New Answers to an Old Question: Essential Underlying ODE versus Inherent ODE.

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Abstract

In the context of linear differential-algebraic equations (DAEs) one finds different associated explicit ordinary differential equations (ODEs), among them *essential underlying* and *inherent explicit regular* ones, abbreviated: EUODEs and IERODEs. EUODEs have been introduced in 1991 for index-2 DAEs in Hessenberg form by means of special transformations. IERODEs result within the framework of the projector based decoupling. Each such explicit ODE is occasionally considered to rule the flow of the DAE.

The question to which extend EUODEs and IERODEs are related to each other has been asked promptly after 1991. For index-2 Hessenberg-form DAEs, answers have been given in 2005, saying that EUODEs represent somehow condensed IERODEs. Recently, EUODEs have been indicated for general arbitrary-index DAEs and it has been proved that they are condensed IERODEs. The understanding of the relation between the IERODE and the EUODEs enables to uncover the stability behavior of the DAE flow.

In the present paper we show that both, the IERODEs and EUODEs of a DAE with arbitrary high index do not at all depend on derivatives of the right-hand side. Further we consider adjoint pairs of DAEs and provide generalizations of the classical Lagrange identity.

1 Introduction

In the context of linear differential-algebraic equations (DAEs) one finds different associated explicit ordinary differential equations (ODEs), among them *essential underlying* ODEs (EUODEs) and *inherent explicit regular* ODEs (IERODEs). EUODEs have been distinguished by Ascher and Petzold ([1]) for index-2 DAEs in Hessenberg form by means of special transformations. IERODEs result within the framework of the projector based decoupling.

Both EUODEs and IERODEs are occasionally considered to rule the flow of the DAE. How are they related to each other? This question has been asked promptly after 1991. First answers have been given by Balla and Vu Hoang Linh ([2, 3]) pointing out that, for index-2 Hessenberg-form DAEs and general index-1 DAEs, an EUODE represents a condensed IERODE. Recently, EUODEs associated with general arbitrary-index DAEs have been distinguished in ([10]). Also in the general case, the EUODEs can be seen as condensed IERODEs,

Any regular linear differential-algebraic equation (DAE) with properly stated leading term features a unique IERODE living in the given space. In contrast, there are several EUODES living in a transformed space with possible minimal dimension. We show that the IERODEs and EUODEs are the only associated explicit ODEs which do not at all depend on derivatives of the right-hand side. This offers the background of a preciser sensitivity analysis and allows to state boundary conditions accurately.

The understanding of the relation between the IERODE and the EUODEs enables to uncover stability properties of the DAE flow such as Lyapunov regularity etc. We particularly concentrate on relevant versions of the Lagrange identity. This should be helpful in the context of numerical boundary value problems, sensitivity analysis, optimization, and control problems, see [1, 4, 6, 7, 9, 12, 15].

We investigate DAEs with properly involved derivative

$$A(t)(Dx)'(t) + B(t)x(t) = q(t), \quad t \in \mathcal{I}. \tag{1}$$

and standard form DAEs

$$E(t)x'(t) + F(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{2}$$

with coefficients

$$E, F \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m)),$$
$$A \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^m)),\ D \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^n)),\ B \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m)),$$

with $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ in the real and complex versions, respectively. The interval $\mathcal{I} \subseteq \mathbb{R}$ is arbitrary, possible infinite. We drop the argument t whenever reasonable. Then the given relations are meant pointwise for all $t \in \mathcal{I}$.

The paper is arranged as follows. In Section 2 we recap known results for index-2 DAEs in Hessenberg form already by means of the notation of the general approach in the following part in order to motivate this approach and to explicate the questions to be considered. In Section 3 we describe the structure of a regular DAE (1). In particular, we deal with its IERODE and EUODEs and show that these are the only associated q-derivative-free explicit ODEs. Adjoint pairs of DAEs are considered in Section 4. We show that, even though the IERODEs of the original DAE and its adjoint equation are not necessarily adjoint to each other, the solutions associated to the DAEs satisfy a Lagrange identity. We specify the results of the previous section in Section 5.

2 Recapping index-2 Hessenberg systems

The system comprising the $m = m_1 + m_2$ equations

$$x_1' + B_{11}x_1 + B_{12}x_2 = q_1, \tag{3}$$
$$B_{21}x_1 = q_2, \tag{4}$$

is said to be a DAE in Hessenberg form of size 2, if the product $B_{21}B_{12}$ remains everywhere nonsingular. This DAE is known to have differentiation index 2 as well as tractability index 2. Writing

$$\underbrace{\begin{bmatrix} I \\ 0 \end{bmatrix}}_{=A}(\underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{=D} x)' + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{bmatrix} x = q, \tag{5}$$

one puts the DAE in the form (1). Owing to the nonsingularity of the product $B_{21}B_{12}$, the direct sum decomposition

$$\ker B_{21} \oplus \operatorname{im} B_{12} = \mathbb{R}^{m_1} \tag{6}$$

is valid and the projector-valued function $\varOmega$ given by

$$\varOmega := B_{12}B_{12}^-, \quad B_{12}^- := (B_{21}B_{12})^{-1}B_{21}, \tag{7}$$

projects pointwise onto $\operatorname{im} B_{12}$ along $\ker B_{21}$.

The usual approach to DAEs consists in extracting explicit ODEs with respect to x or x_1 from a derivative array system. The resulting so-called *underlying ODEs* depend on the special way they are provided. For instance, for (3),(4), one obtains

$$x_1' = -B_{11}x_1 - B_{12}x_2 + q_1,$$
$$x_2' = -\mathfrak{B}^{-1}(\mathfrak{A}B_{11} - \mathfrak{A}')x_1 - \mathfrak{B}^{-1}(\mathfrak{A}B_{12} + \mathfrak{B}')x_2 + \mathfrak{B}^{-1}(\mathfrak{A}\, q_1 - (q_2' - B_{21}q_1)'),$$

with $\mathfrak{A} := -B_{21}B_{11} + B_{21}'$, $\mathfrak{B} := B_{21}B_{12}$. The DAE flow is embedded into the m-dimensional flow of the underlying ODE. Observe that derivatives of components of q encroach on this ODE. Similarly, also the resulting ODE with respect to x_1,

$$x_1' = -((I-\Omega)B_{11} + B_{12}\mathfrak{B}^{-1}B_{21}')x_1 + (I-\Omega)q_1 + B_{12}\mathfrak{B}^{-1}q_2',$$

is affected by the derivative q_2'.

In contrast, an associated explicit ODE of minimal size, which is not at all affected by derivatives of q, has been obtained by special transformations in [1]. We follow the lines of [1], but use quite different notations.

