Yang-Baxter deformations, AdS/CFT, and twist-noncommutative gauge theory

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ABSTRACT: We discuss the AdS/CFT interpretation of homogeneous Yang-Baxter deformations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring as noncommutative deformations of the dual gauge theory, going well beyond the canonical noncommutative case. These homogeneous Yang-Baxter deformations can be of so-called abelian or jordanian type. While abelian deformations have a clear interpretation in string theory and many already had well understood gauge theory duals, jordanian deformations appear novel on both counts. We discuss the symmetry structure of the deformed string from the uniformizing perspective of Drinfeld twists and show how it can be realized on the gauge theory side by considering theories on various noncommutative spaces. We then conjecture that these are the gauge theory duals of our strings, modulo subtleties involving time and singularities. We support this conjecture by a brane construction for two nontrivial examples, corresponding to noncommutative spaces with $[x^- \stackrel{\star}{,} x^i] \sim x^i$ $(i = 1, 2)$. We also briefly discuss a deformation which may be the gravity dual of gauge theory on spacelike κ-Minkowski space.

Contents

1 Introduction

Integrability has led to important insights in many areas of physics, including the AdS/CFT correspondence [1]. Since integrability provides us with powerful tools to study the superstring on $\mathrm{AdS_5} \times \mathrm{S^5}$ and its dual planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) [2, 3], there has been considerable interest in understanding to what extent they can be applied beyond this maximally supersymmetric case. One example of such an extension is planar β-deformed SYM and the associated Lunin-Maldacena background [4–6]. More recently it was understood that the string sigma model can be deformed in a variety of ways while manifestly preserving integrability, and in the present paper we would like to address the AdS/CFT interpretation of large classes of them.

The deformations of the $\text{AdS}_5 \times \text{S}^5$ string we will be considering are based on the construction of [7].[1] In its original form this gives a quantum deformation of the original model [10, 11], whose possible interpretation in terms of string theory and AdS/CFT remains elusive.[2] This "Yang-Baxter" deformation is based on a solution of the modified (inhomogeneous) classical Yang-Baxter equation, but the construction of [7] can be extended to integrable deformations based on the homogeneous classical Yang-Baxter equation [24]. In this setting the homogeneous classical Yang-Baxter equation (CYBE) has many solutions, giving an abundance of integrable deformations. All currently known solutions of the CYBE fall into two classes, known as abelian and jordanian respectively. The abelian solutions (mostly) have nice interpretations in terms of string theory and AdS/CFT, including for example the Lunin-Maldacena background mentioned above [25], and the gravity dual of (canonical) noncommutative SYM [26], which is hence integrable. In fact, it is heuristically clear that abelian solutions correspond to TsT transformations, establishing their status in string theory.[3] Jordanian deformations are more mysterious, and appear to necessarily deform anti-de Sitter space.[4] It is not obvious that the result of these deformations is always a string background, though the metric and B field of the only thus far investigated case are part of one [28, 29]. Also, the interpretation of jordanian deformations in terms of the AdS/CFT correspondence is not known. Given the nice interpretation of their abelian cousins, we would like to shed some light on this, and attempt to give these theories an interpretation in terms of the AdS/CFT (gauge/gravity) correspondence.

After introducing the Yang-Baxter deformed string we will discuss how both abelian and jordanian deformations twist the symmetries of the $\text{AdS}_5 \times \text{S}^5$ string, resulting in a Drinfeld twisted Hopf algebra [30]. This provides a unified picture of the known abelian deformations and extends it to jordanian ones. We then might wonder what type of deformation of $\mathcal{N} = 4$ SYM would similarly carry such a twisted Hopf algebra. Since the symmetries of AdS_5 correspond to spacetime symmetries of $\mathcal{N} = 4$ SYM, the answer to this question naturally lies in the realm of noncommutative field theory[5] in the twist formalism [33, 34], see also e.g. [35–37] and the reviews [38, 39]. In this formulation we start from the usual Hopf algebra based on vector fields on spacetime, a representation of which is carried by the algebra of fields (functions on spacetime). We can then deform this algebra of fields by a Drinfeld twist, so that it naturally carries the representation of a twisted Hopf

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[1]This generalizes earlier work by Klimčík [8, 9].

[2]In the $\text{AdS}_2 \times \text{S}^2$ and $\text{AdS}_3 \times \text{S}^3$ cases some progress has been made on the supergravity front [12], and a suitable maximal deformation limit gives the $\text{AdS}_5 \times \text{S}^5$ mirror model [13–15], which is a solution of supergravity. In general, the maximal deformation limit is closely related to $\text{dS}_5 \times \text{H}^5$ [7, 11, 14–16]. Also, interesting links to the λ-deformation of the non-abelian T dual of the $\text{AdS}_5 \times \text{S}^5$ string [17–19], which have supergravity embeddings [20], have recently been uncovered [21, 22]. These λ-deformations generalize the earlier work of [23].

[3]TsT transformations are also known as Melvin twists, TsT standing for T duality - shift - T duality, where for us a TsT transformation $(x, y)_\beta$ means we T dualize in x, shift y by β times the T dual field, and T dualize back, see e.g. [27] for a brief discussion in the present context.

[4]The required algebraic structure is not compatible with the reality conditions on the sphere ($\mathfrak{su}(4)$). Upon including fermions the story may be more involved, possibly even allowing a new class of deformations.

[5]Noncommutative field theory has a rich history and relevance which we will not attempt to cover here. We refer the reader to the extensive reviews [31, 32].

algebra, resulting in a noncommutative space. While likely technically involved, we believe it should in principle be possible to construct (supersymmetric) gauge theories on such noncommutative spaces using the methods developed in [40–42], see also [31, 32, 37, 39] and references therein, and the discussion below. The matching symmetry structures then suggests that our jordanian deformations of $AdS_5 \times S^5$ could represent gravity duals to this type of noncommutative gauge theories.

Noncommutative field theories can arise in the low energy physics of open strings stretching between D branes [40, 43–46], and our conjecture should be supported by a picture of this type. In the spirit of [1] we would need some brane geometry which in a suitable low energy limit can on the one hand be described in terms of open strings that give us our noncommutative field theory, and on the other as closed strings in a near horizon geometry matching our deformations of $AdS_5 \times S^5$. Already in the abelian case this picture is subtle when dealing with noncommutivity in time however. The 'standard' string construction does not work for temporal-spatial noncommutativity [47], and the appropriate decoupling limit actually gives a noncritical open string theory [47, 48]. However, we can readily deform $AdS_5 \times S^5$ in a way that we would naively interpret as giving this noncommutativity. The resulting spacetime is not the gravity dual of the noncritical string theory [48] however, and is in fact singular. At the same time it is no problem to realize canonical null-spatial noncommutativity [49]; afterall we should be able to boost the spacelike case. Moreover, the model by Hashimoto and Sethi [50] shows that it is in fact possible to realize field theory with nonconstant noncommutativity involving time directly, with a perfectly regular gravity dual. Hence we expect our intuition to apply rather directly at least in the spatial and null cases, in fact possibly in any regular case, and not just for these abelian deformations. We will come back to regularity and the background of [50] later. There is another possible subtlety in case of fully broken supersymmetry which we come back to in the conclusions. Modulo these subtleties then, we would like to conjecture that jordanian deformations of $AdS_5 \times S^5$ are gravity duals of gauge theories on appropriate noncommutative spaces. This discussion clearly calls for a concrete example.

To support our conjecture, we consider two (closely related) jordanian deformations of $AdS_5 \times S^5$ which we can give an explicit embedding in supergravity. We will give deformations of the D3 brane metric that on the one hand have a low energy description in terms of gravity on these jordanian deformed $AdS_5 \times S^5$ spaces, and on the other hand show that we are dealing with D3 branes in a plane wave geometry with a nonconstant B field (though $dB = 0$).[6] The type of noncommutativity we should find for such a background follows from the general results of [52], see also [53] and the earlier work [54–56]. This indeed matches the type of noncommutativity predicted by our Drinfeld twist, which we find to be of the kind

$$[x^- \overset{\star}{,} x^j] = ia\, x^j, \quad j = 1, 2,$$

where x^- is one of the light cone coordinates in the (x^0, x^3) plane. The second deformation has a minus sign for x^2. The first of these deformed models has sixteen real supercharges,

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[6]TsT transformations can also produce (apparently) nonconstant B fields, see e.g. [51]; our B fields need not be associated to isometries however.

the maximal number since we break manifest conformal symmetry, while the second has none.

With our notion of possible dual CFTs in mind, we can try to reverse the question and investigate other noncommutative structures. One noncommutative space that has been studied extensively over the last decades is (generalized) κ-Minkowski space with $[x^\mu \overset{\star}{,} x^\nu] = i\kappa^{-1}(a^\mu x^\nu - a^\nu x^\mu)$. Timelike a^μ gives conventional κ-Minkowski space. This space carries the action of the κ-Poincaré group [57, 58], see also [59], with structures that appear in what is known as doubly special relativity [60, 61] as discussed in [62]. There has been considerable interest in constructing field theories on κ-Minkowski space, see e.g. [63–67] and [68, 69] for gauge theories in particular. The κ-Poincaré group is a proper quantum group however, which is not a structure we expect to reproduce here. Interestingly, κ-Minkowski space can also be obtained in a twisted setting, which might actually be more amenable to field theory constructions [39, 70]. Not surprisingly, to do so we need to go beyond the Poincaré algebra [71], as was considered in [70, 72, 73] by going to $\mathfrak{igl}(1,3)$, and in [39, 74] via the Poincaré-Weyl algebra. While we would find it a little strange to twist based on symmetries that a theory might not physically have, fortunately for us we start with conformal symmetry, hence Poincaré-Weyl symmetry in particular. The corresponding r matrices then yield deformations of $\text{AdS}_5 \times \text{S}^5$ which might have some relation to κ-Minkowski space in its three forms. Since κ-Minkowski noncommutativity involves time however, we should perhaps expect difficulties. In line with this, similarly to the naive canonical temporal-spatial noncommutative case mentioned earlier, the "timelike κ-Minkowski deformation of $\text{AdS}_5 \times \text{S}^5$" is singular, as is the null case. Interestingly, by contrast the spacelike deformation is regular. Given the discussion above we are led to conjecture that it may correspond to a gravity dual of SYM on spacelike κ-Minkowski space (in the twisted sense), though we cannot claim strong direct support. We have not attempted to embed the metrics and B fields of these κ-Minkowski deformations in supergravity (with the exception of the null case which is closely related to our main examples), or consequently tried to investigate a possible D brane picture (in the spacelike case), leaving this for future investigation.

