

Indefinit Eisenstein Lattices: A Modern Ball-Rendezvous with Poincaré, Picard, Hecke, Shimura, Mumford, Deligne, Hirzebruch

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In memory to Vasil Tsanov.

We met us several times in Sveti Constantine at the Geometry Conferences and, moreover, on other places, e.g. in Sofia and Berlin. All times we had a lot of interesting discussions about Mathematics and life. Here I will extend my Sveti Constantine lectures given in spring of 2002.

Abstract

In [Ho02] we counted indefinit metrics (two-dimensional, integrally defined, over Gauß numbers) with a fixed norm (discriminant). We would like to call them also *indefinit class numbers*. In this article we change from Gauß to Eisenstein numbers. ¹ We have to work on the complex two-dimensional unit ball, an Eisenstein lattice on it and the quotient surface. It turns out that the compactified quotient is the complex plane \mathbb{P}^2 . In the first part we present a new proof of this fact. In the second part we construct explicitly a Heegner series with help of Legendre-symbol coefficients. They can be interpreted as "indefinit class numbers" we look for. Geometrically they appear also as number of plane curves with (normed) Eisenstein disc uniformization. ²

¹It is historically interesting that H. Poincaré extended 1883 (see [Poi]) reduction theory of binary forms from $\mathbb{G}l_2(\mathbb{Z})$ -applications to Gauß and Eisenstein integers. It is better explained in Bianchi's paper [Bia]

²Mathematics Subject Classification: 11F11, 11F27, 11F30, 11F55, 11F66, 11G15, 11M41, 11R52, 14C17, 14C20, 14E20, 14G35, 14H30, 14H45, 14J17, 14J25, 20H05, 20H10, 30F45, 32M15, 32S25, 32S45, 44A15 Key words: Arithmetic groups, unit ball, Picard modular surfaces, Kahler-Einstein metric, constant curvature, arithmetic curves, Shimura curves, modular curves, geodesics, modular forms, surface singularities, orbital divisors, rational intersection theory, orbital functionals, Zeta functions

0 Preface

We consider 2-dimensional submetrics of the 3-dimensional hermitian indefinite unimodular diagonal metric. The latter is defined by the sesquilinear form \langle, \rangle with Gram matrix $Dg = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. The complex hermitian space $(\mathcal{C}^3, \langle, \rangle)$ is denoted by $\mathbb{C}^{2,1}$. Let K be an imaginary quadratic number field and $\mathcal{O} = \mathcal{O}_K$ the ring of integers in it. In $\mathbb{C}^{2,1}$ sits the indefinite hermitian lattice $\mathcal{O}^{2,1} = (\mathcal{O}^3, Dg)$. Main objects of our study are indefinite 2-dimensional hermitian sublattices \mathcal{E} of $\mathcal{O}^{2,1}$. We restrict ourselves through the paper to fields K with class number 1, mainly to the field $\mathbb{Q}(\sqrt{-3})$ of Eisenstein numbers. Then \mathcal{E} and its orthogonal \mathcal{O} -line $\mathcal{L} \subset \mathcal{O}^{2,1}$ have 2 respectively 1 \mathcal{O} -generator(s). The latter is uniquely determined up to an \mathcal{O} -unit factor: $\mathcal{L} = \mathcal{O}l$, where l is a primitive $\mathcal{O}^{2,1}$ -vector. The *norm* $n(\mathcal{E})$ is well-defined as $\text{norm } n(l) = \langle l, l \rangle$.

Let $V_- = \{\mathfrak{z} \in \mathbb{C}^{2,1}; n(\mathfrak{z}) < 0\}$ be the set of complex negative norm vectors. The 2-dimensional complex unit ball \mathbb{B} consists of the negative lines in the projective plane:

$$\mathbb{B} = \mathbb{P}V_- = V_-/\mathbb{C}^* \subset \mathbb{P}V = V/\mathbb{C}^* = \mathbb{P}^2.$$

We restrict the group action of $\mathbb{U}((2, 1), \mathbb{C})$ on \mathbb{B} to the arithmetic subgroups $\Gamma \subseteq \mathbb{U}((2, 1), \mathcal{O})$ of finite index. The quotients $\Gamma \backslash \mathbb{B}$ are normal complex algebraic surfaces. The same is true also for their compactifications considered in this article. They are called *Picard modular surfaces*. Refinements and extensions you find in 1.1. For basic definitions and properties I refer to my book [BSA].

We continue with sublattices $\mathcal{E}, \mathcal{L} \subset \mathcal{O}^{2,1}$ as above, but admit all submetrics of V . Tensoring with \mathbb{R} yield a hermitian plane E respectively a line L in V . Excluding positive definiteness, we get non-zero intersections $E_- = E \cap V_-$, $L_- = L \cap V_-$. Their \mathbb{P} -projection to \mathbb{B} is a disc $\mathbb{E} = \mathbb{P}E_- \subset \mathbb{B}$ respectively a point $\lambda = \mathbb{P}L_- = \mathbb{P}L \in \mathbb{B}$. They come with discriminants and norms heritaged from \mathcal{E} or \mathcal{L} , respectively.

In Chapter I we concentrate ourselves to the *Eisenstein Congruence Subgroup* (ECS) $\Gamma(\sqrt{-3}) \subset \mathbb{U}((2, 1), \mathcal{O}_{\mathbb{Q}(\sqrt{-3})})$ and its ball quotient surface $X_{\Gamma(\sqrt{-3})} := \Gamma(\sqrt{-3}) \backslash \mathbb{B}$ with (smallest) compactification $\widehat{X}_{\Gamma(\sqrt{-3})}$. Investigating a special system of partial differential equations E.Picard (1883) saw a close connection between the ball, the ECM and the complex plane. But he could not correctly prove it. It needed a long mathematical process for doing that. With strong results of P.Deligne and G.Mostow around 1970 I was able to prove that $\widehat{X}_{\Gamma(\sqrt{-3})} = \mathbb{P}^2$.³ Here we give a new (more direct) proof without use of Deligne's result. The whole Chapter I is dedicated to the surface classification of $\widehat{X}_{\Gamma(\sqrt{-3})}$. An essential role plays the ramification

³First published together with historical background and motivations in my book [EPD].

locus of the (locally finite) covering $\mathbb{B} \rightarrow X_{\Gamma(\sqrt{-3})}$. It consists of all discs of discriminant -1 . The image of them (branch locus) are six smooth curves. A preview with pictures (restricting to real points, naturally) is presented in Subsection 1.2. It gives a guideline of vision for the later proof steps.

