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Six-loop anomalous dimension of twist-two operators in planar $\mathcal{N} = 4$ SYM theory

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Abstract: We compute the general form of the six-loop anomalous dimension of twist-two operators in planar $\mathcal{N} = 4$ SYM theory. First we find the contribution from the asymptotic Bethe ansatz. Then we reconstruct the wrapping terms from the first 35 even spin values of the full six-loop anomalous dimension computed using the quantum spectral curve of the theory. The obtained anomalous dimension satisfies all known constraints coming from the BFKL equation and the generalized double-logarithmic equation.

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Contents

1 Introduction

The anomalous dimension of composite gauge-invariant operators in $\mathcal{N} = 4$ SYM theory can be calculated with the help of integrability. Integrability in the context of AdS/CFT-correspondence [1–3] was found from the study of the single-trace operators [4] in the leading order of perturbative theory in ref. [5]*. Generalization to higher orders together with the studies of the integrable structures from the superstring theory side, started in ref. [10], allowed formulating all-loop asymptotic Bethe equations [11–22, 25–30]. For the operators of finite length the asymptotic Bethe equations give a non-complete result due to appearance of wrapping effects [31, 32]. Again, the computations of wrapping corrections can be performed using integrability [33–46]. Independent tests of the obtained results were performed with the perturbative field theory computations in refs. [47–50]. Moreover, the results for twist-2 and spin M operators passed very non-trivial tests coming from the Balitsky-Fadin-Kuraev-Lipatov (BFKL) [51–53] and generalized double-logarithmic [54–60] equations which impose all-loop constrains on the structure of the result analytically continued to negative values of M.

In principle, we can do the inverse: we can use constraints from the BFKL and the generalized double-logarithmic equations to reconstruct the wrapping corrections if the part of the full anomalous dimension related to the asymptotic Bethe equations is already

*Earlier, similar integrability was discovered in quantum chromodynamics in the Regge limit [6–8] and for some operators [9].

known. This was done at five loops by one of the authors in ref. [61]. However, one could not obtain any reasonable results at six loops in this way.

Recently, an advanced method for computing the anomalous dimension of operators with arbitrary but fixed twist and spin was developed by two of the authors [62]. This approach is based on a perturbative solution of the quantum spectral curve (QSC) equations [63, 64] and can in principle produce results to any loop order. In this paper we reconstruct a general form of the six-loop anomalous dimension for twist-two operators and use all available constraints to check its correctness.

In section 2 we compute the part of the six-loop anomalous dimension for twist-2 operators coming from the ABA. In section 3 we briefly describe the method of computation of the six-loop anomalous dimension for twist-2 operators with fixed spin. In section 4 we reconstruct the general form of the six-loop anomalous dimension from the first 35 even values, obtained with the method described in section 3. In section 5 we provide the constraints which will be used to verify the obtained result, together with the description of their origin. In Appendices we give the results for the most complicated parts of the six-loop anomalous dimension and its analytic continuation at $M = -2 + \omega$.

2 The six-loop anomalous dimension from Bethe ansatz

Twist-two operators belong to the $\mathfrak{sl}(2)$ sub-sector of the full theory. In this sector, the highest-weight (primary) states consist of two scalar fields $\mathcal{Z}$ and M [†] covariant derivatives $\mathcal{D}$

$$\mathrm{Tr}\,\left(\mathcal{Z}\,\mathcal{D}^M\,\mathcal{Z}\,\right)\,, \tag{2.1}$$

where M should be even. In the spin chain picture which is valid at one loop at weak coupling, such single-trace operators are identified with the states of the non-compact $\mathfrak{sl}(2)$ spin $= -\frac{1}{2}$ length-two Heisenberg magnet with M excitations.

We denote the total scaling dimension of these states as

$$\Delta = 2 + M + \gamma(g)\,, \qquad \text{with} \qquad \gamma(g) = \sum_{\ell=1}^{\infty} \gamma_{2\ell}\, g^{2\ell}\,. \tag{2.2}$$

Here, $\gamma(g)$ is the anomalous part of the dimension depending on the coupling constant

$$g^2 = \frac{\lambda}{16\,\pi^2}\,, \tag{2.3}$$

and $\lambda = N\, g_{\mathrm{YM}}^2$ is the 't Hooft coupling constant. The anomalous dimension $\gamma(g)$ may be determined up to the three-loop order $\mathcal{O}(g^6)$ with help of the asymptotic Bethe ansatz [19].

The asymptotic Bethe equations for the $\mathfrak{sl}(2)$ operators can be found in [20, 30]

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\, \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+}\, \exp\left(2\, i\, \theta(u_k, u_j)\right)\,, \qquad \prod_{k=1}^{M} \frac{x_k^+}{x_k^-} = 1\,, \tag{2.4}$$

[†] M is the value of the Lorentz spin. Hence the notation S is often used instead of M. We chose to use M to not interfere with the notation for harmonic sums.

where the variables $x_k^\pm$ are related to u_k through

$$x_k^\pm = x(u_k^\pm)\,, \qquad u^\pm = u \pm \tfrac{i}{2}\,, \qquad x(u) = \frac{u}{2}\left(1+\sqrt{1-4\,\frac{g^2}{u^2}}\right). \tag{2.5}$$

The dressing phase $\theta(u,v)$ has been conjectured in [30] and shown to be solution of the crossing equation [22] in [23, 24]. To the sixth order in perturbation theory it has the following form

$$\begin{aligned}\theta(u_k,u_j) \;=\; & \left(4\,\zeta_3\,g^6 - 40\,\zeta_5 g^8 + 420\,\zeta_7 g^{10}\right)\left(q_2(u_k)\,q_3(u_j) - q_3(u_k)\,q_2(u_j)\right)\\ & - 8\,\zeta_5 g^{10}\big(q_2(u_k)\,q_5(u_j) - q_5(u_k)\,q_2(u_j)\big)\\ & + 24\,\zeta_5 g^{10}\big(q_3(u_k)\,q_4(u_j) - q_4(u_k)\,q_3(u_j)\big) + \mathcal{O}(g^{12})\,,\end{aligned} \tag{2.6}$$

where $q_r(u)$ are the eigenvalues of the conserved magnon charges, see [20]. The anomalous dimension is given by

$$\gamma^{\rm ABA}(g) = 2\,g^2\sum_{k=1}^{M}\left(\frac{i}{x_k^+} - \frac{i}{x_k^-}\right) = \sum_{l=1}^{\infty} g^{2l}\,\gamma_{2l}^{\rm ABA}(M) \tag{2.7}$$

and it is decomposed at six loops as

$$\gamma_{12}^{\rm ABA}(M) \;=\; \hat{\gamma}_{12} \;=\; \hat{\gamma}_{12}^{\rm rational} + \hat{\gamma}_{12}^{\zeta_3}\,\zeta_3 + \hat{\gamma}_{12}^{\zeta_5}\,\zeta_5 + \hat{\gamma}_{12}^{\zeta_7}\,\zeta_7\,. \tag{2.8}$$

At one loop the Bethe roots u_k are given by the zeros of the Hahn polynomial [28, 65]

$$P_M(u) = {}_3F_2\left(\left.\begin{matrix} -M,\ M+1,\ \tfrac{1}{2}+iu \\ 1,\ 1\end{matrix}\,\right|\,1\right). \tag{2.9}$$

In order to obtain a general expression for the anomalous dimension of twist-two operator for arbitrary M we solve eq. (2.4) perturbatively for fixed values of the spin M and match the coefficients in an appropriate ansatz which assumes the maximal transcendentality principle [67][‡]. The basis for the ansatz is formed from the harmonic sums defined by the following recurrent procedure (see [72])

$$S_a(M) \;=\; \sum_{j=1}^{M}\frac{(\mathrm{sgn}(a))^j}{j^{|a|}}\,, \tag{2.10}$$

$$S_{a_1,\ldots,a_n}(M) \;=\; \sum_{j=1}^{M}\frac{(\mathrm{sgn}(a_1))^j}{j^{|a_1|}}\,S_{a_2,\ldots,a_n}(j)\,. \tag{2.11}$$

The transcendentality k of each sum $S_{a_1,\ldots,a_n}$ is given by the sum of the absolute values of its indices

$$k = |a_1| + \ldots |a_n|\,, \tag{2.12}$$

[‡]The hypothesis about maximal transcendentality principle [67] was confirmed by direct perturbative calculations at the two-loop order [68] and then successfully applied at the three-loop order [69], when corresponding results were obtained in QCD [70, 71].

Contribution	Rational	ζ_3	ζ_5	ζ_3^2	ζ_7	$\zeta_5\zeta_3$	ζ_3^3	ζ_9	Total
Transcendentality of basis	11	8	6	5	4	3	2	2	
γ_{ABA}	8119	577	99		17				8812
$\mathcal{P}_{\text{ABA}}$	1024	128	32		8				1192
γ_{Wrapping}	1393	99	17	7	3	1	1	1	1522
$\mathcal{P}_{\text{Wrapping}}$	256	32	8	4	2	1	1	1	305

Table 1. The number of harmonic sums in the basis for different contributions.

and the transcendentality of a product of harmonic sums is equal to the sum of the transcendentalities of its constituents. According to the maximal transcendentality principle [32, 67–69] the anomalous dimension of twist-two operators can contain only harmonic sums with maximal transcendentality at a given order of perturbative theory. The number of harmonic sums entering into this basis at the ℓ-loop order for the transcendentality $k = 2\ell - 1$ is equal to $((1-\sqrt{2})^k + (1+\sqrt{2})^k)/2$, and at six loops we will need more than 8800 sums, see Table 1.

Fortunately, the generalized Gribov-Lipatov reciprocity [73, 74] enters the game and allows us to significantly reduce the dimension of the basis. Define the reciprocity-respecting function $\mathcal{P}^{\text{ABA}}(M)$ [73–75] by the relation

$$\gamma^{\text{ABA}}(M) = \mathcal{P}^{\text{ABA}}\left(M + \frac{1}{2}\gamma^{\text{ABA}}(M)\right) . \tag{2.13}$$

The reciprocity-respecting splitting function $\mathcal{P}(x)$ [73, 74], related to $\mathcal{P}(M)$ through a Mellin transformation, should satisfy the Gribov-Lipatov relation [76]

$$\mathcal{P}(x) = -\,x\,\mathcal{P}\left(\frac{1}{x}\right) \tag{2.14}$$

at all orders of the perturbation theory.

The practical output which we will use is that the reciprocity function $\mathcal{P}^{\text{ABA}}(M)$ should be expressed only in terms of the binomial sums (see [72])

$$\mathbb{S}_{i_1,\dots,i_k}(N) = (-1)^N \sum_{j=1}^{N} (-1)^j \binom{N}{j}\binom{N+j}{j} S_{i_1,\dots,i_k}(j) \, , \tag{2.15}$$

which is the necessary and sufficient condition to satisfy (2.14) [36][§].

The binomial sums come both from the perturbative field theory calculations of the anomalous dimensions of twist-2 operators [70, 71] and from the solution of the Baxter

[§]Historically, this statement was understood using different but equivalent to the binomial sums basis [74, 77–79]. Relations between the binomial and the nested harmonic sums can be found as the ancillary files of the arXiv version this paper or on the web-page `http://thd.pnpi.spb.ru/~velizh/6loop/`.

equation for the corresponding spin chain [65, 66]. One of the interesting features of these sums is that they are defined only for *positive* values of the indices $i_1, \ldots, i_k$. The number of such sums, which will form the basis, for transcendentality k is equal to 2^{k-1} and at six loops we will about 1000 binomial harmonic sums (see Table 1). Thus, the basis is significantly reduced and, though still huge, can be managed.

In this paper, we first find the result in terms of $\mathcal{P}(M)$ and then reconstruct the anomalous dimension from (2.13): upon substituting the perturbative expansion (2.2), one finds

$$\mathcal{P}^{\rm ABA}(M) = \sum_{l=1}^{\infty} g^{2l}\, \mathcal{P}^{\rm ABA}_{2l}(M)\,. \tag{2.16}$$

From eqs. (2.13) and (2.16) one can find that at six-loop order the reciprocity function $\mathcal{P}_{12}$ is related to the anomalous dimension γ_{12} (see Appendix B of ref. [77] for five loops):

$$\mathcal{P}^{\rm ABA}_{12}(M) = \hat{\mathcal{P}}_{12} \;=\; \hat{\mathcal{P}}^{\rm rational}_{12} + \hat{\mathcal{P}}^{\zeta_3}_{12}\zeta_3 + \hat{\mathcal{P}}^{\zeta_5}_{12}\zeta_5 + \hat{\mathcal{P}}^{\zeta_7}_{12}\zeta_7\,, \tag{2.17}$$

$$\begin{aligned}\hat{\mathcal{P}}^{\rm rational}_{12} \;&=\; \hat{\gamma}^{\rm rational}_{12} - \frac{1}{4}\left(\hat{\gamma}_6^2 + 2\,\hat{\gamma}_4\hat{\gamma}_8^{\rm rational} + 2\,\hat{\gamma}_2\hat{\gamma}_{10}^{\rm rational}\right)' \\ &\quad + \frac{1}{24}\left(\hat{\gamma}_4^3 + 6\,\hat{\gamma}_2\hat{\gamma}_6\hat{\gamma}_4 + 3\,\hat{\gamma}_2^2\hat{\gamma}_8^{\rm rational}\right)'' \\ &\quad - \frac{1}{96}\hat{\gamma}_2^2\left(3\,\hat{\gamma}_4^2 + 2\,\hat{\gamma}_2\hat{\gamma}_6\right)''' + \frac{1}{384}\left(\hat{\gamma}_2^4\hat{\gamma}_4\right)'''' - \frac{1}{23040}\left(\hat{\gamma}_2^6\right)''''' \,,\end{aligned} \tag{2.18}$$

$$\hat{\mathcal{P}}^{\zeta_3}_{12} = \hat{\gamma}^{\zeta_3}_{12} - \frac{1}{2}\left(\hat{\gamma}_4\hat{\gamma}_8^{\zeta_3} + \hat{\gamma}_2\hat{\gamma}_{10}^{\zeta_3}\right)' + \frac{1}{8}\left(\hat{\gamma}_2^2\hat{\gamma}_8^{\zeta_3}\right)''\,, \tag{2.19}$$

$$\hat{\mathcal{P}}^{\zeta_5}_{12} = \hat{\gamma}^{\zeta_5}_{12} - \frac{1}{2}\left(\hat{\gamma}_2\hat{\gamma}_{10}^{\zeta_5}\right)'\,, \tag{2.20}$$

$$\hat{\mathcal{P}}^{\zeta_7}_{12} = \hat{\gamma}^{\zeta_7}_{12}\,, \tag{2.21}$$

where each prime marks a derivative with respect to M.

The rational and ζ_i parts of $\mathcal{P}^{\rm ABA}_{12}(M)$ are computed separately. To compute the rational part, the ABA equations without dressing phase were solved up to six loops for the first 1024 values of M numerically with an accuracy of about 10^{-1000}. The answer should be a linear combination of 1024 binomial harmonic sums, hence we've got enough constraints to fix the coefficients in this combination. These constraints were solved with the help of the `MATHEMATICA` function `LinearSolve`. The obtained coefficients are very close to integers, so their rounding gives the desired result for $\hat{\mathcal{P}}^{\rm rational}_{12}$, which is listed in Appendix A.

For the dressing phase (2.6) the first 40 solutions of the Bethe equations were computed. According to Table 1, this is enough to find the general expression for the functions $\hat{\mathcal{P}}^{\zeta_5}_{12}$ and $\hat{\mathcal{P}}^{\zeta_7}_{12}$, which have the following form:

$$\begin{aligned}\hat{\mathcal{P}}^{\zeta_5}_{12} = -64\Big(&- 8\mathbb{S}_6 58\mathbb{S}_{1,5} - 74\mathbb{S}_{2,4} - 8\mathbb{S}_{3,3} - 8\mathbb{S}_{4,2} + 60\mathbb{S}_{5,1} + 40\mathbb{S}_{1,1,4} - 78\mathbb{S}_{1,2,3} \\ &-38\mathbb{S}_{1,3,2} + 40\mathbb{S}_{1,4,1} - 12\mathbb{S}_{2,1,3} + 88\mathbb{S}_{2,2,2} - 40\mathbb{S}_{2,3,1} - 26\mathbb{S}_{3,2,1} + 26\mathbb{S}_{4,1,1} - 60\mathbb{S}_{1,1,2,2} \\ &+20\mathbb{S}_{1,1,3,1} - 20\mathbb{S}_{1,2,1,2} + 38\mathbb{S}_{1,3,1,1} - 20\mathbb{S}_{2,1,1,2} + 16\mathbb{S}_{2,1,2,1} + 14\mathbb{S}_{2,2,1,1} - 4\mathbb{S}_{3,1,1,1}\Big),\end{aligned} \tag{2.22}$$

$$\hat{\mathcal{P}}^{\zeta_7}_{12} = -3360\left(2\,\mathbb{S}_4 + \mathbb{S}_{1,3} - \mathbb{S}_{2,2} - \mathbb{S}_{1,2,1} - \mathbb{S}_{2,1,1}\right)\,. \tag{2.23}$$