In the next section we introduce our deformed strings and the interpretation of the deformation as a nonlocal gauge transformation. We discuss the resulting symmetry structure of the string in section 3, and how to possibly implement this on the gauge theory side in section 4. These two sections provide intuition and demonstrate interesting structure, but are not essential to understand the results of section 5, where we provide a brane construction illustrating our general conjecture with two concrete examples. In section 6 we take a look at deformations related to κ-Minkowski space and give some discussion regarding the (target space) regularity of the deformations in general. We then conclude with further discussion of our results and associated open questions.

While this paper was in preparation we learned that reference [75] considered some of the r matrices we consider in section 6, in the context of sigma models on four dimensional Minkowski space.

2 Homogeneous Yang-Baxter deformations of the AdS$_5 \times$ S^5 string

The deformations of the AdS$_5 \times$ S^5 superstring that we consider are described by the action [7, 24] [7]

$$S = -\frac{T}{2} \int d\tau d\sigma \frac{1}{2} (\sqrt{h} h^{\alpha\beta} - \epsilon^{\alpha\beta}) \text{sTr}(A_\alpha d_+ J_\beta) \tag{2.1}$$

where $J = (1 - \eta R_g \circ d_+)^{-1}(A)$ with $R_g(X) = g^{-1} R(gXg^{-1})g$. Setting $\eta = 0$ ($R = 0$) gives the undeformed AdS$_5 \times$ S^5 superstring action of [76], which is famously integrable [77]. Now, provided R is antisymmetric,

$$\text{sTr}(R(m)n) = -\text{sTr}(mR(n)), \tag{2.2}$$

and satisfies the classical Yang-Baxter equation (CYBE)

$$[R(m), R(n)] - R([R(m), n] + [m, R(n)]) = 0, \tag{2.3}$$

these deformed models are classically integrable as well, owing to the on shell flatness of the deformed current J. They also have κ symmetry. The operator R is a linear map from a given Lie (super)algebra $\mathfrak{g}$ to itself, which can be conveniently represented as

$$R(m) = (r)^{ij} t_i \, \text{sTr}(t_j m) = \text{sTr}_2(r(1 \otimes m)) \tag{2.4}$$

for some anti-symmetric matrix r

$$r = (r)^{ij} t_i \wedge t_j = \frac{1}{2}(r)^{ij}(t_i \otimes t_j - t_j \otimes t_i) \tag{2.5}$$

with sum implied, and the t_i are the generators of $\mathfrak{g}$. In our case $\mathfrak{g} = \mathfrak{psu}(2,2|4)$ [7, 27]. We will refer to both the operator R and its matrix representation r as the r matrix, where the latter satisfies the CYBE in the form

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{2.6}$$

Here r_{mn} denotes the matrix realization of r acting in spaces m and n in a tensor product, not to be confused with the matrix elements $(r)^{ij}$ with respect to a basis of $\mathfrak{g}$. This is an admittedly abstract construction, but given a concrete r matrix we can expand the above action to get a sigma model on some explicit background. We briefly indicate the general procedure, including our algebra conventions and group parametrization, in appendix A. To directly provide context, the r matrix [26]

$$r = a^2 p_2 \wedge p_3, \tag{2.7}$$

where the p_i denote the translation generators of the conformal group, produces the string sigma model defined on the gravity dual of canonical noncommutative SYM [78, 79], where a is the parameter used in [79].[8]

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[7]Here T is the effective string tension related to the 't Hooft coupling λ as $T = \sqrt{\lambda}/2\pi$, h is the world sheet metric, $\epsilon^{\tau\sigma} = 1$, $A_\alpha = g^{-1}\partial_\alpha g$ with $g \in$ PSU(2,2|4), sTr denotes the supertrace, and $d_\pm = \pm P_1 + 2P_2 \mp P_3$ where the P_i are the projectors onto the ith $\mathbb{Z}_4$ graded components of the semi-symmetric space PSU(2,2|4)/(SO(4,1) $\times$ SO(5)) (super AdS$_5 \times$ S^5).

[8]This and similar statements throughout this paper have strictly speaking only been verified at the bosonic level; while there is no conceptual difference for the fermions, explicitly evaluating the deformed fermionic action is technically involved.

The deformation as a non-local gauge transformation

It turns out that all models of the above type are related to the undeformed $\text{AdS}_5 \times \text{S}^5$ superstring by means of a nonlocal gauge transformation, as was originally found for the Lunin-Maldacena background [6, 80]. Specifically, there is a nonlocal gauge transformation that relates the deformed current J to the undeformed A [81]. Working in conformal gauge, we introduce worldsheet light cone coordinates $\sigma^\pm = (\tau \pm \sigma)/2$ and light cone components of the deformed current J as[9]

$$J_\pm \equiv \frac{1}{1 \mp \eta R_g \circ d_\pm} A_\pm. \tag{2.8}$$

These can be expressed as [81]

$$J_\pm = \tilde{g}^{-1} \partial_\pm \tilde{g}, \tag{2.9}$$

where $\tilde{g}$ is related to g as

$$\tilde{g} = F^{-1} g, \tag{2.10}$$

by the twist F

$$F(\tau, \sigma) = \text{Pexp}\left(-\int_0^\sigma d\hat{\sigma} J_\sigma^g \right) Z, \tag{2.11}$$

with $J_\sigma^g = g J_\sigma g^{-1} - \partial_\sigma g g^{-1}$. In other words, we have

$$J_\pm = \mathscr{G} A_\pm \mathscr{G}^{-1} - \partial_\pm \mathscr{G} \mathscr{G}^{-1}, \tag{2.12}$$

where $\mathscr{G} = g^{-1} F g$ is the nonlocal gauge transformation. There is still some freedom left in the (constant) matrix Z, which as we will soon see, mixes the conserved charges of the model. For future reference we note that

$$J_\pm^g = g(J_\pm - A_\pm)g^{-1} = g\left(\frac{\pm\eta R_g \circ d_\pm}{1 \mp \eta R_g \circ d_\pm} A_\pm \right) g^{-1} = \pm\eta R(g d_\pm J_\pm g^{-1}). \tag{2.13}$$

Conserved charges

The equations of motion of the model take the form [81]

$$\partial_\alpha \Lambda^\alpha + [J_\alpha, \Lambda^\alpha] = 0, \tag{2.14}$$

where we have introduced $d_\pm J_\pm = 2\Lambda_\pm$, i.e.

$$\Lambda^\alpha = \sqrt{h} h^{\alpha\beta} J_\beta^{(2)} - \tfrac{1}{2}\epsilon^{\alpha\beta}(J_\beta^{(1)} - J_\beta^{(3)}). \tag{2.15}$$

These expressions are analogous to the ones for the undeformed model [2], just with $A(g)$ replaced by $J(\tilde{g})$. Given relation (2.9), we can define the conserved current

$$k^\alpha \equiv \tilde{g} \Lambda^a \tilde{g}^{-1}, \tag{2.16}$$

which transforms adjointly under changes of Z. Because we are working in a nonlocal and nonperiodic setting, we cannot generate conserved charges for our closed string from this

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[9]While we follow [81], not all details are identical.

conserved current, except in the undeformed limit of course. Still, we could consider our sigma model on a line with boundary conditions that g becomes constant at large $|\sigma|$, so that we could construct conserved charges out of k as

$$Q \equiv \int_{-\infty}^{\infty} k^{\tau}. \tag{2.17}$$

These charges should generate $\mathfrak{psu}(2,2|4)$, which is (partially) broken by the nonperiodic boundary conditions for the actual string that we will consider however. On top of the manifest symmetry algebra generated by unbroken parts of Q, as in any integrable model we have further conserved charges. These all follow from expanding the monodromy matrix

$$M = \mathrm{Pexp}\left(-\int_{-\pi}^{\pi} d\hat{\sigma} L_{\sigma}^{g}\right) \tag{2.18}$$

where L_{σ}^{g} is the spatial component of the Lax connection of these models, which we take built on J^{g} instead of J but is otherwise given in [24]. What will be important for us is that this monodromy matrix is gauge equivalent to its undeformed counterpart M_0,[10]

$$M = F(\pi)^{-1} M_0 F(-\pi) = Z^{-1} \mathrm{Pexp}\left(\int_{-\pi}^{\pi} d\hat{\sigma} J_{\sigma}^{g}\right) M_0 Z, \tag{2.19}$$

showing how the deformation acts on (twists) the original symmetry algebra. We want to interpret this deformation in the spirit of quantum groups.