In the Sections 2 - 5 (look at the Table of Contents) we collect and prove all details we need for the classification attacks in the last two Sections of the first chapter. There we have to verify the smoothness of the ECS $\widehat{X}_{\Gamma(\sqrt{-3})}$. But also rationality and smoothness of the irreducible branch curve components must be shown. Then the Chern invariants of surface and curves will be determined. Here we used L-series values (proved by means of higher number theory) and Proportionality Theorems, both in [BSA]. Last not least we need a celebrated Theorem of Miyaoka -Yau on classification of surfaces with extreme Chern numbers. At the end we have \mathbb{P}^2 with branch locus consisting of six embedded lines arranged as complete quadrilateral, see Picture in Subsection 1.2.

The main result of Chapter II is Theorem 13.1. It counts the curves on \mathbb{P}^2 of Eisenstein norm N . If, for instance, $N \in \mathbb{N}_+$ is not divisible by 3 then this cardinality is equal to

$$h_N = \sum_{0 < d|N} d^2 \cdot \left[\left(\frac{d}{3} \right) + \left(\frac{N/d}{3} \right) \right]. \quad (1)$$

Thereby $\left(\frac{t}{3} \right)$ is the quadratic rest symbol. A simple modification is necessary, if $3|N$, see Section 13 for details. The h_N 's appear as coefficients of an Heegner series. Its explicit construction for our Eisenstein curves is the main goal of Chapter II, one can say: of the whole article.

More precisely, we call the above plane curves also *Picard-Eisenstein Curves* (PEC). These are the (irreducible) \mathbb{P}^2 -curves \hat{C} , which have along the ball quotient morphism $\mathbb{B} \rightarrow \Gamma(\sqrt{-3}) \backslash \mathbb{B}$ a disc \mathbb{D} of norm N as uniformizing preimage:

$$\begin{array}{ccccc} \mathbb{B} & \twoheadrightarrow & \Gamma(\sqrt{-3}) \backslash \mathbb{B} & \hookrightarrow & \mathbb{P}^2 \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{D} & \twoheadrightarrow & \Gamma(\sqrt{-3}) \backslash \mathbb{D} & \hookrightarrow & \hat{C} \end{array}$$

Basic reference for Chapter II is my article [Ho02], extended in [Ho07]. In Sections 8 and 9 we remember to the construction of "orbital invariants" on commensurability classes of Picard modular surfaces and curves. In this framework we explain the notions of "orbital hights", "orbital curves", "orbital intersection products" of them. ⁴ My work in the just cited papers extended Cogdell's result to all Picard modular surfaces. In the article of 2002 I could

⁴Historical starting point was Cogdell's thesis publication [Cog], dedicated to neat Picard modular congruence subgroups, where the connection with elliptic modular forms of Nebentypus was found.

explicitly found the Heegner series of the Picard-Gauß plane $\mathbb{P}^2 = \hat{X}_{\Gamma(1+i)}$. Explicit knowledge of Koblitz' work on modular forms, see [Kob]. In the mean time we discovered in the work of Erich Hecke (1887 - 1947) explicit constructions of Eisenstein series $\in M_k(n, \chi)$, see [He35],[He40],[He59]. They are of type we look for: weight k , level n , discriminant-Nebentypus χ , see Section 12.

For the field $K = \mathbb{Q}(\sqrt{-3})$, of Eisenstein numbers with Nebentypus-character $\chi_K(n) = \left(\frac{-3}{n}\right)$ the relevant space $M_3(3, \chi_K)$ is 2-dimensional. For a proof we used a dimension-formula of Oesterlé-Cohen [C-O]. On the other hand Hecke's articles present two linearly independent series in $M_3(3, \chi_K)$. So, the latter space of modular forms is generated by two explicitly known Hecke- Eisenstein series E_1, E_2 (reproduced at the end of Section 12. With our combination of arithmetic- geometric methods we calculated the two first coefficients h_1 and h_2 in the general formula 1. Since the Heegner series belongs also to $M_3(3, \chi_K) = \mathbb{C}E_1 + \mathbb{C}E_2$ we can determine precisely its E_1, E_2 - linear combination (see Theorem 13.1 again).

Chapter I (Sections 1 - 7) The Eisenstein Congruence Surface

Chapter II (Sections 8 - 13) Heegner Series of Picard Modular Surfaces

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1 Introduction

1.1 Symbols, Notations

$\mathbb{P}^k = \mathbb{P}_{\mathbb{C}}^k$ denotes the k -dimensional complex projective space. $K = \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic number field, $d \in \mathbb{N}_+$, squarefree. \mathcal{O}_K denotes the ring of integers in K , $h(K)$ the class number of K . The modules/spaces $\mathcal{O}^{n+1}, K^{n+1}, \mathbb{C}^{n+1}$, will be endowed with the unimodular indefinit hermitian metric $\langle \cdot, \cdot \rangle$ with Gram diagonal matrix

$$\begin{pmatrix} +1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & +1 & 0 & \cdot & \cdot & 0 & 0 \\ & & & \ddots & & & \\ & & & & & & \\ 0 & \cdot & \cdot & \cdot & 0 & +1 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 \end{pmatrix}$$

w.r.t. the canonical basis. The corresponding hermitian \mathcal{O} -modules/spaces are denoted by $\mathcal{O}^{n,1}, K^{n,1}$ or $\mathbb{C}^{n,1}$, respectively.