Put $d := m_1 - m_2$. One finds a matrix function $\Gamma_d \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^{m_1}, \mathbb{K}^d))$ such that the columns of Γ_d^* form a basis of $\ker B_{12}^* = (\operatorname{im} B_{12})^{\perp}$. This implies $\ker \Gamma_d = \operatorname{im} B_{12}$, and the matrix function

$$\begin{bmatrix} \Gamma_d \\ B_{21} \end{bmatrix} \quad \text{remains nonsingular.}$$

We denote and verify

$$\Gamma_d^- := \begin{bmatrix} \Gamma_d \\ B_{21} \end{bmatrix}^{-1} \begin{bmatrix} I_d \\ 0 \end{bmatrix}, \quad \Gamma_d\Gamma_d^- = \begin{bmatrix} I_d & 0 \end{bmatrix} \begin{bmatrix} \Gamma_d \\ B_{21} \end{bmatrix} \Gamma_d^- = I_d, \quad \Gamma_d\Gamma_d^-\Gamma_d = \Gamma_d, \quad \Gamma_d^-\Gamma_d\Gamma_d^- = \Gamma_d^-,$$
$$\ker \Gamma_d^-\Gamma_d = \operatorname{im} B_{12}, \quad \operatorname{im} \Gamma_d^-\Gamma_d = \ker B_{21}.$$

This actually means that Γ_d^- is a generalized inverse of Γ_d and $\Gamma_d^-\Gamma_d = I - \Omega$. Furthermore, one can check that

$$\begin{bmatrix} \Gamma_d \\ B_{21} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} = B_{12}(B_{21}B_{12})^{-1}.$$

Defining new variables

$$\eta = \Gamma_d x_1, \quad x_1 = \begin{bmatrix} \Gamma_d \\ B_{21} \end{bmatrix}^{-1} \begin{bmatrix} \eta \\ -q_2 \end{bmatrix} = \Gamma_d^-\eta - \begin{bmatrix} \Gamma_d \\ B_{21} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ q_2 \end{bmatrix} = \Gamma_d^-\eta - B_{12}(B_{21}B_{12})^{-1}q_2,$$

one obtains an explicit ODE with respect to η,

$$\eta' + \Gamma_d B_{11}\Gamma_d^-\eta - \Gamma_d'\Gamma_d^-\eta = \Gamma_d q_1 + (\Gamma_d' - \Gamma_d B_{11})B_{12}(B_{21}B_{12})^{-1}q_2, \tag{8}$$

which is referred to as *essential underlying ODE* (EUODE) in order to distinguish it from other underlying ODEs, emphasizing that it has minimal size d and yields a well-posed boundary-value problem with respect to x_1, see [1]. Below we will indicate such EUODEs for each arbitrary regular DAE.

Having x_1 one computes the component x_2 by

$$x_2 = (B_{21}B_{12})^{-1}B_{21}\{q_1 - (\Gamma_d^-\eta)' + (B_{12}(B_{21}B_{12})^{-1}q_2)' - B_{11}(\Gamma_d^-\eta - B_{12}(B_{21}B_{12})^{-1}q_2)\},$$

which uncovers the structure of the DAE flow. The dynamic degree of freedom equals d.

Since there are different possibilities to choose a basis of $\ker B_{12}^*$, the essential underlying ODE depends on the specially chosen basis. As suggested in [1], one can restrict the options to Γ_d with orthonormal rows.

The projector based approach (e.g. [11]) relies on projector functions such as Ω and decouples the original DAE as it is, devoid of any transformations. Regarding the properties $B_{21} = B_{21}\Omega$ and $\Omega B_{12} = B_{12}$, we immediately derive from equation (3) that

$$(I-\Omega)x_1' + (I-\Omega)B_{11}x_1 = (I-\Omega)q_1, \tag{9}$$

and from (4) that

$$\Omega x_1 = B_{12}(B_{21}B_{12})^{-1}q_2,$$

which leads to

$$x_1 = (I-\Omega)x_1 + B_{12}(B_{21}B_{12})^{-1}q_2,$$
$$x_2 = (B_{21}B_{12})^{-1}B_{21}\{q_1 - ((I-\Omega)x_1)' + (B_{12}(B_{21}B_{12})^{-1}q_2)' - B_{11}(I-\Omega)x_1 - B_{11}B_{12}(B_{21}B_{12})^{-1}q_2\}.$$

Inserting the expression for x_1 into the relation (9) we arrive at the explicit ODE with respect to the only component $u = (I-\Omega)x_1$,

$$\begin{aligned} &u' - (I-\Omega)'u + (I-\Omega)B_{11}(I-\Omega)u \\ &\quad = (I-\Omega)q_1 + (I-\Omega)'B_{12}(B_{21}B_{12})^{-1}q_2 - (I-\Omega)B_{11}B_{12}(B_{21}B_{12})^{-1}q_2, \end{aligned} \tag{10}$$

which is called *inherent explicit regular ODE* (IERODE) and which resides in $\mathbb{K}^{m_1}$. The varying d-dimensional subspace $\operatorname{im}(I-\Omega) \subset \mathbb{K}^{m_1}$ is an invariant subspace of the IERODE, that is, if u_* is a solution of (10) and there is a $\bar{t} \in \mathcal{I}$ with $u(\bar{t}) \in \operatorname{im}(I-\Omega(\bar{t}))$, then one has $u(t) \in \operatorname{im}(I-\Omega(t))$ for all $t \in \mathcal{I}$. This is the basic property of the IERODE in matters of the DAE structure. The IERODE is uniquely defined by the DAE coefficients. It has size $m_1 \geq d$, but solely its flow within the d-dimensional subspace $\operatorname{im}(I-\Omega)$ is relevant for the DAE flow.

The socalled *canonical projector function* of the DAE (3),(4) reads (cf.,[11, p. 107])

$$\Pi_{can} = \begin{bmatrix} I-\Omega & 0 \\ -B_{12}^-(B_{11}-\Omega')(I-\Omega) & 0 \end{bmatrix}$$

and it has constant rank d together with the projector function $(I-\Omega)$. One can check that all solutions of the homogenous DAE with $q=0$ have the form

$$x = \begin{bmatrix} (I-\Omega)x_1 \\ -B_{12}^-(B_{11}-\Omega')(I-\Omega)x_1 \end{bmatrix} = \Pi_{can}x,$$

which means that $\operatorname{im}\Pi_{can}(t)$ represents the d-dimensional time-varying subspace of consistent values of the homogenous DAE.

In the light of our notation the relations

$$\eta = \Gamma_d x_1 = \Gamma_d\Gamma_d^-\Gamma_d x_1 = \Gamma_d(I-\Omega)x_1 = \Gamma_d u,$$
$$u = (I-\Omega)x_1 = \Gamma_d^-\eta,$$

become obvious. Multiplying the EUODE by Γ_d^- results in the IERODE and, conversely, multiplying the IERODE by Γ_d yields the EUODE. In this sense, each EUODE is nothing else an condensed IERODE. This fact was first pointed out in [3] for index-2 DAEs in Hessenberg form and recently verified for general regular DAEs in [10].

The adjoint DAE to the index-2 Hessenberg form DAE (3),(4),

$$\begin{aligned} -y_1' + B_{11}^*y_1 + B_{21}^*y_2 &= p_1, \\ B_{12}^*y_1 &= p_2, \end{aligned}$$

or written in the form (1),

$$-\underbrace{\begin{bmatrix} I \\ 0 \end{bmatrix}}_{=D^*}(\underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{=A^*} y)' + \begin{bmatrix} B_{11}^* & B_{21}^* \\ B_{12}^* & 0 \end{bmatrix} y = p, \tag{11}$$

has also index 2 and the dynamical degree of freedom d. Its canonical projector function reads

$$\Pi_{*\,can} = \begin{bmatrix} I - \Omega^* & 0 \\ B_{21}^{*\,-}(-B_{11}^* - \Omega^{*\prime})(I - \Omega^*) & 0 \end{bmatrix}.$$

It holds that

$$D\Pi_{can}D^- = I - \Omega, \quad A^*\Pi_{*\,can}A^{*\,-} = I - \Omega^* = (D\Pi_{can}D^-)^*,$$

however, the canonical projector function $\Pi_{*\,can}$ differs from Π_{can}^*, except for quite artificial special cases.