3 Twisted symmetry

Deformed symmetry algebras are well known in integrable models, and (both) are intimately tied to the theory of quantum groups. The standard quantum deformation of a Lie (bi)algebra comes from the Drinfeld-Jimbo solution of the modified classical Yang-Baxter equation, with quantum affine algebras as the corresponding quantization of classical affine algebras. In line with this, the original inhomogeneous Yang-Baxter deformation of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring of [7] results in a (standard) quantum deformation of $\mathfrak{psu}(2,2|4)$ [11]. Solutions of the classical Yang-Baxter equation give rise to nonstandard quantizations instead, which can be represented as Drinfeld twists [30]. In our case, by analogy we therefore expect to be dealing with a twisted $\mathfrak{psu}(2,2|4)$ Hopf algebra (and associated twisted Yangian) instead.

3.1 Hopf algebras and Drinfeld twists

There is a standard way to associate a Hopf algebra to a Lie algebra $\mathfrak{g}$. Very briefly, we take the (associative) universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and endow it with a coproduct

$$\begin{aligned}
\Delta(X) &= X \otimes 1 + 1 \otimes X, \text{ for } X \in \mathfrak{g}, \\
\Delta(1) &= 1 \otimes 1.
\end{aligned} \tag{3.1}$$

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[10]At the bosonic level this readily follows by combining the above with the approach of [80]. Eqn. (2.12) tells us that the Lax connection based on J transforms as a gauge field under $\mathscr{G}$. Working based on J^{g} instead (dgg^{-1} as opposed to $g^{-1}dg$ in the undeformed setting), effectively strips g and g^{-1} off of $\mathscr{G}$, leaving F. The fermionic analysis is more involved, but we have no doubt it should go through. In any case we are only explicitly considering the bosonic parts of the deformed models in this paper.

This coproduct is manifestly bilinear, as well as coassociative

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta. \tag{3.2}$$

If we then define a co-unit $\epsilon : \mathfrak{g} \to \mathbb{C}$ as

$$\epsilon(1) = 1, \qquad \epsilon(X) = 0, \tag{3.3}$$

we have a coalgebra. By construction the comultiplication

$$\begin{aligned} \Delta(XY) &= \Delta(X)\Delta(Y), \\ \epsilon(XY) &= \epsilon(X)\epsilon(Y), \end{aligned} \tag{3.4}$$

is an algebra homomorphism of $\mathcal{U}(\mathfrak{g})$ ($\Delta([X,Y]) = [\Delta(X), \Delta(Y)]$), so we are dealing with a bialgebra. Finally, to turn this into a Hopf algebra we define an antipode map s

$$s(X) = -X. \tag{3.5}$$

For details and generalizations we refer to the book [82].

A Drinfeld twist $\mathcal{F}$ is now an invertible element of $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ which satisfies the cocycle condition [30, 83]

$$(\mathcal{F} \otimes 1)(\Delta \otimes 1)\mathcal{F} = (1 \otimes \mathcal{F})(1 \otimes \Delta)\mathcal{F}, \tag{3.6}$$

and the normalization condition

$$(\epsilon \otimes 1)\mathcal{F} = (1 \otimes \epsilon)\mathcal{F} = 1 \otimes 1. \tag{3.7}$$

Since $\mathcal{F}$ should represent a deformation, we also want

$$\mathcal{F} = 1 \otimes 1 + \alpha \mathcal{F}^{(1)} + \mathcal{O}(\alpha^2), \tag{3.8}$$

where α is our deformation parameter. Let us now express $\mathcal{F}$ as a sum of terms in $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$

$$\mathcal{F} = f^\beta \otimes f_\beta, \quad \mathcal{F}^{-1} = \bar{f}^\beta \otimes \bar{f}_\beta, \tag{3.9}$$

where f^β, f_β, $\bar{f}^\beta$, and $\bar{f}_\beta$ all denote in principle distinct elements of $\mathcal{U}(\mathfrak{g})$, and we have an implicit (infinite) sum over β. We can then modify the original coproduct and antipode s of our Hopf algebra to

$$\begin{aligned} \Delta_\mathcal{F}(X) &= \mathcal{F}\Delta(X)\mathcal{F}^{-1}, \\ s_\mathcal{F}(X) &= f^\alpha s(f_\alpha) s(X) s(\bar{f}^\beta) \bar{f}_\beta. \end{aligned} \tag{3.10}$$

The cocycle condition guarantees that the twist preserves coassociativity of the coproduct; this will come back later.

These twists are in one to one correspondence with classical r matrices in the following sense [30, 84]. Firstly, the classical r matrix constructed as

$$r_{12} = \tfrac{1}{2}(\mathcal{F}_{12}^{(1)} - \mathcal{F}_{21}^{(1)}), \tag{3.11}$$

solves the CYBE. Secondly, any twists that have the same classical r matrix result in equivalent quantizations (deformations) of the algebra. Thirdly, a twist exists for any solution of the CYBE (though an explicit construction is not known in general). We will refer to twists with $\mathcal{F}_{12}^{(1)} = r_{12}$ as 'r-symmetric'. In a general integrable model, a twist changes the quantum R matrix (and monodromy matrix) as

$$R_{12} \to R_{12}^{\mathcal{F}} = \mathcal{F}_{21} R_{12} \mathcal{F}_{12}^{-1}. \tag{3.12}$$

3.2 The twist function and gauge fixing

We would now like to identify the twist relevant for our model. Comparing eqs. (2.19) and (3.12) it is natural to relate $F(\pi)$ to $\mathcal{F}_{21}^{-1}$.[11] Also, given the equivalence we just discussed, we are mainly concerned with the leading order expansion of the twist function.[12] By now we of course expect to find exactly the classical r matrix used to deform our model. To see this, let us suggestively rewrite J_σ^g as

$$J_\sigma^g = \tfrac{1}{2}(J_+^g - J_-^g) = \eta R(g(\Lambda_+ + \Lambda_-)g^{-1}) = -2\eta R(g\tilde{g}^{-1}k^\tau \tilde{g}g^{-1}), \tag{3.13}$$

cf. eqs. (2.13), (2.15), and (2.16). We can then evaluate the twist function (2.11) to leading order to find

$$\mathcal{F}(\pi) = \mathrm{Pexp}\left(-\int_{-\pi}^{\pi} d\hat{\sigma} J_\sigma^g\right) Z = \left(1 + 2\eta R(Q_0) + \mathcal{O}(\eta^2)\right) Z. \tag{3.14}$$

and Q_0 are the conserved charges of the undeformed model cf. eqn. (2.17) (now on $[-\pi, \pi)$), generating $\mathfrak{psu}(2,2|4)$. To complete the comparison we need to fix Z as

$$Z = 1 - \eta R(Q_0) + \mathcal{O}(\eta^2), \tag{3.15}$$

which follows by noting that for an r-symmetric quantum twist

$$\mathcal{F}_{12} = 1 + \alpha r_{12} + \mathcal{O}(\alpha^2) = \mathcal{F}_{21}^{-1} \sim F(\pi), \tag{3.16}$$

since $r_{12} = -r_{21}$. We see that in this gauge for Z, the leading order expansion of the twist function is precisely given by the r matrix; the supertrace in R picks out individual charges that act on classical fields taking values in $\mathfrak{psu}(2,2|4)$ via the Poisson bracket. At the quantum level we therefore effectively need to multiply the r matrix by $-i$, which amounts to replacing the deformation parameter η by $-i\eta$ in the twist. Concretely then, at the quantum level we expect to have

$$\mathcal{F} = 1 - i\eta r + \mathcal{O}(\eta^2). \tag{3.17}$$

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[11]Note that the quantum spaces 1 and 2 in the sigma model correspond to the (matrix) algebra $\mathfrak{psu}(2,2|4)$ ($\mathfrak{su}(2,2|4)$) and (classical) field space respectively.

[12]It is not obvious (to the author) that we can always fix Z so that our twist function manifestly gives an (r-symmetric) Drinfeld twist. Investigating this concretely would require extensive studies of the classical dynamics of our deformed models, not to even mention the quantum case. Moreover, at this level the all order gauge fixing of Z presumably depends on the particular r matrix under consideration, while to date there is no classification of all such r matrices. At the same time, since we are dealing with integrable deformations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring and its Yangian symmetry, we clearly should find some deformation (quantization) of this symmetry. Below we discuss concrete examples where this was worked out.

At leading order the twist function really represents a Drinfeld twist, thereby identifying the relevant algebraic structure of our integrable model to all orders. From here on out we incorporate the deformation parameter(s) in the r matrix, and therefore effectively set $\eta = 1$. Let us now see what kind of twists we can do.

3.3 Abelian and jordanian twists

To the author's knowledge, the types of twists that are known in the literature fall into two classes.

Abelian twists are associated to abelian r matrices, which are of the form

$$r = \alpha_{ij} a^i \wedge a^j, \tag{3.18}$$

where all $a \in \mathfrak{g}$ mutually commute. The associated (quantum) Drinfeld-Reshetikhin twist [83] (in our conventions) is given by

$$\mathcal{F} = e^{-ir}. \tag{3.19}$$

It is easy to see that this twist satisfies the cocycle and normalization conditions.

The prototypical example of an abelian r matrix is one based on the Cartan subalgebra of $\mathfrak{g}$. In our case of $\mathfrak{psu}(2,2|4)$ let us consider the r matrix

$$r = -\epsilon^{ijk} \frac{\hat{\gamma}_i}{8} h_j \wedge h_k, \tag{3.20}$$

sums running from one to three, where the h_k denote the cartans of $\mathfrak{su}(4)$. This results in the three parameter generalization of the (real β) Lunin-Maldacena background [4] of Frolov [6].[13] The interpretation of this deformation in terms of a twist by (3.19) has not only been established at the classical level [6, 80], but also quite convincingly at the quantum level, see e.g. [85].