For the reconstruction of $\hat{\mathcal{P}}_{12}^{\zeta_3}$ we used the LLL-algorithm [80], similar to what was done in the previous works [81, 82] by one of the authors (see also ref. [61] for a detailed explanations). Its usage is based on the fact that the coefficients in any anomalous dimension of twist-two operators are usually rather simple numbers, that is the equation for the coefficients is a linear Diophantine equation. Moreover, a lot of binomial harmonic sums which belong to the basis of possible terms are absent in the final expression, i.e. their coefficients are zeros. The LLL-algorithm is realized in many computer algebra systems and for our purposes the MATHEMATICA function LatticeReduce has been used to realize the algorithm. Firstly, we calculate the values of all 128 terms in the basis with transcendentality 8 up to $M = 40$. We hence can write a system of 39 linear equations for 128 coefficients which are assumed to be integers. According to the realization of the LLL-algorithm¶ we add to the 129×129 unit matrix the transpose matrix, obtained from our system of Diophantine equations (coefficients of 128 variables plus one free term), and applying the LatticeReduce function of MATHEMATICA to the obtained $(129 + 39) \times 129$ matrix we get the LLL-reduced matrix, in which we can easily find the result that we are looking for:

$$\begin{aligned}\hat{\mathcal{P}}_{12}^{\zeta_3}= -32\big(&4\,\mathbb{S}_{2,6} - 4\,\mathbb{S}_{7,1} - 8\,\mathbb{S}_{1,1,6} + 12\,\mathbb{S}_{1,2,5} + 3\,\mathbb{S}_{1,3,4} + 3\,\mathbb{S}_{1,4,3} + 3\,\mathbb{S}_{1,5,2} - 13\,\mathbb{S}_{1,6,1}\\
&+8\,\mathbb{S}_{2,1,5} - 18\,\mathbb{S}_{2,2,4} - 7\,\mathbb{S}_{2,3,3} - 7\,\mathbb{S}_{2,4,2} + 20\,\mathbb{S}_{2,5,1} - 11\,\mathbb{S}_{3,1,4} + 8\,\mathbb{S}_{3,2,3} - \mathbb{S}_{3,3,2}\\
&+7\,\mathbb{S}_{3,4,1} - 9\,\mathbb{S}_{4,1,3} + 7\,\mathbb{S}_{4,2,2} + 7\,\mathbb{S}_{4,3,1} - 9\,\mathbb{S}_{5,1,2} + 21\,\mathbb{S}_{5,2,1} - 16\,\mathbb{S}_{6,1,1} - 6\,\mathbb{S}_{1,1,1,5}\\
&+14\,\mathbb{S}_{1,1,2,4} + 8\,\mathbb{S}_{1,1,3,3} + 8\,\mathbb{S}_{1,1,4,2} - 12\,\mathbb{S}_{1,1,5,1} + 5\,\mathbb{S}_{1,2,1,4} - 24\,\mathbb{S}_{1,2,2,3} - 13\,\mathbb{S}_{1,2,3,2}\\
&+19\,\mathbb{S}_{1,2,4,1} - 9\,\mathbb{S}_{1,3,1,3} + 3\,\mathbb{S}_{1,3,2,2} + 9\,\mathbb{S}_{1,3,3,1} - 8\,\mathbb{S}_{1,4,1,2} + 25\,\mathbb{S}_{1,4,2,1} - 19\,\mathbb{S}_{1,5,1,1}\\
&+5\,\mathbb{S}_{2,1,1,4} - 12\,\mathbb{S}_{2,1,2,3} - 7\,\mathbb{S}_{2,1,3,2} + 6\,\mathbb{S}_{2,1,4,1} + 2\,\mathbb{S}_{2,2,1,3} + 26\,\mathbb{S}_{2,2,2,2} - 22\,\mathbb{S}_{2,2,3,1}\\
&+8\,\mathbb{S}_{2,3,1,2} - 33\,\mathbb{S}_{2,3,2,1} + 27\,\mathbb{S}_{2,4,1,1} - 10\,\mathbb{S}_{3,1,1,3} + 10\,\mathbb{S}_{3,1,2,2} + 9\,\mathbb{S}_{3,2,1,2} - 24\,\mathbb{S}_{3,2,2,1}\\
&+10\,\mathbb{S}_{3,3,1,1} - 9\,\mathbb{S}_{4,1,1,2} + 9\,\mathbb{S}_{4,1,2,1} + 23\,\mathbb{S}_{4,2,1,1} - 18\,\mathbb{S}_{5,1,1,1} + 9\,\mathbb{S}_{1,1,1,2,3} + 9\,\mathbb{S}_{1,1,1,3,2}\\
&-6\,\mathbb{S}_{1,1,1,4,1} + 3\,\mathbb{S}_{1,1,2,1,3} - 24\,\mathbb{S}_{1,1,2,2,2} + 13\,\mathbb{S}_{1,1,2,3,1} - 7\,\mathbb{S}_{1,1,3,1,2} + 27\,\mathbb{S}_{1,1,3,2,1}\\
&-20\,\mathbb{S}_{1,1,4,1,1} + 3\,\mathbb{S}_{1,2,1,1,3} - 8\,\mathbb{S}_{1,2,1,2,2} + 3\,\mathbb{S}_{1,2,1,3,1} + 3\,\mathbb{S}_{1,2,2,1,2} - 37\,\mathbb{S}_{1,2,2,2,1}\\
&+27\,\mathbb{S}_{1,2,3,1,1} - 9\,\mathbb{S}_{1,3,1,1,2} + 9\,\mathbb{S}_{1,3,1,2,1} + 25\,\mathbb{S}_{1,3,2,1,1} - 20\,\mathbb{S}_{1,4,1,1,1} + 3\,\mathbb{S}_{2,1,1,1,3}\\
&-8\,\mathbb{S}_{2,1,1,2,2} + 3\,\mathbb{S}_{2,1,1,3,1} - 3\,\mathbb{S}_{2,1,2,1,2} - 5\,\mathbb{S}_{2,1,2,2,1} + 10\,\mathbb{S}_{2,1,3,1,1} + 4\,\mathbb{S}_{2,2,1,1,2}\\
&-6\,\mathbb{S}_{2,2,1,2,1} - 39\,\mathbb{S}_{2,2,2,1,1} + 29\,\mathbb{S}_{2,3,1,1,1} - 9\,\mathbb{S}_{3,1,1,1,2} + 8\,\mathbb{S}_{3,1,1,2,1} + 8\,\mathbb{S}_{3,1,2,1,1}\\
&+21\,\mathbb{S}_{3,2,1,1,1} - 16\,\mathbb{S}_{4,1,1,1,1} + 12\,\mathbb{S}_{1,1,1,2,2,1} - 18\,\mathbb{S}_{1,1,1,3,1,1} + 26\,\mathbb{S}_{1,1,2,2,1,1}\\
&-28\,\mathbb{S}_{1,1,3,1,1,1} + 24\,\mathbb{S}_{1,2,2,1,1,1} - 16\,\mathbb{S}_{1,3,1,1,1,1} + 16\,\mathbb{S}_{2,2,1,1,1,1} - 16\,\mathbb{S}_{3,1,1,1,1,1}\big)\,. \end{aligned} \tag{2.24}$$

We check with $M = 40$ that the obtained expression is indeed correct. The final expression for the six-loop anomalous dimension of twist-two operators from ABA in the canonical basis of the usual harmonic sums (2.11) can be found as the ancillary files of the arXiv version this paper or on the web-page http://thd.pnpi.spb.ru/~velizh/6loop/.

¶See Application in http://reference.wolfram.com/mathematica/ref/LatticeReduce.html

3 Calculations of the six-loop anomalous dimension for finite M

The quantum spectral curve [63, 64] formulates the spectral problem of planar $\mathcal{N} = 4$ SYM in terms of a finite set of Riemann-Hilbert equations. It was derived based on several assumptions, in particular that the theory is integrable. This approach automatically captures all the features of the spectral problem, i.e. no notion of ABA and wrapping contributions is needed. Recently, QSC was solved perturbatively for any operator in the $\mathfrak{sl}(2)$ sector by two of the authors [62], and an efficient `MATHEMATICA`-implementation of this iterative algorithm was provided, allowing up to 10-loop calculations of the simplest operators on a standard laptop. In our work, this procedure has been used to produce the six-loop anomalous dimensions of twist-two operators for finite M. Naturally, the computation time increases significantly with the value of M (15 seconds for $M = 2$ compared to 35 hours for $M = 80$), since one of the algorithm is systematically operating with the Baxter polynomial which is a degree-M polynomial. We refer to [62] for a detailed description of the procedure as well as access to the `MATHEMATICA`-implementation.

To have a sufficient amount of finite M results, we determined $\gamma_{12}(M)$ for the 40 lowest even integer spins, i.e. for $M = 2, 4, \ldots, 80$. We here provide the first five results obtained by the method. The remaining results can be reproduced from the general result below.

$$\begin{aligned}
\gamma_{12}(2) &= 48(-160 - 5472\,\zeta_3 - 432\,\zeta_3^2 + 2340\,\zeta_5 + 3240\,\zeta_3\,\zeta_5 + 1575\,\zeta_7 - 10206\,\zeta_9)\,,\\
\gamma_{12}(4) &= 25\left(\frac{493794415}{118098} - \frac{76738930}{2187}\,\zeta_3 - \frac{545000}{81}\,\zeta_3^2 + \frac{11977223}{729}\,\zeta_5 + \frac{50000}{3}\,\zeta_3\,\zeta_5\right.\\
&\qquad\left. + \frac{1723925}{81}\,\zeta_7 - 37800\,\zeta_9\right)\,,\\
\gamma_{12}(6) &= \frac{49}{25}\left(\frac{1305379290116927483}{14762250000000} - \frac{2757220598468}{3796875}\,\zeta_3 - \frac{80404688}{375}\,\zeta_3^2\right.\\
&\qquad\left. + \frac{1831031111}{5625}\,\zeta_5 + 345744\,\zeta_3\,\zeta_5 + \frac{44325547}{75}\,\zeta_7 - 666792\,\zeta_9\right)\,,\\
\gamma_{12}(8) &= \frac{1}{175}\left(\frac{6338981525904869169696050109 31}{190627211179440000000000} - \frac{12598676610089199091692 1}{400241898000000}\,\zeta_3\right.\\
&\qquad - \frac{467665307183261}{3601500}\,\zeta_3^2 + \frac{232643739682061563 7}{18151560000}\,\zeta_5 + \frac{7932799458}{49}\,\zeta_3\,\zeta_5\\
&\qquad\left. + \frac{19149255965089}{58800}\,\zeta_7 - 281452806\,\zeta_9\right)\,,\\
\gamma_{12}(10) &= \frac{1331}{175}\left(\frac{474005292705148459163445336569621}{216121846904347249152000000000} - 245586\,\zeta_9\right.\\
&\qquad - \frac{124179310062517901797908 17}{472677676700040000000}\,\zeta_3 - \frac{3452378452329719}{23629441500}\,\zeta_3^2\\
&\qquad\left. + \frac{6477832741701863069}{71455431096000}\,\zeta_5 + \frac{604223422}{3969}\,\zeta_3\,\zeta_5 + \frac{1757559752377}{5143824}\,\zeta_7\right)\,.
\end{aligned} \tag{3.1}$$

The first 35 values, i.e. $M \leq 70$, were used to construct the general structure for any M, and it was then checked that the last five results $72 \leq M \leq 80$ are consistent with this structure.

4 Reconstruction of the general form of six-loop anomalous dimension

In this section we describe in detail the method of the reconstruction of the anomalous dimension of twist-two operators from the above results and special numerical algorithms. Similarly to the ABA case, we will look for the reciprocity function $\mathcal{P}^{\rm wrap}$ which is represented in a smaller basis of binomial harmonic sums (2.15) and then reconstruct the conformal dimension using (2.17).

Although we compute the results for the full anomalous dimension it is more suitable and much easier to reconstruct the wrapping correction part (without an ABA part) rather than the full anomalous dimension. At five loops $\gamma_{10}^{\rm wrap}$ and $\mathcal{P}_{10}^{\rm wrap}$ are related as

$$\gamma_{10}^{\rm wrap} = \frac{1}{2}\Big(\gamma_2\gamma_8^{\rm wrap}\Big)' + \mathcal{P}_{10}^{\rm wrap}\,, \tag{4.1}$$

where γ_2 and $\gamma_8^{\rm wrap}$ are given by [34]

$$\gamma_8^{\rm wrap} = \mathcal{P}_8^{\rm wrap} = \mathcal{P}_2^2\,\mathcal{T}_8\,, \tag{4.2}$$

$$\mathcal{T}_8 = \Big(-5\,\zeta_5 + 2\,\mathbb{S}_2\,\zeta_3 + (\mathbb{S}_{2,1,2} - \mathbb{S}_{3,1,1})\Big)\,, \tag{4.3}$$

$$\gamma_2 = \mathcal{P}_2 = 4\,\mathbb{S}_1\,. \tag{4.4}$$

To find $\mathcal{P}_{10}^{\rm wrap}$ we should, in principle, consider the full basis of binomial harmonic sums of given transcendentality. However, at five loops we can write the following general expression for the reciprocity-respecting function $\mathcal{P}_{10}^{\rm wrap}$:

$$\mathcal{P}_{10}^{\rm wrap} = \mathcal{P}_2^2\,\mathcal{T}_{10} + c_1\,\mathcal{P}_2\mathcal{P}_4\mathcal{T}_8 + c_2\,\mathcal{P}_2^4\,\mathcal{T}_8\,, \tag{4.5}$$

$$\mathcal{P}_4 = 8\left(\mathbb{S}_1\mathbb{S}_2 - \mathbb{S}_{2,1} - \mathbb{S}_3\right). \tag{4.6}$$

The appearance of the structure with coefficients c_1 and c_2 is expected from a suggestion that the more general integrable system applicable for the computations of the wrapping corrections represents a set of (at least) two spin-chains, for which ABA is known. During the computations of the wrapping corrections at five loops in ref. [36] we assumed in advance the appearance of such terms, what allowed to find a final result much faster. Indeed, we found in ref. [36]

$$\mathcal{P}_{10}^{\rm wrap} = 2\,\mathcal{P}_2^2\mathcal{T}_{10} + 2\,\mathcal{P}_2\Big(2\,\mathcal{P}_4 + \frac{1}{16}\mathcal{P}_2^3\Big)\Big(-5\,\zeta_5 + 2\,\mathbb{S}_2\,\zeta_3 + (\mathbb{S}_{2,1,2} - \mathbb{S}_{3,1,1})\Big)\,, \tag{4.7}$$

$$\begin{aligned}
\mathcal{T}_{10} = {} & 105\,\zeta_7 - 6\,\mathbb{S}_1\,\zeta_3^2 - 40\,\mathbb{S}_2\,\zeta_5 + 4\big(3\,\mathbb{S}_1\mathbb{S}_{2,1} - 2\,\mathbb{S}_{2,2} + 2\,\mathbb{S}_{3,1} - \mathbb{S}_{2,1,1} - \mathbb{S}_4\big)\,\zeta_3 \\
& + 2\,\Big(\mathbb{S}_1\,(\mathbb{S}_{2,3,1} - \mathbb{S}_{3,1,2}) - \mathbb{S}_{2,1,4} + 2\,\mathbb{S}_{2,2,3} - 5\,\mathbb{S}_{3,1,3} + 2\,\mathbb{S}_{3,2,2} \\
& \quad +2\,\mathbb{S}_{3,3,1} - \mathbb{S}_{4,1,2} + \mathbb{S}_{5,1,1} - 2\,\mathbb{S}_{2,1,2,2} + 2\,\mathbb{S}_{2,1,3,1} - 2\,\mathbb{S}_{2,2,1,2} - 2\,\mathbb{S}_{2,2,2,1} \\
& \quad +2\,\mathbb{S}_{2,3,1,1} - 2\,\mathbb{S}_{3,1,1,2} + 2\,\mathbb{S}_{3,1,2,1} + 2\,\mathbb{S}_{3,2,1,1} - \mathbb{S}_{2,1,1,1,2} + \mathbb{S}_{3,1,1,1,1}\Big)\,.
\end{aligned} \tag{4.8}$$

The advantage of the reconstruction of the wrapping correction part instead of the full anomalous dimension is clear from eq. (4.5): $\mathcal{T}_{10}$ multiplied by $\mathbb{S}_1^2$ has the transcendentality 7 with $2^{9-1-2} = 64$ binomial harmonic sums in the basis instead of the transcendentality 9 with $2^{9-1} = 256$ binomial harmonic sums in the basis for $\mathcal{P}_{10}^{\rm wrap}$, so we will have considerably less binomial harmonic sums in the ansatz.

At six loops $\gamma_{12}^{\rm wrap}$ and $\mathcal{P}_{12}^{\rm wrap}$ are related as

$$\gamma_{12}^{\rm wrap} = \frac{1}{2}\Big(\gamma_4\gamma_8^{\rm wrap} + \gamma_2\gamma_{10}^{\rm wrap}\Big)' - \frac{1}{8}\Big(\gamma_2^2\gamma_8^{\rm wrap}\Big)'' + \mathcal{P}_{12}^{\rm wrap}\,. \tag{4.9}$$

Generalizing eq. (4.5) to the six-loop order we can assume the following general expression for the reciprocity-respecting function $\mathcal{P}_{12}^{\rm wrap}$:

$$\mathcal{P}_{12}^{\rm wrap} = \mathcal{P}_2^2\,\mathcal{T}_{12} + \Big(c_1\,\mathcal{P}_2\mathcal{P}_6 + c_2\,\mathcal{P}_2^3\mathcal{P}_4 + c_3\,\mathcal{P}_2^6 + c_4\,\mathcal{P}_4^2\Big)\mathcal{T}_8 + \Big(c_5\,\mathcal{P}_2\mathcal{P}_4 + c_6\,\mathcal{P}_2^4\Big)\mathcal{T}_{10}\,, \tag{4.10}$$

$$\begin{aligned} &= \mathcal{P}_{12,\,\rm rational}^{\rm wrap} + \zeta_3\mathcal{P}_{12,\,\zeta_3}^{\rm wrap} + \zeta_5\mathcal{P}_{12,\,\zeta_5}^{\rm wrap} + \zeta_3^2\mathcal{P}_{12,\,\zeta_3^2}^{\rm wrap} + \zeta_7\mathcal{P}_{12,\,\zeta_7}^{\rm wrap} + \zeta_3\zeta_5\mathcal{P}_{12,\,\zeta_3\zeta_5}^{\rm wrap} \\ &\quad + \zeta_3^3\mathcal{P}_{12,\,\zeta_3^3}^{\rm wrap} + \zeta_9\mathcal{P}_{12,\,\zeta_9}^{\rm wrap} \end{aligned} \tag{4.11}$$

with $\mathcal{T}_{12}$ and $\mathcal{P}_6$ are given by

$$\mathcal{T}_{12} = \zeta_9\,\mathcal{T}_{\zeta_9} + \zeta_3^3\,\mathcal{T}_{\zeta_3^3} + \zeta_5\,\zeta_3\,\mathcal{T}_{\zeta_5\zeta_3} + \zeta_7\,\mathcal{T}_{\zeta_7} + \zeta_3^2\,\mathcal{T}_{\zeta_3^2} + \zeta_5\,\mathcal{T}_{\zeta_5} + \zeta_3\,\mathcal{T}_{\zeta_3} + \mathcal{T}_{\rm rational}\,, \tag{4.12}$$

$$\mathcal{P}_6 = 16\Big(\mathbb{S}_1^2\mathbb{S}_3 + \mathbb{S}_1\big(\mathbb{S}_{3,1} - \mathbb{S}_{2,2} - 4\,\mathbb{S}_4\big) + 2\big(\mathbb{S}_{2,2,1} - \mathbb{S}_{2,1,2} - \mathbb{S}_{3,2} + 3\,\mathbb{S}_5\big)\Big)\,. \tag{4.13}$$