Letting $q = 0$ and $p = 0$, the associated homogenous IERODEs of the DAE and its adjoint become

$$u' - (I - \Omega)'u + (I - \Omega)B_{11}(I - \Omega)u = 0. \tag{12}$$

and

$$-v' + (I - \Omega^*)'v + (I - \Omega^*)B_{11}^*(I - \Omega^*)v = 0. \tag{13}$$

As first pointed out in [2, 3], the IERODEs (12) and (13) form a classical adjoint pair, only if the projector function Ω is constant, or, equivalently, if the associated subspaces $\operatorname{im} B_{12}$ and $\ker B_{21}$ do not vary with t. This seems to be a drawback of the IERODE, but wrongly. As observes above, only solution components $u = (I - \Omega)u$ and $v = (I - \Omega^*)v$ are relevant for the DAEs, and nevertheless for these components the classical Lagrange identity takes place:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\langle u, v\rangle &= \langle u', v\rangle + \langle u, v'\rangle = 2\langle (I - \Omega)'u, v\rangle = 2\langle (I - \Omega)'(I - \Omega)u, (I - \Omega^*)v\rangle \\ &= 2\langle (I - \Omega)(I - \Omega)'(I - \Omega)u, v\rangle = 0. \end{aligned}$$

Below we will show a corresponding general property for each arbitrary pair of adjoint regular DAEs and their IERODEs.

On the other hand, choosing a condensing matrix function $\Gamma_d \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^{m_1}, \mathbb{K}^d))$, such that $\operatorname{im} \Gamma_d^* = \ker B_{12}^*$, and determining the generalized inverse Γ_d^- so that

$$\Gamma_d^-\Gamma_d = I - \Omega, \ \Gamma_d\Gamma_d^- = I_d,$$

we have at the same time $\operatorname{im} \Gamma_d^- = \ker B_{21}$, i.e., Γ_d^- serves as a basis of $\ker B_{21}^{**} = \ker B_{21}$. Then we may simultaneously condense both IERODEs to EUODEs

$$\eta' + (\Gamma_d B_{11}\Gamma_d^- + \Gamma_d\Gamma_d^{-\prime})\eta = 0$$

and

$$-\zeta' + (\Gamma_d^{-\,*}B_{11}^*\Gamma_d^* + \Gamma_d^{-\,*\prime}\Gamma_d^*)\zeta = 0,$$

which are actually adjoint each to other, without any additional restrictions. Hereby, the IERODE of the adjoint is condensed by means of the special basis Γ_d^- of $\ker B_{21}$, and

$$v = (I - \Omega^*)v = \Gamma_d^*\Gamma_d^{-\,*}v = \Gamma_d^*\Gamma_d\zeta, \quad \zeta = \Gamma_d^{-\,*}v.$$

This result is consistent with the claim (cf. [7, Section 4.3]) that *the EUODE of the adjoint system is the same as the adjoint of the EUODE of the original system.* However, emphasize once again that the EUODEs depend on the specially chosen Γ_d and $\Gamma_{*\,d}$. In particular, one obtains an EUODE of the adjoint DAE by means of any basis $\Gamma_{*\,d}^*$ of $\ker B_{21}$. As pointed out in [3], the above assertion is valid only, if the basis applied for the EUODE of the adjoint DAE is consistent with the basis chosen for the original DAE, that means

$$\Gamma_{*\,d} = \Gamma_d^{-\,*}.$$

3 The structure of regular DAEs with properly involved derivative

In this section we deal with DAEs of the form (1), that is,

$$A(Dx)' + Bx = q. \tag{14}$$

Let the time-varying subspaces

$$\ker A(t) \subseteq \mathbb{K}^n \quad \text{and} \quad \operatorname{im} D(t) \subseteq \mathbb{K}^n, \quad t \in \mathcal{I},$$

be $\mathcal{C}^1$- subspaces. and let the transversality condition

$$\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{K}^n, \quad t \in \mathcal{I}, \tag{15}$$

be valid, which means that the DAE shows a *properly stated leading term.* The decomposition (15) determines the so-called *border projector function* $R \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^n))$ by

$$\ker R(t) = \ker A(t), \quad \operatorname{im} R(t) = \operatorname{im} D(t), \quad t \in \mathcal{I}. \tag{16}$$

Since both involved subspaces are $\mathcal{C}^1$-subspaces, the projector function R is actually continuously differentiable.

We use the projector based analysis to uncover the basic structure of a regular DAE with arbitrarely high tractability index and refer to [11] for details and for general relations with other index notions as well.

Let the DAE (14) be regular in the sense of [11, Definition 2.25]. Recall that regularity is supported by several constant-rank requirements yielding the tractability index $\mu \in \mathbb{N}$ and the *characteristic values*

$$0 \leq r_0 \leq \ldots \leq r_{\mu-1} < r_\mu = m, \quad d = m - \sum_{i=0}^{\mu-1}(m - r_i), \tag{17}$$

of the DAE. Regularity is formally determined by means of *admissible projector functions*

$$P_0, \ldots, P_{\mu-1} \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m))$$

associated with the construction of *admissible matrix functions sequences* starting from $G_0 := AD$ and ending up with a nonsingular G_μ, see [11, Definition 2.6].
The tractability index is consistent with the Kronecker index of regular matrix pencils. For a regular matrix pencil, the characteristic values r_i provide a complete description of the formal structure of the corresponding Weierstraß-Kronecker form. In particular, d is the dimension of the dynamical part ([11, Section 1]).

We use the further matrix functions

$$Q_0 := I - P_0,\ \Pi_0 := P_0,\ Q_i := I - P_i,\ \Pi_i := \Pi_{i-1}P_i \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m)),\ i = 1, \ldots, \mu - 1,$$
$$D\Pi_0 D^-, \ldots, D\Pi_{\mu-1}D^- \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^n)),$$

with the pointwise determined generalized inverse D^- such that

$$DD^-D = D,\ D^-DD^- = D^-,\ DD^- = R,\ D^-D = P_0. \tag{18}$$

The products Π_i and $D\Pi iD^-$ are projector-valued functions again (see [11, Proposition 2.7]).

The sequence of admissible projector functions serves as tool for the decoupling of the DAE itself and the decomposition of the solution x into their characteristic parts, see [11, Section 2.4]. In particular, the component $u = D\Pi_{\mu-1}x$ satisfies the *inherent explicit regular ODE* (IERODE)

$$u' - (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1}G_\mu^{-1}B\Pi_{\mu-1}D^-u = D\Pi_{\mu-1}G_\mu^{-1}q. \tag{19}$$

The components $v_0 = Q_0 x$, $v_1 = \Pi_0 Q_1 x, \ldots, v_{\mu-1} = \Pi_{\mu-2} Q_{\mu-1} x$ satisfy the triangular subsystem involving several differentiations for $\mu > 1$,

$$\begin{bmatrix} 0 & \mathcal{N}_{01} & \cdots & \mathcal{N}_{0,\mu-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & \mathcal{N}_{\mu-2,\mu-1} \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ (Dv_1)' \\ \vdots \\ (Dv_{\mu-1})' \end{bmatrix} \tag{20}$$
$$+ \begin{bmatrix} I & \mathcal{M}_{01} & \cdots & \mathcal{M}_{0,\mu-1} \\ & I & \ddots & \vdots \\ & & \ddots & \mathcal{M}_{\mu-2,\mu-1} \\ & & & I \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{\mu-1} \end{bmatrix} + \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_{\mu-1} \end{bmatrix} D^- u = \begin{bmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \\ \vdots \\ \mathcal{L}_{\mu-1} \end{bmatrix} q.$$