Jordanian twists are associated to the Borel subalgebra of $\mathfrak{g}$, where we take two generators h and e with $[h, e] = e$ to form[14]

$$r = \beta h \wedge e. \tag{3.21}$$

In a matrix realization where $e^2 = 0$, we would have $r^3 = 0$.[15] A compact expression for a representative of the associated (quantum) twist is

$$\mathcal{F} = e^{h \otimes y}, \qquad y = \log(1 - i\beta e). \tag{3.22}$$

Note that this twist is not r-symmetric, but an equivalent r-symmetric version exists [30], given by [84] (see also [86])

$$\mathcal{F} = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{i\beta}{2} \right)^m \sum_{s=0}^{m} (-1)^s \binom{m}{s} e^{m-s} h^{\langle s \rangle} \otimes e^s h^{\langle m-s \rangle}, \tag{3.23}$$

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[13]The Lunin-Maldacena background corresponds to $\hat{\gamma}_1 = \pm\hat{\gamma}_2 = \pm\hat{\gamma}_3 = \hat{\beta}$.

[14]Note that this structure is not compatible with the real Lie algebra $\mathfrak{su}(4)$. Of course we could have equally well chosen h and f with $[h, f] = -f$.

[15]It is possible to have abelian r matrices with a matrix realization where $r^3 = 0$, these are sometimes referred to as abelian-jordanian r matrices [24].

where
$$h^{\langle k \rangle} = h(h+1)\dots(h+k-1), \quad k \in \mathbb{N}^+ \tag{3.24}$$
and $h^{\langle 0 \rangle} = 1$. These twist satisfy the cocycle and normalization conditions.[16] In contrast to the abelian case, flipping the sign of the deformation parameter does not manifestly invert the twist, though again, it does result in a twist that is equivalent to this inverse as a deformation of the algebra. We should additionally note that there are extended jordanian twists [87], see also [86]. Some examples we will consider below fall into this class, but let us not get lost in further details. Regarding both these remarks, we emphasize that for present considerations the essence of the deformation is captured by the r matrix. Finally, note that h cannot carry a physical (length) scale, only e can, while in the abelian case both generators can.

Our example of a jordanian twisted model requires a bit more introduction. The series of papers [88–90] studied the classical symmetry algebra of a Schrödinger deformation of the $\mathrm{AdS}_3 = (\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R)/\mathrm{SL}(2,\mathbb{R})$ sigma model. This deformation breaks the global symmetry of the model to $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{U}(1)_R$, but is still integrable. The complete symmetry algebra was initially found to consist of a Yangian based on $\mathrm{SL}(2,\mathbb{R})_L$, and a quantum affine algebra based on $\mathrm{U}(1)_R$ which was dubbed an exotic symmetry [88, 89]. Being related to the AdS_3 model by a nonlocal gauge transformation [89], the analogue of Z (cf. section 2) was fixed in [90] by comparing the explicit twist function to a jordanian twist, thereby reinterpreting the exotic symmetry as a jordanian twisted Yangian. Since the present Yang-Baxter formulation was not available at the time, the notion of [90] of a relation to jordanian twists appears quite inspired. Of course, this can now be suggestively rephrased. It is easy to verify that the $r = h_R \wedge f_R$ deformation of the AdS_3 sigma model gives exactly the relevant Schrödinger deformed model as also noted in [91] we will come back to this below. Given our earlier discussion, this immediately suggests that there should be a link to a jordanian twist, which is precisely what was established in [88–90].[17]

These examples of abelian and jordanian deformations explicitly demonstrate the interpretation of the non-local gauge transformation in terms of a jordanian or abelian Drinfeld(-Reshetikhin) twist. The same should apply to extended jordanian deformations and their associated twists. These make up all (currently known) homogeneous Yang-Baxter deformations of $\mathrm{AdS}_5 \times \mathrm{S}^5$, including the new ones we consider below. Since strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathcal{N} = 4$ SYM are supposed to represent equivalent representations of $\mathfrak{psu}(2,2|4)$ (and its Yangian in the planar limit), we can now attempt to twist $\mathcal{N} = 4$ SYM the same way.

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[16]In fact, both jordanian and abelian twists satisfy simpler versions of the cocycle condition, see e.g. [87]. When verifying the cocycle condition in a matrix realization with $e^2 = 0$ we should keep in mind that $\Delta(e^2) = \Delta(e)\Delta(e)$.

[17]The jordanian twist of [90] is of the non r-symmetric type of eqn. (3.22), but again we note that an algebraically equivalent r-symmetric one exists. We will not attempt to give an explicit all order gauge fixing of Z that establishes this in the sigma model.

4 Twisted gauge theory and noncommutative geometry

While $PSU(2, 2, |4)$ represents target space (super)symmetries of the $AdS_5 \times S^5$ string, for $\mathcal{N} = 4$ SYM these are associated to spacetime (and R symmetry). Deforming these symmetries will lead us into the realm of noncommutative geometry, so let us briefly describe the relevant structures.

Under the Moyal-Weyl correspondence, noncommutative geometry is described via a deformation of the algebra of functions on a manifold. For instance, provided we can consistently deform this algebra such that two of the coordinate functions in a patch satisfy

$$[x, y] = i\hbar \tag{4.1}$$

we would imagine we are dealing with a noncommutative space of some kind. Now, provided our algebra of functions carries a representation of a Hopf algebra, it is possible to induce a deformation of this algebra from a deformation of the Hopf algebra. This is precisely what we have, and we will use our Drinfeld twists to deform the algebra of (smooth) functions on Minkowski space.[18]

4.1 Drinfeld twists and star products

The space of functions on a manifold forms a module for the Lie algebra of vector fields on this manifold. We can give the universal enveloping algebra of vector fields on Minkowski space, $\mathcal{U}(T\mathcal{M})$, the structure of a Hopf algebra by taking

$$\begin{aligned}
\Delta(\xi) &= \xi \otimes 1 + 1 \otimes \xi, \quad \Delta(1) = 1 \otimes 1, \\
\epsilon(\xi) &= 0, \quad\quad\quad\quad\quad s(\xi) = -\xi,
\end{aligned} \tag{4.2}$$

where $\xi = \xi^\mu \partial_\mu$, just like the case of a finite dimensional Lie algebra $\mathfrak{g}$ we saw above. We say that this Hopf algebra naturally acts on the usual algebra of functions because its algebra and coalgebra structure are respected. At the algebraic level, multiplication in $\mathcal{U}(T\mathcal{M})$ is compatible with the action of vector fields on functions

$$(\xi\zeta)(f) = \xi(\zeta(f)), \tag{4.3}$$

while at the coalgebra level we simply have the product rule

$$\xi(fg) = \xi(\mu(f \otimes g)) = \mu(\Delta(\xi)(f \otimes g)) = \xi(f)g + f\xi(g) \tag{4.4}$$

where $\mu(a \otimes b) = ab$ is just the usual product of functions.

We can now Drinfeld twist this Hopf algebra as in eqs. (3.10), i.e.

$$\begin{aligned}
\Delta_{\mathcal{F}}(\xi) &= \mathcal{F}\Delta(\xi)\mathcal{F}^{-1}, \\
s_{\mathcal{F}}(\xi) &= f^\alpha s(f_\alpha) s(\xi) s(\bar{f}^\beta) \bar{f}_\beta.
\end{aligned} \tag{4.5}$$

---

[18]Our examples of deformations of the $AdS_5 \times S^5$ string will be naturally defined in the Poincaré patch, which is why we focus on Minkowski space. Similar concepts should apply to global anti-de Sitter space and $\mathbb{R} \times S^3$, though the fact that manifest conformal invariance is broken makes this point somewhat subtle as mentioned in e.g. [92]. Note that either way our picture confirms the "guess" for the star product for abelian Cartan-based deformations of AdS_5 given in [92].

The algebra of vector fields remains unchanged, as does the counit, but the coproduct and antipode have changed. This Hopf algebra no longer acts on functions on $\mathcal{M}$, and we can ask what type of deformation of the algebra of functions carries a representation of this Hopf algebra. Not surprisingly, this turns out to be a twist-deformed function algebra; we take functions on $\mathcal{M}$ but with the twisted product

$$\mu_{\mathcal{F}}(f \otimes g) = \mu \circ \mathcal{F}^{-1}(f \otimes g), \tag{4.6}$$

so that as in eqn. (4.4) above we have a 'product rule'

$$\xi(\mu_{\mathcal{F}}(f \otimes g)) = \mu_{\mathcal{F}} \Delta_{\mathcal{F}}(\xi)(f \otimes g). \tag{4.7}$$

As usual, we will denote this twisted product by a (Groenewold-Moyal) star product

$$f \star g \equiv \mu \circ \mathcal{F}^{-1}(f \otimes g), \tag{4.8}$$

where the cocycle condition now ensures associativity. We will encounter examples of twisted products below. The noncommutativity of spacetime can now be read off from

$$[x \overset{\star}{,} y] \equiv x \star y - y \star x. \tag{4.9}$$

Note that if we want to be able to get a Hermitian star product in the sense

$$\overline{f \star g} = \bar{g} \star \bar{f} \tag{4.10}$$

we need to work with the *r*-symmetric version of a jordanian twist. Field theories defined on this type of noncommutative space would carry a twisted Hopf algebra structure that nicely matches the one of our strings of section 3.

4.2 Twisted gauge theory and AdS/CFT

The possibility to so naturally deform (the spacetime of) $\mathcal{N} = 4$ SYM,[19] which would naively give a field theory carrying exactly the same type of twisted Hopf algebra that we encountered for our strings, suggests that such theories may be AdS/CFT (gravity/gauge) dual, like their undeformed counterparts.