Let generally V be an $(n+1)$ -dimensional hermitian \mathbb{C} -vector space of signature $(n,1)$. Looking at norms w.r.t. hermitian sesquilinear form $h = \langle \cdot, \cdot \rangle$ we will speak about negative, positive or cusp vectors v , if the norm $n(v) = \langle v, v \rangle$ is negative, positive, or zero, respectively. We introduce the notations

$$V_- = \{v \in V; n(v) < 0\}, V_+ = \{v \in V; n(v) > 0\}, V_0 = \{v \in V; n(v) = 0\}$$

The complex lines in V through the 0-point will be identified with the points of the n -dimensional projective space \mathbb{P}^n . The negative vectors project along $V \rightarrow \mathbb{P}^n = \mathbb{P}V$ (keep in mind that you have to exclude the \mathcal{O} -vector in V) onto the n -dimensional (complex unit) ball \mathbb{B}^n . The projection maps V_0 onto the boundary $\partial\mathbb{B}$ of the ball. We will always assume, that V is generated by a hermitian K -space V_K with hermitian product h_K , K as above. Notice that $V = V_{\mathbb{C}} = V_K \otimes \mathbb{R}$, $h_K : V_K \times V_K \rightarrow K$.

From now on we concentrate ourselves to the second dimension. We call $\mathbb{B} := \mathbb{B}^2$ the (complex two-dimensional) unit ball. With \mathbb{D} we denote generally a complete linear subdisc of \mathbb{B} . It is defined as non-void intersection of complex projective line L with our fixed ball \mathbb{B} . Consider the projective projections

$$\begin{array}{ccc} V_- & \hookrightarrow & V \\ \downarrow & & \downarrow \\ \mathbb{B} & \hookrightarrow & \mathbb{P}^2 \end{array} \quad (2)$$

again (with three-dimensional complex vector space V . A (complete linear) subdisc \mathbb{D} of our fixed ball \mathbb{B} is a non-void intersection of a projective line

L with \mathbb{B} . Look at the lower rectangle of the following diagram:

$$\mathbb{B} \cap L = \begin{array}{ccc} E \subset & \longrightarrow & V \\ \downarrow & & \downarrow p \\ L \subset & \longrightarrow & \mathbb{P}^2 \\ \uparrow & & \uparrow \\ \mathbb{D} \subset & \longrightarrow & \mathbb{B} \end{array} \quad (3)$$

In the upper part E denotes a subplane of V projecting projectively onto the line L along p .

Definition 1.1 \mathbb{D} is called a K -disc, iff the line embedding in the above diagram is defined over K . It's the same to say, that the L covering the plane E is a K -defined subplane of V . Equivalently: one has two different K -points on L (or on \mathbb{D}).

The action of unitary group $\mathbb{U}((2, 1), \mathbb{C}) \subset \mathbb{G}l_3(\mathbb{C})$ on $V \cong \mathbb{C}_3$ is compatible with the hermitian structure. Therefore it induces an action on V_- . Dividing out the ineffectively acting central diagonal subgroup, the unitary action goes down to that of $\mathbb{P}\mathbb{U}(2, 1)$ on our ball. The corresponding actions are called fractional transformations.

Now fix the imaginary quadratic number field K , also $\mathcal{O} = \mathcal{O}_K$.

Definition 1.2 The arithmetic group $\Gamma^{2,1} = \mathbb{U}((2, 1), \mathcal{O})$ is called the full Picard modular group of the field K . Any subgroup Γ of $\mathbb{U}((2, 1), K)$, commensurable with $\Gamma^{2,1}$, will be called Picard modular group (or P.m. lattice).

Well-known (e.g. from [BSA] and/or references there) are the following

Facts 1.3 The quotient surface $\Gamma \backslash \mathbb{B}$ is an (open, t.m. non-compact) normal complex algebraic surface. The same is true for its Baily-Borel compactification (BB) $\widehat{\Gamma \backslash \mathbb{B}}$. Both types are called Picard modular surfaces. Adding finitely many (normal) points one gets the BB compactification from the open model. These points are called cusp points or cusp singularities.

With the notations

$$\partial_K \mathbb{B} = \partial \mathbb{B} \cap K^2, \quad \widehat{\mathbb{B}} = \mathbb{B} \cup \partial_K \mathbb{B}$$

we get the inclusion diagram with vertical surjections

$$\begin{array}{ccc} \mathbb{B} \subset & \longrightarrow & \widehat{\mathbb{B}} \\ \downarrow & & \downarrow \\ \Gamma \backslash \mathbb{B} \subset & \longrightarrow & \widehat{\Gamma \backslash \mathbb{B}} \end{array}$$

Most important Picard modular subgroups of Γ (as above) are the congruence subgroups of \mathcal{O} -ideals \mathfrak{a} . They are defined as

$$\Gamma(\mathfrak{a}) = \{\gamma \in \Gamma; \gamma \equiv E \pmod{\mathfrak{a}}\},$$

where E is the unit matrix of order 3.

