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<td>128</td>
<td>32</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1192</td>
</tr>
<tr>
<td>$P_{ABA}$</td>
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<td>99</td>
<td>17</td>
<td>7</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1522</td>
</tr>
<tr>
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<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>305</td>
</tr>
</tbody>
</table>

§ Historically, this statement was understood using different but equivalent to the binomial sums basis [74, 77–79]. Relations between the binomial and the nested harmonic sums can be found as the ancillary files of the arXiv version this paper or on the web-page http://thd.pnpi.spb.ru/~velizh/6loop/.
equation for the corresponding spin chain [65, 66]. One of the interesting features of these sums is that they are defined only for positive values of the indices \( i_1, \ldots, i_k \). The number of such sums, which will form the basis, for transcendentality \( k \) is equal to \( 2^{k-1} \) and at six loops we will about 1000 binomial harmonic sums (see Table 1). Thus, the basis is significantly reduced and, though still huge, can be managed.

In this paper, we first find the result in terms of \( \mathcal{P}(M) \) and then reconstruct the anomalous dimension from (2.13): upon substituting the perturbative expansion (2.2), one finds

\[
\mathcal{P}_{ABA}^\text{ABA}(M) = \sum_{l=1}^{\infty} g^{2l} \mathcal{P}_{2l}^\text{ABA}(M). \tag{2.16}
\]

From eqs. (2.13) and (2.16) one can find that at six-loop order the reciprocity function \( \mathcal{P}_{12} \) is related to the anomalous dimension \( \gamma_{12} \) (see Appendix B of ref. [77] for five loops):

\[
\mathcal{P}_{12}^\text{ABA}(M) = \mathcal{P}_{12} = \mathcal{P}_{12}^\text{rational} + \mathcal{P}_{12}^\text{C} \zeta_3 + \mathcal{P}_{12}^\text{C} \zeta_5 + \mathcal{P}_{12}^\text{C} \zeta_7, \tag{2.17}
\]

\[
\mathcal{P}_{12}^\text{rational} = \gamma_{12}^\text{rational} - \frac{1}{4} \left( \hat{\gamma}_2^2 + 2 \hat{\gamma}_4^\text{rational} + 2 \hat{\gamma}_2^\text{rational} \right) \left( \hat{\gamma}_2^\text{rational} \right)''
- \frac{1}{96} \left( 3 \hat{\gamma}_2^4 + 2 \hat{\gamma}_2 \hat{\gamma}_6 \right)'''
+ \frac{1}{384} \left( \hat{\gamma}_2^4 \right)'''' - \frac{1}{23040} \left( \hat{\gamma}_2^6 \right)''''', \tag{2.18}
\]

\[
\mathcal{P}_{12}^\text{C} = \gamma_{12}^\text{C} - \frac{1}{2} \left( \hat{\gamma}_4 \hat{\gamma}_6 \right)'' + \frac{1}{8} \left( \hat{\gamma}_2 \hat{\gamma}_8 \right)'', \tag{2.19}
\]

\[
\mathcal{P}_{12}^\text{C} = \gamma_{12}^\text{C} - \frac{1}{2} \left( \hat{\gamma}_2 \hat{\gamma}_10 \right)', \tag{2.20}
\]

\[
\mathcal{P}_{12}^\text{C} = \gamma_{12}^\text{C}, \tag{2.21}
\]

where each prime marks a derivative with respect to \( M \).

The rational and \( \zeta \) parts of \( \mathcal{P}_{12}^\text{ABA}(M) \) are computed separately. To compute the rational part, the ABA equations without dressing phase were solved up to six loops for the first 1024 values of \( M \) numerically with an accuracy of about 10^{-1000}. The answer should be a linear combination of 1024 binomial harmonic sums, hence we’ve got enough constraints to fix the coefficients in this combination. These constraints were solved with the help of the MATHEMATICA function LinearSolve. The obtained coefficients are very close to integers, so their rounding gives the desired result for \( \mathcal{P}_{12}^\text{rational} \), which is listed in Appendix A.

For the dressing phase (2.6) the first 40 solutions of the Bethe equations were computed. According to Table 1, this is enough to find the general expression for the functions \( \mathcal{P}_{12}^\text{C} \) and \( \mathcal{P}_{12}^\text{C} \), which have the following form:

\[
\mathcal{P}_{12}^\text{C} = -64 \left( -8S_6 58S_{1,5} - 74S_{2,4} - 8S_{3,3} - 8S_{4,2} + 60S_{5,1} + 40S_{1,1,4} - 78S_{1,2,3} 
- 38S_{1,3,2} + 40S_{1,4,1} - 12S_{2,1,3} + 88S_{2,2,2} - 40S_{2,3,1} - 26S_{3,2,1} + 26S_{4,1,1} - 60S_{1,1,2,2} 
+ 20S_{1,1,3,1} - 20S_{1,2,1,2} + 38S_{1,3,1,1} - 20S_{2,1,1,2} + 16S_{2,1,2,1} + 14S_{2,2,1,1} - 4S_{3,1,1,1} \right), \tag{2.22}
\]

\[
\mathcal{P}_{12}^\text{C} = -3360 \left( 2S_4 + S_{1,3} - S_{2,2} - S_{1,2,1} - S_{2,1,1} \right). \tag{2.23}
\]
For the reconstruction of $\hat{\mathcal{P}}_{12}^{\text{Ct}}$ we used the LLL-algorithm [80], similar to what was done in the previous works [81, 82] by one of the authors (see also ref. [61] for a detailed explanations). Its usage is based on the fact that the coefficients in any anomalous dimension of twist-two operators are usually rather simple numbers, that is the equation for the coefficients is a linear Diophantine equation. Moreover, a lot of binomial harmonic sums which belong to the basis of possible terms are absent in the final expression, i.e. their coefficients are zeros. The LLL-algorithm is realized in many computer algebra systems and for our purposes the MATHEMATICA function LatticeReduce has been used to realize the algorithm. Firstly, we calculate the values of all 128 terms in the basis with transcendentality 8 up to $M = 40$. We hence can write a system of 39 linear equations for 128 coefficients which are assumed to be integers. According to the realization of the LLL-algorithm we add to the $129 \times 129$ unit matrix the transpose matrix, obtained from our system of Diophantine equations (coefficients of 128 variables plus one free term), and applying the LatticeReduce function of MATHEMATICA to the obtained $(129 + 39) \times 129$ matrix we get the LLL-reduced matrix, in which we can easily find the result that we are looking for:

$$\hat{\mathcal{P}}_{12}^{\text{Ct}} = -32(4S_{2,6} - 4S_{7,1} - 8S_{1,1,6} + 12S_{1,2,5} + 3S_{1,3,4} + 3S_{1,5,2} - 13S_{1,6,1} + 8S_{2,1,5} - 18S_{2,2,4} - 7S_{2,3,3} - 7S_{2,4,2} + 20S_{2,5,1} - 11S_{3,1,4} + 8S_{3,2,3} - S_{3,3,2} + 7S_{4,1,3} + 7S_{4,2,2} + 7S_{4,3,1} - 9S_{5,1,2} + 21S_{5,2,1} - 16S_{6,1,1} - 6S_{5,1,1,5} + 14S_{1,1,2,4} + 8S_{1,1,3,3} + 8S_{1,1,4,2} - 12S_{1,1,5,1} + 5S_{1,2,1,4} - 24S_{1,2,2,3} - 13S_{1,3,2,3} + 19S_{1,2,4,1} - 9S_{1,3,1,3} + 3S_{1,3,2,2} + 9S_{1,3,3,1} - 8S_{1,4,1,2} + 25S_{1,4,2,1} - 19S_{1,5,1,1} + 5S_{2,1,1,4} - 12S_{2,1,2,3} - 7S_{2,1,3,2} + 6S_{2,1,4,1} + 2S_{2,2,1,3} = 26S_{2,2,2,2} - 22S_{2,3,1,2} + 8S_{2,3,1,2} - 33S_{2,3,2,1} + 27S_{2,4,1,1} - 10S_{3,1,1,3} + 10S_{3,1,2,2} + 9S_{3,2,1,2} - 24S_{3,2,2,1} + 10S_{3,3,1,1} - 9S_{4,1,1,2} + 9S_{4,2,1,2} + 23S_{4,2,1,1} - 18S_{5,1,1,1} + 9S_{1,1,1,2} + 9S_{1,1,2,3} + 9S_{1,1,1,3,2} - 6S_{1,1,1,1,4,1} + 3S_{1,1,1,2,1,3} - 24S_{1,1,2,2,2} + 13S_{1,1,2,3,1} - 7S_{1,1,3,2,1} + 27S_{1,1,1,3,2} + 20S_{1,1,1,1,1,1} + 3S_{1,1,1,2,1,3} - 8S_{1,1,2,1,2,2} + 3S_{1,1,2,1,3,1} + 3S_{1,1,2,2,1,2} - 37S_{1,1,2,2,2,1} + 27S_{1,2,3,1,1} - 9S_{1,3,1,1,2} + 9S_{1,3,1,2,1} + 25S_{1,3,2,1,1} - 20S_{1,4,1,1,1,1} + 3S_{2,1,1,1,1,3} - 8S_{2,1,1,1,2,2} + 3S_{2,1,1,1,3,1} - 3S_{2,1,2,1,2,1} - 5S_{2,1,2,1,2,1} + 10S_{2,1,3,1,1,1} + 4S_{2,2,1,1,2} - 6S_{2,2,1,2,1} - 39S_{2,2,2,1,1} + 29S_{2,3,1,1,1} - 9S_{3,1,1,1,2} + 8S_{3,1,1,2,1} + 8S_{3,1,2,1,1} + 21S_{3,2,1,1,1} - 16S_{4,1,1,1,1} + 12S_{1,1,1,2,2} - 18S_{1,1,1,3,1,1} + 26S_{1,1,2,1,1} - 28S_{1,1,3,1,1,1} + 24S_{1,2,1,1,1,1} - 16S_{1,3,1,1,1,1} + 16S_{2,2,1,1,1,1} - 16S_{3,1,1,1,1,1} ). \quad (2.24)$$

We check with $M = 40$ that the obtained expression is indeed correct. The final expression for the six-loop anomalous dimension of twist-two operators from ABA in the canonical basis of the usual harmonic sums (2.11) can be found as the ancillary files of the arXiv version this paper or on the web-page http://thd.pnpi.spb.ru/~velizh/6loop/.

\*See Application in http://reference.wolfram.com/mathematica/ref/LatticeReduce.html
3 Calculations of the six-loop anomalous dimension for finite $M$

The quantum spectral curve \[63, 64\] formulates the spectral problem of planar $\mathcal{N} = 4$ SYM in terms of a finite set of Riemann-Hilbert equations. It was derived based on several assumptions, in particular that the theory is integrable. This approach automatically captures all the features of the spectral problem, i.e. no notion of ABA and wrapping contributions is needed. Recently, QSC was solved perturbatively for any operator in the $\mathfrak{sl}(2)$ sector by two of the authors \[62\], and an efficient MATHEMATICA-implementation of this iterative algorithm was provided, allowing up to 10-loop calculations of the simplest operators on a standard laptop. In our work, this procedure has been used to produce the six-loop anomalous dimensions of twist-two operators for finite $M$. Naturally, the computation time increases significantly with the value of $M$ (15 seconds for $M = 2$ compared to 35 hours for $M = 80$), since one of the algorithm is systematically operating with the Baxter polynomial which is a degree-$M$ polynomial. We refer to \[62\] for a detailed description of the procedure as well as access to the MATHEMATICA-implementation.