The subspace im $D\Pi_{\mu-1}$ is an invariant subspace for the IERODE. The components $v_0, v_1, \ldots, v_{\mu-1}$ remain within their subspaces im Q_0, im $\Pi_{\mu-2}Q_1, \ldots,$ im $\Pi_0 Q_{\mu-1}$, respectively. The structural decoupling is associated with the decomposition

$$x = D^- u + v_0 + v_1 + \cdots + v_{\mu-1}.$$

All coefficients are continuous and explicitly given in terms of an admissible matrix function sequence as

$$\begin{aligned}
\mathcal{N}_{01} &:= -Q_0 Q_1 D^- \\
\mathcal{N}_{0j} &:= -Q_0 P_1 \cdots P_{j-1} Q_j D^-, && j = 2, \ldots, \mu-1, \\
\mathcal{N}_{i,i+1} &:= -\Pi_{i-1} Q_i Q_{i+1} D^-, \\
\mathcal{N}_{ij} &:= -\Pi_{i-1} Q_i P_{i+1} \cdots P_{j-1} Q_j D^-, && j = i+2, \ldots, \mu-1,\ i = 1, \ldots, \mu-2, \\
\mathcal{M}_{0j} &:= Q_0 P_1 \cdots P_{\mu-1} \mathcal{M}_j D\Pi_{j-1} Q_j, && j = 1, \ldots, \mu-1, \\
\mathcal{M}_{ij} &:= \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} \mathcal{M}_j D\Pi_{j-1} Q_j, && j = i+1, \ldots, \mu-1,\ i = 1, \ldots, \mu-2, \\
\mathcal{L}_0 &:= Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1}, \\
\mathcal{L}_i &:= \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} G_\mu^{-1}, && i = 1, \ldots, \mu-2, \\
\mathcal{L}_{\mu-1} &:= \Pi_{\mu-2} Q_{\mu-1} G_\mu^{-1}, \\
\mathcal{H}_0 &:= Q_0 P_1 \cdots P_{\mu-1} \mathcal{K} \Pi_{\mu-1}, \\
\mathcal{H}_i &:= \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} \mathcal{K} \Pi_{\mu-1}, && i = 1, \ldots, \mu-2, \\
\mathcal{H}_{\mu-1} &:= \Pi_{\mu-2} Q_{\mu-1} \mathcal{K} \Pi_{\mu-1},
\end{aligned}$$

in which

$$\begin{aligned}
\mathcal{K} &:= (I - \Pi_{\mu-1}) G_\mu^{-1} B_{\mu-1} \Pi_{\mu-1} + \sum_{l=1}^{\mu-1} (I - \Pi_{l-1})(P_l - Q_l)(D\Pi_l D^-)' D\Pi_{\mu-1}, \\
\mathcal{M}_j &:= \sum_{k=0}^{j-1} (I - \Pi_k)\{P_k D^- (D\Pi_k D^-)' - Q_{k+1} D^- (D\Pi_{k+1} D^-)'\} D\Pi_{j-1} Q_l D^-, \\
& \quad l = 1, \ldots, \mu-1.
\end{aligned}$$

The IERODE is always uncoupled of the second subsystem, but the latter is tied to the IERODE if among the coefficients $\mathcal{H}_0, \ldots, \mathcal{H}_{\mu-1}$ is at least one who does not vanish. One speaks about a *fine decoupling*, if $\mathcal{H}_1 = \cdots = \mathcal{H}_{\mu-1} = 0$, and about a *complete decoupling*, if $\mathcal{H}_0 = 0$, additionally. A complete decoupling is given, exactly if the coefficient $\mathcal{K}$ vanishes identically. For a regular DAE with sufficiently smooth original data, fine and complete decouplings always exist and can be constructed, see [11, Subsection 2.4.3].

It should be added at this point, that the coefficients of the IERODE depend formally on the special choice of admissible projector functions. However, they are uniquely determined in the scope of fine decouplings.

The so-called *canonical projector function* Π_{can} of a regular DAE (see [11, Definition 2.37]) is actually a generalization of the spectral projector of a regular matrix pencil onto the finite eigenspace along the infinite eigenspace (cf. [11, Section 1.4]). The subspace $\operatorname{im}\Pi_{can}(t) \subseteq \mathbb{K}^m$ consists precisely of all consistent values of the homogeneous DAE at time t.

By means of fine decoupling projector functions $P_0, \ldots, P_{\mu-1}$, the *canonical projector function of the DAE* can be represented as

$$\Pi_{can} = (I - \mathcal{H}_0)\Pi_{\mu-1}.$$

It follows that

$$D\Pi_{\mu-1} = D\Pi_{can}, \quad D\Pi_{\mu-1}D^- = D\Pi_{can}D^-. \tag{21}$$

We emphasize that Π_{can} itself is independent of the choice of the projector functions. Therefore, also $D\Pi_{\mu-1} = D\Pi_{can}$ does not depend of the construction.

Note that fine decoupling projector functions $P_0, \ldots, P_{\mu-1}$ inclose an arbitrarily fixed projector function P_0 along $\ker D$ and the accordingly prescribing generalized inverse D^-. Then it may happen that $\Pi_{can} \neq \Pi_{\mu-1}$. In contrast, complete decoupling projector functions $P_0, \ldots, P_{\mu-1}$ yield the representation $\Pi_{can} = \Pi_{\mu-1}$, which is very useful in theory, but less comfortable in practice since then P_0 and D^- are very special.

The IERODE (19) resulting from a fine decoupling is independent of the special choice of the fine decoupling. We emphasize this fact by rewriting the IERODE as

$$u' - (D\Pi_{can}D^-)'u + D\Pi_{can}G_\mu^{-1}B\Pi_{can}D^-u = D\Pi_{can}G_\mu^{-1}q. \tag{22}$$

Obviously, in the IERODE (22), only inhomogenuities with values in

$$\operatorname{im} D\Pi_{can} = \operatorname{im} D\Pi_{can}G_\mu^{-1}$$

are responsible for the DAE (14).

Theorem 3.1 *Let the DAE (14) be regular with index μ and characteristic values (17). Let its coefficients be sufficiently smooth. Then, the following is valid:*

(1) *The coefficients of the IERODE (22) are uniquely determined by fine decoupling projectors and its inhomogeneity does not inherit any derivative of q.*

(2) *The component $u = D\Pi_{can}x$ of each DAE solution x satisfies the IERODE.*

(3) *d is the dynamical degree of freedom of the DAE and there is an inherent d-dimensional flow located in $\operatorname{im} D\Pi_{can}$, which is not at all affected by derivatives of the inhomogeneity q.*

(4) *Except for the component $u = D\Pi_{can}x$ there is no other solution component satisfying an explicit ODE the inhomogeneity of which does not involve any derivative of q.*

Proof: These assertions (1)-(3) can immediately be concluded from [11, Theorem 2.39] regarding that $D\Pi_{\mu-1} = D\Pi_{can}$, $d = \operatorname{rank}\Pi_{can} = \operatorname{rank} D\Pi_{can}D^-$.