In fact, though not originally phrased in this fashion, there are well known abelian examples of this. To start with, the deformation of the superpotential of SYM that turns it into β deformed SYM [93] can be represented by means of a star product (in SU(4) field space) [4]. The *r* matrix (3.20) gives the corresponding quantum twist, upon representing the Cartan generators of SU(4) via the R charges. Closer to the present context, the quantum twist associated to the *r* matrix (2.7) for canonical noncommutative SYM results indeed in nothing but

$$[x^2 \overset{\star}{,} x^3] \sim ia^2, \tag{4.11}$$

upon realizing $\mathfrak{su}(2,2)$ through vector fields on $\mathbb{R}^{1,3}$, cf. appendix A.2. Similar structures arise in other theories obtained by TsT transformations (and can hence be reproduced in

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[19]We presume it is possible to explicitly construct the associated field theories, at least to say leading order in the deformation parameter(s).

this formalism), such as dipole theories ($r \sim h_i \wedge p_j$ [94]), see e.g. [95], and the noncommutativity obtained by twisting in polar coordinates ($r \sim m_{12} \wedge p_3$) of e.g. [51].

As discussed in the introduction, we expect the interpretation to be less clear when the noncommutativity (the r matrix) involves time, but modulo these subtleties we conjecture that also jordanian deformations of the $\text{AdS}_5 \times \text{S}^5$ superstring have AdS/CFT duals, which take the form of appropriate noncommutative versions of $\mathcal{N} = 4$ SYM. In support of this, let us discuss two (closely related) models where we can give a more constructive description in the spirit of [1].

5 Jordanian twists and branes

The first model we will consider is the jordanian deformation of $\text{AdS}_5 \times \text{S}^5$ studied in [28, 29]. Working in terms of target space light cone coordinates $x^{\pm} = (x^0 \pm x^3)/\sqrt{2}$, this deformation arises from the $\mathfrak{su}(2,2|4)$ r matrix [27]

$$r = a(D - m_{+-}) \wedge p_-, \tag{5.1}$$

where the p, m, and D, denote momenta, Lorentz transformations, and the dilatation generator respectively, cf. appendix A.2, with $[D - m_{+-}, p_-] = 2p_-$. As a deformation of Poincaré AdS, the resulting metric and B field are given by

$$ds^2 = \frac{-2dx^+dx^- + dx_idx^i + dz^2}{z^2} - \frac{a^2(z^2 + x_ix^i)(dx^+)^2}{z^6} + d\Omega_5^2$$
$$B = -\frac{a(x^1dx^1 + x^2dx^2 - zdz)}{z^4} \wedge dx^+, \tag{5.2}$$

where the sums run over $i = 1, 2$, and we note that the zdz term in the B field is a total derivative. Reducing to AdS_3 by dropping x_1 and x_2, this is precisely (equivalent to) the $h_R \wedge f_R$ Schrödinger deformation referred to in section 3.3. A supergravity solution containing the metric and B field of eqs. (5.2) was found in [28], and later shown to match the result of two TsT transformations combined with an S duality [29].[20] Since the isometries involved in the TsT transformation are isometries of the full D3 brane background, we will consider them in this more general setting shortly.

Now up to an overal sign of the B field (except on the total derivative), the same bosonic model arises from the extended r matrix

$$r = a\left((D - m_{+-}) \wedge p_- + 2m_{-i} \wedge p^i\right), \tag{5.3}$$

which is known as the κ-Weyl r matrix [96] (where a is typically denoted κ^{-1}). Importantly, the fermions do distinguish these r matrices; our example model has 16 real supercharges, while the κ-Weyl one has none.[21] We can actually drop either of the two additional terms

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[20]The two TsT transformations produce a C_2 Ramond-Ramond potential from the C_4 potential of $\text{AdS}_5 \times \text{S}^5$, which upon S duality becomes the needed B field. It seems unlikely that we can obtain this B field directly from a (sequence) of TsT transformations, since it is nonzero in a nonisometry direction ($\sqrt{x_ix^i}$).

[21]The manifest symmetry algebra of a given deformed model is spanned by the $t \in \mathfrak{psu}(2,2|4)$ for which $R([t,x]) = [t, R(x)]$ for all $x \in \mathfrak{psu}(2,2|4)$ [27].

and still get a solution of the CYBE.[22] Let us focus on

$$r' = a\left((D - m_{+-}) \wedge p_- + 2m_{-2} \wedge p_2\right), \tag{5.4}$$

which will give our second model. These additional terms only flip signs in the B field, i.e.

$$B' = -a\frac{x^1 dx^1 - x^2 dx^2 - z dz}{z^4} \wedge dx^+, \tag{5.5}$$

but leave the metric invariant. This deformation preserves no supersymmetry, and breaks rotational symmetry in the (x^1, x^2) plane compared to the model above.

In contrast to the first model, it is harder to imagine reproducing this geometry by TsT transformations and S dualities; we need a sum of squares of x^1 and x^2 in the metric, but a difference in the B field. Still, given the relatively simple form of the supergravity equations of motion in this case, we can guess a modified solution that incorporates it, also at the D3 brane level. Hence, we will be able to analyze this second model just like the first, while it has a more 'generic' structure. Note that we cannot be sure that these solutions of supergravity correspond to the deformed coset models without explicitly expanding the fermions and comparing. That being said, the essential parts of our results only rely on the metric and B field.

5.1 Branes and low energy limits

To go from $\text{AdS}_5 \times \text{S}^5$ to (5.2) we do two TsT transformations $(x^-, \zeta)_b$ and $(x^-, \chi)_{\pm b}$ followed by an S duality transformation, where ζ is the polar angle in the (x^1, x^2) plane and χ is the S^1 coordinate of S^5 viewed as an S^1 fibration over $\mathbb{CP}^2$.[23] Applying this to the D3 brane background (see e.g. [79]) gives

$$ds^2 = \frac{1}{\sqrt{f}}(-2dx^+dx^- + dx_i dx^i - b^2(r^2 + f^{-1}x_i x^i)(dx^+)^2) + \sqrt{f}(dr^2 + r^2 d\Omega_5^2),$$

$$B = -bf^{-1}(x^1 dx^1 + x^2 dx^2) \wedge dx^+, \quad C_2 = b\left(f^{-1}(x^2 dx^1 - x^1 dx^2) - r^2(d\chi + \omega)\right) \wedge dx^+,$$

$$C_4 = C_4^0, \tag{5.6}$$

where C_4^0 is the undeformed potential, sums run over $i = 1, 2$,

$$f = 1 + \frac{\alpha'^2 R^4}{r^4}, \tag{5.7}$$

and we have reinstated units in the conventions of [79]. Our coordinates on S^5 and in particular ω are defined in appendix A.4. The dilaton is constant. Note that b has units of inverse length. This solution is symmetric under $x^\pm, b \to -x^\pm, -b$. The solution relevant for the r' deformation is obtained by replacing B and C_2 by[24]

$$B = -bf^{-1}(x^1 dx^1 - x^2 dx^2) \wedge dx^+,$$

$$C_2 = -b\left(f^{-1}(x^2 dx^1 + x^1 dx^2) - \tfrac{2}{\sqrt{3}}r^2(d\chi + \omega)\right) \wedge dx^+. \tag{5.8}$$

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[22] We might be tempted to add them with arbitrary parameters, but this does not work.

[23] The sign choice on $(x^-, \chi)_{\pm b}$ only affects C_2 (the fermions), we explicitly present the $+$ case.

[24] This solution can be generalized to the case where we do not take the TsT parameters equal. Intuitive explanations for the factor of $\sqrt{3}$ are welcome.

Asymptotically far away we can see the original geometry the branes were placed in; given the factor $r^2(dx^+)^2$, we rescale $x^- \to w^2 x^-, x^i \to w x^i, r \to wr$, and consider the limit $w \to \infty$. Up to an overall scale w^2, in cartesian coordinates we find

$$ds^2 = -2dx^+dx^- + dx_l dx^l - b^2(x_l x^l)(dx^+)^2,$$

$$C_2 = b \sum_{k=1}^{4} (x^{2k-1}dx^{2k} - x^{2k}dx^{2k-1}) \wedge dx^+, \tag{5.9}$$

$$B = -b(x^1 dx^1 \pm x^2 dx^2) \wedge dx^+,$$

where the sum in l now runs over eight xs. This is nothing but a plane wave supported by a RR three form, with a B field that has $dB = 0$ but is not constant. We can now look at the effective geometry seen by the open strings stretching between branes placed in this background. This (effective) metric G and noncommutativity parameter θ are obtained from the closed string metric g and B field[25] via [40, 52]

$$G = (g + B)^{-1}g(g - B)^{-1},$$
$$\theta = -2\pi\alpha'(g + B)^{-1}B(g - B)^{-1}. \tag{5.10}$$

giving

$$G_{mn} = \eta_{mn} - \delta_{m+}\delta_{n+}b^2 \sum_{k=3}^{8} x_k x^k,$$

$$\theta = -2\pi\alpha' b(x^1 dx^1 \pm x^2 dx^2) \wedge dx^-, \tag{5.11}$$

To get a finite result in the $\alpha' \to 0$ limit [40] we rescale $x^\pm \to \alpha'^{\pm 1}x^\pm$ to find

$$\theta = -2\pi b(x^1 dx^1 \pm x^2 dx^2) \wedge dx^-. \tag{5.12}$$

In other words, the low energy theory of open strings stretching between D3 branes in this geometry should correspond to a noncommutative version of $\mathcal{N} = 4$ SYM, with noncommutativity of the type

$$[x^- \overset{\star}{,} x^j] = 2\pi i b x^j, \tag{5.13}$$

for $j = 1, 2$, with a minus sign for x^2 in the r' case.