To have a sufficient amount of finite $M$ results, we determined $\gamma_{12}(M)$ for the 40 lowest even integer spins, i.e. for $M = 2, 4, \ldots, 80$. We here provide the first five results obtained by the method. The remaining results can be reproduced from the general result below.

\[
\begin{align*}
\gamma_{12}(2) &= 48(-160 - 5472 \zeta_3 - 432 \zeta_5^2 + 2340 \zeta_5 + 3240 \zeta_3 \zeta_5 + 1575 \zeta_7 - 10206 \zeta_9), \\
\gamma_{12}(4) &= 25 \left( \frac{493794415}{118098} - \frac{76738930}{2187} \zeta_3 - \frac{545000}{81} \zeta_5^2 + \frac{11977223}{729} \zeta_5 + \frac{50000}{3} \zeta_3 \zeta_5 \\
&+ \frac{1723925}{81} \zeta_7 - 37800 \zeta_9 \right), \\
\gamma_{12}(6) &= 49 \left( \frac{1305379290116927483}{14762250000000} - \frac{2757220598468}{3796875} \zeta_3 - \frac{80404688}{375} \zeta_5^2 \\
&+ \frac{1831031111}{5625} \zeta_5 + 345744 \zeta_3 \zeta_5 + \frac{44325547}{75} \zeta_7 - 666792 \zeta_9 \right), \\
\gamma_{12}(8) &= 1 \left( \frac{6338981525904869169669605010931}{1906272111794400000000} - \frac{125986766100891990916921}{40024189800000} \zeta_3 \\
&- \frac{467665307183261}{3601500} \zeta_5^2 + \frac{2326437396820615637}{18151560000} \zeta_5 + \frac{7932799458}{49} \zeta_3 \zeta_5 \\
&+ \frac{19149255965089}{58800} \zeta_7 - 281452806 \zeta_9 \right), \\
\gamma_{12}(10) &= \frac{1331}{175} \left( \frac{474005292705148459163445336569621}{21612184690434724915200000000} - \frac{245586 \zeta_9}{\zeta_3} - \frac{12417931006251790179790817}{4726776767000400000000} \zeta_3 - \frac{3452378452329719}{23629441500} \zeta_5^2 \\
&- \frac{6477832741701863069}{7145543109600} \zeta_5 + \frac{604223422}{3969} \zeta_3 \zeta_5 + \frac{1755759752377}{5143824} \zeta_7 \right). \quad (3.1)
\end{align*}
\]

The first 35 values, i.e. $M \leq 70$, were used to construct the general structure for any $M$, and it was then checked that the last five results $72 \leq M \leq 80$ are consistent with this structure.
4 Reconstruction of the general form of six-loop anomalous dimension

In this section we describe in detail the method of the reconstruction of the anomalous dimension of twist-two operators from the above results and special numerical algorithms. Similarly to the ABA case, we will look for the reciprocity function $\mathcal{P}_{\text{wrap}}$ which is represented in a smaller basis of binomial harmonic sums (2.15) and then reconstruct the conformal dimension using (2.17).

Although we compute the results for the full anomalous dimension it is more suitable and much easier to reconstruct the wrapping correction part (without an ABA part) rather than the full anomalous dimension. At five loops $\gamma_{10}^{\text{wrap}}$ and $\mathcal{P}_{10}^{\text{wrap}}$ are related as

$$\gamma_{10}^{\text{wrap}} = \frac{1}{2} \left( \gamma_2^{\text{wrap}} \right) + \mathcal{P}_{10}^{\text{wrap}},$$

where $\gamma_2$ and $\gamma_8^{\text{wrap}}$ are given by [34]

$$\gamma_8^{\text{wrap}} = \mathcal{P}_8^{\text{wrap}} = \mathcal{P}_2^2 \mathcal{T}_8,$$

$$\mathcal{T}_8 = \left( -5 \zeta_5 + 2 S_2 \zeta_3 + (S_{2,1,2} - S_{3,1,1}) \right),$$

$$\gamma_2 = \mathcal{P}_2 = 4 S_1.$$  \hspace{1cm} (4.1)

To find $\mathcal{P}_{10}^{\text{wrap}}$ we should, in principle, consider the full basis of binomial harmonic sums of given transcendentality. However, at five loops we can write the following general expression for the reciprocity-respecting function $\mathcal{P}_{10}^{\text{wrap}}$:

$$\mathcal{P}_{10}^{\text{wrap}} = \mathcal{P}_2^2 \mathcal{T}_{10} + c_1 \mathcal{P}_2 \mathcal{P}_4 \mathcal{T}_8 + c_2 \mathcal{P}_2^4 \mathcal{T}_8,$$

$$\mathcal{P}_4 = 8 (S_1 S_2 - S_{2,1} - S_3).$$ \hspace{1cm} (4.2)

The appearance of the structure with coefficients $c_1$ and $c_2$ is expected from a suggestion that the more general integrable system applicable for the computations of the wrapping corrections represents a set of (at least) two spin-chains, for which ABA is known. During the computations of the wrapping corrections at five loops in ref. [36] we assumed in advance the appearance of such terms, what allowed to find a final result much faster. Indeed, we found in ref. [36]

$$\mathcal{P}_{10}^{\text{wrap}} = 2 \mathcal{P}_2^2 \mathcal{T}_{10} + 2 \mathcal{P}_2 \left( 2 \mathcal{P}_4 + \frac{1}{16} \mathcal{P}_2^3 \right) \left( -5 \zeta_5 + 2 S_2 \zeta_3 + (S_{2,1,2} - S_{3,1,1}) \right),$$

$$\mathcal{T}_{10} = 105 \zeta_7 - 6 S_1 \zeta_3^2 - 40 S_2 \zeta_5 + 4 (3 S_1 S_{2,1} - 2 S_{2,2} + 3 S_{3,1} - S_{2,1,1} - S_4) \zeta_3$$

$$+ 2 \left( S_1 (S_{2,3,1} - S_{3,1,2}) - S_{2,1,4} + 2 S_{2,2,3} - 5 S_{3,1,3} + 2 S_{3,2,2} + 2 S_{3,1,1,2} + 2 S_{2,1,2,2} + 2 S_{2,1,3,1} - 2 S_{2,2,1,2} - 2 S_{2,2,2,1} \right).$$ \hspace{1cm} (4.3)

The advantage of the reconstruction of the wrapping correction part instead of the full anomalous dimension is clear from eq. (4.5): $\mathcal{T}_{10}$ multiplied by $S_1^2$ has the transcendentality 7 with $2^{9-1-2} = 64$ binomial harmonic sums in the basis instead of the transcendentality 9 with $2^{9} - 1 = 256$ binomial harmonic sums in the basis for $\mathcal{P}_{10}^{\text{wrap}}$, so we will have considerably less binomial harmonic sums in the ansatz.
For this extended ansatz a solution exists (note that we have 35 equations for 8 variables):

\[ \gamma_{12}^{\text{wrap}} = \frac{1}{2} \left( \gamma_4 \gamma_8^{\text{wrap}} + \gamma_2 \gamma_{10}^{\text{wrap}} \right)' - \frac{1}{8} \left( \gamma_2 \gamma_8 \right)^{''} + \mathcal{P}_{12}^{\text{wrap}}. \]  

(4.9)

Generalizing eq. (4.5) to the six-loop order we can assume the following general expression for the reciprocity-respecting function \( \mathcal{P}_{12}^{\text{wrap}} \):

\[
\mathcal{P}_{12}^{\text{wrap}} = \mathcal{P}_2^{\text{wrap}} \mathcal{T}_{12} + \left( c_1 \mathcal{P}_2 \mathcal{P}_6 + c_2 \mathcal{P}_3 \mathcal{P}_4 + c_3 \mathcal{P}_5 \mathcal{P}_6 + c_4 \mathcal{P}_4^2 \right) \mathcal{T}_8 + \left( c_5 \mathcal{P}_2 \mathcal{P}_4 + c_6 \mathcal{P}_2^2 \right) \mathcal{T}_{10},
\]

(4.10)

\[
\mathcal{P}_{12}^{\text{wrap}} = \mathcal{P}_{12, \text{rational}}^{\text{wrap}} + \zeta_3 \mathcal{P}_{12, \zeta_3}^{\text{wrap}} + \zeta_5 \mathcal{P}_{12, \zeta_5}^{\text{wrap}} + \zeta_3 \mathcal{P}_{12, \zeta_3}^{\text{wrap}} + \zeta_5 \mathcal{P}_{12, \zeta_5}^{\text{wrap}} + \zeta_3 \mathcal{P}_{12, \zeta_3}^{\text{wrap}} + \zeta_5 \mathcal{P}_{12, \zeta_5}^{\text{wrap}} + \mathcal{T}_{12}^{\text{rational}},
\]

(4.11)

with \( \mathcal{T}_{12} \) and \( \mathcal{P}_6 \) are given by

\[
\mathcal{T}_{12} = \zeta_9 \mathcal{T}_{\zeta_9} + \zeta_3^2 \mathcal{T}_{\zeta_3} + \zeta_5 \mathcal{T}_{\zeta_5} + \zeta_7 \mathcal{T}_{\zeta_7} + \zeta_3^2 \mathcal{T}_{\zeta_3} + \zeta_5 \mathcal{T}_{\zeta_5} + \zeta_3 \mathcal{T}_{\zeta_3} + \mathcal{T}_{\text{rational}},
\]

(4.12)

\[
\mathcal{P}_6 = 16 \left( S_1^2 S_3 + S_1 (S_{3,1} - S_{2,2} - 4 S_4) + 2 (S_{2,1} - S_{2,1,2} - S_{3,2} + 3 S_5) \right).
\]

(4.13)

Applying the principle of maximal transcendentality we conclude that the transcendentality of the components \( \mathcal{T}_{\zeta_9}, \mathcal{T}_{\zeta_3}, \mathcal{T}_{\zeta_5}, \mathcal{T}_{\zeta_7}, \mathcal{T}_{\zeta_3}, \mathcal{T}_{\zeta_5}, \mathcal{T}_{\text{rational}} \) should be equal to 0, 0, 1, 2, 3, 4, 6 and 9 respectively (see Table 1 for the numbers of the binomial harmonic sums in the corresponding ansatz). The lowest-transcendentality functions, \( \mathcal{T}_{\zeta_9}, \mathcal{T}_{\zeta_3}, \mathcal{T}_{\zeta_5}, \mathcal{T}_{\zeta_7}, \mathcal{T}_{\zeta_3}, \mathcal{T}_{\text{rational}} \) may be obtained exactly with the solution of the equations for the corresponding contribution as we know the full result for even values of \( M \) up to \( M = 70 \) (but we should take into account that the \( \zeta_7 \) contribution is contained in \( \mathcal{T}_{10} \), while \( \zeta_5 \) and \( \zeta_3 \) contributions are contained in both \( \mathcal{T}_6 \) and \( \mathcal{T}_{10} \)). Indeed, from the knowledge of just the Konishi operator \( (M = 2) \) we can find

\[
\mathcal{T}_{\zeta_9} = -3402,
\]

(4.14)

\[
\mathcal{T}_{\zeta_3} = 0,
\]

(4.15)

\[
\mathcal{T}_{\zeta_5} = 360 S_1,
\]

(4.16)

and using the first few values one can easily obtain

\[
\mathcal{P}_{\zeta_3} = -12 \mathcal{P}_2 \left( S_1^4 + 14 S_1 S_2 - 6 (S_{2,1} + S_3) \right).
\]

(4.17)

However already for \( \mathcal{T}_{\zeta_7} \) we can not find any solution, despite that the possible ansatz contains only two binomial harmonic sums: \( S_2 \) and \( S_1^2 \). To find a solution we extend our ansatz such that it is not restricted by eq. (4.10), that is we take the ansatz with all possible binomial harmonic sums with the transcendentality 4:

\[
\text{Basis} \left[ \mathcal{T}_{12, \zeta_7}^{\text{wrap}} \right] = \left\{ S_4, S_{3,1}, S_{2,2}, S_{2,1,1}, S_1 S_3, S_1 S_{2,1}, S_1^2 S_2, S_1^4 \right\}.
\]

(4.18)

For this extended ansatz a solution exists (note that we have 35 equations for 8 variables):

\[
\mathcal{P}_{12, \zeta_7}^{\text{wrap}} = 224 S_1 \left( 12 S_1^3 + 151 S_1 S_2 - 72 S_3 - 52 S_{2,1} \right).
\]