We would like to define arithmetic curves on Picard modular surfaces. For this purpose consider a K -disc \mathbb{D} in \mathbb{B} (in the sense of Definition 1.1) together with a Picard modular lattice Γ . The subgroup of all its elements γ acting on \mathbb{D} itself is denoted by $N_\Gamma(\mathbb{D})$. It is called the *normalizer group* of Γ and \mathbb{D} . Dividing out the ineffective (on \mathbb{D} acting) kernel $Z_\Gamma(\mathbb{D})$ we get the effective on \mathbb{D} acting Fuchsian group $\Gamma_{\mathbb{D}} := N_\Gamma(\mathbb{D})/Z_\Gamma(\mathbb{D})$. The quotient curve $\Gamma_{\mathbb{D}} \backslash \mathbb{D}$ is an (sometimes open, sometimes compact) algebraic curve sitting on $\Gamma \backslash \mathbb{B}$. In general, there exist curve singularities on it. Its closure on any compactification $\overline{\Gamma \backslash \mathbb{B}}$ of $\Gamma \backslash \mathbb{B}$ is denoted by $\overline{\Gamma_{\mathbb{D}} \backslash \mathbb{D}}$. The resolution of singularities of this curve is nothing else but the BB-compactified quotient curve $\widehat{\Gamma_{\mathbb{D}} \backslash \mathbb{D}}$. Altogether we illustrate the situation in the following commutative diagram of embeddings and surjections:

$$\begin{array}{ccccccc}
 \widehat{\mathbb{D}} & \longleftarrow & \mathbb{D} & \longrightarrow & \mathbb{B} & \longrightarrow & \widehat{\mathbb{B}} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Gamma_{\mathbb{D}} \backslash \widehat{\mathbb{D}} & \longleftarrow & \Gamma \backslash \mathbb{D} & \longrightarrow & \Gamma \backslash \mathbb{B} & \longrightarrow & \widehat{\Gamma \backslash \mathbb{B}} \\
 \searrow \text{desing} & & \downarrow \text{clo} & \text{sure} & \downarrow & & \downarrow \text{closure} \\
 & & \overline{\Gamma \backslash \mathbb{D}} & \longrightarrow & \overline{\Gamma \backslash \mathbb{B}} & & \widehat{\Gamma_{\mathbb{D}} \backslash \mathbb{D}} \\
 & & \searrow & & \searrow & & \downarrow \\
 & & & & & & \overline{\Gamma_{\mathbb{D}} \backslash \mathbb{D}}
 \end{array} \tag{4}$$

where $\widehat{\mathbb{D}} := \mathbb{D} \cup \partial_K \mathbb{D}$ with the set $\partial_K \mathbb{D} = \partial \mathbb{D} \cap_K \mathbb{B}$ of K -rational boundary points of our disc.

By the way, the K -boundary points of \mathbb{D} or \mathbb{B} are called cusps of these objects. Their image points along the left or right vertical arrows of the above diagram are called *cuspidal points* of the curve or surface, respectively.

Definition 1.4 *Let \mathbb{D} be a K -disc, Γ a Picard modular group of K and $\Gamma \backslash \mathbb{B}$ the corresponding Picard modular surface. The image curve $\Gamma \backslash \mathbb{D}$ on the surface will be called a K -arithmetic curve or Picard modular curve (PM-curve). We use the same notation for the compactifications of this curve on the surfaces appearing in (4).*

1.2 Preview: Picard modular surfaces of Eisenstein numbers

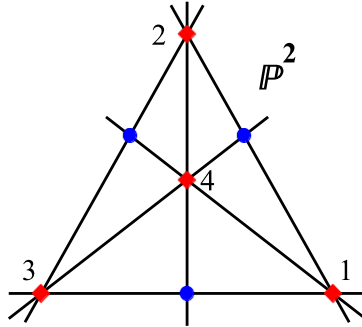
We fix now $K = \mathbb{Q}(\sqrt{-3})$, known as field of Eisenstein numbers. The ring of (integral) Eisenstein numbers is the unique factorization domain $\mathcal{O} = \mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\omega$, where $\omega = e^{\frac{2\pi i}{3}}$ is one of the two primitive third

unit roots. Main objects of this paper will be the congruence subgroup $\Gamma(\sqrt{-3}) := \Gamma^{2,1}(1 - \omega)$ of the principal \mathcal{O} -ideal $(1 - \omega)$ generated by the \mathcal{O} -prime element $1 - \omega$ of norm 3.

Theorem 1.5 ([EPD]). *The BB compactification $\widehat{\Gamma(\sqrt{-3})} \backslash \mathbb{B}$ is the projective complex plane \mathbb{P}^2 . There are precisely four cusp points on it. The six lines through pairs of them is the (compactified) branch locus of the quotient morphism $\mathbb{B} \rightarrow \Gamma(\sqrt{-3}) \backslash \mathbb{B}$.*

We visualize the (compactified) branch curve in Picture ref1.5 below (real part). The configuration is known as *complete quadrilateral* on the complex plane \mathbb{P}^2 . The ramification locus on \mathbb{B} consists of infinitely many discs: the $\Gamma(\sqrt{-3})$ -orbit of the six ramification discs drawn in Picture ref1.6 below (real cut). Representative $\Gamma(\sqrt{-3})$ -cusps, covering (all four) plane cusp points along the cusp-extended quotient morphism (see Diagram ref3), are marked by little red diamonds in the pictures.

Picture 1.1: Branch locus Quadrilateral

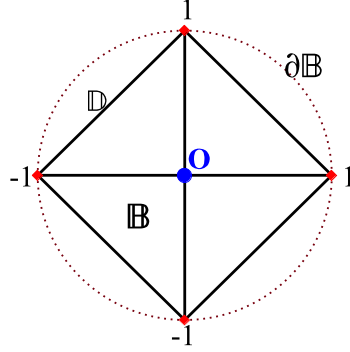


(5)

We fix a numeration (1,2,3,4) of the cusp points. It allows us to denote each branch line by the two cusp points lying on it. For instance $L_{1,2} = L_{2,1}$ is the line through the points "1" and "2".