If now instead we consider the near horizon low energy limit by replacing $r \to \alpha' R^2/z$ and $b \to a/\alpha'$ and taking $\alpha' \to 0$, by construction we get our deformed AdS$_5 \times$ S^5 metric of eqn. (5.2)

$$(\alpha' R^2)^{-1}ds^2 = \frac{-2dx^+dx^- + dx_i dx^i + dz^2}{z^2} - \frac{a^2 R^4(z^2 + x_i x^i)(dx^+)^2}{z^6} + d\Omega_5^2 \tag{5.14}$$

with the B field of eqs. (5.2) and (5.5)

$$(\alpha' R^2)^{-1}B = aR^2 \frac{(x^1 dx^1 \pm x^2 dx^2)}{z^4} \wedge dx^+, \tag{5.15}$$

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[25]Note that our conventions differ by a factor of $2\pi\alpha'(i)$ on the B field from those of [40].

In our Drinfeld twist picture for the first model we get[26]

$$[x^\mu \stackrel{\star}{,} x^\nu] = \mu_{\mathcal{F}}(x^\mu \otimes x^\nu - x^\nu \otimes x^\mu) = 2iaR^2((D - m_{+-}) \wedge p_-)(x^\mu \otimes x^\nu), \qquad (5.16)$$

which explicitly gives

$$[x^- \stackrel{\star}{,} x^i] = iaR^2 x^i, \qquad (5.17)$$

cf. appendix A.2. This nicely matches the noncommutativity structure of eqn. (5.13). Adding the extra term of r' precisely introduces a sign for x^2. We believe this to be a good indication that these deformations of $\text{AdS}_5 \times \text{S}^5$ provide gravity dual descriptions of noncommutative $\mathcal{N} = 4$ $\text{U}(N)$ SYM, noncommutative in the sense of eqn. (5.13), involving spatial directions and a single null one. Note that the parameters in eqs. (5.13) and (5.17) are related precisely by the effective string tension $T = \sqrt{\lambda}/2\pi$.[27] The equations themselves apply in the opposite domains of weakly coupled gauge theory and classical string theory respectively. Correspondingly, note that just like the canonical noncommative case [78, 79] the metric approaches that of undeformed $\text{AdS}_5 \times \text{S}^5$ at large z (the infrared regime of the dual field theory), but differs for $z \sim \sqrt{a}R$ or $z \sim (a^2 R^4(x_1^2 \pm x_2^2))^{1/3}$ depending on the region of space we consider, cf. eqs. (5.13) and (5.17).

Now that we have further support for our conjecture in two nontrivial cases, let us look at some other possibly interesting structures.

6 κ-Minkowksi r matrices and singular backgrounds

Generalized κ-Minkowksi space corresponds to

$$[x^\mu \stackrel{\star}{,} x^\nu] = i\kappa^{-1}(a^\mu x^\nu - a^\nu x^\mu), \qquad (6.1)$$

with $|a_\mu a^\mu| = 1$, where a timelike a^μ gives true κ-Minkowski space with $[x^0 \stackrel{\star}{,} x^j] = i\kappa^{-1}x^j$, but we could also consider spacelike or null κ-Minkowski space. These will prove instructive examples, (despite) having precisely the type of noncommutativity we might expect to be difficult to give a dual interpretation. The timelike case can be obtained from a (quantum) Drinfeld twist based on the r matrix [39, 74]

$$r = \kappa^{-1} D \wedge p_0, \qquad (6.2)$$

while the spacelike and null case are similarly represented by

$$r = \kappa^{-1} D \wedge p_3, \qquad (6.3)$$

and

$$r = \kappa^{-1} D \wedge p_-, \qquad (6.4)$$

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[26]Here we are working with $\mathcal{F}^{-1} = 1 + ir + \mathcal{O}(a^2)$ in the r-symmetrized form (3.23). Higher order terms do not contribute to this commutator given the realization of the p_μ as differential operators.

[27]This precisely matches our expectations based on for example the Lunin-Maldacena(-Frolov) background, where we can ask what r matrix would result in e.g. $\Phi_2\Phi_3 \to e^{i\pi\gamma_1}\Phi_2\Phi_3$ in the dual field theory. Given that the Φ_j transform in the fundamental of SO(6) and hence have charge $2i$ under the anti-hermitian h_j of SU(4), we would expect $r = \pi T\gamma_1(2i)^{-2}(2h_2 \wedge h_3) = -8^{-1}\sqrt{\lambda}\gamma_1\epsilon^{1jk}h_j \wedge h_k$, precisely the $\hat{\gamma}_1$ term in the r matrix (3.20) under the usual identification $\hat{\gamma}_i/\sqrt{\lambda} = \gamma_i$.

respectively.[28]

The corresponding timelike κ-deformation of AdS$_5$ is given by

$$ds^2 = \frac{z^2(-dt^2 + dr^2 + dz^2) - \kappa^{-2}(dr - rz^{-1}dz)^2}{z^4 - \kappa^{-2}(z^2 + r^2)} + \frac{r^2(d\theta^2 + \sin^2\theta d\phi^2)}{z^2},$$
$$B = \kappa^{-1}\frac{zdz \wedge dt + rdr \wedge dt}{z^4 - \kappa^{-2}(z^2 + r^2)},$$
(6.5)

where we have introduced spherical coordinates (r, θ, ϕ) on $\mathbb{R}^3$ and denote x^0 by t. The spacelike κ deformation analogously yields

$$ds^2 = \frac{\nu^2(-d\beta^2 + \cosh^2\beta d\xi^2)}{z^2} + \frac{z^2(dx_3^2 + d\nu^2 + dz^2) + \kappa^{-2}(d\nu - \nu z^{-1}dz)^2}{z^4 + \kappa^{-2}(z^2 + \nu^2)},$$
$$B = \kappa^{-1}\frac{zdz \wedge dx^3 + \nu d\nu \wedge dx^3}{z^4 + \kappa^{-2}(z^2 + \nu^2)},$$
(6.6)

where we introduced 'hyperbolic coordinates' on $\mathbb{R}^{1,2}$ via $t = \nu\sinh\beta$, $x^1 = \nu\cosh\beta\sin\xi$, and $x^2 = \nu\cosh\beta\cos\xi$. Finally the null deformation reads

$$ds^2 = \frac{z^2(-2dx^+dx^- + d\rho^2 + dz^2) - z^{-2}((zdx^+ - x^+dz)^2 + (x^+d\rho - \rho dx^+)^2)}{z^4 - \kappa^{-2}(x^+)^2} + \frac{\rho^2 d\zeta^2}{z^2},$$
$$B = \kappa^{-1}\frac{-zdz \wedge dx^+ - x^+dx^+ \wedge dx^- - \rho d\rho \wedge dx^+}{z^4 - \kappa^{-2}(x^+)^2},$$
(6.7)

where we took polar coordinates ρ and ζ on the (x_1, x_2) plane. Note that here the zdz term in the B field is a total derivative. The integrable sigma models associated to these spaces have no (manifest) supersymmetry.

The last of these deformed spaces is actually a special case of a background given in [27], based on the r matrix

$$r = (a\,D - c\,m_{+-}) \wedge p_-,$$
(6.8)

giving the null κ-Minkowski r matrix at $c = 0$, and containing our main example of eqn. (5.1) at $c = a$. We can express this r matrix as a sum of the r matrix for our main example and the abelian r matrix $(D + m_{+-}) \wedge p_-$. On its own this abelian r matrix corresponds to a (formal) TsT transformation in (y, x^-), where y is the coordinate associated with the boost-dilation (Schrödinger dilation) $x^+ \to e^{2\alpha}x^+$, $z \to e^\alpha z$, $\rho \to e^\alpha\rho$ generated by $D + m_{+-}$ [27]. Since y and x^- are still isometry coordinates of the deformed geometry (5.2), we can append this TsT transformation to the sequence of the previous section, and we readily find that this indeed gives the general background of [27] and in particular the null κ-Minkowski deformed AdS$_5$ of eqn. (6.7), which is thereby embedded in supergravity.[29] We have not investigated this for the timelike and spacelike κ-Minkowski deformations. Also, since this TsT transformation involves isometries of AdS$_5$ that are not isometries of the brane background, we cannot directly use this trick there.

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[28]Also in this null case we have the bosonically equivalent $r = \kappa^{-1}(m_{+-} \wedge p_- + m_{i-} \wedge p^i)$.

[29]At the algebraic level the corresponding picture is that $(D + m_{+-}) \wedge p_-$ is subordinate [86] to $(D - m_{+-}) \wedge p_-$, so that their sum is a solution of the CYBE, and the total twist factorizes into a product of an abelian and a jordanian piece.

At large z these spaces approach undeformed $\text{AdS}_5 \times \text{S}^5$, but differences become apparent as we decrease, and in particular the timelike and null κ-Minkowski deformations of $\text{AdS}_5 \times \text{S}^5$ become very different; they are singular. In the timelike case we encounter a singularity at $z^2 = (\kappa^{-2} + \sqrt{\kappa^{-4} + \kappa^{-2}r^2})/2$, while for the null case we do so at $z^2 = \kappa^{-1}|x^+|$. Here we can draw an analogy to the abelian temporal-spatial noncommutative case and the model by Hashimoto and Sethi mentioned in the introduction.