(4.19)
Comparing this result to $P_3$ from eq. (4.6) one can see that the coefficients of $S_1S_3$ and $S_1S_2,1$ are different, so our simple assumption about a general form of $P_{12}^{\text{wrap}}$ is not correct and we should extend all bases for $\zeta_5$, $\zeta_3$ and rational contributions to $P_{12}^{\text{wrap}}$. We have found that a minimal extension can be performed by expanding all terms with the brackets in eq. (4.10), that is we should expand $P_2P_6T_8$, $P_2^2P_4T_8$, $P_2^2T_8$, $P_2^4T_8$, $P_2P_4T_{10}$, $P_2^2T_{10}$ and add all possible terms from these expansions to the corresponding ansatz for $T_{\zeta_5}$, $T_{\zeta_3}$ or $T_{\text{rational}}$ contributions. The ansatz for the $\zeta_5$ contribution to $P_{12}^{\text{wrap}}$ will contain the following binomial harmonic sums:

$$
\text{Basis} [P_{12,\zeta_5}^{\text{wrap}}] = \left\{ S_1^3S_4, S_1^2S_3, 1, S_1^2S_{2,2}, S_1^2S_{2,1,1}, S_1^3S_3, S_1^3S_2, 1, S_1^4S_2, S_6, S_1S_2S_3, S_1S_2S_{2,1}, S_1S_{2,2,1}, S_1S_{2,1,2}, S_1S_{3,2}, S_1S_5, (S_1S_2 - S_3 - S_2)^2 \right\},
$$

(4.20)

where the first line corresponds to the basis for $T_{\zeta_5}$ while the second line comes from the expansion of eq. (4.10). Note that for the $\zeta_5$ contribution we have found that the last term in eq. (4.20), which corresponds to $P_2^4$, may not be expanded and this property is also correct for the other contributions. So we have 16 binomial harmonic sums in the ansatz for the $\zeta_5$ contribution and the coefficients of these sums can be found from the known values. The final result for the $\zeta_5$ contribution is given by:

$$
P_{12,\zeta_5}^{\text{wrap}} = 8 \left( \frac{2044}{3} S_1S_2S_{2,1} - \frac{1760}{3} S_1^3S_{2,1} - \frac{436}{3} S_1^3S_2 - \frac{25}{3} S_6 - 80(-S_{2,1} + S_1S_2 - S_3)^2 
- 160S_1^3S_3 + 1024S_2^2S_4 + 656S_2^2S_{2,2} - 2024S_1^2S_{3,1} + 320S_2^2S_{2,1,1} 
- 320S_1S_{2,2,1} + 608 S_1S_2S_3 - 960 S_1S_3 \right).
$$

(4.21)

For the $\zeta_3$ contribution our minimal ansatz consists of $2^{8-1-2} + 19 = 51$ binomial harmonic sums, while we have only 35 values. To find the coefficients in the ansatz we use the same method as we used for the reconstruction of the $\zeta_3$ contribution to the ABA part of the anomalous dimension in eq. (2.24). With the help of the \texttt{LatticeReduce} function from \textsc{Mathematica}, which realizes the LLL-algorithm, we have found the following general expression for the $\zeta_3$ contribution to $P_{12}^{\text{wrap}}$:

$$
P_{12,\zeta_3}^{\text{wrap}} = 32 \left( \frac{5}{6} S_1^3S_2 + 12 S_1S_{2,1} - \frac{10}{3} S_1^4S_{2,1,1} + \frac{46}{3} S_1^3S_2 - \frac{49}{3} S_1^3S_2 - \frac{10}{3} S_1^4S_4 
+ \frac{248}{3} S_1^3S_{2,1} + 8 S_1^3S_2S_3 - 8 S_1^3S_{2,3} + 16 S_1^3S_{3,2} - 16 S_1^3S_{4,1} - 80 S_1^3S_{2,2,1} - 24 S_1^3S_{2,1,1,1} 
- 66 S_1^2S_3^2S_2 - \frac{332}{3} S_2^2S_3S_{2,2} - 66 S_1^2S_{2,2}S_2 + \frac{82}{3} S_1^2S_3^2 - 118 S_1^2S_2S_4 + 20 S_1^2S_6 + 12 S_2^2S_4 
+ \frac{116}{3} S_1^2S_3 + \frac{152}{3} S_1^2S_4 + 22 S_1^2S_5 + 32 S_1^2S_2S_{2,1,1} + \frac{92}{3} S_1^2S_{2,1,3} - 22 S_1^2S_{2,3,1} 
+ 14 S_1^2S_{4,1,1} + 34 S_1^2S_{2,2,1,1} + 20 S_1^2S_{2,1,1,1,1} + 24 S_1^2S_{2,3,1} + 96 S_1^2S_2S_5 - 48 S_1^2S_2S_{2,1} 
- 48 S_1^2S_2S_{2,2} + 32 S_1^2S_{2,1} + 32 S_1S_2S_{2,2,1} + 32 S_1S_3 (2S_2 - 2S_3 + S_{2,1,1} + S_4) 
+ \frac{16}{3} S_1S_{2,1}^2 (2S_2 - 2S_{3,1} + S_{2,1,1} + S_4) + 8 S_2 (S_1S_2 - S_{2,1} - S_3)^2 \right) .
$$

(4.22)
The last part is the rational contribution. In this case our minimal ansatz consists of \(2^{11−1−2} + 67 = 323\) binomial harmonic sums. The available 35 values are enough to find all coefficients in this ansatz with the help of the LLL-algorithm, but for this purpose we used a C++ implementation of the LLL-algorithm in the form of fpLLL-program [84]. After about one hour of computational time we obtained the following general expression for the rational contribution to \(P_{12}^{\text{wrap}}\):

\[
P_{12,\text{rational}}^{\text{wrap}} = -\frac{32}{3} \left( \frac{5}{4} S_1 S_3 S_{1,1,1} - \frac{5}{4} S_1 S_{2,1,2} - 6 S_1 S_2 S_{3,1,2} + 6 S_1 S_{1,1,2} - 25 S_1 S_2 S_{2,1,2} + 5 S_1 S_{2,1,4} \\
-4 S_1 S_{2,2,3} + 33 S_1 S_{3,3,1} + 7 S_3 S_{3,1,3} - 4 S_3 S_{3,2,2} - 12 S_3 S_{3,3,1} + 5 S_3 S_{4,1,1} - 5 S_3 S_{5,1,1} \\
+13 S_2 S_{2,2,1,2} + S_2 S_{2,1,3,1} + S_2 S_{2,2,1,2} + 12 S_2 S_{2,2,2,1} - 13 S_2 S_{2,3,1,1} + 4 S_2 S_{3,1,1,2} \\
-13 S_2 S_{3,1,2,1} - 13 S_2 S_{3,2,1,1} + 5 S_4 S_{2,1,1,2} - 5 S_4 S_{3,1,1,1,1} - 27 S_5 S_{3,2,1,2} \\
+10 S_5 S_{2,1,2,1,2} + 2 S_5 S_{2,2,4,4} - 55 S_5 S_{2,3,2,1,1} + 13 S_5 S_{2,5,1,1} + 37 S_5 S_{3,3,1,1} \\
-18 S_5 S_{3,1,1,1} + 54 S_5 S_{3,1,2,1,2} - 18 S_5 S_{3,1,4,1} - 21 S_5 S_{3,2,3,1} + 15 S_5 S_{4,1,1} - 13 S_5 S_{4,1,3} \\
+15 S_5 S_{4,1,3,1} - 12 S_5 S_{5,1,2} + 22 S_5 S_{2,1,3,2} + 15 S_5 S_{2,1,4,1} - 23 S_5 S_{2,2,1,3} + 6 S_5 S_{2,2,2,2} \\
+51 S_5 S_{2,2,3,1} - 10 S_5 S_{3,2,3,1,2} + 24 S_5 S_{3,2,3,1,2} - 27 S_5 S_{3,3,1,1,3} - 45 S_5 S_{3,3,1,2,2} + 24 S_5 S_{3,3,1,3,1} \\
-29 S_5 S_{3,3,2,1,2,1} - 7 S_5 S_{3,3,2,1,2,1} + S_5 S_{3,3,3,1,1} + 12 S_5 S_{3,2,1,1,1,1} - 4 S_5 S_{2,2,1,2,1,2} + 33 S_5 S_{2,1,2,1,2} \\
+4 S_5 S_{2,1,3,1,1,1} - S_5 S_{2,2,2,1,1,1} + 12 S_5 S_{3,3,1,1,1,1} - 12 S_5 S_{3,1,1,1,1,2} - 14 S_5 S_{3,1,1,1,2,1} \\
-14 S_5 S_{3,1,2,1,1,2} + 12 S_5 S_{3,1,2,1,1,2} + 37 S_5 S_{2,2,2,1,2} - 48 S_5 S_{3,1,2,1,1,2} + 13 S_5 S_{3,1,2,1,1,2} \\
+45 S_5 S_{2,1,4,1} - 30 S_5 S_{2,1,6} - 25 S_5 S_{2,2,3} + 16 S_5 S_{2,2,5} - 17 S_5 S_{2,3,3,1} + 53 S_5 S_{2,2,3,1} \\
-150 S_5 S_{2,3,1,1} - 66 S_5 S_{2,3,3,1,1} + 67 S_5 S_{2,3,3,1,1} + 5 S_5 S_{2,1,1,3,1,1} + 25 S_5 S_{3,3,1,1} \\
-50 S_5 S_{2,1,3,1,2} + 29 S_5 S_{2,3,3,1,3} - S_5 S_{2,3,3,1,5} + 10 S_5 S_{2,3,3,2,2} + 52 S_5 S_{2,3,3,2,4} - 57 S_5 S_{2,3,4,3,1} \\
+24 S_5 S_{3,3,5,1} + 15 S_5 S_{4,1,4,1,2} + 30 S_5 S_{4,1,4,1,4} - 10 S_5 S_{4,1,4,2,3} - 4 S_5 S_{4,1,4,4,1} - 22 S_5 S_{5,5,1,1} \\
+20 S_5 S_{5,1,3,1} + 20 S_5 S_{5,2,2,2} - 4 S_5 S_{5,3,1,1} - 30 S_5 S_{6,1,2,2} + 30 S_5 S_{7,1,1,1} + 52 S_5 S_{2,6,2,1,2} \\
-18 S_5 S_{2,2,1,2,4} - 23 S_5 S_{2,2,1,3,1,3} - 31 S_5 S_{2,2,1,3,3,3} - 18 S_5 S_{2,1,4,2} + 3 S_5 S_{2,1,5,1,1} + 86 S_5 S_{2,2,2,1,2,1} \\
+18 S_5 S_{2,2,1,4,1} - 45 S_5 S_{2,2,2,2,1,1} - S_5 S_{2,2,2,2,3,2} + 10 S_5 S_{2,2,2,3,2,2} - 44 S_5 S_{2,2,4,1,1} - 47 S_5 S_{2,3,2,3,1,1} \\
-4 S_5 S_{2,3,1,1,3} + 14 S_5 S_{2,3,2,2,2} + 6 S_5 S_{2,3,3,3,1} - 18 S_5 S_{2,4,1,2,1} + 18 S_5 S_{2,5,1,1,1} + 25 S_5 S_{3,3,1,1,2} \\
+44 S_5 S_{3,1,1,4} - 62 S_5 S_{3,3,1,2,1,2} + 2 S_5 S_{3,3,1,3,2} - 14 S_5 S_{3,3,1,4,1} - 52 S_5 S_{3,3,2,1,1,1} - 14 S_5 S_{3,3,2,1,3} \\
+30 S_5 S_{3,2,2,2,2} - 13 S_5 S_{3,3,2,2,2} + 5 S_5 S_{3,3,3,1,2,1} + 23 S_5 S_{3,3,3,2,1,1} + 9 S_5 S_{3,4,1,1,1} + 10 S_5 S_{4,1,1,1,3} \\
-48 S_5 S_{4,1,2,2,2} + 2 S_5 S_{4,1,3,1,3} + 35 S_5 S_{4,2,1,2,1} - 23 S_5 S_{4,2,2,2,1} + 14 S_5 S_{4,3,3,1,1} - 20 S_5 S_{5,1,1,1,2} \\
+18 S_5 S_{5,1,2,1,1} + 18 S_5 S_{5,2,1,1,1} + 38 S_5 S_{5,2,1,1,1,1} - 6 S_5 S_{5,2,1,1,1,2} + 4 S_5 S_{5,2,1,1,1,4} - 2 S_5 S_{5,2,1,2,1,3} \\
-27 S_5 S_{2,1,2,2,2} + 22 S_5 S_{2,1,2,3,1,2} - 25 S_5 S_{2,1,3,1,1,2} + 7 S_5 S_{2,1,3,2,1,2} - 12 S_5 S_{2,1,4,1,1,1} 
\]
corresponding results, coming from the above six-loop anomalous dimension. We will use

\[ N \]

In this section we will discuss the known weak-coupling constraints on the six-loop anomalous dimension. The result for the full anomalous dimension can be found in the ancillary files of the arXiv version of the paper and on web-page: http://thd.pnpi.spb.ru/~velizh/6loop/.