To verify (4) we turn to the unfolded structure associated with a complete decoupling. The system (20) has exactly one solution $v_0, \ldots, v_{\mu-1}$. It can be solved successively for $v_{\mu-1}, \ldots, v_0$ in terms of q and possibly its derivatives. If $\mu > 1$, part of the solution components actually depend on derivatives of q. There are also solution components depending only on q such as $v_{\mu-1} = \mathcal{L}_{\mu-1}q$ and $T_{\mu-2}v_{\mu-2} = T_{\mu-2}\{\mathcal{L}_{\mu-2}q - \mathcal{M}_{\mu-2\,\mu-1}\mathcal{L}_{\mu-1}q\}$, in which $T_{\mu-2}$ denotes a projection along $\operatorname{im}\mathcal{N}_{\mu-2\,\mu-1}$. This makes clear, that, in any case, each derivative v_i' necessarily depends at least on one component of a derivative of q. This extends also to any linear combination of the v_i' and u, since these are living in separated subspaces. □

Example 3.2 *The constant coefficient DAE,*

$$\begin{aligned} x_1' - \alpha x_1 - x_2 &= q_1, \\ x_3' + x_2 &= q_2, \\ x_4' + x_3 &= q_3, \\ x_5' + x_4 &= q_4, \\ x_5 &= q_5, \end{aligned}$$

written in the form (14) reads

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x \right)' + \begin{bmatrix} -\alpha & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = q,$$

with $m = 5$, $n = 4$. The following associated matrix sequence (cf. [11, Subsection 1.2.3]) is admissible and provides $\mu = 4$ and the characteristic values $r_0 = r_1 = r_2 = r_3 = 4$, $r_4 = 5$, $d = 1$:

$$G_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Q_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} -\alpha & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Q_1 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \Pi_0 Q_1 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 1 & -1 & \alpha & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 & 0 & 0 & 1+\alpha & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \Pi_1 Q_2 = \begin{bmatrix} 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_3 = \begin{bmatrix} 1 & -1 & \alpha & -\alpha^2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Q_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1-\alpha-\alpha^2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \Pi_2 Q_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & -\alpha^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G_4 = \begin{bmatrix} 1 & -1 & \alpha & -\alpha^2 & \alpha^3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \Pi_3 = \begin{bmatrix} 1 & 0 & 1 & -\alpha & \alpha^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, D\Pi_3 D^- = \begin{bmatrix} 1 & 1 & -\alpha & \alpha^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Additionally, it follows that

$$\begin{aligned} Q_3 G_4^{-1} B_0 \Pi_3 &= 0, & Q_2 P_3 G_4^{-1} B_0 \Pi_3 &= 0, \\ Q_1 P_2 P_3 G_4^{-1} B_0 \Pi_3 &= 0, & Q_0 P_1 P_2 P_3 G_4^{-1} B_0 \Pi_3 &= 0, \end{aligned}$$

and

$$\Pi_3 G_4^{-1} B_0 \Pi_3 = -\alpha \Pi_3. \tag{23}$$

Therefore, these projector functions P_0, P_1, P_2, P_3 provide a complete decoupling of the given DAE and Π_3 coincides with the canonical projector Π_{can}. The projector functions Q_0, $\Pi_0 Q_1$, $\Pi_1 Q_2$ and $\Pi_2 Q_3$ represent the variables x_2, x_3, x_4 and x_5, respectively. The projector $D\Pi_3 D^-$ and the coefficient (23) determine the IERODE with respect to the component $u = D\Pi_3 x$, namely

$$u' - \alpha u = D\Pi_3 G_4^{-1} q.$$

Dropping the three zero rows it results that

$$\begin{aligned}(x_1 + x_3 - \alpha x_4 + \alpha^2 x_5)' - \alpha(x_1 + x_3 - \alpha x_4 + \alpha^2 x_5)\\ = q_1 + q_2 - \alpha q_3 + \alpha^2 q_4 - \alpha^3 q_5.\end{aligned} \tag{24}$$

Obviously, no derivative of the inhomogeneity q encroaches into the IERODE.

In contrast, each associated explicit ODE with respect to the original components x_i depend on derivatives of q. In particular, the component x_1 satisfies the ODE

$$x_1' - \alpha x_1 = q_1 + q_2 - (q_3 - (q_4 - q_5')')'. \tag{25}$$

□

The property of the IERODE to be free of all derivatives of q allows to state boundary conditions for the DAE accurately, see [12], which seizes the suggestion of [1] in essence. However, recall that though the respective flow resides in the d-dimensional subspace $\operatorname{im} D\Pi_{can} \subseteq \mathbb{K}^n$ the IERODE itself lives in $\mathbb{K}^n$. Next we will see that the IERODE can be condensed to minimal size d, and the resulting EUODE is also free of derivatives of q.

Each regular DAE can be transformed into a form with decoupled fast and slow parts, similar to the Weierstraß-Kronecker form of a regular matrix pencil. We quote the respective result from [10, 11]:

Theorem 3.3 *Each regular DAE (14) with index μ and characteristic values (17) can be transformed by pointwise nonsingular matrix functions*

$$L \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m)), \quad K \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m)),$$

and a refactorization of the leading term by H with

$$\begin{aligned}&H \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^s, \mathbb{K}^n)), \quad H^- \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^s)), \quad n, s \geq r := \operatorname{rank} D,\\ &HH^-H = H, \quad H^-HH^- = H^-, \quad RHH^-R = R,\end{aligned}$$

into the structured form

$$\tilde{A}(\tilde{D}\tilde{x})' + \tilde{B}\tilde{x} = Lq, \tag{26}$$

with

$$\tilde{A} = LAH = \begin{bmatrix} I_d & 0 \\ 0 & N \end{bmatrix}, \quad \tilde{D} = H^- DK = \begin{bmatrix} I_d & 0 \\ 0 & P_N \end{bmatrix},$$

$$\tilde{B} = LBK - LAH(H^-R)'DK = \begin{bmatrix} W & 0 \\ 0 & I_{m-d} \end{bmatrix},$$

$$N = \begin{bmatrix} 0 & N_{0\,1} & \dots & N_{0\,\mu-1} \\ & \ddots & \ddots & \\ & & 0 & N_{\mu-2\,\mu-1} \\ & & & 0 \end{bmatrix}, P_N = \begin{bmatrix} 0_{m-r_0} & & & \\ & I_{m-r_1} & & \\ & & \ddots & \\ & & & I_{m-r_{\mu-1}} \end{bmatrix},$$

in which the entries $N_{i-1\,i}$ have size $(m - r_{i-1}) \times (m - r_i)$ and full rank $m - r_i$, $i = 1, \dots, \mu - 1$.

We recall that the transformation matrices stated in Theorem 3.3 have the following explicit form (cf. [11, p. 146])

$$L = \begin{bmatrix} I_d & 0 \\ 0 & (I+\tilde{\mathcal{M}})^{-1} \end{bmatrix} \begin{bmatrix} \Gamma_d D\Pi_{can} \\ \Gamma_0 Q_0 \\ \vdots \\ \Gamma_{\mu-1} D\Pi_{\mu-2} Q_{\mu-1} \end{bmatrix} G_\mu^{-1}, \quad K = \left(\begin{bmatrix} \Gamma_d D\Pi_{can} \\ \Gamma_0 Q_0 \\ \vdots \\ \Gamma_{\mu-1} D\Pi_{\mu-2} Q_{\mu-1} \end{bmatrix} \right)^{-1},$$

at which completely decoupling projector functions are used and the functions Γ_i, $i = 0, 1, \ldots, \mu-1$, are defined as in [11, p. 143]. Though the matrix function G_μ and its inverse may depend on the special choice of the completely decoupling projector functions, the expressions $\Pi_{can} G_\mu^{-1}$ and $G_\mu \Pi_{can} = AD\Pi_{can}$ are invariant.

The transformed DAE (26) comprises the explicit ODE

$$\eta' + W\eta = \Gamma_d D\Pi_{can} G_\mu^{-1} q, \tag{27}$$

with size d. Its inhomogeneity does not comprise derivatives of q. The ODE (27) is said to be an *essential underlying ODE* (EUODE) of the DAE (14) in [10] after the respective notion introduced for index-2 Hessenberg form DAEs in [1, 7, 3].

Recall that the IERODE lives in $\mathbb{K}^n$, $n \geq d$. The IERODE is unique in the scope of fine decouplings. Its coefficients are expressed in terms of the original DAE. In contrast, the EUODE has minimal size d, but it is accessible by suitable transformations only. These transformations are not uniquely determined, so that several EUODEs accrue, cf. also Theorem 3.4 below.