Applying the principle of maximal transcendentality we conclude that the transcendentality of the components $\mathcal{T}_{\zeta_9}$, $\mathcal{T}_{\zeta_3^3}$, $\mathcal{T}_{\zeta_3\zeta_5}$, $\mathcal{T}_{\zeta_7}$, $\mathcal{T}_{\zeta_3^2}$, $\mathcal{T}_{\zeta_5}$, $\mathcal{T}_{\zeta_3}$, $\mathcal{T}_{\rm rational}$ should be equal to 0, 0, 1, 2, 3, 4, 6 and 9 respectively (see Table 1 for the numbers of the binomial harmonic sums in the corresponding ansatz). The lowest-transcendentality functions, $\mathcal{T}_{\zeta_9}$, $\mathcal{T}_{\zeta_3^3}$, $\mathcal{T}_{\zeta_3\zeta_5}$, $\mathcal{T}_{\zeta_7}$, $\mathcal{T}_{\zeta_3^2}$, $\mathcal{T}_{\zeta_5}$ may be obtained exactly with the solution of the equations for the corresponding contribution as we know the full result for even values of M up to $M = 70$ (but we should take into account that the ζ_7 contribution is contained in $\mathcal{T}_{10}$, while ζ_5 and ζ_3 contributions are contained in both $\mathcal{T}_8$ and $\mathcal{T}_{10}$). Indeed, from the knowledge of just the Konishi operator ($M = 2$) we can find

$$\mathcal{T}_{\zeta_9} = -3402\,, \tag{4.14}$$

$$\mathcal{T}_{\zeta_3^3} = 0\,, \tag{4.15}$$

$$\mathcal{T}_{\zeta_3\zeta_5} = 360\,\mathbb{S}_1\,, \tag{4.16}$$

and using the first few values one can easily obtain

$$\mathcal{P}_{\zeta_3^2} = -12\,\mathcal{P}_2^2\Big(\mathbb{S}_1^3 + 14\,\mathbb{S}_1\mathbb{S}_2 - 6\big(\mathbb{S}_{2,1} + \mathbb{S}_3\big)\Big)\,. \tag{4.17}$$

However already for $\mathcal{T}_{\zeta_7}$ we can not find any solution, despite that the possible ansatz contains only two binomial harmonic sums: $\mathbb{S}_2$ and $\mathbb{S}_1^2$. To find a solution we extend our ansatz such that it is not restricted by eq. (4.10), that is we take the ansatz with all possible binomial harmonic sums with the transcendentality 4:

$$\mathrm{Basis}\big[\mathcal{P}_{12,\,\zeta_7}^{\rm wrap}\big] = \Big\{\mathbb{S}_4,\,\mathbb{S}_{3,1},\,\mathbb{S}_{2,2},\,\mathbb{S}_{2,1,1},\,\mathbb{S}_1\mathbb{S}_3,\,\mathbb{S}_1\mathbb{S}_{2,1},\,\mathbb{S}_1^2\mathbb{S}_2,\,\mathbb{S}_1^4\Big\}. \tag{4.18}$$

For this extended ansatz a solution exists (note that we have 35 equations for 8 variables):

$$\mathcal{P}_{12,\,\zeta_7}^{\rm wrap} = 224\,\mathbb{S}_1\Big(12\,\mathbb{S}_1^3 + 151\,\mathbb{S}_1\mathbb{S}_2 - 72\,\mathbb{S}_3 - 52\,\mathbb{S}_{2,1}\Big)\,. \tag{4.19}$$

Comparing this result to $\mathcal{P}_4$ from eq. (4.6) one can see that the coefficients of $\mathbb{S}_1\mathbb{S}_3$ and $\mathbb{S}_1\mathbb{S}_{2,1}$ are different, so our simple assumption about a general form of $\mathcal{P}_{12}^{\text{wrap}}$ is not correct and we should extend all bases for ζ_5, ζ_3 and rational contributions to $\mathcal{P}_{12}^{\text{wrap}}$. We have found that a minimal extension can be performed by expanding all terms with the brackets in eq. (4.10), that is we should expand $\mathcal{P}_2\mathcal{P}_6\mathcal{T}_8$, $\mathcal{P}_2^3\mathcal{P}_4\mathcal{T}_8$, $\mathcal{P}_2^6\mathcal{T}_8$, $\mathcal{P}_4^2\mathcal{T}_8$, $\mathcal{P}_2\mathcal{P}_4\mathcal{T}_{10}$, $\mathcal{P}_2^4\mathcal{T}_{10}$ and add all possible terms from these expansions to the corresponding ansatz for $\mathcal{T}_{\zeta_5}$, $\mathcal{T}_{\zeta_3}$ or $\mathcal{T}_{\text{rational}}$ contributions. The ansatz for the ζ_5 contribution to $\mathcal{P}_{12}^{\text{wrap}}$ will contain the following binomial harmonic sums:

$$\begin{aligned}\text{Basis}\big[\mathcal{P}_{12,\zeta_5}^{\text{wrap}}\big] = \Big\{&\mathbb{S}_1^2\mathbb{S}_4,\, \mathbb{S}_1^2\mathbb{S}_{3,1},\, \mathbb{S}_1^2\mathbb{S}_{2,2},\, \mathbb{S}_1^2\mathbb{S}_{2,1,1},\, \mathbb{S}_1^3\mathbb{S}_3,\, \mathbb{S}_1^3\mathbb{S}_{2,1},\, \mathbb{S}_1^4\mathbb{S}_2,\, \mathbb{S}_1^6,\\ &\mathbb{S}_1\mathbb{S}_2\mathbb{S}_3,\, \mathbb{S}_1\mathbb{S}_2\mathbb{S}_{2,1},\, \mathbb{S}_1\mathbb{S}_{2,2,1},\, \mathbb{S}_1\mathbb{S}_{2,1,2},\, \mathbb{S}_1\mathbb{S}_{3,2},\, \mathbb{S}_1\mathbb{S}_5,\, \big(\mathbb{S}_1\mathbb{S}_2-\mathbb{S}_3-\mathbb{S}_{2,1}\big)^2\Big\}, \end{aligned} \tag{4.20}$$

where the first line corresponds to the basis for $\mathcal{T}_{\zeta_5}$ while the second line comes from the expansion of eq. (4.10). Note that for the ζ_5 contribution we have found that the last term in eq. (4.20), which corresponds to $\mathcal{P}_4^2$, may not be expanded and this property is also correct for the other contributions. So we have 16 binomial harmonic sums in the ansatz for the ζ_5 contribution and the coefficients of these sums can be found from the known values. The final result for the ζ_5 contribution is given by:

$$\begin{aligned}\mathcal{P}_{12,\zeta_5}^{\text{wrap}} = 8\Big(&\frac{2944}{3}\mathbb{S}_1\mathbb{S}_2\mathbb{S}_{2,1} - \frac{1760}{3}\mathbb{S}_1^3\mathbb{S}_{2,1} - \frac{436}{3}\mathbb{S}_1^4\mathbb{S}_2 - \frac{25}{3}\mathbb{S}_1^6 - 80\big(-\mathbb{S}_{2,1}+\mathbb{S}_1\mathbb{S}_2-\mathbb{S}_3\big)^2\\ &-160\,\mathbb{S}_1^3\mathbb{S}_3 + 1024\,\mathbb{S}_1^2\mathbb{S}_4 + 656\,\mathbb{S}_1^2\mathbb{S}_{2,2} - 2024\,\mathbb{S}_1^2\mathbb{S}_{3,1} + 320\,\mathbb{S}_1^2\mathbb{S}_{2,1,1}\\ &-320\,\mathbb{S}_1\mathbb{S}_{2,2,1} + 608\,\mathbb{S}_1\mathbb{S}_2\mathbb{S}_3 - 960\,\mathbb{S}_1\mathbb{S}_5\Big). \end{aligned} \tag{4.21}$$

For the ζ_3 contribution our minimal ansatz consists of $2^{8-1-2}+19=51$ binomial harmonic sums, while we have only 35 values. To find the coefficients in the ansatz we use the same method as we used for the reconstruction of the ζ_3 contribution to the ABA part of the anomalous dimension in eq. (2.24). With the help of the `LatticeReduce` function from `MATHEMATICA`, which realizes the `LLL`-algorithm, we have found the following general expression for the ζ_3 contribution to $\mathcal{P}_{12}^{\text{wrap}}$:

$$\begin{aligned}\mathcal{P}_{12,\zeta_3}^{\text{wrap}} = 32\Bigg(&\frac{5}{6}\mathbb{S}_1^6\mathbb{S}_2 + 12\,\mathbb{S}_1^5\mathbb{S}_{2,1} - \frac{10}{3}\mathbb{S}_1^4\mathbb{S}_{2,1,1} + \frac{46}{3}\mathbb{S}_1^4\mathbb{S}_{2,2} + \frac{49}{3}\mathbb{S}_1^4\mathbb{S}_2^2 - \frac{10}{3}\mathbb{S}_1^4\mathbb{S}_4\\ &+\frac{248}{3}\mathbb{S}_1^3\mathbb{S}_2\mathbb{S}_{2,1} + 8\,\mathbb{S}_1^3\mathbb{S}_2\mathbb{S}_3 - 8\,\mathbb{S}_1^3\mathbb{S}_{2,3} + 16\,\mathbb{S}_1^3\mathbb{S}_{3,2} - 16\,\mathbb{S}_1^3\mathbb{S}_{4,1} - 80\,\mathbb{S}_1^3\mathbb{S}_{2,2,1} - 24\,\mathbb{S}_1^3\mathbb{S}_{2,1,1,1}\\ &-66\,\mathbb{S}_1^2\mathbb{S}_{2,1}^2 - \frac{332}{3}\mathbb{S}_1^2\mathbb{S}_3\mathbb{S}_{2,1} - 66\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{2,2} + \frac{82}{3}\mathbb{S}_1^2\mathbb{S}_2^3 - 118\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_4 + 20\,\mathbb{S}_1^2\mathbb{S}_6 + 12\,\mathbb{S}_1^2\mathbb{S}_{2,4}\\ &+\frac{116}{3}\mathbb{S}_1^2\mathbb{S}_{3,3} + \frac{152}{3}\mathbb{S}_1^2\mathbb{S}_{4,2} + 22\,\mathbb{S}_1^2\mathbb{S}_{5,1} - 32\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{2,1,1} + \frac{92}{3}\mathbb{S}_1^2\mathbb{S}_{2,1,3} - 22\,\mathbb{S}_1^2\mathbb{S}_{2,3,1}\\ &+14\,\mathbb{S}_1^2\mathbb{S}_{4,1,1} + 34\,\mathbb{S}_1^2\mathbb{S}_{2,2,1,1} + 20\,\mathbb{S}_1^2\mathbb{S}_{2,1,1,1,1} + 24\,\mathbb{S}_1\mathbb{S}_2^2\mathbb{S}_3 + 96\,\mathbb{S}_1\mathbb{S}_2\mathbb{S}_5 - 48\,\mathbb{S}_1\mathbb{S}_2^2\mathbb{S}_{2,1}\\ &-48\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{2,2} + 32\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{2,2} + 32\,\mathbb{S}_1\mathbb{S}_2\mathbb{S}_{2,2,1} + 32\,\mathbb{S}_1\mathbb{S}_3\big(2\,\mathbb{S}_{2,2} - 2\,\mathbb{S}_{3,1} + \mathbb{S}_{2,1,1} + \mathbb{S}_4\big)\\ &+\frac{16}{3}\mathbb{S}_1\mathbb{S}_{2,1}\big(2\,\mathbb{S}_{2,2} - 2\,\mathbb{S}_{3,1} + \mathbb{S}_{2,1,1} + \mathbb{S}_4\big) + 8\,\mathbb{S}_2\big(\mathbb{S}_1\mathbb{S}_2 - \mathbb{S}_{2,1} - \mathbb{S}_3\big)^2\Bigg). \end{aligned} \tag{4.22}$$

The last part is the rational contribution. In this case our minimal ansatz consists of $2^{11-1-2}+67=323$ binomial harmonic sums. The available 35 values are enough to find all coefficients in this ansatz with the help of the LLL-algorithm, but for this purpose we used a C++ implementation of the LLL-algorithm in the form of fplll-program [84]. After about one hour of computations we obtained the following general expression for the rational contribution to $\mathcal{P}_{12}^{\text{wrap}}$:

$$
\begin{aligned}
\mathcal{P}_{12,\,\text{rational}}^{\text{wrap}} = -\frac{32}{3}\Bigg(& \frac{5}{4}\,\mathbb{S}_1^6\mathbb{S}_{3,1,1} - \frac{5}{4}\,\mathbb{S}_1^6\mathbb{S}_{2,1,2} - 6\,\mathbb{S}_1^5\mathbb{S}_{2,3,1} + 6\,\mathbb{S}_1^5\mathbb{S}_{3,1,2} - 25\,\mathbb{S}_1^4\mathbb{S}_2\mathbb{S}_{2,1,2} + 5\,\mathbb{S}_1^4\mathbb{S}_{2,1,4} \\
& -4\,\mathbb{S}_1^4\mathbb{S}_{2,2,3} + 33\,\mathbb{S}_1^4\mathbb{S}_2\mathbb{S}_{3,1,1} + 7\,\mathbb{S}_1^4\mathbb{S}_{3,1,3} - 4\,\mathbb{S}_1^4\mathbb{S}_{3,2,2} - 12\,\mathbb{S}_1^4\mathbb{S}_{3,3,1} + 5\,\mathbb{S}_1^4\mathbb{S}_{4,1,2} - 5\,\mathbb{S}_1^4\mathbb{S}_{5,1,1} \\
& +13\,\mathbb{S}_1^4\mathbb{S}_{2,1,2,2} + \mathbb{S}_1^4\mathbb{S}_{2,1,3,1} + \mathbb{S}_1^4\mathbb{S}_{2,2,1,2} + 12\,\mathbb{S}_1^4\mathbb{S}_{2,2,2,1} - 13\,\mathbb{S}_1^4\mathbb{S}_{2,3,1,1} + 4\,\mathbb{S}_1^4\mathbb{S}_{3,1,1,2} \\
& -13\,\mathbb{S}_1^4\mathbb{S}_{3,1,2,1} - 13\,\mathbb{S}_1^4\mathbb{S}_{3,2,1,1} + 5\,\mathbb{S}_1^4\mathbb{S}_{2,1,1,1,2} - 5\,\mathbb{S}_1^4\mathbb{S}_{3,1,1,1,1} - 27\,\mathbb{S}_1^3\mathbb{S}_3\mathbb{S}_{2,1,2} \\
& +10\,\mathbb{S}_1^3\mathbb{S}_{2,1}\mathbb{S}_{2,1,2} + 2\,\mathbb{S}_1^3\mathbb{S}_{2,2,4} - 55\,\mathbb{S}_1^3\mathbb{S}_2\mathbb{S}_{2,3,1} + 12\,\mathbb{S}_1^3\mathbb{S}_{2,5,1} + 37\,\mathbb{S}_1^3\mathbb{S}_3\mathbb{S}_{3,1,1} \\
& -18\,\mathbb{S}_1^3\mathbb{S}_{2,1}\mathbb{S}_{3,1,1} + 54\,\mathbb{S}_1^3\mathbb{S}_2\mathbb{S}_{3,1,2} - 18\,\mathbb{S}_1^3\mathbb{S}_{3,1,4} - 21\,\mathbb{S}_1^3\mathbb{S}_{3,2,3} + 15\,\mathbb{S}_1^3\mathbb{S}_{3,4,1} - 13\,\mathbb{S}_1^3\mathbb{S}_{4,1,3} \\
& +15\,\mathbb{S}_1^3\mathbb{S}_{4,3,1} - 12\,\mathbb{S}_1^3\mathbb{S}_{5,1,2} + 22\,\mathbb{S}_1^3\mathbb{S}_{2,1,3,2} + 15\,\mathbb{S}_1^3\mathbb{S}_{2,1,4,1} - 23\,\mathbb{S}_1^3\mathbb{S}_{2,2,1,3} + 6\,\mathbb{S}_1^3\mathbb{S}_{2,2,2,2} \\
& +51\,\mathbb{S}_1^3\mathbb{S}_{2,2,3,1} - 10\,\mathbb{S}_1^3\mathbb{S}_{2,3,1,2} + 24\,\mathbb{S}_1^3\mathbb{S}_{2,3,2,1} - 27\,\mathbb{S}_1^3\mathbb{S}_{3,1,1,3} - 45\,\mathbb{S}_1^3\mathbb{S}_{3,1,2,2} + 24\,\mathbb{S}_1^3\mathbb{S}_{3,1,3,1} \\
& -29\,\mathbb{S}_1^3\mathbb{S}_{3,2,1,2} - 7\,\mathbb{S}_1^3\mathbb{S}_{3,2,2,1} + \mathbb{S}_1^3\mathbb{S}_{3,3,1,1} + 12\,\mathbb{S}_1^3\mathbb{S}_{2,1,1,3,1} - 4\,\mathbb{S}_1^3\mathbb{S}_{2,1,2,1,2} + 33\,\mathbb{S}_1^3\mathbb{S}_{2,1,2,2,1} \\
& +4\,\mathbb{S}_1^3\mathbb{S}_{2,1,3,1,1} - \mathbb{S}_1^3\mathbb{S}_{2,2,2,1,1} + 12\,\mathbb{S}_1^3\mathbb{S}_{2,3,1,1,1} - 12\,\mathbb{S}_1^3\mathbb{S}_{3,1,1,1,2} - 14\,\mathbb{S}_1^3\mathbb{S}_{3,1,1,2,1} \\
& -14\,\mathbb{S}_1^3\mathbb{S}_{3,1,2,1,1} + 124\,\mathbb{S}_1^2\mathbb{S}_4\mathbb{S}_{2,1,2} + 37\,\mathbb{S}_1^2\mathbb{S}_{2,2}\mathbb{S}_{2,1,2} - 48\,\mathbb{S}_1^2\mathbb{S}_{3,1}\mathbb{S}_{2,1,2} + 13\,\mathbb{S}_1^2\mathbb{S}_{2,1,1}\mathbb{S}_{2,1,2} \\
& +45\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{2,1,4} - 30\,\mathbb{S}_1^2\mathbb{S}_{2,1,6} - 25\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{2,2,3} + 16\,\mathbb{S}_1^2\mathbb{S}_{2,2,5} - 17\,\mathbb{S}_1^2\mathbb{S}_3\mathbb{S}_{2,3,1} + 53\,\mathbb{S}_1^2\mathbb{S}_{2,1}\mathbb{S}_{2,3,1} \\
& -150\,\mathbb{S}_1^2\mathbb{S}_4\mathbb{S}_{3,1,1} - 66\,\mathbb{S}_1^2\mathbb{S}_{2,2}\mathbb{S}_{3,1,1} + 67\,\mathbb{S}_1^2\mathbb{S}_{3,1}\mathbb{S}_{3,1,1} + 5\,\mathbb{S}_1^2\mathbb{S}_{2,1,1}\mathbb{S}_{3,1,1} + 25\,\mathbb{S}_1^2\mathbb{S}_3\mathbb{S}_{3,1,2} \\
& -50\,\mathbb{S}_1^2\mathbb{S}_{2,1}\mathbb{S}_{3,1,2} + 29\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{3,1,3} - 18\,\mathbb{S}_1^2\mathbb{S}_{3,1,5} + 10\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{3,2,2} + 52\,\mathbb{S}_1^2\mathbb{S}_{3,2,4} - 57\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{3,3,1} \\
& +24\,\mathbb{S}_1^2\mathbb{S}_{3,5,1} + 15\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{4,1,2} + 30\,\mathbb{S}_1^2\mathbb{S}_{4,1,4} - 10\,\mathbb{S}_1^2\mathbb{S}_{4,2,3} - 4\,\mathbb{S}_1^2\mathbb{S}_{4,4,1} - 22\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{5,1,1} \\
& +20\,\mathbb{S}_1^2\mathbb{S}_{5,1,3} + 20\,\mathbb{S}_1^2\mathbb{S}_{5,2,2} - 4\,\mathbb{S}_1^2\mathbb{S}_{5,3,1} - 30\,\mathbb{S}_1^2\mathbb{S}_{6,1,2} + 30\,\mathbb{S}_1^2\mathbb{S}_{7,1,1} + 52\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{2,1,2,2} \\
& -18\,\mathbb{S}_1^2\mathbb{S}_{2,1,2,4} - 23\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{2,1,3,1} - 31\,\mathbb{S}_1^2\mathbb{S}_{2,1,3,3} - 18\,\mathbb{S}_1^2\mathbb{S}_{2,1,4,2} + 3\,\mathbb{S}_1^2\mathbb{S}_{2,1,5,1} + 86\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{2,2,1,2} \\
& +18\,\mathbb{S}_1^2\mathbb{S}_{2,2,1,4} - 4\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{2,2,2,1} - \mathbb{S}_1^2\mathbb{S}_{2,2,2,3} + 10\,\mathbb{S}_1^2\mathbb{S}_{2,2,3,2} - 44\,\mathbb{S}_1^2\mathbb{S}_{2,2,4,1} - 47\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{2,3,1,1} \\
& -4\,\mathbb{S}_1^2\mathbb{S}_{2,3,1,3} + 14\,\mathbb{S}_1^2\mathbb{S}_{2,3,2,2} + 6\,\mathbb{S}_1^2\mathbb{S}_{2,3,3,1} - 18\,\mathbb{S}_1^2\mathbb{S}_{2,4,1,2} + 18\,\mathbb{S}_1^2\mathbb{S}_{2,5,1,1} + 25\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{3,1,1,2} \\
& +44\,\mathbb{S}_1^2\mathbb{S}_{3,1,1,4} - 62\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{3,1,2,1} + 2\,\mathbb{S}_1^2\mathbb{S}_{3,1,3,2} - 14\,\mathbb{S}_1^2\mathbb{S}_{3,1,4,1} - 52\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{3,2,1,1} - 14\,\mathbb{S}_1^2\mathbb{S}_{3,2,1,3} \\
& +30\,\mathbb{S}_1^2\mathbb{S}_{3,2,2,2} - 13\,\mathbb{S}_1^2\mathbb{S}_{3,2,3,1} - 5\,\mathbb{S}_1^2\mathbb{S}_{3,3,1,2} + 23\,\mathbb{S}_1^2\mathbb{S}_{3,3,2,1} + 9\,\mathbb{S}_1^2\mathbb{S}_{3,4,1,1} + 10\,\mathbb{S}_1^2\mathbb{S}_{4,1,1,3} \\
& -48\,\mathbb{S}_1^2\mathbb{S}_{4,1,2,2} + 2\,\mathbb{S}_1^2\mathbb{S}_{4,1,3,1} + 35\,\mathbb{S}_1^2\mathbb{S}_{4,2,1,2} - 23\,\mathbb{S}_1^2\mathbb{S}_{4,2,2,1} + 14\,\mathbb{S}_1^2\mathbb{S}_{4,3,1,1} - 20\,\mathbb{S}_1^2\mathbb{S}_{5,1,1,2} \\
& +18\,\mathbb{S}_1^2\mathbb{S}_{5,1,2,1} + 18\,\mathbb{S}_1^2\mathbb{S}_{5,2,1,1} + 38\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{2,1,1,1,2} - 6\,\mathbb{S}_1^2\mathbb{S}_{2,1,1,1,4} + 4\,\mathbb{S}_1^2\mathbb{S}_{2,1,1,4,1} - 2\,\mathbb{S}_1^2\mathbb{S}_{2,1,2,1,3} \\
& -27\,\mathbb{S}_1^2\mathbb{S}_{2,1,2,2,2} + 22\,\mathbb{S}_1^2\mathbb{S}_{2,1,2,3,1} - 25\,\mathbb{S}_1^2\mathbb{S}_{2,1,3,1,2} + 7\,\mathbb{S}_1^2\mathbb{S}_{2,1,3,2,1} - 12\,\mathbb{S}_1^2\mathbb{S}_{2,1,4,1,1}
\end{aligned}
$$

$$
\begin{aligned}
&+38\,\mathbb{S}_1^2\mathbb{S}_{2,2,1,1,3}-60\,\mathbb{S}_1^2\mathbb{S}_{2,2,1,2,2}+48\,\mathbb{S}_1^2\mathbb{S}_{2,2,1,3,1}-34\,\mathbb{S}_1^2\mathbb{S}_{2,2,2,1,2}-2\,\mathbb{S}_1^2\mathbb{S}_{2,2,2,2,1}\\
&-6\,\mathbb{S}_1^2\mathbb{S}_{2,2,3,1,1}-14\,\mathbb{S}_1^2\mathbb{S}_{2,3,1,1,2}+25\,\mathbb{S}_1^2\mathbb{S}_{2,3,1,2,1}+42\,\mathbb{S}_1^2\mathbb{S}_{2,3,2,1,1}-56\,\mathbb{S}_1^2\mathbb{S}_2\mathbb{S}_{3,1,1,1,1}\\
&-8\,\mathbb{S}_1^2\mathbb{S}_{3,1,1,1,3}-20\,\mathbb{S}_1^2\mathbb{S}_{3,1,1,2,2}+13\,\mathbb{S}_1^2\mathbb{S}_{3,1,1,3,1}-26\,\mathbb{S}_1^2\mathbb{S}_{3,1,2,1,2}+9\,\mathbb{S}_1^2\mathbb{S}_{3,1,2,2,1}\\
&+18\,\mathbb{S}_1^2\mathbb{S}_{3,1,3,1,1}-9\,\mathbb{S}_1^2\mathbb{S}_{3,2,1,1,2}+42\,\mathbb{S}_1^2\mathbb{S}_{3,2,1,2,1}+23\,\mathbb{S}_1^2\mathbb{S}_{3,2,2,1,1}+7\,\mathbb{S}_1^2\mathbb{S}_{3,3,1,1,1}\\
&-6\,\mathbb{S}_1^2\mathbb{S}_{4,1,1,1,2}+6\,\mathbb{S}_1^2\mathbb{S}_{5,1,1,1,1}-18\,\mathbb{S}_1^2\mathbb{S}_{2,1,1,1,2,2}-10\,\mathbb{S}_1^2\mathbb{S}_{2,1,1,1,3,1}-7\,\mathbb{S}_1^2\mathbb{S}_{2,1,1,2,1,2}\\
&-4\,\mathbb{S}_1^2\mathbb{S}_{2,1,1,2,2,1}-11\,\mathbb{S}_1^2\mathbb{S}_{2,1,1,3,1,1}-20\,\mathbb{S}_1^2\mathbb{S}_{2,1,2,1,1,2}-36\,\mathbb{S}_1^2\mathbb{S}_{2,1,2,2,1,1}+6\,\mathbb{S}_1^2\mathbb{S}_{2,1,3,1,1,1}\\
&-18\,\mathbb{S}_1^2\mathbb{S}_{2,2,1,1,1,2}-7\,\mathbb{S}_1^2\mathbb{S}_{2,2,2,1,1,1}+18\,\mathbb{S}_1^2\mathbb{S}_{2,3,1,1,1,1}+14\,\mathbb{S}_1^2\mathbb{S}_{3,1,1,1,1,2}+18\,\mathbb{S}_1^2\mathbb{S}_{3,1,1,1,2,1}\\
&+13\,\mathbb{S}_1^2\mathbb{S}_{3,1,1,2,1,1}+8\,\mathbb{S}_1^2\mathbb{S}_{3,1,2,1,1,1}+18\,\mathbb{S}_1^2\mathbb{S}_{3,2,1,1,1,1}-30\,\mathbb{S}_1^2\mathbb{S}_{2,1,1,1,1,1,2}+30\,\mathbb{S}_1^2\mathbb{S}_{3,1,1,1,1,1,1}\\
&+68\,\mathbb{S}_1\mathbb{S}_{2,1,2}^2-144\,\mathbb{S}_1\mathbb{S}_5\mathbb{S}_{2,1,2}+68\,\mathbb{S}_1\mathbb{S}_{3,2}\mathbb{S}_{2,1,2}-48\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{2,1,4}-8\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{2,1,4}\\
&-48\,\mathbb{S}_1\mathbb{S}_{2,1,2}\mathbb{S}_{2,2,1}+24\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{2,2,3}+18\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{2,2,3}+144\,\mathbb{S}_1\mathbb{S}_5\mathbb{S}_{3,1,1}-68\,\mathbb{S}_1\mathbb{S}_{3,2}\mathbb{S}_{3,1,1}\\
&-68\,\mathbb{S}_1\mathbb{S}_{2,1,2}\mathbb{S}_{3,1,1}+48\,\mathbb{S}_1\mathbb{S}_{2,2,1}\mathbb{S}_{3,1,1}-24\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{3,1,3}-46\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{3,1,3}+24\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{3,2,2}\\
&-8\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{3,2,2}-2\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{3,3,1}+44\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{3,3,1}-48\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{4,1,2}+18\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{4,1,2}\\
&+48\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{5,1,1}+8\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{5,1,1}-24\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{2,1,2,2}-64\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{2,1,2,2}-2\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{2,1,3,1}\\
&+24\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{2,1,3,1}-24\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{2,2,1,2}-90\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{2,2,1,2}+2\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{2,2,2,1}-44\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{2,2,2,1}\\
&+24\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{2,3,1,1}+64\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{2,3,1,1}-24\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{3,1,1,2}+8\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{3,1,1,2}-2\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{3,1,2,1}\\
&+64\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{3,1,2,1}+24\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{3,2,1,1}+64\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{3,2,1,1}-48\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{2,1,1,1,2}\\
&-8\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{2,1,1,1,2}+48\,\mathbb{S}_1\mathbb{S}_3\mathbb{S}_{3,1,1,1,1}+8\,\mathbb{S}_1\mathbb{S}_{2,1}\mathbb{S}_{3,1,1,1,1}\\
&-12\Big(\mathbb{S}_1\mathbb{S}_2-\mathbb{S}_3-\mathbb{S}_{2,1}\Big)^2\mathbb{S}_{2,1,2}+12\Big(\mathbb{S}_1\mathbb{S}_2-\mathbb{S}_3-\mathbb{S}_{2,1}\Big)^2\mathbb{S}_{3,1,1}\Bigg),
\end{aligned}
\tag{4.23}
$$

So, collecting all terms we can obtain the general expression for $\mathcal{P}_{12}^{\rm wrap}$. The general result for the full anomalous dimension can be found in the ancillary files of the arXiv version of the paper and on web-page: http://thd.pnpi.spb.ru/~velizh/6loop/.

5 Weak-coupling constraints

In this section we will discuss the known weak-coupling constraints on the six-loop anomalous dimension of twist-two operators in $\mathcal{N}=4$ SYM theory and compare them with the corresponding results, coming from the above six-loop anomalous dimension. We will use three classes of constraints, which are provided by the BFKL equation and by the generalized double-logarithmic equation at $M=-2+\omega$ and at $M=-r+\omega$, where $r=4, 6, 8, \ldots$.

A splitting function $P(x)$, which is related to the anomalous dimension through a

Mellin transformation

$$\gamma(M) = \int_0^1 dx\, x^M P(x)\,, \tag{5.1}$$

contains some powers of $\ln x$ in each order of perturbative theory when x tends to zero. This region of small x is very interesting from the experimental point of view as there are some theoretical methods which allow to sum all such large logarithms in all orders of perturbative theory. Mostly small-x physics is related to the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [51–53] and double-logarithmic equations [57–59]. So, as a splitting function is related to the anomalous dimension through a Mellin transformation, an expansion near $x = 0$ will correspond to some poles in the anomalous dimension, which appear when the argument of the anomalous dimension gets a negative value, as the harmonic sums (2.11), being a generalization of the Ψ-function, have poles at negative values of their argument.

5.1 BFKL equation

The relation between the anomalous dimension of twist-two operators and the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [51–53] and its next-to-leading logarithm approximation (NLLA) generalization [85, 86] emerges upon analytic continuation of the function $\gamma(g, M)$ to complex values of M. This is straightforward in the one-loop case since

$$\gamma_2(M) = 8\, g^2\, S_1(M) = 8\, g^2\, \left(\Psi(M+1) - \Psi(1)\right)\,, \tag{5.2}$$

where $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$ is the digamma function. At any loop order one expects singularities at all negative integer values of M. The first in this series of singular points,

$$M = -1 + \omega\,, \tag{5.3}$$

corresponds to the so-called BFKL pomeron. In the above formula ω should be considered infinitesimally small. The BFKL equation relates $\gamma(g)$ and ω in the vicinity of the point $M = -1 + \omega$. It predicts that, if expanded in g, the ℓ-loop anomalous dimension $\gamma_{2\ell}(\omega)$ exhibits poles in ω. Moreover, the residues and the order of the poles can be derived directly from the BFKL equation. The BFKL equation has been formulated up to the next-to-leading logarithm approximation (NLLA) and determines the leading and next-to-leading poles of $\gamma_{2\ell}(\omega)$. The NLLA BFKL equation for twist-two operators in $\mathcal{N} = 4$ SYM theory in the dimensional reduction scheme can be written as follows [85, 86]

$$\frac{\omega}{-4\, g^2} = \chi(\gamma) - g^2\, \delta(\gamma)\,, \tag{5.4}$$

where

$$\chi(\gamma) = \Psi\left(-\frac{\gamma}{2}\right) + \Psi\left(1 + \frac{\gamma}{2}\right) - 2\,\Psi\left(1\right)\,, \tag{5.5}$$

$$\begin{aligned} \delta(\gamma) &= 4\,\chi''(\gamma) + 6\,\zeta_3 + 2\,\zeta_2\,\chi(\gamma) + 4\,\chi(\gamma)\,\chi'(\gamma) \\ &\quad - \frac{\pi^3}{\sin\frac{\pi\gamma}{2}} - 4\,\Phi\left(-\frac{\gamma}{2}\right) - 4\,\Phi\left(1 + \frac{\gamma}{2}\right)\,. \end{aligned} \tag{5.6}$$

The function $\Phi(\gamma)$ is given by

$$\Phi(\gamma) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\gamma)^2}\Big[\Psi\,(k+\gamma+1) - \Psi(1)\Big]. \tag{5.7}$$

Upon using the expansion (2.2), one easily determines the leading singularity structure. Perturbatively expanding the anomalous dimension in the argument of the right-hand sided functions $\chi(\gamma)$ and $\delta(\gamma)$ in eq. (5.4) one can find a relation between the anomalous dimension at $M=-1+\omega$ and highest poles order by order, which can be written as:

$$\begin{aligned}\gamma =\ & \big(2+0\,\omega+\mathcal{O}(\omega^2)\big)\left(\frac{-4\,g^2}{\omega}\right) - \big(0+0\,\omega+\mathcal{O}(\omega^2)\big)\left(\frac{-4\,g^2}{\omega}\right)^2 \\ & + \big(0+\,\zeta_3\,\omega+\mathcal{O}(\omega^2)\big)\left(\frac{-4\,g^2}{\omega}\right)^3 - \left(4\,\zeta_3+\frac{5}{4}\,\zeta_4\,\omega+\mathcal{O}(\omega^2)\right)\left(\frac{-4\,g^2}{\omega}\right)^4 \\ & - \left(0+\Big(2\,\zeta_2\,\zeta_3+16\,\zeta_5\Big)\,\omega+\mathcal{O}(\omega^2)\right)\left(\frac{-4g^2}{\omega}\right)^5 \\ & - \left(\Big(0\,\zeta_2\,\zeta_3+4\,\zeta_5\Big)+\Big(3\,\zeta_3^2-\frac{143}{48}\,\zeta_6\Big)\,\omega+\mathcal{O}(\omega^2)\right)\left(\frac{-4g^2}{\omega}\right)^6 \pm \ldots . \end{aligned} \tag{5.8}$$

The last line of the above equation gives the prediction for the analytic continuation of the six-loop anomalous dimension at $M=-1+\omega$ and our full result satisfies these constraints.