5 Weak-coupling constraints

In this section we will discuss the known weak-coupling constraints on the six-loop anomalous dimension of twist-two operators in \( \mathcal{N} = 4 \) SYM theory and compare them with the corresponding results, coming from the above six-loop anomalous dimension. We will use three classes of constraints, which are provided by the BFKL equation and by the generalized double-logarithmic equation at \( M = -2 + \omega \) and at \( M = -r + \omega \), where \( r = 4, 6, 8, \ldots \).

A splitting function \( P(x) \), which is related to the anomalous dimension through a

\[ +38 S_1^2 S_{2,2,1,1,3} - 60 S_1^2 S_{2,2,1,2,2} + 48 S_1^2 S_{2,2,1,3,1} - 34 S_1^2 S_{2,2,1,1,2} - 2 S_1^2 S_{2,2,2,1} \\
-6 S_1^2 S_{2,2,3,1,1} - 14 S_1^2 S_{2,3,1,1,2} + 25 S_1^2 S_{3,1,1,1,2} + 42 S_1^2 S_{3,3,1,1,2} - 56 S_1^2 S_{2,3,1,1,1,1} \\
-8 S_1^2 S_{3,1,1,1,3} - 20 S_1^2 S_{3,1,1,2,2} + 13 S_1^2 S_{3,1,1,3,1} - 26 S_1^2 S_{3,1,2,1,2} + 9 S_1^2 S_{3,1,2,2,1} \\
+18 S_1^2 S_{3,1,3,1,1} - 9 S_1^2 S_{3,2,1,1,2} + 42 S_1^2 S_{3,3,1,2,1} + 23 S_1^2 S_{3,3,2,1,1} + 7 S_1^2 S_{3,3,3,1,1} \\
-6 S_1^2 S_{4,1,1,1,1,1} + 6 S_1^2 S_{5,1,1,1,1,1} - 18 S_1^2 S_{2,1,1,1,1,2,2} - 10 S_1^2 S_{2,1,1,1,1,3,1} - 7 S_1^2 S_{2,1,1,2,1,1,2} \\
-4 S_1^2 S_{2,1,1,2,2,1} - 11 S_1^2 S_{2,1,1,3,1,1} - 20 S_1^2 S_{2,1,2,1,1,2} - 36 S_1^2 S_{2,1,2,2,1,1,1} + 6 S_1^2 S_{2,1,3,1,1,1,1} \\
-18 S_1^2 S_{2,2,2,1,1,1,2} - 7 S_1^2 S_{2,2,2,2,1,1,1} + 18 S_1^2 S_{2,3,1,1,1,1,1} + 14 S_1^2 S_{3,1,1,1,1,1,2} + 18 S_1^2 S_{3,1,1,1,1,2,1} \\
+13 S_1^2 S_{3,1,2,1,1,1,1} + 8 S_1^2 S_{3,2,1,1,1,1,1} + 18 S_1^2 S_{3,3,1,1,1,1,1} - 30 S_1^2 S_{2,1,1,1,1,1,1,1} + 30 S_1^2 S_{3,1,1,1,1,1,1,1,1} \\
+68 S_1^2 S_{2,1,2} - 144 S_1 S_2 S_{2,1,2,2,1} + 68 S_1 S_2 S_{2,2,2,1,2} - 48 S_1 S_2 S_{2,2,3,1,2} - 8 S_1 S_2 S_{2,2,4,1,2} \\
-48 S_1 S_2 S_{2,2,5,1,1} + 24 S_1 S_2 S_{2,2,6,1,1} + 18 S_1 S_2 S_{2,2,7,1,1} + 144 S_1 S_2 S_{2,2,8,1,1} - 68 S_1 S_2 S_{2,2,9,1,1} \\
-68 S_1 S_2 S_{2,3,1,1,1,2} + 48 S_1 S_2 S_{2,3,1,2,1,1} - 24 S_1 S_2 S_{3,1,1,3,1} - 46 S_1 S_2 S_{3,1,2,1,3} + 24 S_1 S_2 S_{3,2,1,3,2} \\
-8 S_1 S_2 S_{3,2,2,1,2} - 2 S_1 S_2 S_{3,3,1,3,1} + 44 S_1 S_2 S_{3,3,1,3,3} - 48 S_1 S_2 S_{3,3,1,4,2} + 18 S_1 S_2 S_{3,3,4,1,2} \\
+48 S_1 S_2 S_{3,5,1,1,1} + 8 S_1 S_2 S_{3,5,1,1,1,1} - 24 S_1 S_2 S_{3,5,1,2,2,2} - 64 S_1 S_2 S_{3,5,1,2,2,2} - 2 S_1 S_2 S_{3,5,1,3,1,1} \\
+24 S_1 S_2 S_{3,5,1,3,1,3} - 24 S_1 S_2 S_{3,5,1,3,2,1} - 2 S_1 S_2 S_{3,5,1,3,2,2} + 44 S_1 S_2 S_{3,5,1,3,2,2} \\
+24 S_1 S_2 S_{3,5,1,3,2,2} + 64 S_1 S_2 S_{3,5,1,3,2,2} + 8 S_1 S_2 S_{3,5,1,3,2,2} - 2 S_1 S_2 S_{3,5,1,3,2,2} \\
+64 S_1 S_2 S_{3,5,1,3,2,2} + 64 S_1 S_2 S_{3,5,1,3,2,2} - 2 S_1 S_2 S_{3,5,1,3,2,2} \\
-8 S_1 S_2 S_{3,5,1,3,2,2} + 48 S_1 S_2 S_{3,5,1,3,2,2} + 8 S_1 S_2 S_{3,5,1,3,2,2} \\
-12 \left( S_1 S_2 - S_3 - S_2,1 \right)^2 S_{2,1,1,2} + 12 \left( S_1 S_2 - S_3 - S_2,1 \right)^2 S_{3,1,1,1} \right), \quad (4.23)
Mellin transformation
\[ \gamma(M) = \int_0^1 dx x^M P(x), \] (5.1)
contains some powers of \( \ln x \) in each order of perturbative theory when \( x \) tends to zero. This region of small \( x \) is very interesting from the experimental point of view as there are some theoretical methods which allow to sum all such large logarithms in all orders of perturbative theory. Mostly small-\( x \) physics is related to the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation \([51–53]\) and double-logarithmic equations \([57–59]\). So, as a splitting function is related to the anomalous dimension through a Mellin transformation, an expansion near \( x = 0 \) will correspond to some poles in the anomalous dimension, which appear when the argument of the anomalous dimension gets a negative value, as the harmonic sums \((2.11)\), being a generalization of the \( \Psi \)-function, have poles at negative values of their argument.

5.1 BFKL equation
The relation between the anomalous dimension of twist-two operators and the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation \([51–53]\) and its next-to-leading logarithm approximation (NLLA) generalization \([85, 86]\) emerges upon analytic continuation of the function \( \gamma(g, M) \) to complex values of \( M \). This is straightforward in the one-loop case since
\[ \gamma_2(M) = 8g^2 S_1(M) = 8g^2 (\Psi(M + 1) - \Psi(1)), \] (5.2)
where \( \Psi(x) = \frac{d}{dx} \log \Gamma(x) \) is the digamma function. At any loop order one expects singularities at all negative integer values of \( M \). The first in this series of singular points,
\[ M = -1 + \omega, \] (5.3)
corresponds to the so-called BFKL pomeron. In the above formula \( \omega \) should be considered infinitesimally small. The BFKL equation relates \( \gamma(g) \) and \( \omega \) in the vicinity of the point \( M = -1 + \omega \). It predicts that, if expanded in \( g \), the \( \ell \)-loop anomalous dimension \( \gamma_{2\ell}(\omega) \) exhibits poles in \( \omega \). Moreover, the residues and the order of the poles can be derived directly from the BFKL equation. The BFKL equation has been formulated up to the next-to-leading logarithm approximation (NLLA) and determines the leading and next-to-leading poles of \( \gamma_{2\ell}(\omega) \). The NLLA BFKL equation for twist-two operators in \( \mathcal{N} = 4 \) SYM theory in the dimensional reduction scheme can be written as follows \([85, 86]\)
\[ \frac{\omega}{-4g^2} = \chi(\gamma) - g^2 \delta(\gamma), \] (5.4)
where
\[ \chi(\gamma) = \Psi\left(-\frac{\gamma}{2}\right) + \Psi\left(1 + \frac{\gamma}{2}\right) - 2\Psi(1), \] (5.5)
\[ \delta(\gamma) = 4\chi''(\gamma) + 6\zeta_3 + 2\zeta_2\chi(\gamma) + 4\chi(\gamma)\chi'(\gamma) \]
\[ -\frac{\pi^3}{\sin \frac{\pi}{2}} - 4\Phi\left(-\frac{\gamma}{2}\right) - 4\Phi\left(1 + \frac{\gamma}{2}\right), \] (5.6)
The function $\Phi(\gamma)$ is given by

$$
\Phi(\gamma) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\gamma)^2} \left[ \Psi(k+\gamma+1) - \Psi(1) \right].
$$

(5.7)

Upon using the expansion (2.2), one easily determines the leading singularity structure. Perturbatively expanding the anomalous dimension in the argument of the right-hand sided functions $\chi(\gamma)$ and $\delta(\gamma)$ in eq. (5.4) one can find a relation between the anomalous dimension at $M = -1 + \omega$ and highest poles order by order, which can be written as:

$$
\gamma = \left( 2 + 0 \omega + O(\omega^2) \right) \left( -\frac{4g^2}{\omega} \right) - \left( 0 + 0 \omega + O(\omega^2) \right) \left( -\frac{4g^2}{\omega} \right)^2
$$

$$
+ \left( 0 + \zeta_3 \omega + O(\omega^2) \right) \left( -\frac{4g^2}{\omega} \right)^3 - \left( 4 \zeta_3 + \frac{5}{4} \zeta_4 \omega + O(\omega^2) \right) \left( -\frac{4g^2}{\omega} \right)^4
$$

$$
- \left( 0 + \left( 2\zeta_2 \zeta_3 + 16 \zeta_5 \right) \omega + O(\omega^2) \right) \left( -\frac{4g^2}{\omega} \right)^5
$$

$$
- \left( 0 \zeta_2 \zeta_3 + 4 \zeta_5 \right) + \left( 3 \zeta_3^2 - \frac{143}{48} \zeta_6 \right) \omega + O(\omega^2) \right) \left( -\frac{4g^2}{\omega} \right)^6 \pm \ldots.
$$

(5.8)

The last line of the above equation gives the prediction for the analytic continuation of the six-loop anomalous dimension at $M = -1 + \omega$ and our full result satisfies these constraints.