As described in Section 2 for index-2 DAEs in Hessenberg form (3), (4), and accordingly (5), each EUODE can be seen as condensed IERODE. Recall that in Section 2 the relations

$$\operatorname{im} \varGamma_d^* = \ker B_{12}^* = (\operatorname{im} B_{12})^\perp = (\ker (I - \varOmega))^\perp = \operatorname{im} (I - \varOmega)^* = \operatorname{im} (D\Pi_{can} D^-)^*$$

have been used to construct the condensing transform $\varGamma_d$. In the general case we can proceed similarly as described in [11, Section 2.8] and [10].

We take a closed look at the general condensing. Since the projector function $D\Pi_{can}D^-$ is continuously differentiable and has rank d, so is $(D\Pi_{can}D^-)^*$, and $\operatorname{im} (D\Pi_{can}D^-)^*$ is spanned by d continuously differentiable basis functions. This means that there is a matrix function such that

$$\varGamma_d \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^d)), \quad \operatorname{im} \varGamma_d^* = \operatorname{im} (D\Pi_{can}D^-)^*, \quad \ker \varGamma_d^* = \{0\}. \tag{28}$$

Then we determine the pointwise generalized inverse $\varGamma_d^- \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^d, \mathbb{K}^n))$ by

$$\varGamma_d \varGamma_d^- \varGamma_d = \varGamma_d, \quad \varGamma_d^- \varGamma_d \varGamma_d^- = \varGamma_d^-, \quad \varGamma_d^- \varGamma_d = D\Pi_{can}D^-, \quad \varGamma_d \varGamma_d^- = I_d. \tag{29}$$

Letting $\eta = \varGamma_d u$ for the solutions $u = D\Pi_{can}D^- u = \varGamma_d^- \varGamma_d u$ of the IERODE leads to an EUODE (27) in which (cf. [11, Section 2.8])

$$W = -\varGamma_d' \varGamma_d^- + \varGamma_d D\Pi_{can} G_\mu^{-1} B \Pi_{can} D^- \varGamma_d^- = -\varGamma_d' \varGamma_d^- + \varGamma_d D\Pi_{can} G_\mu^{-1} B D^- \varGamma_d^-. \tag{30}$$

The next theorem records and accents the result obtained.

Theorem 3.4 *Each EUODE (27) of a regular DAE (14) with index μ and characteristic values (17) can be represented as a condension of the IERODE (22) by means of a matrix function (28) and the generalized inverse (29). The coefficient W of the associated EUODE and the IERODE are related by (30).*

We emphasize once again that, though the IERODE is unique in the context of fine decouplings, the EUODE actually depends on the choice of the basis functions of $\operatorname{im} (D\Pi_{can}D^-)^*$.

Example 3.5 *We continue to consider the index-4 DAE from Example 3.2. Seemingly, most authors would take the ODE (25) for an EUODE, but this is a mistake.*

Here Π_3 coincides with the canonical projector Π_{can} and we have

$$\Pi_{can} = \begin{bmatrix} 1 & 0 & 1 & -\alpha & \alpha^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, D\Pi_{can}D^- = \begin{bmatrix} 1 & 1 & -\alpha & \alpha^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathrm{im}\,(D\Pi_{can}D^-)^* = \mathrm{im} \begin{bmatrix} 1 \\ 1 \\ -\alpha \\ \alpha^2 \end{bmatrix}.$$

Choosing $\Gamma_d = [1\ 1\ \ -\alpha\ \alpha^2]$ yields $\eta = \Gamma_d u = x_1 + x_3 - \alpha x_4 + \alpha^2 x_5$. The associated EUODE $\eta' - \alpha\eta = \Gamma_d D\Pi_3 G_4^{-1} q$ reads in detail

$$\begin{aligned}(x_1 + x_3 - \alpha x_4 + \alpha^2 x_5)' - \alpha(x_1 + x_3 - \alpha x_4 + \alpha^2 x_5)\\ = q_1 + q_2 - \alpha q_3 + \alpha^2 q_4 - \alpha^3 q_5,\end{aligned}$$

which coincides with (24). This EUODE is actually a condensed IERODE. □

One can see the diversity of EUODEs as a drawback, however, as in [10, Section 6] concerning stability issues, one can even take advantage from the arbitrariness of the basis Γ_d^*.

Theorem 3.6 *Let the DAE (14) be regular with index μ and characteristic values (17). Put $q = 0$. Let the condensing transformation Γ_d and Γ_d^- be determined by (28),(29). Then the following is valid:*

(1) *If the matrix functions $D\Pi_{can}$ and $\Pi_{can}D^-$ remain bounded, the Lyapunov spectrum of the DAE coincides with that of the IERODE with respect to $\mathrm{im}\, D\Pi_{can}$.*

(2) *If the matrix functions $\Gamma_d D\Pi_{can}$ and $\Pi_{can}D^-\Gamma_d^-$ remain bounded, the Lyapunov spectra of the DAE and the EUODE coincide.*

(3) *Γ_d can always be chosen so that the EUODE preserves the Lyapunov spectrum of the DAE.*

Proof: The assertions (1) and (2) can immediately be concluded from the relations given for $q = 0$:

$$\begin{aligned} x &= \Pi_{can}D^- u, \quad u = D\Pi_{can}x,\\ x &= \Pi_{can}D^-\Gamma_d^-\eta, \quad \eta = \Gamma_d D\Pi_{can}x. \end{aligned}$$

(3) As shown in [10], the relations (28),(29) are fulfilled for $\Gamma_d = U^*\Pi_{can}D^-$, $\Gamma_d^- = DU$, in which U is an orthonormal basis of $\mathrm{im}\,\Pi_{can}$, and one has $x = U\eta$. □

Note that Theorem 3.6 (3) demands an unlimited further flexibility for the choice of the basis Γ_d^*. If the choice is restricted to orthonormal bases as originally proposed in [1], assertion (3) is no longer true. In general, $\Gamma_d^* = D^{-\,*}\Pi_{can}^* U$ should be expected to fail to be orthonormal.

4 Adjoint pairs

Pairs of explicit ODEs and their adjoints,

$$x' + Bx = 0, \quad -y' + B^*y = 0,$$

feature properties being useful in theory and practical treatment. For instance, the Lagrange identity means that the product $\langle x(t), y(t)\rangle$ of each solution pair remains constant, and the Perron Theorem states that the original ODE is Lyapunov regular with Lyapunov exponents $\lambda_1, \dots, \lambda_m$ exactly if the adjoint is Lyapunov regular with Lyapunov exponents $-\lambda_1, \dots, -\lambda_m$. These properties are usuaally utilized when formulating and solving boundary-value problems, in sensibility analysis, optimization and control.

Aiming at corresponding results we consider the DAE with properly stated leading term,

$$A(Dx)' + Bx = q, \tag{31}$$

together with its adjoint equation (cf. [5]),

$$-D^*(A^*y)' + B^*y = p. \tag{32}$$

The DAE (32) has a properly stated leading term at the same time as (31), with the associated border projector function R^*.

For any solution pair $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{K}^m)$ and $y \in \mathcal{C}^1_{A^*}(\mathcal{I}, \mathbb{K}^m)$ of the DAEs (31) and (32), respectively, and a function $w \in \mathcal{C}^1(\mathcal{I}, \mathbb{K})$ satisfying the condition

$$w'(t) = -\langle q(t), y(t)\rangle + \langle x(t), p(t)\rangle, \quad t \in \mathcal{I},$$

one obtains

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}(\langle D(t)x(t),\, A(t)^*y(t)\rangle + w(t)) &= \langle (Dx)'(t),\, A(t)^*y(t)\rangle + \langle D(t)x(t),\, (A^*y)'(t)\rangle + w'(t)\\ &= \langle A(t)(Dx)'(t),\, y(t)\rangle + \langle x(t),\, D(t)^*(A^*y)'(t)\rangle + w'(t)\\ &= \langle -B(t)x(t) + q(t),\, y(t)\rangle + \langle x(t),\, B(t)^*y(t) - p(t)\rangle + w'(t) = 0, \quad t \in \mathcal{I},\end{aligned}$$

such that

$$\langle D(t)x(t),\, A(t)^*y(t)\rangle + w(t) = \text{constant}, \quad t \in \mathcal{I}.$$

This implies the *Lagrange identity* for the homogeneous DAEs (31) with $q = 0$ and (32) with $p = 0$:

$$\langle D(t)x(t),\, A(t)^*y(t)\rangle = \text{constant}, \quad t \in \mathcal{I}. \tag{33}$$

In contrast to the explicit ODEs, for DAEs and even for implicit ODEs, one can no longer expect a Lagrange identity with respect to $\langle x(t), y(t)\rangle$ as the next example indicates.