5.2 Generalized double-logarithmic equation: $M=-2+\omega$

Another class of constraints on the anomalous dimension of twist-two operators follows from the double-logarithmic asymptotics of the scattering amplitudes. The double-logarithmic asymptotics of the scattering amplitudes were investigated in QED and QCD in the papers [54–56] and [57–59] (see also `arXiv` version of ref. [67]). It corresponds to summing the leading terms $(\alpha \ln^2 s)^k$ in all orders of perturbation theory. In the combination with a Mellin transformation, the double-logarithmic asymptotics allow to predict the singular part of anomalous dimensions near the point $M=-2$. For our purpose and in our notation the double-logarithmic equation has the following form

$$\gamma\,(2\,\omega+\gamma) = -16\,g^2\,. \tag{5.9}$$

The solution of this equation gives a prediction for the highest pole (g^{2k}/ω^{2k-1}) in all orders of perturbative theory:

$$\begin{aligned}\gamma = -\omega+\omega\sqrt{1-\frac{16g^2}{\omega^2}} = 2\,\frac{(-4\,g^2)}{\omega} - 2\,\frac{(-4\,g^2)^2}{\omega^3} + 4\,\frac{(-4\,g^2)^3}{\omega^5} - 10\,\frac{(-4\,g^2)^4}{\omega^7} \\ +28\,\frac{(-4\,g^2)^5}{\omega^9} - 84\,\frac{(-4\,g^2)^6}{\omega^{11}} + \ldots . \end{aligned} \tag{5.10}$$

The investigation of the analytic properties of the anomalous dimension of twist-2 operators in $\mathcal{N}=4$ SYM theory led to the suggestion about a simple generalization of the double-logarithmic equation [60][‖]. The main idea was that in eq. (5.9) the corrections to

[‖] For the first time, a such generalization was suggested by Lev N. Lipatov and Andrei Onishchenko at 2004, but was not published. Then, it was improved by Lev N. Lipatov in ref. [32].

the leading order equation will modify only the right-hand side and that such modification admit, besides an expansion in the coupling constant g^2, only the appearance of a regular terms depending on ω (and, possibly, γ). Substituting the results for the analytic continuation of the anomalous dimension of twist-2 operators near $M = -2 + \omega$ into eq. (5.9) we indeed find the following form of the generalized double-logarithmic equation [60]

$$\gamma\,(2\,\omega + \gamma) = \sum_{k=1} \sum_{m=0} \mathfrak{C}^k_m\, \omega^m\, g^{2k}\,. \tag{5.11}$$

The coefficients $\mathfrak{C}^k_m$ can be found in Appendix of ref. [61], but for the test of the six-loop anomalous dimension we need only the fact that the perturbative expansion of the left hand side of the above equation (5.11) does not contain any poles near $M = -2$, which is indeed correct for the result obtained in this paper. We present the result for the analytic continuation of the six-loop anomalous dimension at $M = -2 + \omega$ in Appendix B, and one can see that there are about *one hundred* terms up to g^{12}/w^2, which can be checked by the generalized double-logarithmic equation (5.11), so we have very strong test for the correctness of our general result for the six-loop anomalous dimension. Note also that with the generalized double-logarithmic equation (5.11) we can control even the ζ_9 term in the six-loop anomalous dimension coming from eq. (4.14) as its analytic continuation is proportional to ζ_9/ω^2, what is impossible with the BFKL equation (5.5).

5.3 Generalized double-logarithmic equation: $M = -r + \omega,\ r = 2,\ 4,\ 6,\ \dots$

In ref. [60] we have found another generalization of the double-logarithmic equation, which hold true not only for $M = -2+\omega$, but for all other negative even values $M = -r+\omega\,,\ r = 2, 4, 6, \dots$. This generalization is related to the analytic properties of the reciprocity-respecting anomalous dimension $\mathcal{P}(M)$, which near $M = -r + \omega\,,\ r = 2, 4, 6, \dots$ can be written as:

$$\mathcal{P}_{\mathrm{DL}}(\omega, r) = 2 \sum_{k=1} \sum_{m=0} \mathcal{D}^k_m(r)\, \omega^m \left(\frac{-4\, g^2}{\omega}\right)^k \tag{5.12}$$

where some coefficients $\mathcal{D}^k_m(r)$ can be found in ref. [60]. For the test of the six-loop anomalous dimension we need only the fact that according to eq. (5.12) the six-loop reciprocity-respecting anomalous dimension $\mathcal{P}_{12}(M)$ does not contain any poles higher than $1/\omega^6$, which is indeed correct for the obtained result.

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A Appendix A

We have found the following result for $\hat{\mathcal{P}}_{12}(M)$ from eq. (2.17):

$$
\begin{aligned}
\frac{\hat{\mathcal{P}}_{12}^{\text{rational}}}{128} = &-180\,\mathbb{S}_{2,2,7} + 540\,\mathbb{S}_{3,1,7} - 592\,\mathbb{S}_{3,2,6} + 644\,\mathbb{S}_{4,1,6} - 684\,\mathbb{S}_{4,2,5} + 700\,\mathbb{S}_{5,1,5} \\
&-700\,\mathbb{S}_{5,2,4} + 636\,\mathbb{S}_{6,1,4} - 444\,\mathbb{S}_{6,2,3} + 176\,\mathbb{S}_{7,1,3} - 96\,\mathbb{S}_{7,2,2} - 228\,\mathbb{S}_{1,2,2,6} \\
&-4\,\mathbb{S}_{1,2,7,1} + 700\,\mathbb{S}_{1,3,1,6} - 728\,\mathbb{S}_{1,3,2,5} + 732\,\mathbb{S}_{1,4,1,5} - 728\,\mathbb{S}_{1,4,2,4} + 660\,\mathbb{S}_{1,5,1,4} \\
&-548\,\mathbb{S}_{1,5,2,3} + 344\,\mathbb{S}_{1,6,1,3} - 224\,\mathbb{S}_{1,6,2,2} + 52\,\mathbb{S}_{1,7,1,2} - 28\,\mathbb{S}_{1,7,2,1} + 4\,\mathbb{S}_{2,1,2,6} \\
&-4\,\mathbb{S}_{2,1,7,1} - 640\,\mathbb{S}_{2,2,1,6} + 988\,\mathbb{S}_{2,2,2,5} + 78\,\mathbb{S}_{2,2,3,4} + 78\,\mathbb{S}_{2,2,4,3} + 78\,\mathbb{S}_{2,2,5,2} \\
&-70\,\mathbb{S}_{2,2,6,1} - 864\,\mathbb{S}_{2,3,1,5} + 874\,\mathbb{S}_{2,3,2,4} + 2\,\mathbb{S}_{2,3,5,1} - 820\,\mathbb{S}_{2,4,1,4} + 698\,\mathbb{S}_{2,4,2,3} \\
&+2\,\mathbb{S}_{2,4,4,1} - 478\,\mathbb{S}_{2,5,1,3} + 326\,\mathbb{S}_{2,5,2,2} + 2\,\mathbb{S}_{2,5,3,1} - 106\,\mathbb{S}_{2,6,1,2} + 72\,\mathbb{S}_{2,6,2,1} \\
&-16\,\mathbb{S}_{2,7,1,1} + 592\,\mathbb{S}_{3,1,1,6} - 676\,\mathbb{S}_{3,1,2,5} - 242\,\mathbb{S}_{3,1,3,4} - 242\,\mathbb{S}_{3,1,4,3} - 242\,\mathbb{S}_{3,1,5,2} \\
&+282\,\mathbb{S}_{3,1,6,1} - 768\,\mathbb{S}_{3,2,1,5} + 1144\,\mathbb{S}_{3,2,2,4} + 260\,\mathbb{S}_{3,2,3,3} + 260\,\mathbb{S}_{3,2,4,2} - 288\,\mathbb{S}_{3,2,5,1} \\
&-290\,\mathbb{S}_{3,3,1,4} + 230\,\mathbb{S}_{3,3,2,3} - 138\,\mathbb{S}_{3,4,1,3} + 84\,\mathbb{S}_{3,4,2,2} - 14\,\mathbb{S}_{3,5,1,2} + 8\,\mathbb{S}_{3,5,2,1} \\
&+660\,\mathbb{S}_{4,1,1,5} - 746\,\mathbb{S}_{4,1,2,4} - 278\,\mathbb{S}_{4,1,3,3} - 278\,\mathbb{S}_{4,1,4,2} + 286\,\mathbb{S}_{4,1,5,1} - 762\,\mathbb{S}_{4,2,1,4} \\
&+1092\,\mathbb{S}_{4,2,2,3} + 296\,\mathbb{S}_{4,2,3,2} - 290\,\mathbb{S}_{4,2,4,1} - 138\,\mathbb{S}_{4,3,1,3} + 84\,\mathbb{S}_{4,3,2,2} - 14\,\mathbb{S}_{4,4,1,2} \\
&+8\,\mathbb{S}_{4,4,2,1} + 636\,\mathbb{S}_{5,1,1,4} - 726\,\mathbb{S}_{5,1,2,3} - 302\,\mathbb{S}_{5,1,3,2} + 276\,\mathbb{S}_{5,1,4,1} - 470\,\mathbb{S}_{5,2,1,3} \\
&+778\,\mathbb{S}_{5,2,2,2} - 274\,\mathbb{S}_{5,2,3,1} - 14\,\mathbb{S}_{5,3,1,2} + 8\,\mathbb{S}_{5,3,2,1} + 368\,\mathbb{S}_{6,1,1,3} - 492\,\mathbb{S}_{6,1,2,2} \\
&+240\,\mathbb{S}_{6,1,3,1} - 144\,\mathbb{S}_{6,2,1,2} + 16\,\mathbb{S}_{6,2,2,1} + 80\,\mathbb{S}_{7,1,1,2} - 32\,\mathbb{S}_{7,1,2,1} - 32\,\mathbb{S}_{7,2,1,1} \\
&-8\,\mathbb{S}_{1,1,1,2,6} + 8\,\mathbb{S}_{1,1,1,7,1} - 8\,\mathbb{S}_{1,1,2,1,6} - 236\,\mathbb{S}_{1,1,2,2,5} + 4\,\mathbb{S}_{1,1,2,3,4} + 4\,\mathbb{S}_{1,1,2,4,3} \\
&+4\,\mathbb{S}_{1,1,2,5,2} - 28\,\mathbb{S}_{1,1,2,6,1} + 788\,\mathbb{S}_{1,1,3,1,5} - 776\,\mathbb{S}_{1,1,3,2,4} - 4\,\mathbb{S}_{1,1,3,5,1} + 700\,\mathbb{S}_{1,1,4,1,4} \\
&-620\,\mathbb{S}_{1,1,4,2,3} - 4\,\mathbb{S}_{1,1,4,4,1} + 440\,\mathbb{S}_{1,1,5,1,3} - 324\,\mathbb{S}_{1,1,5,2,2} - 4\,\mathbb{S}_{1,1,5,3,1} + 128\,\mathbb{S}_{1,1,6,1,2} \\
&-96\,\mathbb{S}_{1,1,6,2,1} + 32\,\mathbb{S}_{1,1,7,1,1} - 8\,\mathbb{S}_{1,2,1,1,6} + 20\,\mathbb{S}_{1,2,1,2,5} + 4\,\mathbb{S}_{1,2,1,3,4} + 4\,\mathbb{S}_{1,2,1,4,3} \\
&+4\,\mathbb{S}_{1,2,1,5,2} - 24\,\mathbb{S}_{1,2,1,6,1} - 736\,\mathbb{S}_{1,2,2,1,5} + 1070\,\mathbb{S}_{1,2,2,2,4} + 82\,\mathbb{S}_{1,2,2,3,3} + 82\,\mathbb{S}_{1,2,2,4,2} \\
&-22\,\mathbb{S}_{1,2,2,5,1} - 870\,\mathbb{S}_{1,2,3,1,4} + 784\,\mathbb{S}_{1,2,3,2,3} - 2\,\mathbb{S}_{1,2,3,3,2} + 18\,\mathbb{S}_{1,2,3,4,1} - 600\,\mathbb{S}_{1,2,4,1,3} \\
&+454\,\mathbb{S}_{1,2,4,2,2} + 18\,\mathbb{S}_{1,2,4,3,1} - 214\,\mathbb{S}_{1,2,5,1,2} + 176\,\mathbb{S}_{1,2,5,2,1} - 76\,\mathbb{S}_{1,2,6,1,1} + 704\,\mathbb{S}_{1,3,1,1,5} \\
&-794\,\mathbb{S}_{1,3,1,2,4} - 302\,\mathbb{S}_{1,3,1,3,3} - 302\,\mathbb{S}_{1,3,1,4,2} + 306\,\mathbb{S}_{1,3,1,5,1} - 794\,\mathbb{S}_{1,3,2,1,4} + 1174\,\mathbb{S}_{1,3,2,2,3} \\
&+312\,\mathbb{S}_{1,3,2,3,2} - 304\,\mathbb{S}_{1,3,2,4,1} - 198\,\mathbb{S}_{1,3,3,1,3} + 140\,\mathbb{S}_{1,3,3,2,2} + 2\,\mathbb{S}_{1,3,3,3,1} - 54\,\mathbb{S}_{1,3,4,1,2} \\
&+38\,\mathbb{S}_{1,3,4,2,1} - 12\,\mathbb{S}_{1,3,5,1,1} + 664\,\mathbb{S}_{1,4,1,1,4} - 756\,\mathbb{S}_{1,4,1,2,3} - 310\,\mathbb{S}_{1,4,1,3,2} + 292\,\mathbb{S}_{1,4,1,4,1}
\end{aligned}
$$