### 5.2 Generalized double-logarithmic equation: $M = -2 + \omega$

Another class of constraints on the anomalous dimension of twist-two operators follows from the double-logarithmic asymptotics of the scattering amplitudes. The double-logarithmic asymptotics of the scattering amplitudes were investigated in QED and QCD in the papers [54–56] and [57–59] (see also arXiv version of ref. [67]). It corresponds to summing the leading terms $(\alpha \ln^2 s)^k$ in all orders of perturbation theory. In the combination with a Mellin transformation, the double-logarithmic asymptotics allow to predict the singular part of anomalous dimensions near the point $M = -2$. For our purpose and in our notation the double-logarithmic equation has the following form

$$
\gamma(2 \omega + \gamma) = -16g^2.
$$

(5.9)

The solution of this equation gives a prediction for the highest pole $(g^{2k}/\omega^{2k-1})$ in all orders of perturbative theory:

$$
\gamma = -\omega + \omega \sqrt{1 - \frac{16g^2}{\omega^2}} = 2 \left( -\frac{4g^2}{\omega} \right) - 2 \left( -\frac{4g^2}{\omega^3} \right)^2 + 4 \left( -\frac{4g^2}{\omega^5} \right)^3 - 10 \left( -\frac{4g^2}{\omega^7} \right)^4
$$

$$
+ 28 \left( -\frac{4g^2}{\omega^9} \right)^5 - 84 \left( -\frac{4g^2}{\omega^{11}} \right)^6 + \ldots
$$

(5.10)

The investigation of the analytic properties of the anomalous dimension of twist-2 operators in $\mathcal{N} = 4$ SYM theory led to the suggestion about a simple generalization of the double-logarithmic equation [60]. The main idea was that in eq. (5.9) the corrections to

\footnote{For the first time, a such generalization was suggested by Lev N. Lipatov and Andrei Onishchenko at 2004, but was not published. Then, it was improved by Lev N. Lipatov in ref. [32].}
the leading order equation will modify only the right-hand side and that such modification admit, besides an expansion in the coupling constant $g^2$, only the appearance of a regular terms depending on $\omega$ (and, possibly, $\gamma$). Substituting the results for the analytic continuation of the anomalous dimension of twist-2 operators near $M = -2 + \omega$ into eq. (5.9) we indeed find the following form of the generalized double-logarithmic equation [60]

$$\gamma(2\omega + \gamma) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} c_k^m \omega^m g^{2k}. \tag{5.11}$$

The coefficients $c_k^m$ can be found in Appendix of ref. [61], but for the test of the six-loop anomalous dimension we need only the fact that the perturbative expansion of the left hand side of the above equation (5.11) does not contain any poles near $M = -2$, which is indeed correct for the result obtained in this paper. We present the result for the analytic continuation of the six-loop anomalous dimension at $M = -2 + \omega$ in Appendix B, and one can see that there are about one hundred terms up to $g^{12}/\omega^2$, which can be checked by the generalized double-logarithmic equation (5.11), so we have very strong test for the correctness of our general result for the six-loop anomalous dimension. Note also that with the generalized double-logarithmic equation (5.11) we can control even the $\zeta_9$ term in the six-loop anomalous dimension coming from eq. (4.14) as its analytic continuation is proportional to $\zeta_9/\omega^2$, what is impossible with the BFKL equation (5.5).

5.3 Generalized double-logarithmic equation: $M = -r + \omega, \ r = 2, 4, 6, \ldots$.

In ref. [60] we have found another generalization of the double-logarithmic equation, which hold true not only for $M = -2 + \omega$, but for all other negative even values $M = -r + \omega, \ r = 2, 4, 6, \ldots$. This generalization is related to the analytic properties of the reciprocity-respecting anomalous dimension $\mathcal{P}(M)$, which near $M = -r + \omega, \ r = 2, 4, 6, \ldots$ can be written as:

$$\mathcal{P}_{DL}(\omega, r) = 2 \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \mathcal{D}_m^k(r) \omega^m \left(-\frac{4 g^2}{\omega}\right)^k \tag{5.12}$$

where some coefficients $\mathcal{D}_m^k(r)$ can be found in ref. [60]. For the test of the six-loop anomalous dimension we need only the fact that according to eq. (5.12) the six-loop reciprocity-respecting anomalous dimension $\mathcal{P}_{12}(M)$ does not contain any poles higher than $1/\omega^6$, which is indeed correct for the obtained result.