Example 4.1 *The scalar homogenous implicit ODE from [7, Section 4]*

$$\mathrm{e}^t x'(t) + \frac{1}{2}\mathrm{e}^t x(t) = 0$$

can be written in the form (1) with $m = n = 1$, $A(t) = \mathrm{e}^{\mathrm{t}}$, $D(t) = 1$, and $B(t) = \frac{1}{2}\mathrm{e}^t$. The adjoint ODE is

$$-(\mathrm{e}^t y(t))' + \frac{1}{2}\mathrm{e}^t y(t) = 0.$$

The solutions are simply

$$x(t) = \mathrm{e}^{-\frac{1}{2}t}c_x, \quad \text{and} \;\; y(t) = \mathrm{e}^{-\frac{1}{2}t}c_y,$$

with constants c_x, c_y. The product $x(t)y(t) = \mathrm{e}^{-t}c_x c_y$ fails to be time-invariant, but

$$\langle D(t)x(t),\, A(t)^*y(t)\rangle = \mathrm{e}^{-\frac{1}{2}t}c_x\mathrm{e}^t\mathrm{e}^{-\frac{1}{2}t}c_y = c_x c_y$$

is so. □

Recall that the IERODE and the EUODEs of the DAE (31) have the form

$$u' - (D\Pi_{can}D^-)'u + D\Pi_{can}G_\mu^{-1}B\Pi_{can}D^-u = D\Pi_{can}G_\mu^{-1}\,q, \tag{34}$$

and

$$\eta' + W\eta = \Gamma_d D\Pi_{can}G_\mu^{-1}q. \tag{35}$$

Applying the decoupling procedure to the adjoint DAE (32) accordingly we derive its IERODE as

$$v' - (A^*\Pi_{*\,can}A^{*\,-})'v + A^*\Pi_{*\,can}G_{*\,\mu}^{-1}B^*\Pi_{*\,can}A^{*\,-}v = A^*\Pi_{*\,can}G_{*\,\mu}^{-1}\,p, \tag{36}$$

in which the asterisk-index indicates the matrix functions associated with the coefficients of the DAE (32), i.e., $A_* := -D^*$, $D_* := A^*$, and $B_* := B^*$.

Furthermore, the EUODEs of the adjoint DAE (32) have the form

$$\zeta' + W_*\zeta = \Gamma_{*\,d}A^*\Pi_{*\,can}G_{*\,\mu}^{-1}\,p. \tag{37}$$

The next theorem generalizes and completes results from [2, 3] concerning index-1 and index-2 DAEs. It elucidates the respective observations recorded for index-2 DAEs in Hessenberg form in Section 2.

Theorem 4.2 *The following assertions are valid for each regular DAE (31) with index μ and characteristic values (17):*

(1) *The adjoint DAE (32) is regular with the same index and characteristic values.*

(2) $A^*\Pi_{*\,can}A^{*\,-} = (D\Pi_{can}D^-)^*$.

(3) *The IERODE (36) of the adjoint DAE can be described in terms of the original DAE as*

$$-v' + (D\Pi_{can}D^-)^{*\,\prime}v + (D\Pi_{can}G_\mu^{-1}B\Pi_{can}D^-)^*v = D^{-\,*}\Pi_{can}^*\,p, \tag{38}$$

(4) *The adjoint of the IERODE of the DAE (31) with $q = 0$ and the IERODE of the adjoint DAE (32) with $p = 0$ coincide precisely if $D\Pi_{can}D^-$ is time-invariant.*

(5) *The EUODEs of the DAE (31) and the EUODEs of the adjoint DAE (32) built with consistent bases $\Gamma_{*\,d} = \Gamma_d^{-\,*}$ are adjoint each to other. It holds that $W_* = -W^*$ and the EUODE (37) can be rewritten as*

$$-\zeta' + W^*\zeta = \Gamma_d^{-\,*}D^{-\,*}\Pi_{can}^*\,p. \tag{39}$$

Proof: Assertion (1) is a special case of [10, Theorem 3(1)]. Assertion (2) is given by [10, Lemma 3]. Assertion (3) immediately implies (4). (5) can be derived from (3) by straightforward calculations. So it remains to check (3). We apply completely decoupling projector functions and regard that the IERODE is independent of the special choice of fine decoupling projector functions. By analogous arguments as used for the relation $\Pi_{*\,can} = G_\mu^{-\,*}\Pi_{can}^*D^*A^*$ in the proof of [10, Lemma 3] we find the relation

$$\Pi_{can} = G_{*\,\mu}^{-\,*}\Pi_{*\,can}^*D_*^*A_*^* = -G_{*\,\mu}^{-\,*}\Pi_{*\,can}^*AD,$$

thus $D^{-\,*}\Pi_{can}^* = -A^*\Pi_{*\,can}G_{*\,\mu}^{-1}$, and further

$$\begin{aligned} A^*\Pi_{*\,can}G_{*\,\mu}^{-1}B^*\Pi_{*\,can}A^{*\,-} &= -D^{-\,*}\Pi_{can}^*B^*\Pi_{*\,can}A^{*\,-} \\ &= -D^{-\,*}\Pi_{can}^*B^*G_\mu^{-\,*}\Pi_{can}^*D^*A^*A^{*\,-} \\ &= -D^{-\,*}\Pi_{can}^*B^*G_\mu^{-\,*}\Pi_{can}^*D^* = -(D\Pi_{can}G_\mu^{-1}B\Pi_{can}D^-)^*. \end{aligned}$$

This yields the equation

$$v' - (D\Pi_{can}D^-)^{*\,\prime}v - (D\Pi_{can}G_\mu^{-1}B\Pi_{can}D^-)^*v = -D^{-\,*}\Pi_{can}^*\,p,$$

Finally, multiplying this equation by -1 gives (38). □

One might see in Theorem 4.2(4) a drawback of the IERODE, but this is unsubstantiated. One must regard that only the flows proceeding in the corresponding invariant subspaces im $D\Pi_{can}D^-$ and im $A^*\Pi_{can}A^{*\,-} = \mathrm{im}\,(D\Pi_{can}D^-)^*$ are relevant for the DAEs.