Corollary 1.6 *The branch/ramification index at each of the six lines/discs is equal to 3.*

Picture 1.2: Covering ramification discs



(6)

Here the numbers ± 1 are nothing else but the x -coordinates on the horizontal disc or the y -coordinates on the vertical one, respectively. More precisely, we marked the cusps by (affine) coordinates.

$$\kappa_1 = (1, 0), \kappa_2 = (0, 1), \kappa_3 = (-1, 0), \kappa_4 = (0, -1) \quad (7)$$

The disc joining the two cusps κ_i and κ_j , $1 \leq i < j \leq 4$, will be denoted by $\mathbb{D}_{i,j}$. Blue points represent (\mathbb{R} -visible) intersection points of two ramification discs. Also their projections along the quotient morphism $\mathbb{B} \rightarrow \Gamma(\sqrt{-3}) \backslash \mathbb{B} = \mathbb{P}^2$ are marked by the same color. On \mathbb{P}^2 (all) three of them are \mathbb{R} -visible, drawn in Picture 1.2.

Remark 1.7 *In this paper we will give a new proof of the statements of this subsection. It is - other than in my book [EPD] - essentially supported by the branch locus. My doctor-grandmother Emmy Noether emphasized a century before today the important role of ramifications. It's my aim to improve it geometrically in dimension 2.*

The proof will be finished at the end of Section 7.

Remark 1.8 *With*

$$G = \begin{pmatrix} \frac{\sqrt{-3}}{2} & 0 & -\frac{\sqrt{-3}}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, N = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{-3}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{-3}} & 0 & 0 \end{pmatrix}$$

the hermitian spaces $\mathbb{C}^{2,1} = (\mathbb{C}^3, Dg)$ and (\mathbb{C}^3, N) are isometric. Namely the Gram matrices Dg and N are conjugated to each other: ${}^t\overline{G} \cdot N \cdot G = Dg$. It defines the isometry

$$\Theta : V = \mathbb{C}^{2,1} \xrightarrow{\sim} (\mathbb{C}^3, N) = V' \quad , \quad \mathbf{c} \mapsto G\mathbf{c}$$

It induces an isomorphism of unitary groups:

$$\mathbb{U}((2, 1), \mathbb{C}) = \mathbb{U}(Dg, \mathbb{C}) \xrightarrow{\sim} \mathbb{U}(N, \mathbb{C}) \quad , \quad A \mapsto G^{-1}AG$$

It restricts to the unitary subgroups with coefficients in $K = \mathbb{Q}(\sqrt{-3})$. The image of the Picard modular Eisenstein lattice $\Gamma(\sqrt{-3})$ lands in the commensurability class of $\Delta = \mathbb{U}(N, \mathcal{O})$, say $\Gamma' = G^{-1}\Gamma(\sqrt{-3})G$.

Cogdell established the Heegner series for neat principal congruence subgroups $\Delta(\alpha), \alpha \in \mathcal{O}_K$. I explained ([Ho02], [Ho07]) how Heegner's construction extends to all members of the commensurability class of Δ . We pull the pair (V', Γ') back along the isometry Θ to $(V, \Gamma(\sqrt{-3}))$. Along this way we get the same Heegner series for $\Gamma(\sqrt{-3})$ and G' . We get also the same quotient surface \mathbb{P}^2 for both arithmetic groups. Also the plane curve interpretations are the same.

2 Unimodular sublattices

We fix an imaginary quadratic number field K , for simplicity of class number 1. Moreover, the ring $\mathcal{O} = \mathcal{O}_K$ of K -integers is assumed to be a unique factorization domain (Gauß and Eisenstein numbers belong to this class). For any hermitian \mathcal{O} -sublattice Λ of $\mathcal{O}^{2,1}$ of \mathcal{O} -rank 1, 2 or 3 we define the dual lattice as \mathcal{O} -module

$$\Lambda^\# = \{\mathfrak{x} \in \Lambda \otimes K; \langle \mathfrak{x}, \mathfrak{l} \rangle \in \mathcal{O}; \text{ for all } \mathfrak{l} \in \Lambda\}.$$

Notice that $\Lambda \subseteq \Lambda^\#$ with equality if and only if Λ is unimodular. Equivalently: The discriminant $dcr(\Lambda)$ is equal to ± 1 . Thereby the discriminant of Λ is defined to be the determinant of the Gram-Matrix of an \mathcal{O} -basis of Λ with respect to the hermitian form. It is defined up to multiplication with a norm unit of \mathcal{O} .

Two subsets M, N of $K^{2,1}$ are said to be *orthogonal*, iff $\langle m, n \rangle = 0$ for all $m \in M, n \in N$. We write $M \perp N$ in this case. Orthogonal direct sums of lattices are written as $M \boxplus N$. The orthogonal complement of $M \subseteq \mathcal{O}^{2,1}$ in $\mathcal{O}^{2,1}$ is the sublattice

$$M^\perp = \{\mathfrak{n} \in \mathcal{O}^{2,1}; \mathfrak{n} \perp M\}.$$

Obviously, the discriminants of orthogonal lattices $M, N \subseteq \mathcal{O}^{2,1}$ behave multiplicatively:

$$dcr(M \boxplus N) = dcr(M) \cdot dcr(N). \quad (8)$$

Two sublattices M, N of Λ are called *orthogonal complementary* (in $\mathcal{O}^{2,1}$), iff $M \cap N = \{\mathfrak{o}\}$, $M^\perp = N$ and $N^\perp = M$.

Fact 2.1 ([Apo], Prop. 6.1). *If M and N are orthogonal complementary sublattices of $\mathcal{O}^{2,1}$, then $M^\# / M \cong N^\# / N$ as \mathcal{O} -modules.*

Corollary 2.2 *Under the above conditions it holds that: M is unimodular if and only if N is unimodular.*

Corollary 2.3 *Let M be an indefinit rank-2 sublattice of $\mathcal{O}^{2,1}$ and G its orthogonal complement in $\mathcal{O}^{2,1}$, say $G = \mathcal{O}\mathfrak{c}$. Then $-dcr(M) = n(\mathfrak{c}) > 0$.*

Hint. Express $M^\# : M$ in terms of the discriminant of M , do the same with $G^\# : G$ and compare.