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\[ \frac{\hat{p}_{12}^{\text{rational}}}{128} = -180 S_{2,2,7} + 540 S_{3,1,7} - 592 S_{3,2,6} + 644 S_{4,1,6} - 684 S_{4,2,5} + 700 S_{5,1,5} \\
-700 S_{5,2,4} + 636 S_{6,1,4} - 444 S_{6,2,3} + 176 S_{7,1,3} - 96 S_{7,2,2} - 228 S_{1,2,2,6} \\
-4 S_{1,2,7,1} + 700 S_{1,3,1,6} - 728 S_{1,3,2,5} + 732 S_{1,4,1,5} - 728 S_{1,4,2,4} + 660 S_{1,5,1,4} \\
-548 S_{1,5,2,3} + 344 S_{1,6,1,3} - 224 S_{1,6,2,2} + 52 S_{1,7,1,2} - 28 S_{1,7,2,1} + 4 S_{2,1,2,6} \\
-4 S_{2,1,7,1} - 640 S_{2,2,1,6} + 988 S_{2,2,2,5} + 782 S_{2,2,3,4} + 78 S_{2,2,4,3} + 78 S_{2,2,5,2} \\
-70 S_{2,2,6,1} - 864 S_{2,3,1,5} + 874 S_{2,3,2,4} + 2 S_{2,3,5,1} - 820 S_{2,4,1,4} + 698 S_{2,4,2,3} \\
+2 S_{2,4,4,1} - 478 S_{2,5,1,3} + 326 S_{2,5,2,2} + 2 S_{2,5,3,1} - 106 S_{2,6,1,2} + 72 S_{2,6,2,1} \\
-16 S_{2,7,1,1} + 592 S_{3,1,1,6} - 676 S_{3,1,2,5} - 242 S_{3,1,3,4} - 242 S_{3,1,4,3} - 242 S_{3,1,5,2} \\
+282 S_{3,1,6,1} - 768 S_{3,2,1,5} + 1144 S_{3,2,2,4} + 260 S_{3,2,3,3} + 260 S_{3,2,4,2} - 288 S_{3,2,5,1} \\
-290 S_{3,3,1,4} + 230 S_{3,3,2,3} - 138 S_{3,4,1,3} + 84 S_{3,4,2,2} - 14 S_{3,5,1,2} + 8 S_{3,5,2,1} \\
+660 S_{4,1,1,5} - 746 S_{4,1,2,4} - 278 S_{4,1,3,3} - 278 S_{4,1,4,2} + 286 S_{4,1,5,1} - 762 S_{4,2,1,4} \\
+1092 S_{4,2,2,3} + 296 S_{4,2,3,2} - 290 S_{4,2,4,1} - 138 S_{4,3,1,3} + 84 S_{4,3,2,2} - 14 S_{4,4,1,2} \\
+8 S_{4,4,2,1} + 636 S_{5,1,1,4} - 726 S_{5,1,2,3} - 302 S_{5,1,3,2} + 276 S_{5,1,4,1} - 470 S_{5,2,1,3} \\
+778 S_{5,2,2,2} - 274 S_{5,2,3,1} - 14 S_{5,3,1,2} + 8 S_{5,3,2,1} + 368 S_{5,6,1,3} - 492 S_{6,1,2,2} \\
+240 S_{6,1,3,1} - 144 S_{6,2,1,2} + 16 S_{6,2,2,1} + 80 S_{7,1,1,2} - 32 S_{7,1,2,1} - 32 S_{7,2,1,1} \\
-8 S_{1,1,1,2,6} + 8 S_{1,1,1,2,7} - 8 S_{1,1,2,1,6} - 236 S_{1,1,2,2,5} + 4 S_{1,1,2,3,4} + 4 S_{1,1,2,4,3} \\
+4 S_{1,1,2,5,2} - 28 S_{1,1,2,6,1} + 788 S_{1,1,3,1,5} - 776 S_{1,1,3,2,4} - 4 S_{1,1,3,5,1} + 700 S_{1,1,4,1,4} \\
-620 S_{1,1,4,2,3} - 4 S_{1,1,4,4,1} + 440 S_{1,1,5,1,3} - 324 S_{1,1,5,2,2} - 4 S_{1,1,5,3,1} + 128 S_{1,1,6,1,2} \\
-96 S_{1,1,6,2,1} + 32 S_{1,1,7,1,1} - 8 S_{1,2,1,1,6} + 20 S_{1,2,1,2,5} + 4 S_{1,2,1,3,4} + 4 S_{1,2,1,4,3} \\
+4 S_{1,2,1,5,2} - 24 S_{1,2,1,6,1} - 736 S_{1,2,2,1,5} + 1070 S_{1,2,2,2,4} + 82 S_{1,2,2,3,3} + 82 S_{1,2,2,4,2} \\
-22 S_{1,2,2,5,1} - 870 S_{1,2,3,1,4} + 784 S_{1,2,3,2,3} - 2 S_{1,2,3,3,2} + 18 S_{1,2,3,4,1} - 600 S_{1,2,4,1,3} \\
+454 S_{1,2,4,2,2} + 18 S_{1,2,4,3,1} - 214 S_{1,2,5,1,2} + 176 S_{1,2,5,2,1} - 76 S_{1,2,6,1,1} + 704 S_{1,3,1,1,5} \\
-794 S_{1,3,1,2,4} - 302 S_{1,3,1,3,3} - 302 S_{1,3,1,4,2} + 306 S_{1,3,1,5,1} - 794 S_{1,3,2,1,4} + 1174 S_{1,3,2,2,3} \\
+312 S_{1,3,2,3,2} - 304 S_{1,3,2,4,1} - 198 S_{1,3,3,1,3} + 140 S_{1,3,3,2,2} + 2 S_{1,3,3,3,1} - 54 S_{1,3,4,1,2} \\
+38 S_{1,3,4,2,1} - 12 S_{1,3,5,1,1} + 664 S_{1,4,1,1,4} - 756 S_{1,4,1,2,3} - 310 S_{1,4,1,3,2} + 292 S_{1,4,1,4,1}
\[-576 S_{1,4,2,1,3} + 904 S_{1,4,2,2,2} - 286 S_{1,4,2,3,1} - 54 S_{1,4,3,1,2} + 38 S_{1,4,3,2,1} - 12 S_{1,4,4,1,1} + 456 S_{1,5,1,1,3} - 568 S_{1,5,1,2,2} + 258 S_{1,5,1,3,1} - 244 S_{1,5,2,1,2} + 110 S_{1,5,2,2,1} - 12 S_{1,5,3,1,1} + 172 S_{1,6,1,1,2} - 92 S_{1,6,1,2,1} + 120 S_{1,6,2,1,1} + 56 S_{1,7,1,1,1} - 8 S_{2,1,1,1,1} + 60 S_{2,1,1,1,2} + 4 S_{2,1,1,4,3} + 4 S_{2,1,1,5,2} - 24 S_{2,1,1,6,1} + 16 S_{2,1,2,1,5} + 68 S_{2,1,2,2,4} - 14 S_{2,1,2,3,3} - 14 S_{2,1,2,4,2} + 56 S_{2,1,2,5,1} - 340 S_{2,1,3,1,4} + 316 S_{2,1,3,2,3} - 2 S_{2,1,3,3,2} + 16 S_{2,1,3,4,1} - 260 S_{2,1,4,1,3} + 212 S_{2,1,4,2,2} + 16 S_{2,1,4,3,1} - 122 S_{2,1,5,1,2} + 112 S_{2,1,5,2,1} - 60 S_{2,1,6,1,1} - 640 S_{2,2,1,1,5} + 704 S_{2,2,1,2,4} + 264 S_{2,2,1,3,3} + 264 S_{2,2,1,4,2} - 240 S_{2,2,1,5,1} + 1226 S_{2,2,2,1,4} - 1816 S_{2,2,2,2,3} - 381 S_{2,2,2,3,2} + 245 S_{2,2,2,4,1} + 638 S_{2,2,3,1,3} - 540 S_{2,2,3,2,2} - 18 S_{2,2,3,3,1} + 289 S_{2,2,4,1,2} - 179 S_{2,2,4,2,1} + 56 S_{2,2,5,1,1} - 810 S_{2,3,1,1,4} + 918 S_{2,3,1,2,3} + 367 S_{2,3,1,3,2} - 339 S_{2,3,1,4,1} + 742 S_{2,3,2,1,3} - 1136 S_{2,3,2,2,2} + 322 S_{2,3,2,3,1} + 93 S_{2,3,3,1,2} - 77 S_{2,3,3,2,1} + 36 S_{2,3,4,1,1} - 600 S_{2,4,1,1,3} + 730 S_{2,4,1,2,2} - 312 S_{2,4,1,3,1} + 358 S_{2,4,2,1,2} - 200 S_{2,4,2,2,1} + 36 S_{2,4,3,1,1} - 258 S_{2,5,1,1,2} + 136 S_{2,5,1,2,1} + 206 S_{2,5,2,1,1} - 116 S_{2,6,1,1,1} + 620 S_{3,1,1,1,5} - 700 S_{3,1,1,2,4} - 260 S_{3,1,1,3,3} - 260 S_{3,1,1,4,2} + 280 S_{3,1,1,5,1} - 266 S_{3,1,2,1,4} + 602 S_{3,1,2,2,3} + 293 S_{3,1,2,3,2} - 307 S_{3,1,2,4,1} + 50 S_{3,1,3,1,3} + 84 S_{3,1,3,2,2} - 110 S_{3,1,3,3,1} + 65 S_{3,1,4,1,2} - 371 S_{3,1,4,2,1} + 348 S_{3,1,5,1,1} - 738 S_{3,2,1,1,4} + 832 S_{3,2,1,2,3} + 329 S_{3,2,1,3,2} - 309 S_{3,2,1,4,1} + 550 S_{3,2,2,1,3} - 1126 S_{3,2,2,2,2} + 456 S_{3,2,2,3,1} - 313 S_{3,2,3,1,2} + 347 S_{3,2,3,2,1} - 342 S_{3,2,4,1,1} + 198 S_{3,3,1,1,3} + 248 S_{3,3,1,2,2} - 108 S_{3,3,1,3,1} + 102 S_{3,3,2,1,2} - 48 S_{3,3,2,2,1} + 4 S_{3,3,3,1,1} - 68 S_{3,4,1,1,2} + 36 S_{3,4,1,2,1} + 44 S_{3,4,2,1,1} - 16 S_{3,5,1,1,1} + 616 S_{4,1,1,1,4} - 700 S_{4,1,1,2,3} - 284 S_{4,1,1,3,2} + 272 S_{4,1,1,4,1} - 232 S_{4,1,2,1,3} + 594 S_{4,1,2,2,2} - 304 S_{4,1,2,3,1} + 60 S_{4,1,3,1,2} - 362 S_{4,1,3,2,1} + 344 S_{4,1,4,1,1} - 564 S_{4,2,1,1,3} + 684 S_{4,2,1,2,2} - 288 S_{4,2,1,3,1} + 280 S_{4,2,2,1,2} + 188 S_{4,2,2,2,1} - 336 S_{4,2,3,1,1} - 68 S_{4,3,1,1,2} + 36 S_{4,3,1,2,1} + 44 S_{4,3,2,1,1} - 16 S_{4,4,1,1,1} + 452 S_{5,1,1,1,3} - 556 S_{5,1,1,2,2} + 242 S_{5,1,1,3,1} - 132 S_{5,1,2,1,2} - 192 S_{5,1,2,2,1} + 330 S_{5,1,3,1,1} - 262 S_{5,2,1,1,2} + 140 S_{5,2,1,2,1} + 98 S_{5,2,2,1,1} - 16 S_{5,3,1,1,1} + 188 S_{6,1,1,1,2} - 96 S_{6,1,2,1,1} - 128 S_{6,2,1,1,1} + 64 S_{7,1,1,1,1} + 16 S_{7,1,1,1,1,6} - 24 S_{7,1,1,1,1,5} - 8 S_{1,1,1,1,3,4} - 8 S_{1,1,1,1,5,2} + 32 S_{1,1,1,1,6,1} - 16 S_{1,1,1,2,1,5} - 200 S_{1,1,1,2,2,4} + 20 S_{1,1,1,2,3,3} + 20 S_{1,1,1,2,4,2} - 68 S_{1,1,1,2,5,1} + 744 S_{1,1,1,3,1,4} - 668 S_{1,1,1,3,2,3} + 4 S_{1,1,1,3,3,2} - 24 S_{1,1,1,3,4,1} + 508 S_{1,1,1,4,1,3} - 392 S_{1,1,1,4,2,2} - 24 S_{1,1,1,4,3,1} + 192 S_{1,1,1,5,1,2}
\[-176 S_{1,1,5,2,1} + 80 S_{1,1,1,1,6,1,1} - 16 S_{1,1,2,1,1,1,5} + 28 S_{1,1,2,1,2,1,4} + 16 S_{1,1,2,1,3,3,1} + 16 S_{1,2,1,1,2,1,4,2} - 44 S_{1,1,2,1,5,1,1} - 740 S_{1,1,2,1,2,1,4,1} + 974 S_{1,1,2,1,2,2,3,1} + 64 S_{1,1,2,1,2,3,2,3} + 38 S_{1,1,2,2,4,1} - 678 S_{1,1,2,3,1,3,1} + 534 S_{1,1,2,3,2,2,2} + 50 S_{1,1,2,3,3,3,1} - 308 S_{1,1,2,4,1,1,2} + 310 S_{1,1,2,4,2,1,1} - 164 S_{1,1,2,5,1,1,1} + 712 S_{1,1,3,1,1,1,4} - 810 S_{1,1,3,1,2,1,3} - 328 S_{1,1,3,1,3,1,2} + 310 S_{1,1,3,1,4,1,1} - 654 S_{1,1,3,2,1,3,1} + 1006 S_{1,1,3,2,2,2,2} - 282 S_{1,1,3,2,3,2,3} - 96 S_{1,1,3,3,3,1,2} + 90 S_{1,1,3,3,2,1,2} - 48 S_{1,1,3,4,1,1,1} + 520 S_{1,1,4,1,1,1,3} - 628 S_{1,1,4,1,2,1,2} + 276 S_{1,1,4,1,3,1,1} - 332 S_{1,1,4,2,1,1,2} + 200 S_{1,1,4,2,2,1,1} - 48 S_{1,1,4,3,1,1,1} + 244 S_{1,1,5,1,1,1,2} - 128 S_{1,1,5,1,2,1,1} - 220 S_{1,1,5,2,1,1,1} + 136 S_{1,1,6,1,1,1,1} - 16 S_{1,2,1,1,1,1,5} + 28 S_{1,2,1,1,1,2,4} + 16 S_{1,2,1,1,1,3,3} + 16 S_{1,2,1,1,1,4,2} - 44 S_{1,2,1,1,5,1,1} + 12 S_{1,2,1,2,1,4,1} + 36 S_{1,2,1,2,2,3,1} - 30 S_{1,2,1,2,3,2} + 80 S_{1,2,1,2,4,1,1} - 276 S_{1,2,1,3,1,3,3} + 220 S_{1,2,1,3,2,2,1} + 34 S_{1,2,1,3,3,3,1} - 148 S_{1,2,1,4,1,1,2} + 172 S_{1,2,1,4,2,1,1} - 100 S_{1,2,1,5,1,1,1} - 664 S_{1,2,2,1,1,1,4} + 742 S_{1,2,2,1,2,1,3} + 294 S_{1,2,2,1,3,1,2} - 244 S_{1,2,2,1,4,1,1} + 1078 S_{1,2,2,2,1,3} - 1582 S_{1,2,2,2,2,1,3} + 178 S_{1,2,2,2,2,3,1} + 398 S_{1,2,2,2,3,1,2} - 362 S_{1,2,2,3,2,1,1} + 186 S_{1,2,2,4,1,1,1} - 682 S_{1,2,3,1,1,3} + 804 S_{1,2,3,1,2,2} - 324 S_{1,2,3,1,3,1} + 474 S_{1,2,3,2,1,2} - 336 S_{1,2,3,2,2,1} + 