Let u and v be solutions of the IERODEs (34) and (38) belonging to the relevant subspaces such that $u = D\Pi_{can}D^- u$, $v = (D\Pi_{can}D^-)^* v$, and let the function ω satisfy the condition

$$\omega'(t) = \langle u(t), (D^{-\,*}\Pi_{can}^* p)(t)\rangle - \langle (D\Pi_{can}G_\mu^{-1} q)(t), v(t)\rangle, \quad t \in \mathcal{I}.$$

We derive

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}(\langle u(t), v(t)\rangle + \omega(t)) &= \langle u'(t), v(t)\rangle + \langle u(t), v'(t)\rangle + w'(t)\\
&= \langle (D\Pi_{can}D^-)'(t)u(t), v(t)\rangle + \langle u(t), (D\Pi_{can}D^-)^{*\prime}(t)v(t)\rangle\\
&= 2\langle (D\Pi_{can}D^-)'(t)u(t), v(t)\rangle\\
&= 2\langle (D\Pi_{can}D^-)'(t)(D\Pi_{can}D^-)(t)u(t), (D\Pi_{can}D^-)^*(t)v(t)\rangle\\
&= 2\langle \underbrace{(D\Pi_{can}D^-)(t)(D\Pi_{can}D^-)'(t)(D\Pi_{can}D^-)(t)}_{=0} u(t), v(t)\rangle = 0, \quad t \in \mathcal{I},
\end{aligned}$$

such that

$$\langle u(t), v(t)\rangle + \omega(t) = \text{constant}, \quad t \in \mathcal{I}. \tag{40}$$

This leads to the following theorem:

Theorem 4.3 *Let the DAE (31) be regular with index μ and characteristic values (17). Then, for each solution pair u and v of the corresponding IERODES (34) and (38), which proceed within the associated invariant subspaces $\operatorname{im} D\Pi_{can}D^-$ and $\operatorname{im} A^*\Pi_{*\,can}A^{*\,-}$, respectively, the identity (40) is valid. In particular, for the homogenous case with $q = 0$, $p = 0$, the classical Lagrange identity,*

$$\langle u(t), v(t)\rangle = \mathit{constant}, \quad t \in \mathcal{I}. \tag{41}$$

comes true.

In [10], Lyapunov regularity of arbitrary-index DAEs is introduced and analyzed by means of special spectrum preserving EUODEs. In particular, the following statement is verified in [10]:

Theorem 4.4 *Let the DAE (31) be regular with index μ and characteristic values (17). Then, if $AD\Pi_{can}$, $\Pi_{can}(AD)^-$, $\Gamma_d D\Pi_{can}$, and $\Pi_{can}D^-\Gamma_d^-$ remain bounded, the DAE and its adjoint are Lyapunov regular at the same time and the Perron identity is valid.*

It is demonstrated in [10] by examples that, in essence, here one cannot do unless those strong boundedness conditions. Even though the statement Theorem 3.6 (3) sounds promising it is not really helpful in this respect, since, in general, the spectrum preserving bases Γ_d and $\Gamma_{*\,d}$ do not fulfill the consistency condition of Theorem 4.2 (5).

5 Standard form DAEs

Now we turn to the standard form DAE

$$Ex' + Fx = q \tag{42}$$

and its adjoint (cf. [6])

$$-(E^*y)' + F^*y = p. \tag{43}$$

We assume that the time-varying subspace

$$\ker E(t) \subseteq \mathbb{K}^m, \quad t \in \mathcal{I},$$

is a $\mathcal{C}^1$- subspace, which means that the orthoprojector function E^+E is continuously differentiable. Then, E has constant rank. We refer to [11, 14] for elaborated discussions concerning this constant-rank condition and the possibilities of avoiding this condition in the class of quasi-regular DAEs. Here we deal with regular DAEs only. Regularity, the tractability index and the characteristic values of a standard form DAE (42) are given by means of any *proper factorization* $E =: AD$, $A \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^m))$, $D \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^n))$ such that $\ker E(t) = \ker D(t)$, $t \in \mathcal{I}$ and the transversality condition (15) is valid, see [11, Section 2.7]. At this place, it should be noted, that refactorizations do not change the index and the characteristic values, thus regularity.

Since $\ker E$ is a $\mathcal{C}^1$-subspace, there are proper factorizations $E =: AD$. One can simply use $A = E$, $D = P$, with a projector function P along $\ker E$, for instance, $P = E^+E$, as applied already in [8]. Then we rewrite the standard form DAE (42) as DAE with properly stated leading term,

$$A(Dx)' + (F - AD')x = q. \tag{44}$$

The DAE (43) is obviously out of the scope of a standard form DAE. Often, additionally supposing that E and y are continuously differentiable, one turns to the standard form DAE

$$-E^*y' + (F^* - E^{*\prime})y = p. \tag{45}$$

Having said that, applying the proper factorization $E^* = (AD)^* = D^*A^*$, to equation (45) or to equation (43) leads to

$$-D^*(A^*y)' + (F^* - D^{*\prime}A^*)y = p, \tag{46}$$

which is the correct adjoint counterpart of the DAE (44).

Replacing both DAEs (42) and (43) by proper versions (44) and (46), one can apply all results mentioned in Sections 3 and 4. The IERODEs of a standard form DAE (42) are given via proper factorizations of the leading coefficient E. Each factorization generates a special IERODE. Again, the EUODEs arise as condensed IERODEs and, in turn, various bases can be chosen for that purposes. We emphasize that all these IERODEs and EUODEs are not affected by derivatives of the homogeneity q. What concerns adjoint pairs of DAEs (42) and (43), the material from Section 4 naturally applies, if consistent factorizations yielding (44) and (46) are used.

The next statement represents a consequence of [10, Theorem 3(1)] concerning so-called factorization-adjoint pairs of DAEs. It specifies and extends [9, Theorem 3.5].

Theorem 5.1 *For the standard form DAE (42)with sufficiently smooth coefficients the following comes true:*

(1) *If the DAE (42) is regular with index μ and characteristic values (17), then so is its adjoint (43) and vice versa. In particular, an adjoint pair share in the dynamical degree of freedom d.*

(2) *If the DAEs (42) and (43) are regular, then they possess EUODEs being adjoint each to other in the classical sense.*

As already demonstrated by Example 4.1, the classical Lagrange identity does not come true for the solutions DAEs (42) and (43), however, we can take use of the identity (33) via factorizations. Let x and y be solutions of the homogenous DAEs (42) with $q = 0$ and (43) with $p = 0$. Then we have

$$\langle E(t)x(t), y(t)\rangle = \langle A(t)D(t)x(t), y(t)\rangle = \langle D(t)x(t), A(t)^*y(t)\rangle = \text{ constant},$$

which coincides with the generalized Lagrange identity obtained in [4, 15].

5.1 List of symbols and abbreviations

$\mathbb{K}$	set of real numbers $\mathbb{R}$ and set of complex numbers $\mathbb{C}$
$\mathcal{L}(\mathbb{K}^s, \mathbb{K}^n)$	set of $\mathbb{K}$-valued $n \times s$ - matrices and linear operators from $\mathbb{K}^s$ to $\mathbb{K}^n$
$\mathcal{C}(\mathcal{I}, X)$	space of continuous functions mapping $\mathcal{I}$ into X
$\mathcal{C}^1(\mathcal{I}, X)$	space of continuously differentiable functions mapping $\mathcal{I}$ into X
$\mathcal{C}^1_M(\mathcal{I}, X)$	$\{x \in \mathcal{C}(\mathcal{I}, X) : Mx \in \mathcal{C}^1(\mathcal{I}, Y)$, with $M \in \mathcal{L}(X, Y)$
K^*	adjoint matrix
K^-	generalized inverse, $KK^-K = K$, $K^-KK^- = K^-$
K^+	Moore-Penrose inverse
K^{*-}	$[K^*]^-$
K^{-*}	$[K^-]^*$
K^{-*-}	$[[K^-]^*]^-$
$\ker K$	nullspace (kernel) of K
$\operatorname{im} K$	image (range) of K
$\langle \cdot, \cdot \rangle$	scalar product in $\mathbb{K}^m$
$\lvert \cdot \rvert$	vector and matrix norms
$\oplus$	direct sum
DAE	differential-algebraic equation
ODE	ordinary differential equation
IVP	initial value problem
IERODE	inherent explicit regular ODE
EUODE	essential underlying ODE

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