$$-576\,\mathbb{S}_{1,4,2,1,3}+904\,\mathbb{S}_{1,4,2,2,2}-286\,\mathbb{S}_{1,4,2,3,1}-54\,\mathbb{S}_{1,4,3,1,2}+38\,\mathbb{S}_{1,4,3,2,1}-12\,\mathbb{S}_{1,4,4,1,1}$$
$$+456\,\mathbb{S}_{1,5,1,1,3}-568\,\mathbb{S}_{1,5,1,2,2}+258\,\mathbb{S}_{1,5,1,3,1}-244\,\mathbb{S}_{1,5,2,1,2}+110\,\mathbb{S}_{1,5,2,2,1}-12\,\mathbb{S}_{1,5,3,1,1}$$
$$+172\,\mathbb{S}_{1,6,1,1,2}-92\,\mathbb{S}_{1,6,1,2,1}-120\,\mathbb{S}_{1,6,2,1,1}+56\,\mathbb{S}_{1,7,1,1,1}-8\,\mathbb{S}_{2,1,1,1,6}+20\,\mathbb{S}_{2,1,1,2,5}$$
$$+4\,\mathbb{S}_{2,1,1,3,4}+4\,\mathbb{S}_{2,1,1,4,3}+4\,\mathbb{S}_{2,1,1,5,2}-24\,\mathbb{S}_{2,1,1,6,1}+16\,\mathbb{S}_{2,1,2,1,5}+68\,\mathbb{S}_{2,1,2,2,4}$$
$$-14\,\mathbb{S}_{2,1,2,3,3}-14\,\mathbb{S}_{2,1,2,4,2}+56\,\mathbb{S}_{2,1,2,5,1}-340\,\mathbb{S}_{2,1,3,1,4}+316\,\mathbb{S}_{2,1,3,2,3}-2\,\mathbb{S}_{2,1,3,3,2}$$
$$+16\,\mathbb{S}_{2,1,3,4,1}-260\,\mathbb{S}_{2,1,4,1,3}+212\,\mathbb{S}_{2,1,4,2,2}+16\,\mathbb{S}_{2,1,4,3,1}-122\,\mathbb{S}_{2,1,5,1,2}+112\,\mathbb{S}_{2,1,5,2,1}$$
$$-60\,\mathbb{S}_{2,1,6,1,1}-640\,\mathbb{S}_{2,2,1,1,5}+704\,\mathbb{S}_{2,2,1,2,4}+264\,\mathbb{S}_{2,2,1,3,3}+264\,\mathbb{S}_{2,2,1,4,2}-240\,\mathbb{S}_{2,2,1,5,1}$$
$$+1226\,\mathbb{S}_{2,2,2,1,4}-1816\,\mathbb{S}_{2,2,2,2,3}-381\,\mathbb{S}_{2,2,2,3,2}+245\,\mathbb{S}_{2,2,2,4,1}+638\,\mathbb{S}_{2,2,3,1,3}-540\,\mathbb{S}_{2,2,3,2,2}$$
$$-18\,\mathbb{S}_{2,2,3,3,1}+289\,\mathbb{S}_{2,2,4,1,2}-179\,\mathbb{S}_{2,2,4,2,1}+56\,\mathbb{S}_{2,2,5,1,1}-810\,\mathbb{S}_{2,3,1,1,4}+918\,\mathbb{S}_{2,3,1,2,3}$$
$$+367\,\mathbb{S}_{2,3,1,3,2}-339\,\mathbb{S}_{2,3,1,4,1}+742\,\mathbb{S}_{2,3,2,1,3}-1136\,\mathbb{S}_{2,3,2,2,2}+322\,\mathbb{S}_{2,3,2,3,1}+93\,\mathbb{S}_{2,3,3,1,2}$$
$$-77\,\mathbb{S}_{2,3,3,2,1}+36\,\mathbb{S}_{2,3,4,1,1}-600\,\mathbb{S}_{2,4,1,1,3}+730\,\mathbb{S}_{2,4,1,2,2}-312\,\mathbb{S}_{2,4,1,3,1}+358\,\mathbb{S}_{2,4,2,1,2}$$
$$-200\,\mathbb{S}_{2,4,2,2,1}+36\,\mathbb{S}_{2,4,3,1,1}-258\,\mathbb{S}_{2,5,1,1,2}+136\,\mathbb{S}_{2,5,1,2,1}+206\,\mathbb{S}_{2,5,2,1,1}-116\,\mathbb{S}_{2,6,1,1,1}$$
$$+620\,\mathbb{S}_{3,1,1,1,5}-700\,\mathbb{S}_{3,1,1,2,4}-260\,\mathbb{S}_{3,1,1,3,3}-260\,\mathbb{S}_{3,1,1,4,2}+280\,\mathbb{S}_{3,1,1,5,1}-266\,\mathbb{S}_{3,1,2,1,4}$$
$$+602\,\mathbb{S}_{3,1,2,2,3}+293\,\mathbb{S}_{3,1,2,3,2}-307\,\mathbb{S}_{3,1,2,4,1}+50\,\mathbb{S}_{3,1,3,1,3}+84\,\mathbb{S}_{3,1,3,2,2}-110\,\mathbb{S}_{3,1,3,3,1}$$
$$+65\,\mathbb{S}_{3,1,4,1,2}-371\,\mathbb{S}_{3,1,4,2,1}+348\,\mathbb{S}_{3,1,5,1,1}-738\,\mathbb{S}_{3,2,1,1,4}+832\,\mathbb{S}_{3,2,1,2,3}+329\,\mathbb{S}_{3,2,1,3,2}$$
$$-309\,\mathbb{S}_{3,2,1,4,1}+550\,\mathbb{S}_{3,2,2,1,3}-1126\,\mathbb{S}_{3,2,2,2,2}+456\,\mathbb{S}_{3,2,2,3,1}-31\,\mathbb{S}_{3,2,3,1,2}+347\,\mathbb{S}_{3,2,3,2,1}$$
$$-342\,\mathbb{S}_{3,2,4,1,1}-198\,\mathbb{S}_{3,3,1,1,3}+248\,\mathbb{S}_{3,3,1,2,2}-108\,\mathbb{S}_{3,3,1,3,1}+102\,\mathbb{S}_{3,3,2,1,2}-48\,\mathbb{S}_{3,3,2,2,1}$$
$$+4\,\mathbb{S}_{3,3,3,1,1}-68\,\mathbb{S}_{3,4,1,1,2}+36\,\mathbb{S}_{3,4,1,2,1}+44\,\mathbb{S}_{3,4,2,1,1}-16\,\mathbb{S}_{3,5,1,1,1}+616\,\mathbb{S}_{4,1,1,1,4}$$
$$-700\,\mathbb{S}_{4,1,1,2,3}-284\,\mathbb{S}_{4,1,1,3,2}+272\,\mathbb{S}_{4,1,1,4,1}-232\,\mathbb{S}_{4,1,2,1,3}+594\,\mathbb{S}_{4,1,2,2,2}-304\,\mathbb{S}_{4,1,2,3,1}$$
$$+60\,\mathbb{S}_{4,1,3,1,2}-362\,\mathbb{S}_{4,1,3,2,1}+344\,\mathbb{S}_{4,1,4,1,1}-564\,\mathbb{S}_{4,2,1,1,3}+684\,\mathbb{S}_{4,2,1,2,2}-288\,\mathbb{S}_{4,2,1,3,1}$$
$$+280\,\mathbb{S}_{4,2,2,1,2}+188\,\mathbb{S}_{4,2,2,2,1}-336\,\mathbb{S}_{4,2,3,1,1}-68\,\mathbb{S}_{4,3,1,1,2}+36\,\mathbb{S}_{4,3,1,2,1}+44\,\mathbb{S}_{4,3,2,1,1}$$
$$-16\,\mathbb{S}_{4,4,1,1,1}+452\,\mathbb{S}_{5,1,1,1,3}-556\,\mathbb{S}_{5,1,1,2,2}+242\,\mathbb{S}_{5,1,1,3,1}-132\,\mathbb{S}_{5,1,2,1,2}-192\,\mathbb{S}_{5,1,2,2,1}$$
$$+330\,\mathbb{S}_{5,1,3,1,1}-262\,\mathbb{S}_{5,2,1,1,2}+140\,\mathbb{S}_{5,2,1,2,1}+98\,\mathbb{S}_{5,2,2,1,1}-16\,\mathbb{S}_{5,3,1,1,1}+188\,\mathbb{S}_{6,1,1,1,2}$$
$$-96\,\mathbb{S}_{6,1,1,2,1}-96\,\mathbb{S}_{6,1,2,1,1}-128\,\mathbb{S}_{6,2,1,1,1}+64\,\mathbb{S}_{7,1,1,1,1}+16\,\mathbb{S}_{1,1,1,1,1,6}-24\,\mathbb{S}_{1,1,1,1,2,5}$$
$$-8\,\mathbb{S}_{1,1,1,1,3,4}-8\,\mathbb{S}_{1,1,1,1,4,3}-8\,\mathbb{S}_{1,1,1,1,5,2}+32\,\mathbb{S}_{1,1,1,1,6,1}-16\,\mathbb{S}_{1,1,1,2,1,5}-200\,\mathbb{S}_{1,1,1,2,2,4}$$
$$+20\,\mathbb{S}_{1,1,1,2,3,3}+20\,\mathbb{S}_{1,1,1,2,4,2}-68\,\mathbb{S}_{1,1,1,2,5,1}+744\,\mathbb{S}_{1,1,1,3,1,4}-668\,\mathbb{S}_{1,1,1,3,2,3}+4\,\mathbb{S}_{1,1,1,3,3,2}$$
$$-24\,\mathbb{S}_{1,1,1,3,4,1}+508\,\mathbb{S}_{1,1,1,4,1,3}-392\,\mathbb{S}_{1,1,1,4,2,2}-24\,\mathbb{S}_{1,1,1,4,3,1}+192\,\mathbb{S}_{1,1,1,5,1,2}$$

$$
\begin{aligned}
&-176\,\mathbb{S}_{1,1,1,5,2,1}+80\,\mathbb{S}_{1,1,1,6,1,1}-16\,\mathbb{S}_{1,1,2,1,1,5}+28\,\mathbb{S}_{1,1,2,1,2,4}+16\,\mathbb{S}_{1,1,2,1,3,3}\\
&+16\,\mathbb{S}_{1,1,2,1,4,2}-44\,\mathbb{S}_{1,1,2,1,5,1}-740\,\mathbb{S}_{1,1,2,2,1,4}+974\,\mathbb{S}_{1,1,2,2,2,3}+64\,\mathbb{S}_{1,1,2,2,3,2}\\
&+38\,\mathbb{S}_{1,1,2,2,4,1}-678\,\mathbb{S}_{1,1,2,3,1,3}+534\,\mathbb{S}_{1,1,2,3,2,2}+50\,\mathbb{S}_{1,1,2,3,3,1}-308\,\mathbb{S}_{1,1,2,4,1,2}\\
&+310\,\mathbb{S}_{1,1,2,4,2,1}-164\,\mathbb{S}_{1,1,2,5,1,1}+712\,\mathbb{S}_{1,1,3,1,1,4}-810\,\mathbb{S}_{1,1,3,1,2,3}-328\,\mathbb{S}_{1,1,3,1,3,2}\\
&+310\,\mathbb{S}_{1,1,3,1,4,1}-654\,\mathbb{S}_{1,1,3,2,1,3}+1006\,\mathbb{S}_{1,1,3,2,2,2}-282\,\mathbb{S}_{1,1,3,2,3,1}-96\,\mathbb{S}_{1,1,3,3,1,2}\\
&+90\,\mathbb{S}_{1,1,3,3,2,1}-48\,\mathbb{S}_{1,1,3,4,1,1}+520\,\mathbb{S}_{1,1,4,1,1,3}-628\,\mathbb{S}_{1,1,4,1,2,2}+276\,\mathbb{S}_{1,1,4,1,3,1}\\
&-332\,\mathbb{S}_{1,1,4,2,1,2}+200\,\mathbb{S}_{1,1,4,2,2,1}-48\,\mathbb{S}_{1,1,4,3,1,1}+244\,\mathbb{S}_{1,1,5,1,1,2}-128\,\mathbb{S}_{1,1,5,1,2,1}\\
&-220\,\mathbb{S}_{1,1,5,2,1,1}+136\,\mathbb{S}_{1,1,6,1,1,1}-16\,\mathbb{S}_{1,2,1,1,1,5}+28\,\mathbb{S}_{1,2,1,1,2,4}+16\,\mathbb{S}_{1,2,1,1,3,3}\\
&+16\,\mathbb{S}_{1,2,1,1,4,2}-44\,\mathbb{S}_{1,2,1,1,5,1}+12\,\mathbb{S}_{1,2,1,2,1,4}+36\,\mathbb{S}_{1,2,1,2,2,3}-30\,\mathbb{S}_{1,2,1,2,3,2}\\
&+80\,\mathbb{S}_{1,2,1,2,4,1}-276\,\mathbb{S}_{1,2,1,3,1,3}+220\,\mathbb{S}_{1,2,1,3,2,2}+34\,\mathbb{S}_{1,2,1,3,3,1}-148\,\mathbb{S}_{1,2,1,4,1,2}\\
&+172\,\mathbb{S}_{1,2,1,4,2,1}-100\,\mathbb{S}_{1,2,1,5,1,1}-664\,\mathbb{S}_{1,2,2,1,1,4}+742\,\mathbb{S}_{1,2,2,1,2,3}+294\,\mathbb{S}_{1,2,2,1,3,2}\\
&-244\,\mathbb{S}_{1,2,2,1,4,1}+1078\,\mathbb{S}_{1,2,2,2,1,3}-1582\,\mathbb{S}_{1,2,2,2,2,2}+178\,\mathbb{S}_{1,2,2,2,3,1}+398\,\mathbb{S}_{1,2,2,3,1,2}\\
&-362\,\mathbb{S}_{1,2,2,3,2,1}+186\,\mathbb{S}_{1,2,2,4,1,1}-682\,\mathbb{S}_{1,2,3,1,1,3}+804\,\mathbb{S}_{1,2,3,1,2,2}-324\,\mathbb{S}_{1,2,3,1,3,1}\\
&+474\,\mathbb{S}_{1,2,3,2,1,2}-336\,\mathbb{S}_{1,2,3,2,2,1}+94\,\mathbb{S}_{1,2,3,3,1,1}-356\,\mathbb{S}_{1,2,4,1,1,2}+188\,\mathbb{S}_{1,2,4,1,2,1}\\
&+346\,\mathbb{S}_{1,2,4,2,1,1}-232\,\mathbb{S}_{1,2,5,1,1,1}+632\,\mathbb{S}_{1,3,1,1,1,4}-724\,\mathbb{S}_{1,3,1,1,2,3}-300\,\mathbb{S}_{1,3,1,1,3,2}\\
&+290\,\mathbb{S}_{1,3,1,1,4,1}-232\,\mathbb{S}_{1,3,1,2,1,3}+616\,\mathbb{S}_{1,3,1,2,2,2}-320\,\mathbb{S}_{1,3,1,2,3,1}+60\,\mathbb{S}_{1,3,1,3,1,2}\\
&-380\,\mathbb{S}_{1,3,1,3,2,1}+362\,\mathbb{S}_{1,3,1,4,1,1}-610\,\mathbb{S}_{1,3,2,1,1,3}+728\,\mathbb{S}_{1,3,2,1,2,2}-302\,\mathbb{S}_{1,3,2,1,3,1}\\
&+346\,\mathbb{S}_{1,3,2,2,1,2}+134\,\mathbb{S}_{1,3,2,2,2,1}-332\,\mathbb{S}_{1,3,2,3,1,1}-112\,\mathbb{S}_{1,3,3,1,1,2}+64\,\mathbb{S}_{1,3,3,1,2,1}\\
&+100\,\mathbb{S}_{1,3,3,2,1,1}-60\,\mathbb{S}_{1,3,4,1,1,1}+496\,\mathbb{S}_{1,4,1,1,1,3}-598\,\mathbb{S}_{1,4,1,1,2,2}+260\,\mathbb{S}_{1,4,1,1,3,1}\\
&-152\,\mathbb{S}_{1,4,1,2,1,2}-200\,\mathbb{S}_{1,4,1,2,2,1}+348\,\mathbb{S}_{1,4,1,3,1,1}-342\,\mathbb{S}_{1,4,2,1,1,2}+188\,\mathbb{S}_{1,4,2,1,2,1}\\
&+196\,\mathbb{S}_{1,4,2,2,1,1}-60\,\mathbb{S}_{1,4,3,1,1,1}+260\,\mathbb{S}_{1,5,1,1,1,2}-136\,\mathbb{S}_{1,5,1,1,2,1}-136\,\mathbb{S}_{1,5,1,2,1,1}\\
&-244\,\mathbb{S}_{1,5,2,1,1,1}+160\,\mathbb{S}_{1,6,1,1,1,1}-16\,\mathbb{S}_{2,1,1,1,1,5}+28\,\mathbb{S}_{2,1,1,1,2,4}+16\,\mathbb{S}_{2,1,1,1,3,3}\\
&+16\,\mathbb{S}_{2,1,1,1,4,2}-44\,\mathbb{S}_{2,1,1,1,5,1}+12\,\mathbb{S}_{2,1,1,2,1,4}+36\,\mathbb{S}_{2,1,1,2,2,3}-30\,\mathbb{S}_{2,1,1,2,3,2}\\
&+80\,\mathbb{S}_{2,1,1,2,4,1}-276\,\mathbb{S}_{2,1,1,3,1,3}+220\,\mathbb{S}_{2,1,1,3,2,2}+34\,\mathbb{S}_{2,1,1,3,3,1}-148\,\mathbb{S}_{2,1,1,4,1,2}\\
&+172\,\mathbb{S}_{2,1,1,4,2,1}-100\,\mathbb{S}_{2,1,1,5,1,1}+12\,\mathbb{S}_{2,1,2,1,1,4}-26\,\mathbb{S}_{2,1,2,1,2,3}-18\,\mathbb{S}_{2,1,2,1,3,2}\\
&+40\,\mathbb{S}_{2,1,2,1,4,1}+284\,\mathbb{S}_{2,1,2,2,1,3}-298\,\mathbb{S}_{2,1,2,2,2,2}-108\,\mathbb{S}_{2,1,2,2,3,1}+222\,\mathbb{S}_{2,1,2,3,1,2}\\
&-280\,\mathbb{S}_{2,1,2,3,2,1}+180\,\mathbb{S}_{2,1,2,4,1,1}-280\,\mathbb{S}_{2,1,3,1,1,3}+322\,\mathbb{S}_{2,1,3,1,2,2}-120\,\mathbb{S}_{2,1,3,1,3,1}
\end{aligned}
$$

$$+218\,\mathbb{S}_{2,1,3,2,1,2}-184\,\mathbb{S}_{2,1,3,2,2,1}+62\,\mathbb{S}_{2,1,3,3,1,1}-166\,\mathbb{S}_{2,1,4,1,1,2}+88\,\mathbb{S}_{2,1,4,1,2,1}$$

$$+184\,\mathbb{S}_{2,1,4,2,1,1}-132\,\mathbb{S}_{2,1,5,1,1,1}-592\,\mathbb{S}_{2,2,1,1,1,4}+656\,\mathbb{S}_{2,2,1,1,2,3}+256\,\mathbb{S}_{2,2,1,1,3,2}$$

$$-214\,\mathbb{S}_{2,2,1,1,4,1}+216\,\mathbb{S}_{2,2,1,2,1,3}-550\,\mathbb{S}_{2,2,1,2,2,2}+198\,\mathbb{S}_{2,2,1,2,3,1}+78\,\mathbb{S}_{2,2,1,3,1,2}$$

$$+164\,\mathbb{S}_{2,2,1,3,2,1}-212\,\mathbb{S}_{2,2,1,4,1,1}+990\,\mathbb{S}_{2,2,2,1,1,3}-1131\,\mathbb{S}_{2,2,2,1,2,2}+389\,\mathbb{S}_{2,2,2,1,3,1}$$

$$-892\,\mathbb{S}_{2,2,2,2,1,2}+604\,\mathbb{S}_{2,2,2,2,2,1}+36\,\mathbb{S}_{2,2,2,3,1,1}+415\,\mathbb{S}_{2,2,3,1,1,2}-187\,\mathbb{S}_{2,2,3,1,2,1}$$

$$-348\,\mathbb{S}_{2,2,3,2,1,1}+220\,\mathbb{S}_{2,2,4,1,1,1}-626\,\mathbb{S}_{2,3,1,1,1,3}+747\,\mathbb{S}_{2,3,1,1,2,2}-313\,\mathbb{S}_{2,3,1,1,3,1}$$

$$+196\,\mathbb{S}_{2,3,1,2,1,2}+204\,\mathbb{S}_{2,3,1,2,2,1}-386\,\mathbb{S}_{2,3,1,3,1,1}+461\,\mathbb{S}_{2,3,2,1,1,2}-257\,\mathbb{S}_{2,3,2,1,2,1}$$

$$-318\,\mathbb{S}_{2,3,2,2,1,1}+100\,\mathbb{S}_{2,3,3,1,1,1}-356\,\mathbb{S}_{2,4,1,1,1,2}+190\,\mathbb{S}_{2,4,1,1,2,1}+190\,\mathbb{S}_{2,4,1,2,1,1}$$

$$+360\,\mathbb{S}_{2,4,2,1,1,1}-248\,\mathbb{S}_{2,5,1,1,1,1}+592\,\mathbb{S}_{3,1,1,1,1,4}-668\,\mathbb{S}_{3,1,1,1,2,3}-264\,\mathbb{S}_{3,1,1,1,3,2}$$

$$+260\,\mathbb{S}_{3,1,1,1,4,1}-228\,\mathbb{S}_{3,1,1,2,1,3}+560\,\mathbb{S}_{3,1,1,2,2,2}-292\,\mathbb{S}_{3,1,1,2,3,1}+66\,\mathbb{S}_{3,1,1,3,1,2}$$

$$-364\,\mathbb{S}_{3,1,1,3,2,1}+342\,\mathbb{S}_{3,1,1,4,1,1}-228\,\mathbb{S}_{3,1,2,1,1,3}+259\,\mathbb{S}_{3,1,2,1,2,2}-95\,\mathbb{S}_{3,1,2,1,3,1}$$

$$+42\,\mathbb{S}_{3,1,2,2,1,2}+426\,\mathbb{S}_{3,1,2,2,2,1}-368\,\mathbb{S}_{3,1,2,3,1,1}+59\,\mathbb{S}_{3,1,3,1,1,2}-147\,\mathbb{S}_{3,1,3,1,2,1}$$

$$-408\,\mathbb{S}_{3,1,3,2,1,1}+388\,\mathbb{S}_{3,1,4,1,1,1}-594\,\mathbb{S}_{3,2,1,1,1,3}+695\,\mathbb{S}_{3,2,1,1,2,2}-275\,\mathbb{S}_{3,2,1,1,3,1}$$