94 S_{1,2,3,3,1,1,1} - 356 S_{1,2,4,1,1,1,2} + 188 S_{1,2,4,1,1,2,1} + 346 S_{1,2,4,2,1,1,1} - 232 S_{1,2,5,1,1,1,1} + 632 S_{1,3,1,1,1,1,4} - 724 S_{1,3,1,1,2,3} - 300 S_{1,3,1,1,3,1,2} + 290 S_{1,3,1,1,4,1,1} - 232 S_{1,3,1,2,1,3,1} + 616 S_{1,3,1,2,2,2} - 320 S_{1,3,1,2,3,1} + 60 S_{1,3,1,3,1,2,1} - 380 S_{1,3,1,3,1,3,1} + 362 S_{1,3,1,4,1,1,1} - 610 S_{1,3,2,1,1,1,3} + 728 S_{1,3,2,1,2,1,2} - 302 S_{1,3,2,1,3,1,1} + 346 S_{1,3,2,2,1,2,1} + 134 S_{1,3,2,2,2,1,1} - 332 S_{1,3,2,3,1,1,1} - 112 S_{1,3,3,1,1,1,2} + 64 S_{1,3,3,1,2,1,2} + 100 S_{1,3,3,2,1,1,1} - 60 S_{1,3,4,1,1,1,1} + 496 S_{1,4,1,1,1,1,3} - 598 S_{1,4,1,1,2,1,2,2} + 260 S_{1,4,1,1,3,1,3,1} - 152 S_{1,4,1,2,1,1,2} - 200 S_{1,4,1,2,2,1,1} + 348 S_{1,4,1,3,1,1,1} - 342 S_{1,4,2,1,1,1,2} + 188 S_{1,4,2,1,2,1,2} + 196 S_{1,4,2,2,1,1,1} - 60 S_{1,4,3,1,1,1,1} + 260 S_{1,5,1,1,1,1,2} - 136 S_{1,5,1,1,2,1,1} - 136 S_{1,5,1,2,1,1,1} - 244 S_{1,5,1,2,1,1,1} + 160 S_{1,5,1,1,1,1,1} - 16 S_{2,1,1,1,1,1,5} + 28 S_{2,1,1,1,1,2,4} + 16 S_{2,1,1,1,1,3,3} + 16 S_{2,1,1,1,1,4,2} - 44 S_{2,1,1,1,5,1,1} + 12 S_{2,1,1,2,1,4,1} + 36 S_{2,1,1,2,2,3,1} - 30 S_{2,1,1,2,3,2} + 80 S_{2,1,1,2,4,1,1} - 276 S_{2,1,1,3,1,3} + 220 S_{2,1,1,3,2,2,1} + 34 S_{2,1,1,3,3,3,1} - 148 S_{2,1,1,4,1,1,2} + 172 S_{2,1,1,4,2,1,1} - 100 S_{2,1,1,5,1,1,1} + 12 S_{2,1,1,2,1,4,1} - 26 S_{2,1,1,2,2,3,1} - 18 S_{2,1,1,2,3,2} + 40 S_{2,1,2,1,1,4,1} + 284 S_{2,1,2,2,1,3} - 298 S_{2,1,2,2,2,2} - 108 S_{2,1,2,2,3,1} + 22 S_{2,1,2,3,1,2} - 280 S_{2,1,2,3,2,1} + 180 S_{2,1,2,4,1,1} - 280 S_{2,1,3,1,1,1,3} + 322 S_{2,1,3,1,2,2} - 120 S_{2,1,3,1,3,1} - 18 -
\[-162 S_{1,1,2,1,3,1,2} + 216 S_{1,1,2,1,3,2,1} - 124 S_{1,1,2,1,4,1,1} - 584 S_{1,1,2,1,1,1,3} + 680 S_{1,1,2,1,2,1,2} - 276 S_{1,1,2,2,1,3,1} + 762 S_{1,1,2,2,2,1,2} - 848 S_{1,1,2,2,2,2,1} + 298 S_{1,1,2,2,3,1,1} - 432 S_{1,1,2,3,1,1,2} + 222 S_{1,1,3,1,1,1,2} + 482 S_{1,1,3,2,1,1,1} - 348 S_{1,1,4,1,1,1,1} + 528 S_{1,1,3,1,1,1,3} - 638 S_{1,1,3,1,1,2,2} + 284 S_{1,1,3,1,1,3,1} - 156 S_{1,1,3,1,1,2,1} - 216 S_{1,1,3,1,1,2,2} + 358 S_{1,1,3,1,1,1,3} - 392 S_{1,1,3,1,2,1,2} + 206 S_{1,1,3,2,1,2,1} + 266 S_{1,1,3,2,2,1,1} - 104 S_{1,1,3,3,1,1,1} + 304 S_{1,1,4,1,1,1,2} - 156 S_{1,1,4,1,1,1,1} - 156 S_{1,1,4,1,1,2,1,1} - 328 S_{1,1,4,2,1,1,1} + 240 S_{1,1,5,1,1,1,1} + 12 S_{1,2,1,1,2,1,3} + 12 S_{1,2,1,1,3,1,2} - 28 S_{1,2,1,1,4,1,1} + 18 S_{1,2,1,2,1,2,2} + 60 S_{1,2,1,2,1,2,3,1} - 162 S_{1,2,1,1,3,1,2} + 14 S_{1,2,1,2,1,3,1} + 202 S_{1,2,2,1,2,2,1} - 390 S_{1,2,1,2,2,1,2} + 208 S_{1,2,2,1,2,3,1,1} + 98 S_{1,2,2,1,2,3,2,1} + 236 S_{1,2,2,1,2,3,2,1} - 176 S_{1,2,1,4,1,1,1} - 512 S_{1,2,2,1,1,1,3} + 604 S_{1,2,2,1,1,2,2} - 254 S_{1,2,2,1,1,3,1} + 156 S_{1,2,2,1,2,1,2} + 114 S_{1,2,2,1,2,2,1} - 220 S_{1,2,2,1,3,1,1} + 710 S_{1,2,2,2,1,1,2} - 364 S_{1,2,2,2,1,2,1} - 892 S_{1,2,2,2,2,1,1} + 386 S_{1,2,2,2,3,1,1} - 418 S_{1,2,3,1,1,1,2} + 210 S_{1,2,3,1,1,2,1} + 210 S_{1,2,3,1,2,1,1} + 462 S_{1,2,3,2,1,1,1} - 340 S_{1,2,4,1,1,1,1} + 480 S_{1,3,1,1,1,1,3} - 576 S_{1,3,1,1,1,2,2} + 250 S_{1,3,1,1,1,1,3,1} - 152 S_{1,3,1,1,2,1,2} - 194 S_{1,3,1,1,2,1,2,1} + 350 S_{1,3,1,1,3,1,1} - 152 S_{1,3,1,1,2,1,1,2} + 86 S_{1,3,1,1,2,1,1,2} - 254 S_{1,3,1,2,1,1,1} + 382 S_{1,3,1,3,1,1,1} - 382 S_{1,3,2,1,1,1,2} + 210 S_{1,3,2,1,1,2,1} + 210 S_{1,3,2,1,2,1,1} + 274 S_{1,3,2,2,1,1,1} - 116 S_{1,3,3,1,1,1,1} + 300 S_{1,4,1,1,1,1,2} - 156 S_{1,4,1,1,1,2,1} - 156 S_{1,4,1,1,2,1,1} - 156 S_{1,4,1,2,1,1,1} - 340 S_{1,4,2,1,1,1,1} + 240 S_{1,5,1,1,1,1,1} + 12 S_{2,1,1,1,1,3,2} + 12 S_{2,1,1,1,1,4,1} + 18 S_{2,1,1,1,1,2,2} + 60 S_{2,1,1,1,2,3,1} - 162 S_{2,1,1,1,3,1,2} + 216 S_{2,1,1,1,3,2,1} - 124 S_{2,1,1,1,4,1,1} - 8 S_{2,1,1,2,1,2,2} + 14 S_{2,1,1,2,1,3,1} + 202 S_{2,1,1,2,2,1,2} - 390 S_{2,1,1,2,2,2,1} + 208 S_{2,1,1,2,3,1,1} - 188 S_{2,1,1,3,1,1,2} + 98 S_{2,1,1,3,1,2,1} + 236 S_{2,1,1,3,2,1,1} - 176 S_{2,1,1,4,1,1,1} - 8 S_{2,1,2,1,1,2,2} + 14 S_{2,1,2,1,1,1,3} - 64 S_{2,1,2,1,2,2,1} + 90 S_{2,1,2,1,3,1,1} + 198 S_{2,1,2,2,1,1,2} - 96 S_{2,1,2,2,1,2,1} - 466 S_{2,1,2,2,2,1,1} + 284 S_{2,1,2,3,1,1,1} - 176 S_{2,1,2,3,1,1,2} + 86 S_{2,1,2,3,1,2,1} + 86 S_{2,1,3,1,1,1,2} + 208 S_{2,1,3,1,1,2,1} - 156 S_{2,1,4,1,1,1,1} - 480 S_{2,2,1,1,1,1,3} + 552 S_{2,2,1,1,1,2,2} - 216 S_{2,2,1,1,1,3,1} + 152 S_{2,2,1,1,2,1,2} + 82 S_{2,2,1,1,2,2,1} - 196 S_{2,2,1,1,3,1,1} + 152 S_{2,2,1,2,1,1,2} - 86 S_{2,2,1,2,1,2,1} + 48 S_{2,2,1,2,2,1,1} - 180 S_{2,2,1,3,1,1,1} + 64 S_{2,2,2,1,1,1,2} - 350 S_{2,2,2,1,1,2,1} - 350 S_{2,2,2,1,2,1,1} - 770 S_{2,2,2,2,1,1,1} + 270 S_{2,2,3,1,1,1,1} - 384 S_{2,3,1,1,1,1,2} + 210 S_{2,3,1,1,1,2,1} + 210 S_{2,3,1,1,2,1,1} + 210 S_{2,3,1,2,1,1,1} + 470 S_{2,3,2,1,1,1,1} - 340 S_{2,4,1,1,1,1,1} + 480 S_{3,1,1,1,1,1,3} - 556 S_{3,1,1,1,1,1,2} + 222 S_{3,1,1,1,1,1,3} - 152 S_{3,1,1,1,1,2,1,2} - 162 S_{3,1,1,1,1,2,2,1} + 316 S_{3,1,1,1,1,3,1,1} - 152 S_{3,1,1,1,2,1,1,2} + 86 S_{3,1,1,1,2,1,2,1} \]
\[ -234S_{3,1,1,2,2,1,1} + 376S_{3,1,1,3,1,1,1} - 152S_{3,1,2,1,1,1,2} + 86S_{3,1,2,1,1,2,1} + 86S_{3,1,2,1,2,1,1} \\
-274S_{3,1,2,2,1,1,1} + 402S_{3,1,3,1,1,1,1} - 384S_{3,2,1,1,1,1,2} + 210S_{3,2,1,1,1,2,1} + 210S_{3,2,1,1,2,1,1} \\
+210S_{3,2,2,1,1,1,1} + 270S_{3,2,2,2,1,1,1} - 116S_{3,3,1,1,1,1,1} + 300S_{4,1,1,1,1,1,2} - 156S_{4,1,1,1,1,2,1} \\
-156S_{4,1,1,2,1,1,1} - 156S_{4,1,2,1,1,1,1} - 340S_{4,2,1,1,1,1,1} + 240S_{5,1,1,1,1,1,1} \\
-64S_{1,1,1,1,2,2,2} - 40S_{1,1,1,1,2,3,1} + 240S_{1,1,1,1,3,1,2} - 296S_{1,1,1,1,3,2,1} \\
+136S_{1,1,1,1,4,1,1} - 344S_{1,1,1,1,2,2,1,2} + 540S_{1,1,1,1,2,2,1,2,1} - 220S_{1,1,1,1,2,3,1,1} \\
+336S_{1,1,1,1,3,1,1,2} - 164S_{1,1,1,1,3,1,2,1} - 380S_{1,1,1,1,3,2,1,1} + 256S_{1,1,1,1,4,1,1,1} \\
+48S_{1,1,1,2,2,1,1} - 76S_{1,1,1,2,3,1,1,1} - 376S_{1,1,1,2,2,1,1,2} + 172S_{1,1,1,2,2,1,2,1} \\
+780S_{1,1,1,2,2,2,1,1} - 496S_{1,1,1,2,3,1,1,1} + 340S_{1,1,1,3,1,1,1,1} - 156S_{1,1,1,3,1,1,2,1,1} \\
-156S_{1,1,1,3,2,1,1,1} - 360S_{1,1,1,3,2,1,1,2} + 280S_{1,1,1,4,1,1,1,1} + 48S_{1,1,1,2,1,1,2,1} \\
-76S_{1,1,2,1,1,1,3,1,1} + 204S_{1,1,2,1,2,2,1,1} - 252S_{1,1,2,1,2,3,1,1,1} - 336S_{1,1,2,2,1,1,1,2} \\
+156S_{1,1,2,2,1,2,1,1} + 156S_{1,1,2,2,1,2,1,2} + 748S_{1,1,2,2,2,1,1,1} - 380S_{1,1,2,2,3,1,1,1} \\
+300S_{1,1,3,1,1,1,1,2} - 156S_{1,1,3,1,1,1,2,1} - 156S_{1,1,3,1,1,2,1,1} - 156S_{1,1,3,1,2,1,1,1} \\
-380S_{1,1,3,2,1,1,1,1} + 280S_{1,1,4,1,1,1,1,1} + 48S_{1,2,1,1,1,2,2,1} - 76S_{1,2,1,1,1,3,1,1} \\
+204S_{1,2,1,1,2,2,1,1} - 252S_{1,2,1,1,3,1,1,1} + 232S_{1,2,1,2,1,1,1,1} - 156S_{1,2,1,3,1,1,1,1} \\
-300S_{1,2,2,1,1,1,1,2} + 156S_{1,2,2,1,1,2,1,1} + 156S_{1,2,2,1,2,1,1,1} + 156S_{1,2,2,1,2,1,2,1} \\
+636S_{1,2,2,2,1,1,1,1} - 380S_{1,2,3,1,1,1,1,1} + 300S_{1,3,1,1,1,1,1,2} - 156S_{1,3,1,1,1,1,2,1} \\
-156S_{1,3,1,1,1,2,1,1} - 156S_{1,3,1,1,2,1,1,1} - 156S_{1,3,1,2,1,1,1,1} - 380S_{1,3,2,1,1,1,1,1} \\
+280S_{1,4,1,1,1,1,1,1} + 48S_{2,1,1,1,1,2,2,1} - 76S_{2,1,1,1,1,3,1,1} + 204S_{2,1,1,1,2,2,1,1} \\
-252S_{2,1,1,1,3,1,1,1} + 232S_{2,1,2,2,1,1,1,1} - 156S_{2,1,3,1,1,1,1,1} + 156S_{2,2,2,1,1,1,1,1} \\
-156S_{2,2,1,3,1,1,1,1,1} - 300S_{2,2,2,1,1,1,2,1} + 156S_{2,2,2,1,1,2,1,1} + 156S_{2,2,2,1,2,1,1,1} \\
+156S_{2,2,2,1,2,1,2,1,1} + 156S_{2,2,1,2,1,1,1,1} + 636S_{2,2,2,1,1,1,1,1} - 380S_{2,3,1,1,1,1,1,1} \\
+300S_{3,1,1,1,1,1,1,2} - 156S_{3,1,1,1,1,2,1} - 156S_{3,1,1,1,2,1,1} - 156S_{3,1,1,2,1,1,1} \\
-156S_{3,1,2,1,1,1,1} - 156S_{3,1,2,1,1,2,1} - 380S_{3,2,1,1,1,1,1,1} + 280S_{4,1,1,1,1,1,1,1} \\
-256S_{1,1,1,1,1,2,2,1,1} + 392S_{1,1,1,1,1,3,1,1,1} - 416S_{1,1,1,1,2,2,1,1,1} + 280S_{1,1,1,1,1,3,1,1,1,1} \\
-280S_{1,1,1,2,2,1,1,1,1} + 280S_{1,1,1,3,1,1,1,1,1} - 280S_{1,1,2,2,1,1,1,1,1} + 280S_{1,1,3,1,1,1,1,1,1} + 280S_{3,1,1,1,1,1,1,1,1} \cdot (A.1)
\]
B Appendix B