The usual argument that gives a canonical noncommutative field theory description of open strings breaks down in the temporal-spatial case due to a critical value of the electric field (B_{0i}) on the brane [47, 48]. Beyond this value string pair production destabilizes the background, while we would need to cross it to get a finite noncommutative $\alpha' \to 0$ limit. This critical value is reflected by a pole in the open string noncommutativity θ. While it might appear tempting to link this to the poles we see in the B fields above, we should keep in mind that at this stage we have no reason to assume these spaces are actually gravity duals, and even if they were we would be dealing with a 'near horizon', closed string quantity. What we can do however, is construct the deformation of $\text{AdS}_5 \times \text{S}^5$ that we would naively associate to $[x^0 \stackrel{\star}{,} x^1] \sim a^2$. The resulting geometry is just a (formal) TsT transformation $(x^1, x^0)_{a^2}$ of $\text{AdS}_5 \times \text{S}^5$ (in flat space this generates the desired electric field), which due to the shift of time by a spatial coordinate generates a singularity at $z \sim a$. This singularity is presumably an indication that our considerations break down, or may be the counterpart to fundamental problems with a naively constructed dual field theory, though we should note that it is apparently possible to formulate unitary field theories with noncommutativity involving time [97, 98]. In either case the singularity would indicate that a dual interpretation of the timelike and null κ-Minkowski cases above is rather unclear and presumably problematic. The spacelike case would suggestively pass this test however.

At this stage we should briefly discuss the model by Hashimoto and Sethi [50]. Their noncommutative field theory and associated gravity dual are obtained by TsT transforming a stack of D3 branes, and doing a (singular) coordinate change, hence we expect this model to fit in our framework. In [75] it was noted that the four dimensional metric that the branes see can be reproduced by their truncated Yang-Baxter deformation of Minkowski space. We have checked that the corresponding abelian r matrix

$$r = am_{1-} \wedge p_2, \tag{6.9}$$

reproduces also the corresponding gravity dual as a deformation of $\text{AdS}_5 \times \text{S}^5$, upon identifying $x^+ = a^{-1}y^+$, $x^1 = y^+ \tilde{y}$, $x^- = ay^- + \frac{a}{2}y^+\tilde{y}^2$, $x^2 = -\tilde{z}$ [50, 75], and a as the $\tilde{R}$ of [50]. The associated noncommutative field theory has

$$[x^1 \stackrel{\star}{,} x^2] = iax^+, \qquad [x^- \stackrel{\star}{,} x^2] = iax^1, \tag{6.10}$$

which our Drinfeld twist picture beautifully postdicts. While this structure unequivocally involves time by containing both light cone coordinates, this theory does arise from an appropriate low energy open string limit [50]. The associated gravity dual is regular.

Combining the fact that it is thus possible to obtain time-involving noncommutative field by typical string arguments, and the fact that spacelike κ-Minkowski deformed

AdS$_5 \times$ S^5 is regular, we are tempted to conjecture that it is the gravity dual of SYM on spacelike κ-Minkowski space. Of course, this comes with disclaimers: we have not embedded this space in supergravity, or consequently attempted to find a brane picture for the associated noncommutativity, and we may well run into difficulties besides regularity, also more directly on the field theory side. It would be very interesting to investigate this further. The regularity of the spacelike case is clearly related to the fact that when time appears in the noncommutative structure, it does so homogeneously, but we have no deeper interpretation to offer at this time. Let us emphasize again that we are considering κ-Minkowski space in the twisted setting, so that the associated symmetry structure is not that of the κ-Poincaré group.[30]

7 Conclusions

Jordanian deformations of the AdS$_5 \times$ S^5 superstring have an interesting deformed symmetry algebra of a type that can occur in $\mathcal{N} = 4$ SYM if defined on an appropriate noncommutative space. While the situation is subtle when time is involved, we conjectured that in the null and spatial cases, as well as at least some further regular cases, these two classes of deformed theories should be gauge/gravity duals of one another. We demonstrated these ideas on two examples with noncommutativity of the type

$$[x^- \overset{\star}{,} x^j] = (\pm)ia\, x^j, \quad j = 1, 2, \tag{7.1}$$

matching our conjecture with a brane picture. Let us emphasize that this noncommutativity is different from "isometric" ones obtained by TsT transformations, such as

$$[x^- \overset{\star}{,} x^1] = ic\, x^2, \qquad [x^- \overset{\star}{,} x^2] = -ic\, x^1$$

for $r = c\, m_{12} \wedge p_-$. Of course, these noncommutative versions of SYM should be integrable in the planar limit.[31] Given the concrete AdS/CFT interpretation we have found for these two jordanian deformations, it may be interesting to do various 'classic' AdS/CFT computations regarding correlation functions and Wilson loops in these deformed dual geometries, (as well as) to understand the 'boundary' geometry of these deformed spaces in relation to the noncommutativity (7.1).

We also studied κ-Minkowski–related deformations of AdS$_5 \times$ S^5, where in particular the spacelike case shows interesting features. Based on the regularity of the spacelike deformation, though lacking direct support in the form of a brane picture, we conjectured that this deformation may be dual to the corresponding noncommutative version of SYM. Finding further support for, or subtleties with this conjecture is certainly an important direction for future investigation.[32]

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[30]Interestingly, the null κ-Poincaré group can be viewed as a (different) Drinfeld twist, see e.g. [99].

[31]Still, it is the integrability preserving nature of these deformations as opposed to their inherent integrability that appears to be of main relevance for our story; the twists arose from nonlocal gauge transformations relating deformed currents to the undeformed one, preserving flatness, hence integrability.

[32]Though outside the scope of the present paper, regarding these κ deformations it is interesting to recall that the κ-Poincaré group was originally obtained as a contraction limit of SO$_q(3,2)$, leading us to wonder whether a similar contraction limit of SO$_q(4,2)$ and PSU$_q(2,2|4)$ could have an interesting implementation in the quantum deformed AdS$_5 \times$ S^5 sigma model.

The fact that the second of our main examples as well as the (spacelike) κ-Minkowski case are not supersymmetric may lead to subtleties. Let us elaborate on this point by example. As we already implicitly saw above, there is a 'natural' deformation of SYM for the three parameter generalization of the Lunin-Maldacena background [6, 100] (which our construction would indeed exactly suggest). This 'γ_i-deformed' SYM has no supersymmetry, and its classical conformal symmetry is broken at the quantum level even in the planar limit [101], leading to questions about its AdS/CFT interpretation. However, without supersymmetry some of the string modes may become tachyonic at the quantum level and lead to a deformation of AdS$_5$ as the analogue of breaking conformal invariance. Hence, despite lacking a clean AdS/CFT interpretation, there may well be a duality, and correspondingly it appears to be possible to match spectra between the two theories at least in the planar limit, for a large class of states, see [85, 102] and references therein. Similar subtleties may arise for nonsupersymmetric Jordanian deformations.

Moreover regarding supersymmetry, while we presently only considered bosonic generators, we could actually try to 'supersymmetrize' our r matrices by adding fermionic bilinears, see e.g. [103]. Provided such r matrices exist here, we suppose they would directly modify the RR fluxes of the background, affecting the supersymmetry properties of the background without affecting its metric and (bosonic) B field. If purely fermionic abelian cases exist they presumably correspond to some sort of fermionic analogue of a TsT transformation, based on fermionic T duality [104, 105]. Note that the simple r matrix based (super)symmetry analysis [27] would change correspondingly.

Given their integrability in the planar limit, it would also be interesting to understand whether singular backgrounds can be given a meaningful interpretation in general, even just as strings. The free string picture at least is rather clear for the already problematic abelian $p_0 \wedge p_1$ deformation at least, but less so for generic (jordanian) ones.

In general, it would be great to understand whether all homogeneous Yang-Baxter deformations of AdS$_5 \times$ S^5 can be embedded in supergravity. At the abelian level there appears to be a clear link to TsT transformations, which if proven would manifest this. For jordanian deformations there is no clear relation to supergravity solution generating transformations; are they always embeddable, and if so do they correspond to a new solution generating technique, or can they be fully understood in terms of known ones? Of course, in trying to interpret these deformations in terms of supergravity it may be helpful to have an explicit classification of solutions to the Yang-Baxter equation over $\mathfrak{psu}(2,2|4)$. No such classification exists for generic simple Lie algebras, but solutions over the Poincaré algebra have been classified [106]. In that particular case the analysis was likely facilitated by the semidirect product structure of the Poincaré algebra and the isomorphism of the Lorentz algebra to $\mathfrak{sl}(2,\mathbb{C})$ however. We have investigated some solutions of [106]; the interested reader can find one such Poincaré based (singular) deformation of AdS$_5 \times$ S^5 in appendix A.5. Finally, since much of the tools based on integrability in AdS/CFT rely on an exact S matrix approach, it may prove fruitful to look for further light cone gauge compatible deformations of global anti-de Sitter space.