$$+192\,\mathbb{S}_{3,2,1,2,1,2}+172\,\mathbb{S}_{3,2,1,2,2,1}-362\,\mathbb{S}_{3,2,1,3,1,1}+353\,\mathbb{S}_{3,2,2,1,1,2}-173\,\mathbb{S}_{3,2,2,1,2,1}$$

$$+158\,\mathbb{S}_{3,2,2,2,1,1}-346\,\mathbb{S}_{3,2,3,1,1,1}-114\,\mathbb{S}_{3,3,1,1,1,2}+66\,\mathbb{S}_{3,3,1,1,2,1}+66\,\mathbb{S}_{3,3,1,2,1,1}$$

$$+106\,\mathbb{S}_{3,3,2,1,1,1}-64\,\mathbb{S}_{3,4,1,1,1,1}+480\,\mathbb{S}_{4,1,1,1,1,3}-568\,\mathbb{S}_{4,1,1,1,2,2}+236\,\mathbb{S}_{4,1,1,1,3,1}$$

$$-152\,\mathbb{S}_{4,1,1,2,1,2}-178\,\mathbb{S}_{4,1,1,2,2,1}+330\,\mathbb{S}_{4,1,1,3,1,1}-152\,\mathbb{S}_{4,1,2,1,1,2}+86\,\mathbb{S}_{4,1,2,1,2,1}$$

$$-236\,\mathbb{S}_{4,1,2,2,1,1}+366\,\mathbb{S}_{4,1,3,1,1,1}-346\,\mathbb{S}_{4,2,1,1,1,2}+190\,\mathbb{S}_{4,2,1,1,2,1}+190\,\mathbb{S}_{4,2,1,2,1,1}$$

$$+198\,\mathbb{S}_{4,2,2,1,1,1}-64\,\mathbb{S}_{4,3,1,1,1,1}+264\,\mathbb{S}_{5,1,1,1,1,2}-136\,\mathbb{S}_{5,1,1,1,2,1}-136\,\mathbb{S}_{5,1,1,2,1,1}$$

$$-136\,\mathbb{S}_{5,1,2,1,1,1}-248\,\mathbb{S}_{5,2,1,1,1,1}+160\,\mathbb{S}_{6,1,1,1,1,1}-16\,\mathbb{S}_{1,1,1,1,1,1,2,4}-16\,\mathbb{S}_{1,1,1,1,1,1,3,3}$$

$$-16\,\mathbb{S}_{1,1,1,1,1,1,4,2}+40\,\mathbb{S}_{1,1,1,1,1,1,5,1}-140\,\mathbb{S}_{1,1,1,1,1,2,2,3}+28\,\mathbb{S}_{1,1,1,1,1,2,3,2}-72\,\mathbb{S}_{1,1,1,1,1,2,4,1}$$

$$+540\,\mathbb{S}_{1,1,1,1,1,3,1,3}-420\,\mathbb{S}_{1,1,1,1,1,3,2,2}-32\,\mathbb{S}_{1,1,1,1,1,3,3,1}+232\,\mathbb{S}_{1,1,1,1,1,4,1,2}-248\,\mathbb{S}_{1,1,1,1,1,4,2,1}$$

$$+120\,\mathbb{S}_{1,1,1,1,1,5,1,1}+12\,\mathbb{S}_{1,1,1,1,2,1,2,3}+12\,\mathbb{S}_{1,1,1,1,2,1,3,2}-28\,\mathbb{S}_{1,1,1,1,2,1,4,1}-608\,\mathbb{S}_{1,1,1,1,2,2,1,3}$$

$$+730\,\mathbb{S}_{1,1,1,1,2,2,2,2}+44\,\mathbb{S}_{1,1,1,1,2,2,3,1}-374\,\mathbb{S}_{1,1,1,1,2,3,1,2}+436\,\mathbb{S}_{1,1,1,1,2,3,2,1}-240\,\mathbb{S}_{1,1,1,1,2,4,1,1}$$

$$+576\,\mathbb{S}_{1,1,1,1,3,1,1,3}-682\,\mathbb{S}_{1,1,1,1,3,1,2,2}+288\,\mathbb{S}_{1,1,1,1,3,1,3,1}-406\,\mathbb{S}_{1,1,1,1,3,2,1,2}+284\,\mathbb{S}_{1,1,1,1,3,2,2,1}$$

$$-96\,\mathbb{S}_{1,1,1,1,3,3,1,1}+300\,\mathbb{S}_{1,1,1,1,4,1,1,2}-148\,\mathbb{S}_{1,1,1,1,4,1,2,1}-300\,\mathbb{S}_{1,1,1,1,4,2,1,1}+200\,\mathbb{S}_{1,1,1,1,5,1,1,1}$$

$$+12\,\mathbb{S}_{1,1,2,1,1,1,2,3}+12\,\mathbb{S}_{1,1,2,1,1,1,3,2}-28\,\mathbb{S}_{1,1,2,1,1,1,4,1}+18\,\mathbb{S}_{1,1,2,1,2,2,2}+60\,\mathbb{S}_{1,1,2,1,2,3,1}$$

$$-162\,\mathbb{S}_{1,1,2,1,3,1,2}+216\,\mathbb{S}_{1,1,2,1,3,2,1}-124\,\mathbb{S}_{1,1,2,1,4,1,1}-584\,\mathbb{S}_{1,1,2,2,1,1,3}+680\,\mathbb{S}_{1,1,2,2,1,2,2}$$
$$-276\,\mathbb{S}_{1,1,2,2,1,3,1}+762\,\mathbb{S}_{1,1,2,2,2,1,2}-848\,\mathbb{S}_{1,1,2,2,2,2,1}+298\,\mathbb{S}_{1,1,2,2,3,1,1}-432\,\mathbb{S}_{1,1,2,3,1,1,2}$$
$$+222\,\mathbb{S}_{1,1,2,3,1,2,1}+482\,\mathbb{S}_{1,1,2,3,2,1,1}-348\,\mathbb{S}_{1,1,2,4,1,1,1}+528\,\mathbb{S}_{1,1,3,1,1,1,3}-638\,\mathbb{S}_{1,1,3,1,1,2,2}$$
$$+284\,\mathbb{S}_{1,1,3,1,1,3,1}-156\,\mathbb{S}_{1,1,3,1,2,1,2}-216\,\mathbb{S}_{1,1,3,1,2,2,1}+358\,\mathbb{S}_{1,1,3,1,3,1,1}-392\,\mathbb{S}_{1,1,3,2,1,1,2}$$
$$+206\,\mathbb{S}_{1,1,3,2,1,2,1}+266\,\mathbb{S}_{1,1,3,2,2,1,1}-104\,\mathbb{S}_{1,1,3,3,1,1,1}+304\,\mathbb{S}_{1,1,4,1,1,1,2}-156\,\mathbb{S}_{1,1,4,1,1,2,1}$$
$$-156\,\mathbb{S}_{1,1,4,1,2,1,1}-328\,\mathbb{S}_{1,1,4,2,1,1,1}+240\,\mathbb{S}_{1,1,5,1,1,1,1}+12\,\mathbb{S}_{1,2,1,1,1,2,3}+12\,\mathbb{S}_{1,2,1,1,1,3,2}$$
$$-28\,\mathbb{S}_{1,2,1,1,1,4,1}+18\,\mathbb{S}_{1,2,1,1,2,2,2}+60\,\mathbb{S}_{1,2,1,1,2,3,1}-162\,\mathbb{S}_{1,2,1,1,3,1,2}+216\,\mathbb{S}_{1,2,1,1,3,2,1}$$
$$-124\,\mathbb{S}_{1,2,1,1,4,1,1}-8\,\mathbb{S}_{1,2,1,2,1,2,2}+14\,\mathbb{S}_{1,2,1,2,1,3,1}+202\,\mathbb{S}_{1,2,1,2,2,1,2}-390\,\mathbb{S}_{1,2,1,2,2,2,1}$$
$$+208\,\mathbb{S}_{1,2,1,2,3,1,1}-188\,\mathbb{S}_{1,2,1,3,1,1,2}+98\,\mathbb{S}_{1,2,1,3,1,2,1}+236\,\mathbb{S}_{1,2,1,3,2,1,1}-176\,\mathbb{S}_{1,2,1,4,1,1,1}$$
$$-512\,\mathbb{S}_{1,2,2,1,1,1,3}+604\,\mathbb{S}_{1,2,2,1,1,2,2}-254\,\mathbb{S}_{1,2,2,1,1,3,1}+156\,\mathbb{S}_{1,2,2,1,2,1,2}+114\,\mathbb{S}_{1,2,2,1,2,2,1}$$
$$-220\,\mathbb{S}_{1,2,2,1,3,1,1}+710\,\mathbb{S}_{1,2,2,2,1,1,2}-364\,\mathbb{S}_{1,2,2,2,1,2,1}-892\,\mathbb{S}_{1,2,2,2,2,1,1}+386\,\mathbb{S}_{1,2,2,3,1,1,1}$$
$$-418\,\mathbb{S}_{1,2,3,1,1,1,2}+210\,\mathbb{S}_{1,2,3,1,1,2,1}+210\,\mathbb{S}_{1,2,3,1,2,1,1}+462\,\mathbb{S}_{1,2,3,2,1,1,1}-340\,\mathbb{S}_{1,2,4,1,1,1,1}$$
$$+480\,\mathbb{S}_{1,3,1,1,1,1,3}-576\,\mathbb{S}_{1,3,1,1,1,2,2}+250\,\mathbb{S}_{1,3,1,1,1,3,1}-152\,\mathbb{S}_{1,3,1,1,2,1,2}-194\,\mathbb{S}_{1,3,1,1,2,2,1}$$
$$+350\,\mathbb{S}_{1,3,1,1,3,1,1}-152\,\mathbb{S}_{1,3,1,2,1,1,2}+86\,\mathbb{S}_{1,3,1,2,1,2,1}-254\,\mathbb{S}_{1,3,1,2,2,1,1}+382\,\mathbb{S}_{1,3,1,3,1,1,1}$$
$$-382\,\mathbb{S}_{1,3,2,1,1,1,2}+210\,\mathbb{S}_{1,3,2,1,1,2,1}+210\,\mathbb{S}_{1,3,2,1,2,1,1}+274\,\mathbb{S}_{1,3,2,2,1,1,1}-116\,\mathbb{S}_{1,3,3,1,1,1,1}$$
$$+300\,\mathbb{S}_{1,4,1,1,1,1,2}-156\,\mathbb{S}_{1,4,1,1,1,2,1}-156\,\mathbb{S}_{1,4,1,1,2,1,1}-156\,\mathbb{S}_{1,4,1,2,1,1,1}-340\,\mathbb{S}_{1,4,2,1,1,1,1}$$
$$+240\,\mathbb{S}_{1,5,1,1,1,1,1}+12\,\mathbb{S}_{2,1,1,1,1,2,3}+12\,\mathbb{S}_{2,1,1,1,1,3,2}-28\,\mathbb{S}_{2,1,1,1,1,4,1}+18\,\mathbb{S}_{2,1,1,1,2,2,2}$$
$$+60\,\mathbb{S}_{2,1,1,1,2,3,1}-162\,\mathbb{S}_{2,1,1,1,3,1,2}+216\,\mathbb{S}_{2,1,1,1,3,2,1}-124\,\mathbb{S}_{2,1,1,1,4,1,1}-8\,\mathbb{S}_{2,1,1,2,1,2,2}$$
$$+14\,\mathbb{S}_{2,1,1,2,1,3,1}+202\,\mathbb{S}_{2,1,1,2,2,1,2}-390\,\mathbb{S}_{2,1,1,2,2,2,1}+208\,\mathbb{S}_{2,1,1,2,3,1,1}-188\,\mathbb{S}_{2,1,1,3,1,1,2}$$
$$+98\,\mathbb{S}_{2,1,1,3,1,2,1}+236\,\mathbb{S}_{2,1,1,3,2,1,1}-176\,\mathbb{S}_{2,1,1,4,1,1,1}-8\,\mathbb{S}_{2,1,2,1,1,2,2}+14\,\mathbb{S}_{2,1,2,1,1,3,1}$$
$$-64\,\mathbb{S}_{2,1,2,1,2,2,1}+90\,\mathbb{S}_{2,1,2,1,3,1,1}+198\,\mathbb{S}_{2,1,2,2,1,1,2}-96\,\mathbb{S}_{2,1,2,2,1,2,1}-466\,\mathbb{S}_{2,1,2,2,2,1,1}$$
$$+284\,\mathbb{S}_{2,1,2,3,1,1,1}-176\,\mathbb{S}_{2,1,3,1,1,1,2}+86\,\mathbb{S}_{2,1,3,1,1,2,1}+86\,\mathbb{S}_{2,1,3,1,2,1,1}+208\,\mathbb{S}_{2,1,3,2,1,1,1}$$
$$-156\,\mathbb{S}_{2,1,4,1,1,1,1}-480\,\mathbb{S}_{2,2,1,1,1,1,3}+552\,\mathbb{S}_{2,2,1,1,1,2,2}-216\,\mathbb{S}_{2,2,1,1,1,3,1}+152\,\mathbb{S}_{2,2,1,1,2,1,2}$$
$$+82\,\mathbb{S}_{2,2,1,1,2,2,1}-196\,\mathbb{S}_{2,2,1,1,3,1,1}+152\,\mathbb{S}_{2,2,1,2,1,1,2}-86\,\mathbb{S}_{2,2,1,2,1,2,1}+48\,\mathbb{S}_{2,2,1,2,2,1,1}$$
$$-180\,\mathbb{S}_{2,2,1,3,1,1,1}+648\,\mathbb{S}_{2,2,2,1,1,1,2}-350\,\mathbb{S}_{2,2,2,1,1,2,1}-350\,\mathbb{S}_{2,2,2,1,2,1,1}-770\,\mathbb{S}_{2,2,2,2,1,1,1}$$
$$+270\,\mathbb{S}_{2,2,3,1,1,1,1}-384\,\mathbb{S}_{2,3,1,1,1,1,2}+210\,\mathbb{S}_{2,3,1,1,1,2,1}+210\,\mathbb{S}_{2,3,1,1,2,1,1}+210\,\mathbb{S}_{2,3,1,2,1,1,1}$$
$$+470\,\mathbb{S}_{2,3,2,1,1,1,1}-340\,\mathbb{S}_{2,4,1,1,1,1,1}+480\,\mathbb{S}_{3,1,1,1,1,1,3}-556\,\mathbb{S}_{3,1,1,1,1,2,2}+222\,\mathbb{S}_{3,1,1,1,1,3,1}$$
$$-152\,\mathbb{S}_{3,1,1,1,2,1,2}-162\,\mathbb{S}_{3,1,1,1,2,2,1}+316\,\mathbb{S}_{3,1,1,1,3,1,1}-152\,\mathbb{S}_{3,1,1,2,1,1,2}+86\,\mathbb{S}_{3,1,1,2,1,2,1}$$