□

Examples 2.4 *A central role will play the following six norm 1 vectors $\mathfrak{d}_{i,j}$ together with their orthogonal planes $E_{i,j}$, $1 \leq i < j \leq 4$, (of discriminant -1):*

$$\begin{aligned} E_{1,2} \perp \mathfrak{d}_{1,2} &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, E_{1,3} \perp \mathfrak{d}_{1,3} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, E_{1,4} \perp \mathfrak{d}_{1,4} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \\ E_{2,3} \perp \mathfrak{d}_{2,3} &= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, E_{2,4} \perp \mathfrak{d}_{2,4} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, E_{3,4} \perp \mathfrak{d}_{3,4} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \end{aligned} \quad (9)$$

The unimodular plane $E_{i,j}$ is generated by the i -th and j -th column C_i, C_j , respectively, of the following "cusp matrix" (all 4 columns are cusp vectors):

$$C = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad (10)$$

The projective projections of the C_i (see Diagram 3, Section 1) are the four cusps $\kappa_i = \mathbb{P}C_i$, $i = 1..4$. We visualized them in Picture 1.2 at the end of Section 1. Consequently, the \mathbb{P} -projections of the planes $E_{i,j}$ are six projective lines $L_{i,j}$ whose intersections with the ball \mathbb{B} are the discs $\mathbb{D}_{i,j}$ drawn in Picture 1.2.

3 Counting Special Points

3.1 Cusp points on $\widehat{\Gamma} \backslash \mathbb{B}$

We remember to the following

Theorem 3.1 *(Feustel [Fe1], elegant proof in [Zi]) Let $\Gamma_K^{2,1}$ be the full Picard modular group of an arbitrary imaginary quadratic number field K . Then the quotient surface $\widehat{\Gamma}_K^{2,1} \backslash \mathbb{B}$ has precisely $h(K)$ cusp points.*

We concentrate us to the field of Eisenstein numbers again, omitting some indices. E.g. write simply Γ for the full Picard modular group $\mathbb{U}((2, 1), \mathcal{O})$, where $\mathcal{O} = \mathbb{Z}[\omega]$ is the ring of integral Eisenstein numbers.

Lemma 3.2 *The factor group $\Gamma/\Gamma(\sqrt{-3})$ is isomorphic to the (doubled) symmetric group $\pm S_4$. It is geometrically represented as orthogonal group $\mathbb{O}((2, 1), \mathbb{F}_3)$, where $\mathbb{F} = \mathbb{F}_3$ denotes the finite field $\mathcal{O}/(1 - \omega)$ consisting of three elements $0, +1, -1$, say.*

Proof. The action of Γ on $\mathcal{O}^{2,1}$ goes modulo $(1 - \omega)$ down to an orthogonal action on $\mathbb{F}^{2,1}$ with ineffective kernel $\Gamma(\sqrt{-3})$. So we have an embedding

$$\Gamma/\Gamma(\sqrt{-3}) \hookrightarrow \mathbb{U}((2, 1), \mathbb{F}) = \mathbb{O}((2, 1), \mathbb{F}) \cong \pm S_4 \quad (11)$$

It is well-known that the projective group $\mathbb{P}\mathbb{O}((2, 1), \mathbb{F})$ is isomorphic to the symmetric group S_4 , see e.g. [Dieu]. It appears as permutation group of the \mathbb{F} -points

$$(1 : 0 : 1), (0 : 1 : 1), (-1 : 0 : 1), (0 : -1 : 1), \quad (12)$$

which are images along the (projectivized) residue map $\mathcal{O}^3 \rightarrow \mathbb{F}^3$ of the cusp vectors sitting in the cusp matrix C in (10). In affine coordinates the corresponding $\Gamma(\sqrt{-3})$ -cusp points are listed in (7) and drawn in Picture 1.2. The Γ -elements

$$\Sigma_{12} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & -3 \end{pmatrix}, \Sigma_{13} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \Sigma_{14} = \begin{pmatrix} -1 & 2 & -2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

act via residue map on the \mathbb{F} -cusp vectors $C_i \pmod{3}$ as S_4 -transpositions (12), (1, 3), (1, 4) (up to sign). These three elements generate S_4 . Therefore $\Gamma/\Gamma(\sqrt{-3}) \cong \pm S_4$, hence, the inclusion (11) is the identical map. The Lemma is proved. □

Corollary 3.3 *The Eisenstein congruence surface $\Gamma(\sqrt{-3}) \backslash \widehat{\mathbb{B}}$ has precisely four cusp points. Lifted representants on $\partial\mathbb{B}$ are the (red) \mathbb{B} -boundary points drawn in Picture 1.2.*

Proof. From Feustel's Theorem 3.1 we know that the cusp orbit of the full Picard modular group Γ consists of only one element, take $\Gamma\kappa_1$. The preimage of this cusp point on the congruence surface are the (projectivized) orbits

$$\Gamma(\sqrt{-3}) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \Gamma(\sqrt{-3}) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \Gamma(\sqrt{-3}) \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \Gamma(\sqrt{-3}) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

see (12) as well as Pictures (1.2) and (1.2).

3.2 \mathbb{B} -points of maximal negative norm -1

Each K -line $L = K\mathfrak{a} \in K^3$ can be generated by a primitive vector $\mathfrak{p} \in \mathcal{O}^3$, which is unique up to multiplication with a 6-th unit root.

Definition 3.4 *With these notations, the norm $N = n(L)$ is uniquely defined as $n(\mathfrak{p}) \in \mathbb{Z}$. \mathfrak{p} is called a norm N vector. If the norm is negative, then $p = \mathbb{P}\mathfrak{p} = L$ is a ball point. We endow p and also its image point P on the Eisenstein congruence surface with the same norm $n(P) = n(p) = n(\mathfrak{p}) = N$. We speak then also about norm N points p or P .*

It is clear that only K -points with integral negative norms exist on the ball \mathbb{B} and on the Picard modular quotient surfaces.