Acknowledgements

A Algebra, coordinates, and the sigma model action

A.1 Matrix realization of $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$

In this paper we are mainly concerned with the bosonic subalgebra $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$ of $\mathfrak{psu}(2,2|4)$. For details on the material presented here, as well as its supersymmetric extension, we refer to the pedagogical review [2] whose conventions we follow. We will only briefly list the facts we need, beginning with the γ matrices

$$
\begin{aligned}
\gamma^0 &= i\sigma_3 \otimes \sigma_0, \quad \gamma^1 = \sigma_2 \otimes \sigma_2, \quad \gamma^2 = -\sigma_2 \otimes \sigma_1, \\
\gamma^3 &= \sigma_1 \otimes \sigma_0, \quad \gamma^4 = \sigma_2 \otimes \sigma_3, \qquad \gamma^5 = -i\gamma^0,
\end{aligned}
\tag{A.1}
$$

where $\sigma_0 = 1_{2\times2}$ and the remaining σ_i are the Pauli matrices. With these matrices the generators of $\mathfrak{so}(4,1)$ in the spinor representation are given by $m^{ij} = \frac{1}{4}[\gamma^i, \gamma^j]$ where the indices run from zero to four, while for $\mathfrak{so}(5)$ we can give the same construction with indices running from one to five. The algebra $\mathfrak{su}(2,2)$ is spanned by these generators of $\mathfrak{so}(4,1)$ together with the γ^i for $i = 0, \ldots, 4$, while $\mathfrak{su}(4)$ is spanned by the combination of $\mathfrak{so}(5)$ and $i\gamma^j$ for $j = 1, \ldots, 5$. Concretely, these generators satisfy

$$
m^\dagger \gamma^5 + \gamma^5 m = 0
\tag{A.2}
$$

for $m \in \mathfrak{su}(2,2)$, and

$$
n^\dagger + n = 0
\tag{A.3}
$$

for $n \in \mathfrak{su}(4)$. This means that we are dealing with the canonical group metric $\gamma^5 = \text{diag}(1,1,-1,-1)$ for $SU(2,2)$, and that $e^{\alpha n}$ and $e^{\alpha m}$ give group elements for real α.

The generator Ω of the $\mathbb{Z}_4$ automorphism of $\mathfrak{psu}(2,2|4)$ acts on these bosonic subalgebras as

$$
\Omega(m) = -Km^t K,
\tag{A.4}
$$

where $K = -\gamma^2\gamma^4$, which leaves the subalgebras $\mathfrak{so}(4,1)$ and $\mathfrak{so}(5)$ invariant.

We denote the Cartan generators of $\mathfrak{su}(4)$ by h_j, $j = 1, 2, 3$, taken to be

$$
h_1 = \text{diag}(i, i, -i, -i), \quad h_2 = \text{diag}(i, -i, i, -i), \quad h_3 = \text{diag}(i, -i, -i, i).
\tag{A.5}
$$

Our group parametrization below will use the Poincaré translation generators

$$
p^\mu = \frac{1}{2}(\gamma^\mu - \gamma^\mu\gamma^4),
\tag{A.6}
$$

as well as the special conformal generators

$$k^\mu = \frac{1}{2}(\gamma^\mu + \gamma^\mu \gamma^4). \tag{A.7}$$

Both the p and the k are nilpotent (as matrices)

$$p^\mu p^\nu = k^\mu k^\nu = 0. \tag{A.8}$$

A.2 Vector field realization of $\mathfrak{su}(2,2)$

We can represent the conformal algebra (we work with anti-Hermitian generators) in terms of differential operators as

$$
\begin{aligned}
M_{\mu\nu} &= x_\mu \partial_\nu - x_\nu \partial_\mu, & K_\mu &= x_\alpha x^\alpha \partial_\mu - 2x_\mu x^\nu \partial_\nu \\
P_\mu &= \partial_\mu & D &= -x^\mu \partial_\mu,
\end{aligned} \tag{A.9}
$$

which satisfy

$$
\begin{aligned}
[M_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu, & [M_{\mu\nu}, K_\rho] &= \eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu, \\
[M_{\mu\nu}, D] &= 0, & [D, P_\mu] = P_\mu, & [D, K_\mu] = -K_\mu, \\
[P_\mu, K_\nu] &= 2M_{\mu\nu} + 2\eta_{\mu\nu} D, & [M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu\rho} M_{\nu\sigma} + \text{perms}.
\end{aligned} \tag{A.10}
$$

These generators realize the matrix generators $m(M)$, $p(P)$ and $k(K)$ of the previous section as vector fields, with D corresponding to $\frac{1}{2}\gamma_4$. In the main text we refer to $\frac{1}{2}\gamma_4$ as D directly, and use m, p, and k to denote the generators in any appropriate realization.

A.3 Bosonic sigma model action

The concrete metrics and B fields in this paper can be read off from the bosonic part of the deformed sigma model. In these bosonic models we work with the coset representative

$$g = \begin{pmatrix} g_a & 0 \\ 0 & g_s \end{pmatrix}, \tag{A.11}$$

with

$$g_a = e^{x_\mu p^\mu} e^{\log z D} = (1 + x_\mu p^\mu) e^{\log z D}, \tag{A.12}$$

and for completeness

$$g_s = e^{\phi^i h_i} e^{-\frac{\xi}{2}\gamma^1 \gamma^3} e^{\frac{i}{2}\arcsin r \gamma^1}. \tag{A.13}$$

Substituting this group element in the undeformed version of eqn. (2.1), gives the (bosonic) string action of $\mathrm{AdS}_5 \times \mathrm{S}^5$ in Poincaré coordinates, i.e. an action of the form

$$S = -\frac{T}{2} \int \mathrm{d}\tau \mathrm{d}\sigma \left(g_{MN} dx^M dx^N - B_{MN} dx^M \wedge dx^N \right), \tag{A.14}$$

where in this case g is the metric of $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the B field is zero. A coset representative that gives global AdS can be found in e.g. [2] or [10]. The metric and B field corresponding

to the deformed action (2.1) can be readily extracted following [10, 28]. We need to construct the current J, which is defined as in the main text via

$$J = (1 - \eta R_g \circ d)^{-1}(A) \tag{A.15}$$

The operator $1 - \eta R_g \circ d$ can indeed be inverted (at least for general values of the group coordinates, and generally perturbatively in η) [10, 11]. In practice, for the bosonic action we only need $J^{(2)}$ (grades one and three are fermionic), where we notice that we have

$$P_2(A) = A^{(2)} = P_2\left((1 - 2\eta R_g \circ P_2)(J)\right) = P_2\left((1 - 2\eta R_g)(J^{(2)})\right). \tag{A.16}$$

This equation can be readily solved for $J^{(2)}$ (using a computer), and substituted in the action, from which the metric and B field can be read off by comparing to eqn. (A.14) above. To be clear, cf. eqn. (A.4) we have $P_2(X) = \frac{1}{2}(X - \Omega(X))$. In practice it is useful to expand eqn. (A.16) over the basis of the grade two components, rather than work directly with the matrix realization of $\mathfrak{su}(2,2|4)$.

A.4 S^5 as S^1 over $\mathbb{CP}^2$

To describe S^5 as an S^1 fibration over $\mathbb{CP}^2$ we can embed a five sphere of radius one in $\mathbb{R}^6$ with cartesian coordinates y_j via

$$
\begin{aligned}
y_1 + iy_2 &= e^{i\chi} \sin\mu \cos\tfrac{\theta}{2} e^{\frac{i}{2}(\psi+\phi)}, \\
y_3 + iy_4 &= e^{i\chi} \sin\mu \sin\tfrac{\theta}{2} e^{\frac{i}{2}(\psi-\phi)}, \\
y_5 + iy_6 &= e^{i\chi} \cos\mu.
\end{aligned}
\tag{A.17}
$$

This results in the metric

$$ds^2 = (d\chi + \omega)^2 + d\mu^2 + \sin^2\mu(s_1^2 + s_2^2 + \cos^2\mu\, s_3^2), \tag{A.18}$$

where $\omega = \sin^2\mu\, s_3$, and

$$
\begin{aligned}
s_1 &= \frac{1}{2}(\cos\psi d\theta + \sin\psi \sin\theta d\phi), \\
s_2 &= \frac{1}{2}(\sin\psi d\theta - \cos\psi \sin\theta d\phi), \\
s_3 &= \frac{1}{2}(d\psi + \cos\theta d\phi).
\end{aligned}
\tag{A.19}
$$

Note that

$$d\chi + \omega = y_1 dy_2 - y_2 dy_1 + y_3 dy_4 - y_4 dy_3 + y_5 dy_6 - y_6 dy_5, \tag{A.20}$$

which for $\mathbb{R}^6$ in terms of hyperspherical coordinates in the fibered sense really means

$$r^2(d\chi + \omega) = x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3 + x_5 dx_6 - x_6 dx_5, \tag{A.21}$$

where r is the radial coordinate and the x_i are (unconstrained) cartesian coordinates. Note that in the main text the indices will be shifted by two since x_1 and x_2 are already used.

A.5 A Poincaré based deformation of AdS$_5 \times$ S^5

To give a (random) example of the sort of deformations one can get by considering Poincaré r matrices we consider the jordanian r matrix [86, 106]

$$r = -am_{02} \wedge (m_{12} + m_{01}), \tag{A.22}$$

would naively correspond to noncommutativity of the type

$$[x^m \overset{\star}{,} x^n] = ia\epsilon^{mn}{}_p x^p (x^0 + x^2), \tag{A.23}$$

where indices m,n and p run from zero to two and ϵ is totally antisymmetric with $\epsilon^{012} = 1 = -\epsilon_0{}^{12}$. The corresponding deformation of AdS$_5 \times$ S^5 is given by

$$
\begin{aligned}
ds^2 &= \frac{z^2\nu^2(-d\beta^2 + \cosh^2\beta d\xi^2)}{z^4 - a^2(\sinh\beta + \cos\xi\cosh\beta)^2\nu^4} + \frac{d\nu^2 + dx_3^2 + dz^2}{z^2}, \\
B &= a\frac{(\sinh\beta + \cos\xi\cosh\beta)\cosh\beta\nu^4}{z^4 - a^2(\sinh\beta + \cos\xi\cosh\beta)^2\nu^4}d\xi \wedge d\beta,
\end{aligned}
\tag{A.24}
$$

using the hyperbolic coordinates of the spacelike κ-Minkowski deformation in the main text. We chose this example out of many, including abelian ones, where the deformation parameter has dimensions of inverse length squared. Also this space appears to be singular.

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