$$
\begin{aligned}
&-234\,\mathbb{S}_{3,1,1,2,2,1,1}+376\,\mathbb{S}_{3,1,1,3,1,1,1}-152\,\mathbb{S}_{3,1,2,1,1,1,2}+86\,\mathbb{S}_{3,1,2,1,1,2,1}+86\,\mathbb{S}_{3,1,2,1,2,1,1}\\
&-274\,\mathbb{S}_{3,1,2,2,1,1,1}+402\,\mathbb{S}_{3,1,3,1,1,1,1}-384\,\mathbb{S}_{3,2,1,1,1,1,2}+210\,\mathbb{S}_{3,2,1,1,1,2,1}+210\,\mathbb{S}_{3,2,1,1,2,1,1}\\
&+210\,\mathbb{S}_{3,2,1,2,1,1,1}+270\,\mathbb{S}_{3,2,2,1,1,1,1}-116\,\mathbb{S}_{3,3,1,1,1,1,1}+300\,\mathbb{S}_{4,1,1,1,1,1,2}-156\,\mathbb{S}_{4,1,1,1,1,2,1}\\
&-156\,\mathbb{S}_{4,1,1,1,2,1,1}-156\,\mathbb{S}_{4,1,1,2,1,1,1}-156\,\mathbb{S}_{4,1,2,1,1,1,1}-340\,\mathbb{S}_{4,2,1,1,1,1,1}+240\,\mathbb{S}_{5,1,1,1,1,1,1}\\
&-64\,\mathbb{S}_{1,1,1,1,1,2,2,2}-40\,\mathbb{S}_{1,1,1,1,1,2,3,1}+240\,\mathbb{S}_{1,1,1,1,1,3,1,2}-296\,\mathbb{S}_{1,1,1,1,1,3,2,1}\\
&+136\,\mathbb{S}_{1,1,1,1,1,4,1,1}-344\,\mathbb{S}_{1,1,1,1,2,2,1,2}+540\,\mathbb{S}_{1,1,1,1,2,2,2,1}-220\,\mathbb{S}_{1,1,1,1,2,3,1,1}\\
&+336\,\mathbb{S}_{1,1,1,1,3,1,1,2}-164\,\mathbb{S}_{1,1,1,1,3,1,2,1}-380\,\mathbb{S}_{1,1,1,1,3,2,1,1}+256\,\mathbb{S}_{1,1,1,1,4,1,1,1}\\
&+48\,\mathbb{S}_{1,1,1,2,1,2,2,1}-76\,\mathbb{S}_{1,1,1,2,1,3,1,1}-376\,\mathbb{S}_{1,1,1,2,2,1,1,2}+172\,\mathbb{S}_{1,1,1,2,2,1,2,1}\\
&+780\,\mathbb{S}_{1,1,1,2,2,2,1,1}-496\,\mathbb{S}_{1,1,1,2,3,1,1,1}+340\,\mathbb{S}_{1,1,1,3,1,1,1,2}-156\,\mathbb{S}_{1,1,1,3,1,1,2,1}\\
&-156\,\mathbb{S}_{1,1,1,3,1,2,1,1}-360\,\mathbb{S}_{1,1,1,3,2,1,1,1}+280\,\mathbb{S}_{1,1,1,4,1,1,1,1}+48\,\mathbb{S}_{1,1,2,1,1,2,2,1}\\
&-76\,\mathbb{S}_{1,1,2,1,1,3,1,1}+204\,\mathbb{S}_{1,1,2,1,2,2,1,1}-252\,\mathbb{S}_{1,1,2,1,3,1,1,1}-336\,\mathbb{S}_{1,1,2,2,1,1,1,2}\\
&+156\,\mathbb{S}_{1,1,2,2,1,1,2,1}+156\,\mathbb{S}_{1,1,2,2,1,2,1,1}+748\,\mathbb{S}_{1,1,2,2,2,1,1,1}-380\,\mathbb{S}_{1,1,2,3,1,1,1,1}\\
&+300\,\mathbb{S}_{1,1,3,1,1,1,1,2}-156\,\mathbb{S}_{1,1,3,1,1,1,2,1}-156\,\mathbb{S}_{1,1,3,1,1,2,1,1}-156\,\mathbb{S}_{1,1,3,1,2,1,1,1}\\
&-380\,\mathbb{S}_{1,1,3,2,1,1,1,1}+280\,\mathbb{S}_{1,1,4,1,1,1,1,1}+48\,\mathbb{S}_{1,2,1,1,1,2,2,1}-76\,\mathbb{S}_{1,2,1,1,1,3,1,1}\\
&+204\,\mathbb{S}_{1,2,1,1,2,2,1,1}-252\,\mathbb{S}_{1,2,1,1,3,1,1,1}+232\,\mathbb{S}_{1,2,1,2,2,1,1,1}-156\,\mathbb{S}_{1,2,1,3,1,1,1,1}\\
&-300\,\mathbb{S}_{1,2,2,1,1,1,1,2}+156\,\mathbb{S}_{1,2,2,1,1,1,2,1}+156\,\mathbb{S}_{1,2,2,1,1,2,1,1}+156\,\mathbb{S}_{1,2,2,1,2,1,1,1}\\
&+636\,\mathbb{S}_{1,2,2,2,1,1,1,1}-380\,\mathbb{S}_{1,2,3,1,1,1,1,1}+300\,\mathbb{S}_{1,3,1,1,1,1,1,2}-156\,\mathbb{S}_{1,3,1,1,1,1,2,1}\\
&-156\,\mathbb{S}_{1,3,1,1,1,2,1,1}-156\,\mathbb{S}_{1,3,1,1,2,1,1,1}-156\,\mathbb{S}_{1,3,1,2,1,1,1,1}-380\,\mathbb{S}_{1,3,2,1,1,1,1,1}\\
&+280\,\mathbb{S}_{1,4,1,1,1,1,1,1}+48\,\mathbb{S}_{2,1,1,1,1,2,2,1}-76\,\mathbb{S}_{2,1,1,1,1,3,1,1}+204\,\mathbb{S}_{2,1,1,1,2,2,1,1}\\
&-252\,\mathbb{S}_{2,1,1,1,3,1,1,1}+232\,\mathbb{S}_{2,1,1,2,2,1,1,1}-156\,\mathbb{S}_{2,1,1,3,1,1,1,1}+156\,\mathbb{S}_{2,1,2,2,1,1,1,1}\\
&-156\,\mathbb{S}_{2,1,3,1,1,1,1,1}-300\,\mathbb{S}_{2,2,1,1,1,1,1,2}+156\,\mathbb{S}_{2,2,1,1,1,1,2,1}+156\,\mathbb{S}_{2,2,1,1,1,2,1,1}\\
&+156\,\mathbb{S}_{2,2,1,1,2,1,1,1}+156\,\mathbb{S}_{2,2,1,2,1,1,1,1}+636\,\mathbb{S}_{2,2,2,1,1,1,1,1}-380\,\mathbb{S}_{2,3,1,1,1,1,1,1}\\
&+300\,\mathbb{S}_{3,1,1,1,1,1,1,2}-156\,\mathbb{S}_{3,1,1,1,1,1,2,1}-156\,\mathbb{S}_{3,1,1,1,1,2,1,1}-156\,\mathbb{S}_{3,1,1,1,2,1,1,1}\\
&-156\,\mathbb{S}_{3,1,1,2,1,1,1,1}-156\,\mathbb{S}_{3,1,2,1,1,1,1,1}-380\,\mathbb{S}_{3,2,1,1,1,1,1,1}+280\,\mathbb{S}_{4,1,1,1,1,1,1,1}\\
&-256\,\mathbb{S}_{1,1,1,1,1,2,2,1,1}+392\,\mathbb{S}_{1,1,1,1,1,3,1,1,1}-416\,\mathbb{S}_{1,1,1,1,2,2,1,1,1}+280\,\mathbb{S}_{1,1,1,1,3,1,1,1,1}\\
&-280\,\mathbb{S}_{1,1,1,2,2,1,1,1,1}+280\,\mathbb{S}_{1,1,1,3,1,1,1,1,1}-280\,\mathbb{S}_{1,1,2,2,1,1,1,1,1}+280\,\mathbb{S}_{1,1,3,1,1,1,1,1,1}\\
&-280\,\mathbb{S}_{1,2,2,1,1,1,1,1,1}+280\,\mathbb{S}_{1,3,1,1,1,1,1,1,1}-280\,\mathbb{S}_{2,2,1,1,1,1,1,1,1}+280\,\mathbb{S}_{3,1,1,1,1,1,1,1,1}\,. \qquad \text{(A.1)}
\end{aligned}
$$

B Appendix B

Analytic continuation of all harmonic sums up to transcendentality 11 was performed with the help of the HARMPOL package [87] for FORM [88] and using DATAMINE [89] tables for the substitution of the multiple zeta functions, or multiple polylogarithms at $x = 1$ through usual Euler zeta-functions ζ_i and the minimal numbers of multiple zeta-functions. The results of the analytic continuation for the full planar six-loop anomalous dimension of twist-2 operators in $\mathcal{N} = 4$ SYM theory is the following:

$$
\begin{aligned}
\gamma(-2+\omega) \;=\; & 8g^2\Big[-\frac{1}{\omega}+1+\omega(1+\zeta_2)+\omega^2(1-\zeta_3)+\omega^3(1+\zeta_4)+\omega^4(1-\zeta_5) \\
&\qquad +\omega^5(1+\zeta_6)+\omega^6(1-\zeta_7)+\omega^7(1+\zeta_8)+\omega^8(1-\zeta_9)\Big] \\
&+g^4\Big[-\frac{32}{\omega^3}+\frac{64}{\omega^2}+\frac{32\zeta_2+32}{\omega}+-16\zeta_3+\omega(32\zeta_2-16\zeta_3-20\zeta_4-32) \\
&\quad +\omega^2(-64\zeta_3\zeta_2+64\zeta_2-48\zeta_3+12\zeta_4+116\zeta_5-64) \\
&\quad +\omega^3\left(40\zeta_3^2-80\zeta_3+96\zeta_2+44\zeta_4-8\zeta_5-\frac{115\zeta_6}{3}-96\right) \\
&\quad +\omega^4\left(-128\zeta_5\zeta_2+128\zeta_2-112\zeta_3-76\zeta_3\zeta_4+76\zeta_4-40\zeta_5+5\zeta_6+\frac{579\zeta_7}{2}-128\right) \\
&\quad +\omega^5\Big(\frac{512\mathrm{h}_{53}}{57}-\frac{2560\mathrm{h}_{71}}{19}+160\zeta_2-144\zeta_3+108\zeta_4+\frac{3768\zeta_3\zeta_5}{19}-72\zeta_5+37\zeta_6 \\
&\qquad -3\zeta_7-\frac{16485\zeta_8}{76}-160\Big) \\
&\quad +\omega^6\Big(-192\zeta_7\zeta_2+192\zeta_2-176\zeta_3+140\zeta_4-140\zeta_4\zeta_5-104\zeta_5-69\zeta_3\zeta_6+69\zeta_6 \\
&\qquad -35\zeta_7+\frac{7\zeta_8}{4}+\frac{1057\zeta_9}{2}-192\Big)\Big] \\
&+g^6\Big[-\frac{256}{\omega^5}+\frac{768}{\omega^4}+\frac{512\zeta_2}{\omega^3}+\frac{-768\zeta_2-384\zeta_3-512}{\omega^2}+\frac{-256\zeta_2+576\zeta_3-416\zeta_4-768}{\omega} \\
&\quad +448\zeta_2\zeta_3+384\zeta_3+80\zeta_4+192\zeta_5-768+\omega\big(-336\zeta_3^2-704\zeta_2\zeta_3+192\zeta_3-352\zeta_4 \\
&\quad +272\zeta_5-1052\zeta_6-512\big)+\omega^2\big(432\zeta_3^2-832\zeta_2\zeta_3+104\zeta_4\zeta_3-256\zeta_2-336\zeta_4 \\
&\quad +1744\zeta_2\zeta_5+1184\zeta_5+1616\zeta_6-1146\zeta_7\big)+\omega^3\big(944\zeta_2\zeta_3^2+656\zeta_3^2-1216\zeta_2\zeta_3 \\
&\quad -120\zeta_4\zeta_3-\frac{44416\zeta_5\zeta_3}{19}-192\zeta_3-\frac{8192\mathrm{h}_{53}}{171}+\frac{40960\mathrm{h}_{71}}{57}-768\zeta_2+128\zeta_4+1840\zeta_5 \\
&\quad -1312\zeta_2\zeta_5+\frac{268\zeta_6}{3}-584\zeta_7-\frac{12315\zeta_8}{19}+768\big)+\omega^4\big(-432\zeta_3^3-384\zeta_2\zeta_3^2+1008\zeta_3^2 \\
&\quad -1856\zeta_2\zeta_3-680\zeta_4\zeta_3+\frac{50736\zeta_5\zeta_3}{19}-\frac{6664\zeta_6\zeta_3}{3}-384\zeta_3+\frac{10240\mathrm{h}_{53}}{57}-\frac{51200\mathrm{h}_{71}}{19} \\
&\quad -1536\zeta_2+1040\zeta_4-1376\zeta_2\zeta_5-132\zeta_4\zeta_5+2240\zeta_5-\frac{1816\zeta_6}{3}+5728\zeta_2\zeta_7+2190\zeta_7 \\
&\quad +\frac{90622\zeta_8}{171}-\frac{57904\zeta_9}{9}+1792\big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+g^8\Big[-\frac{2560}{\omega^7}+\frac{10240}{\omega^6}+\frac{7168\zeta_2-5120}{\omega^5}+\frac{-18432\zeta_2-5632\zeta_3-10240}{\omega^4}\\
&\quad+\frac{14336\zeta_3-13440\zeta_4-7680}{\omega^3}+\frac{9216\zeta_3\zeta_2+8192\zeta_2+1536\zeta_3+14848\zeta_4+3200\zeta_5}{\omega^2}\\
&\quad+\frac{-4416\zeta_3^2-14080\zeta_2\zeta_3-6144\zeta_3+9216\zeta_2+3712\zeta_4-960\zeta_5-7504\zeta_6+10240}{\omega}\\
&\quad+6464\zeta_3^2-3328\zeta_2\zeta_3-2400\zeta_4\zeta_3-9728\zeta_3+6144\zeta_2+1024\zeta_4-1664\zeta_2\zeta_5-4160\zeta_5\\
&\quad+\frac{45008\zeta_6}{3}+2472\zeta_7+20480+\omega\Big(1408\zeta_2\zeta_3^2+1664\zeta_3^2+3072\zeta_2\zeta_3-4512\zeta_4\zeta_3\\
&\quad+\frac{10912\zeta_5\zeta_3}{3}-10240\zeta_3+\frac{4096\mathrm{h}_{53}}{27}-\frac{20480\mathrm{h}_{71}}{9}+2048\zeta_2+2944\zeta_4+5120\zeta_2\zeta_5\\
&\quad-6400\zeta_5+\frac{40640\zeta_6}{3}-8336\zeta_7+\frac{453500\zeta_8}{27}+28160\Big)+\omega^2\Big(352\zeta_3^3+3456\zeta_2\zeta_3^2\\
&\quad-1408\zeta_3^2+5120\zeta_2\zeta_3-3776\zeta_4\zeta_3-\frac{365344\zeta_5\zeta_3}{57}-2520\zeta_6\zeta_3-8704\zeta_3+5632\zeta_4\\
&\quad-\frac{372736\mathrm{h}_{53}}{513}+\frac{1863680\mathrm{h}_{71}}{171}+10880\zeta_2\zeta_5+9496\zeta_4\zeta_5-7680\zeta_5+5760\zeta_6\\
&\quad-38912\zeta_2\zeta_7-29288\zeta_7-\frac{7149836\zeta_8}{513}+\frac{46024\zeta_9}{3}+30720\Big)\Big]\\
&+g^{10}\Big[-\frac{28672}{\omega^9}+\frac{143360}{\omega^8}+\frac{102400\zeta_2-143360}{\omega^7}+\frac{-368640\zeta_2-81920\zeta_3-143360}{\omega^6}\\
&\quad+\frac{163840\zeta_2+292864\zeta_3-289792\zeta_4}{\omega^5}\\
&\quad+\frac{194560\zeta_3\zeta_2+286720\zeta_2-110592\zeta_3+622080\zeta_4+40960\zeta_5+172032}{\omega^4}\\
&\quad+\frac{286720-87040\zeta_3^2-473088\zeta_2\zeta_3-241664\zeta_3+184320\zeta_2+37888\zeta_4-46080\zeta_5+\frac{137984}{3}\zeta_6}{\omega^3}\\
&\quad+\frac{209408\zeta_3^2+12288\zeta_2\zeta_3-250368\zeta_4\zeta_3-196608\zeta_3-150528\zeta_4-24064\zeta_2\zeta_5-81920\zeta_5}{\omega^2}\\
&\quad+\frac{\frac{767872}{3}\zeta_6+36736\zeta_7+286720}{\omega^2}+\frac{126976\zeta_2\zeta_3^2+10240\zeta_3^2+208896\zeta_2\zeta_3+270592\zeta_4\zeta_3}{\omega}\\
&\quad+\frac{\frac{1537280}{57}\zeta_5\zeta_3-55296\zeta_3+\frac{843776}{513}\mathrm{h}_{53}-\frac{4218880}{171}\mathrm{h}_{71}-163840\zeta_2-144384\zeta_4-75264\zeta_2\zeta_5}{\omega}\\
&\quad+\frac{143360-71680\zeta_5-\frac{39424}{3}\zeta_6-178752\zeta_7+\frac{144246544}{513}\zeta_8}{\omega}-41344\zeta_3^3-210432\zeta_2\zeta_3^2-83456\zeta_3^2\\
&\quad+215040\zeta_2\zeta_3+84480\zeta_4\zeta_3+\frac{5259520}{57}\zeta_5\zeta_3-271040\zeta_6\zeta_3+118784\zeta_3-245760\zeta_2-87552\zeta_4\\
&\quad-\frac{1163264}{513}\mathrm{h}_{53}+\frac{5816320}{171}\mathrm{h}_{71}+26624\zeta_2\zeta_5-87968\zeta_4\zeta_5-20480\zeta_5-\frac{657664}{3}\zeta_6-83200\zeta_2\zeta_7\\
&\quad+4032\zeta_7-\frac{101468128}{513}\zeta_8-\frac{39344\zeta_9}{9}-143360\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+g^{12}\Bigg[-\frac{344064}{\omega^{11}}+\frac{2064384}{\omega^{10}}+\frac{1490944\zeta_2-3096576}{\omega^9}+\frac{-6881280\zeta_2-1204224\zeta_3-1376256}{\omega^8}\\
&\quad+\frac{6307840\zeta_2+5529600\zeta_3-5625856\zeta_4+2064384}{\omega^7}\\
&\quad+\frac{3768320\zeta_3\zeta_2+5734400\zeta_2-4792320\zeta_3+17889280\zeta_4+546816\zeta_5+4128768}{\omega^6}\\
&\quad+\frac{3440640-1640448\zeta_3^2-12771328\zeta_2\zeta_3-5038080\zeta_3-6133760\zeta_4-1071104\zeta_5+3989504\zeta_6}{\omega^5}\\
&\quad+\frac{5523456\zeta_3^2+5349376\zeta_2\zeta_3-8453120\zeta_4\zeta_3-737280\zeta_3-5505024\zeta_2-10680320\zeta_4}{\omega^4}\\
&\quad+\frac{591360\zeta_7-667648\zeta_2\zeta_5-1267712\zeta_5-2685952\zeta_6}{\omega^4}+\frac{3540992\zeta_2\zeta_3^2-1994752\zeta_3^2}{\omega^3}\\
&\quad+\frac{8953856\zeta_2\zeta_3+17525760\zeta_4\zeta_3+\frac{13147136}{19}\zeta_5\zeta_3+4276224\zeta_3+\frac{1638400}{57}\mathrm{h}_{53}-\frac{8192000}{19}\mathrm{h}_{71}}{\omega^3}\\
&\quad+\frac{-8028160\zeta_2-7004160\zeta_4-849920\zeta_2\zeta_5+100352\zeta_5-\frac{15717376}{3}\zeta_6-3243520\zeta_7}{\omega^3}\\
&\quad+\frac{\frac{745997696}{171}\zeta_8-5160960}{\omega^3}+\frac{-1100800\zeta_3^3-8599552\zeta_2\zeta_3^2-4012032\zeta_3^2+5283840\zeta_2\zeta_3}{\omega^2}\\
&\quad+\frac{495616\zeta_4\zeta_3+\frac{47233024}{57}\zeta_5\zeta_3-\frac{5374720}{3}\zeta_6\zeta_3+7618560\zeta_3-\frac{37683200}{513}\mathrm{h}_{53}+\frac{188416000}{171}\mathrm{h}_{71}}{\omega^2}\\
&\quad+\frac{-6881280\zeta_2-1880064\zeta_4+2869248\zeta_2\zeta_5-1059328\zeta_4\zeta_5+2027520\zeta_5-\frac{18255872}{3}\zeta_6}{\omega^2}\\
&\quad+\frac{-1978112\zeta_2\zeta_7+1971200\zeta_7-\frac{6224639872}{513}\zeta_8-\frac{607616}{9}\zeta_9-10321920}{\omega^2}\Bigg]
\end{aligned}
\tag{B.1}
$$

where

$$
\mathrm{h}_{71}=H_{-7,-1}(1),\qquad \mathrm{h}_{53}=H_{-5,-3}(1)\,, \tag{B.2}
$$

and $H_{i_1,\ldots,i_k}(x)$ are the harmonic polylogarithms [87].

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