Proposition 3.5 *The -1 vectors in $\mathcal{O}^{2,1}$ fill precisely one Γ -orbit, namely $\Gamma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. This orbit splits into three $\Gamma(\sqrt{-3})$ -orbits.*

Proof. Otremba proved in [Otr] that, especially for Eisenstein numbers, there is only one isometry class of positive definit unimodular hermitian \mathcal{O} -lattices of rank 2, see also Hashimoto [Has]. Take the simplest one E generated by the first two of the canonical \mathcal{O}^3 -basis vectors

$$\mathfrak{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathfrak{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathfrak{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

If \mathfrak{c} denotes an arbitrary norm -1 vector in $\mathcal{O}^{2,1}$, then its orthogonal complement there, denoted by F , is a rank 2 sublattice of $\mathcal{O}^{2,1}$ with discriminant $+1$ by Corollary 2.3. According to Otremba/Hashimoto there are lattice isomorphisms

$$E \cong F, \mathcal{O}\mathfrak{e}_3 \cong \mathcal{O}\mathfrak{c}, \text{ finally : } \mathcal{O}^{2,1} = E \boxplus \mathcal{O}\mathfrak{e}_3 \cong F \boxplus \mathcal{O}\mathfrak{c}.$$

So we extended the second isometry to a rank 3 one $\gamma \in \Gamma$, hence $\mathfrak{c} \in \Gamma\mathfrak{e}_3$.

For the second statement we must check the orbit $\Gamma\mathfrak{e}_3$ modulo $\Gamma(\sqrt{-3})$. We deal with the action of the factor group S_4 on the residue space $\mathbb{F}^{2,1}$. The Klein's four group $K4 \subset S_4$ fixes \mathfrak{e}_3 . The factor group $S_3 \cong S_4/K4$ moves effectively this vector. Geometrically, on the Eisenstien congruence surface, there are three moved points, visualized in Picture 1.2 by blue bullets. Pulling them back to the ball we get the explicit splitting

$$\Gamma\mathfrak{e}_3 = \pm\Gamma(\sqrt{-3}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sqcup \pm\Gamma(\sqrt{-3}) \begin{pmatrix} 1 \\ 1 \\ 1-\omega \end{pmatrix} \sqcup \pm\Gamma(\sqrt{-3}) \begin{pmatrix} 1 \\ -1 \\ 1-\omega \end{pmatrix}.$$

Namely, the three -1 vectors inside are not pairwise $\Gamma(\sqrt{-3})$ -equivalent. □

3.3 Norms of K -discs and their quotient curves

We denote by \mathcal{E} now an indefinite rank-2 sublattice of $\mathcal{O}^{2,1}$. Its orthogonal complement in $\mathcal{O}^{2,1}$ is an \mathcal{O} -line \mathcal{L} . We know that $n(\mathcal{L}) = -dcr(\mathcal{E}) > 0$ by Corollary 2.3. The complex plane $E = \mathbb{R} \otimes \mathcal{E}$ defines the K -disc $\mathbb{D} = \mathbb{P}E \cap \mathbb{B}$ in the ball. Similarly, we set $L = \mathbb{R} \otimes \mathcal{L}$.

Definition 3.6 *With the above notations we call $n(E) := n(\mathcal{E}) =: n(\mathbb{D})$ the norm of the K -disc \mathbb{D} . Also the image curve $\Gamma(\sqrt{-3})\backslash\mathbb{D}$ (as well any closure/compactification) on the Eisenstein Congruence Surface will be endowed with the same norm.*

Observe, that conversely a given K -disc \mathbb{D} determines uniquely the line L as projective (algebraic) closure of the disc in the projective plane $\mathbb{P}^2 \supset \mathbb{B}$. Furthermore this line is uniquely lifted to the plane $E \subset V$, see Diagram 3 in Section 1. Moreover, we get with $\mathcal{E} = E \cap \mathcal{O}^{2,1}$ the corresponding indefinit rank 2 sublattice of $\mathcal{O}^{2,1}$. Its absolute discriminant is the norm of \mathbb{D} (see Cor. 2.3). Moreover, given an arithmetic curve C on the Eisenstein Congruence Surface, say $C = \Gamma(\sqrt{-3})\backslash\mathbb{D}$, then the covering K -disc \mathbb{D} is determined up to $\Gamma(\sqrt{-3})$ -equivalence. But the disc-norm is stable under $\Gamma(\sqrt{-3})$ -transformations. So also *curve norm* $n(C)$ is well-defined.

Of special interest are the discs and their quotient curves of norm $+1$. Parallel to Proposition 3.5 we prove

Proposition 3.7 *The $+1$ vectors in $\mathcal{O}^{2,1}$ fill precisely one Γ -orbit. (For simplicity you can take $\Gamma\mathfrak{e}_1$). This orbit splits into six $\Gamma(\sqrt{-3})$ -orbits, generated e.g. by the following six norm 1 vectors (arranged in three orthogonally intersecting pairs):*

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \perp \begin{pmatrix} \omega \\ \omega^2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \perp \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \perp \begin{pmatrix} -1 \\ \omega \\ \omega^2 \end{pmatrix} \quad (13)$$

Proof. Obviously, they are not pairwise $(1 - \omega)$ -congruent to each other. Therefore they generate different $\Gamma(\sqrt{-3})$ -orbits of norm 1 vectors. The symmetric group S_4 acts on their residue classes modulo $(1 - \omega)$ with ineffective kernel $K4$. The residue vectors coincide with the six \mathbb{F} -vectors $\mathfrak{d}_{i,j} \in \mathbb{F}^{2,1}$, $1 \leq i < j \leq 4$ listed in (9). There are no more 1-vectors in $\mathbb{F}^{2,1}$. Therefore the six ones in (13) form a complete set of representatives of 1 vectors modulo $\Gamma(\sqrt{-3})$. □