Analytic continuation of all harmonic sums up to transcendentality 11 was performed with the help of the HAMPOI package [87] for FORM [88] and using DATAMINE [89] tables for the substitution of the multiple zeta functions, or multiple polylogarithms at $x = 1$ through usual Euler zeta-functions $\zeta_n$ and the minimal numbers of multiple zeta-functions. The results of the analytic continuation for the full planar six-loop anomalous dimension of twist-2 operators in $\mathcal{N} = 4$ SYM theory is the following:

$$\gamma(-2 + \omega) = 8g^2 \left[ -\frac{1}{\omega^2} + 1 + \omega(1 + \zeta_2) + \omega^2(1 - \zeta_3) + \omega^3(1 + \zeta_4) + \omega^4(1 - \zeta_5) + \omega^5(1 + \zeta_6) + \omega^6(1 - \zeta_7) + \omega^7(1 + \zeta_8) + \omega^8(1 - \zeta_9) \right]$$

$$+ g^4 \left[ -\frac{32}{\omega^4} + \frac{64}{\omega^2} + \frac{32\zeta_2 + 32}{\omega} + 16\zeta_3 + \omega(32\zeta_2 - 16\zeta_3 - 20\zeta_4 - 32) + \omega^2(-64\zeta_3\zeta_2 + 64\zeta_2 - 48\zeta_4 + 12\zeta_4 + 116\zeta_5 - 64) + \omega^3\left(40\zeta_3^2 - 80\zeta_3 + 96\zeta_4 + 44\zeta_4 - 8\zeta_5 - \frac{115\zeta_6}{3} - 96\right) + \omega^4\left(-128\zeta_5\zeta_2 + 128\zeta_2 - 112\zeta_3 - 76\zeta_3\zeta_4 + 76\zeta_4 - 40\zeta_5 + 5\zeta_6 + \frac{579\zeta_7}{2} - 128\right) + \omega^5\left(\frac{512h_{53}}{57} - \frac{2560h_{71}}{19} + 160\zeta_2 - 144\zeta_3 + 108\zeta_4 + \frac{3768\zeta_3\zeta_5}{19} - 72\zeta_5 + 37\zeta_6 - 3\zeta_7 - \frac{16485\zeta_8}{76} - 160\right) + \omega^6\left(-192\zeta_5\zeta_2 + 192\zeta_2 - 176\zeta_3 + 140\zeta_4 - 140\zeta_4\zeta_5 + 104\zeta_5 - 69\zeta_3\zeta_6 + 69\zeta_6 - 35\zeta_7 + \frac{76\zeta_8}{4} + \frac{1057\zeta_9}{2} - 192\right) \right]$$

$$+ g^6 \left[ -\frac{256}{\omega^6} + \frac{768}{\omega^4} + \frac{512\zeta_2}{\omega^3} + \frac{-768\zeta_2 - 384\zeta_3 - 512}{\omega^2} + \frac{-256\zeta_2 + 576\zeta_3 - 416\zeta_4 - 768}{\omega} + 448\zeta_2\zeta_3 + 384\zeta_3 + 80\zeta_4 + 192\zeta_5 - 768 + \omega(-336\zeta_3^2 - 704\zeta_2\zeta_3 + 192\zeta_3 - 352\zeta_4 + 272\zeta_5 - 1052\zeta_6 - 512) + \omega^2(432\zeta_3^2 - 832\zeta_2\zeta_3 + 104\zeta_4\zeta_3 - 256\zeta_2 - 336\zeta_4 + 1744\zeta_3\zeta_5 + 1184\zeta_5 + 1616\zeta_6 - 1146\zeta_7) + \omega^3(944\zeta_2\zeta_3^2 + 656\zeta_3^2 - 1216\zeta_2\zeta_3 - 120\zeta_4\zeta_3 - \frac{44416\zeta_5\zeta_3}{19} - 192\zeta_3 - \frac{8192h_{53}}{171} + \frac{40960h_{71}}{57} - 768\zeta_2 + 128\zeta_4 + 1840\zeta_5 - 1312\zeta_2\zeta_5 + \frac{268\zeta_6}{3} - 584\zeta_7 - \frac{12315\zeta_8}{19} + 768) + \omega^4(-32\zeta_3^3 - 384\zeta_2\zeta_3^2 + 1008\zeta_3^2 - 1856\zeta_2\zeta_3 - 680\zeta_4\zeta_3 + \frac{50736\zeta_5\zeta_3}{19} - 6664\zeta_6\zeta_3 + 384\zeta_3 + \frac{10240h_{53}}{57} - \frac{512000h_{71}}{19} - 1536\zeta_2 + 1040\zeta_4 - 1376\zeta_2\zeta_5 - 132\zeta_4\zeta_5 + 2240\zeta_5 - \frac{1816\zeta_6}{3} + 5728\zeta_2\zeta_7 + 2190\zeta_7 + \frac{90622\zeta_8}{171} - \frac{57904\zeta_9}{9} + 1792) \right]$$
\[+g^{8}\left[ - \frac{2560}{\omega^5} + \frac{10240}{\omega^6} + \frac{7168\zeta_2 - 5120}{\omega^5} + \frac{-18432\zeta_2 - 5632\zeta_3 - 10240}{\omega^4} + \frac{14336\zeta_3 - 13440\zeta_4 - 7680}{\omega^5} + \frac{9216\zeta_3\zeta_2 + 8192\zeta_2 + 1536\zeta_3 + 14848\zeta_4 + 3200\zeta_5}{\omega^2} + \frac{-4416\zeta_3^2 - 14080\zeta_2\zeta_3 - 6144\zeta_3 + 9216\zeta_2 + 3712\zeta_4 - 960\zeta_5 - 7504\zeta_6 + 10240}{\omega} + \frac{6464\zeta_3^2 - 3328\zeta_2\zeta_3 - 2400\zeta_4\zeta_3 - 9728\zeta_3 + 6144\zeta_2 + 1024\zeta_4 - 1664\zeta_2\zeta_5 - 4160\zeta_5}{\omega} + \frac{45008\zeta_6}{3} + 2472\zeta_7 + 20480 + \omega\left( 1408\zeta_2\zeta_3^2 + 1664\zeta_2^3 + 3072\zeta_2\zeta_3 - 4512\zeta_4\zeta_3 \right) + \frac{10912\zeta_5\zeta_3}{3} - 10240\zeta_3 + \frac{4096h_{53}}{27} - \frac{20480h_{71}}{9} + 2048\zeta_2 + 2944\zeta_4 + 5120\zeta_2\zeta_5 - 6400\zeta_5 + \frac{40640\zeta_6}{3} - 8336\zeta_7 + \frac{453500\zeta_8}{27} + 28160 \right) + \omega^2 \left( \frac{352\zeta_3^3 + 3456\zeta_2\zeta_3^2}{\omega} - 1408\zeta_3^2 + 5120\zeta_2\zeta_3 - 3776\zeta_4\zeta_3 - \frac{365344\zeta_5\zeta_3}{57} - 2520\zeta_6\zeta_5 - 8704\zeta_3 + 5632\zeta_4 - \frac{372736h_{53}}{513} + \frac{1863680h_{71}}{171} + 10880\zeta_2\zeta_5 + 9496\zeta_4\zeta_5 - 7680\zeta_5 + 5760\zeta_6 - 38912\zeta_2\zeta_7 - 29288\zeta_7 - \frac{7149836\zeta_8}{513} + \frac{46024\zeta_4^2}{3} + 30720 \right) \]

\[+g^{10}\left[ - \frac{28672}{\omega^9} + \frac{143360}{\omega^8} + \frac{102400\zeta_2 - 143360}{\omega^7} + \frac{-368640\zeta_2 - 81920\zeta_3 - 143360}{\omega^6} + \frac{163840\zeta_2 + 292864\zeta_3 - 289792\zeta_4}{\omega^5} + \frac{194560\zeta_3\zeta_2 + 286720\zeta_2 - 110592\zeta_3 + 622080\zeta_4 + 40960\zeta_5 + 172032}{\omega^4} + \frac{286720 - 87040\zeta_3^2 - 473088\zeta_2\zeta_3 - 241664\zeta_4 + 184320\zeta_2 + 37888\zeta_4 - 46080\zeta_5 + \frac{137984\zeta_6}{3}}{\omega^3} + \frac{209408\zeta_4^2 + 12288\zeta_2\zeta_3 - 250368\zeta_4\zeta_3 - 196608\zeta_3 - 150528\zeta_4 - 24064\zeta_2\zeta_5 - 81920\zeta_5}{\omega^2} + \frac{76872\zeta_6 + 36736\zeta_7 + 286720}{\omega^2} + \frac{126976\zeta_2\zeta_3^2 + 10240\zeta_3^2 + 208896\zeta_2\zeta_3 + 270592\zeta_4\zeta_3}{\omega} + \frac{1537280\zeta_5\zeta_3 - 55296\zeta_6 + \frac{843776}{513}h_{53} - \frac{4218880}{171}h_{71} - 163840\zeta_2 - 144384\zeta_4 - 75264\zeta_2\zeta_5}{\omega} + \frac{143360 - 71680\zeta_5 - \frac{39424}{3}\zeta_6 - 178752\zeta_7 + \frac{144246544}{513}{\zeta}_8}{\omega} - 41344\zeta_3^3 - 210432\zeta_2\zeta_3^2 - 83456\zeta_3^2 + \frac{215040\zeta_2\zeta_3 + 84480\zeta_4\zeta_3 + \frac{5259520}{57}\zeta_5\zeta_3 - 271040\zeta_6\zeta_3 + 118784\zeta_3 - 245760\zeta_2 - 87552\zeta_4 - \frac{1163264}{513}h_{53} + \frac{5816320}{171}h_{71} + 26624\zeta_2\zeta_5 - 87968\zeta_4\zeta_5 - 20480\zeta_5 - \frac{657664}{3}\zeta_6 - 83200\zeta_2\zeta_7 + 4032\zeta_7 - \frac{101468128}{513}\zeta_8 - \frac{39344\zeta_9}{9} - 143360 \right] \]
\[+g^{12}\left[-\frac{344064}{\omega^1} + \frac{2064384}{\omega^2} + \frac{1490944\zeta_2 - 3096576}{\omega^3} - \frac{6881280\zeta_2 - 1204224\zeta_3 - 1376256}{\omega^4} \right. \\
+ \frac{6307840\zeta_2 + 5529600\zeta_3 - 5625856\zeta_4 + 2064384}{\omega^5} \\
\left. + \frac{3768320\zeta_2 + 5734400\zeta_3 - 4792320\zeta_4 + 17889280\zeta_4 + 546816\zeta_5 + 4128768}{\omega^6} \\
+ \frac{3440640 - 1640448\zeta_3^2 - 12771328\zeta_2\zeta_3 - 5038080\zeta_3 - 6133760\zeta_4 - 1071104\zeta_5 + 3989504\zeta_6}{\omega^7} \\
+ \frac{5523456\zeta_3^2 + 5349376\zeta_2\zeta_3 - 8453120\zeta_4\zeta_3 - 737280\zeta_3 - 5505024\zeta_2 - 10680320\zeta_4}{\omega^8} \\
\left. + \frac{591360\zeta_7 - 667648\zeta_2\zeta_5 - 1267712\zeta_5 - 2685952\zeta_6 + 3540992\zeta_2\zeta_3^2 - 1994752\zeta_3^2}{\omega^9} \\
+ \frac{8953856\zeta_2\zeta_3 + 17525760\zeta_4\zeta_3 + \frac{13147136}{19}\zeta_5\zeta_3 + 4276224\zeta_3 + \frac{1638400}{57}\zeta_5 + \frac{8192000}{19}\zeta_7}{\omega^{10}} \\
+ \frac{-8028160\zeta_2 - 7004160\zeta_4 - 849920\zeta_2\zeta_5 + 100352\zeta_5 - \frac{15717376}{3}\zeta_6 - 3243520\zeta_7}{\omega^{11}} \\
+ \frac{\frac{745907696}{171}\zeta_8 - 5160960}{\omega^{12}} - \frac{1100800\zeta_3^3 - 8599552\zeta_2\zeta_3^2 - 4012032\zeta_3^2 + 5283840\zeta_2\zeta_3}{\omega^{13}} \\
\left. + \frac{495616\zeta_4\zeta_3 + \frac{47233024}{57}\zeta_5\zeta_3 - \frac{5374720}{3}\zeta_6\zeta_3 + 7618560\zeta_3 - \frac{37085200}{513}\zeta_5 + \frac{18841600}{171}\zeta_7}{\omega^{14}} \\
\right] (B.1) \\
\text{where} \\
\begin{align*}
  h_{71} &= H_{-7,-1}(1), \\
  h_{53} &= H_{-5,-3}(1), \\
\end{align*} (B.2) \\
and \( H_{i_1,\ldots,i_k}(x) \) are the harmonic polylogarithms \cite{87}. 

References


[65] V. Dippel, unpublished