Remark 3.8 *Each of the three ortho-pairs indicated in (13) generate a positive definite subplane in $\mathcal{O}^{2,1}$ of minimal discriminant $+1$. They have orthovectors of norm -1 , namely:*

$$\begin{pmatrix} \omega \\ -\omega^2 \\ \omega - \omega^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \omega \\ 1 - \omega \end{pmatrix} \quad (14)$$

Visualisation. The six discs $\mathbb{D}_{i,j}$ have been defined at the end of Section 2 by their norm 1 vectors in (13). They are drawn in Picture 1.2. Their quotient curves on the congruence surface appear in Picture 1.2. The three -1 points on \mathbb{B} , projected from the vectors in (14), are intersections of the disc pairs with orthogonal vectors listed in (14). Only one of them is visible (real

coordinates). But along the quotient map $\mathbb{B} \rightarrow \Gamma(\sqrt{-3}) \backslash \mathbb{B}$ all three points will be visible (as blue bullets) on the quotient surface. By Proposition 3.5 these are all -1 points on the congruence surface. Each of them is an intersection point of two norm 1 curves.

3.4 Stabelizing subgroups

Lemma 3.9 *The stabilizer group of e_3 in $\Gamma(\sqrt{-3})$ is the diagonal group*

$$\left\{ \begin{pmatrix} \omega^i & 0 & 0 \\ 0 & \omega^j & 0 \\ 0 & 0 & \omega^k \end{pmatrix}, 1 \leq i, j, k \leq 3 \right\}$$

The projective stabilizer of the (blue) -1 point $O \in \mathbb{B}$, is represented by

$$\langle \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle \times \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle = \left\{ \begin{pmatrix} \omega^i & 0 & 0 \\ 0 & \omega^j & 0 \\ 0 & 0 & 1 \end{pmatrix}, 1 \leq i, j \leq 3 \right\} \quad (15)$$

The projective stabilizer of any -1 point on \mathbb{B} is isomorphic to the bicyclic group $K9 := Z_3 \times Z_3$ of order 9.

Proof. Obviously, the Γ -stabilizer of \mathfrak{e}_3 is $\begin{pmatrix} U & \mathfrak{o} \\ \mathfrak{t}_\mathfrak{o} & \langle -\omega \rangle \end{pmatrix}$ with $U = \mathbb{U}(2, \mathcal{O})$ and \mathfrak{o} the 2-dimensional zero vector. Intersection with the congruence subgroup yield the diagonal group in Lemma 3.9 and finally the projective $K9$ -representants in (14). □

Definition 3.10 *For any Picard modular group Δ we call $\delta \in \Delta$ a Δ -elliptic element iff it has finite order and three different eigenvalues. The notation will also be used for the projectivized elements. The point $Q \in \mathbb{B}$ is called a Δ -elliptic point iff the stabelizing (isotropy, stationary) group $\text{Stab}_\Delta(Q)$ contains a Δ -elliptic element.*

Examples 3.11 . *The central point $O \in \mathbb{B}$ is a $\Gamma(\sqrt{-3})$ -elliptic point because its stationary group contains the Gw -elliptic element $\text{diag}(\omega, \omega^2, 1)$. The elliptic property can be easily transported via Proposition 3.5 to all (blue) -1 points of \mathbb{B} .*

Remark 3.12 *Any elliptic $\delta \in \Delta$ has precisely one fixed point $Q \in \mathbb{B}$.*

Proof. It is easy to see that the eigenbasis of δ consists of an orthogonal basis of V . This is only possible with two positively normed vectors and a negative one, say $q \in V_-$. Then $Q = \mathbb{P}q$ is the point we looked for. □

Definition 3.13 *With the notations of Definition 3.10 we call δ a Δ -reflection, iff it has finite order and precisely two different eigenvalues, where the eigenline of the simple one belongs to V_+ .*

Remark 3.14 *Eigenvectors of different eigenvalues are orthogonal to each other. Therefore the eigenplane $E \subset \mathbb{C}^{2,1}$ of the double eigenvalue is indefinite being orthogonal to a positively normed vector. It follows that E projects and restricts to a disc $\mathbb{D} = \mathbb{B} \cap \mathbb{P}E$ on our ball.*

Definition 3.15 *Such \mathbb{D} as above is called a Δ -reflection disc (precisely: of the reflection δ). The centralizer (group) of \mathbb{D} is the ineffectively on \mathbb{D} acting subgroup*

$$Z_{\Delta}(\mathbb{D}) = \{\gamma \in \Delta; \gamma|_{\mathbb{D}} = id|_{\mathbb{D}}\} \quad (16)$$

Example 3.16 *The element $\text{diag}(\omega, 1, 1)$ acts identical on the horizontal disc $\mathbb{D}_{2,4}$, and $\text{diag}(1, \omega, 1)$ acts inefficiently on the vertical coordinate disc $\mathbb{D}_{1,3}$.*

Compare the notations at the end of Section 2 and the Visualisation 1.2.

Lemma 3.17 *All -1 points Q on \mathbb{B} are intersection points of two $\Gamma(\sqrt{-3})$ -reflection discs \mathbb{D}, \mathbb{D}' . The stationary groups are isomorphic to $K9 = Z_3 \times Z_3$, with centralizer groups $Z_{\Gamma(\sqrt{-3})}(\mathbb{D}_i) \cong Z_3$ of the \mathbb{D} and \mathbb{D}' as generators.*

Proof. The situation around O described in Lemma 3.9 is transported with the help of Γ to each -1 point Q (see Proposition 3.5).

4 All elements of finite order

Let K be an imaginary quadratic number field and $\gamma \in \mathbb{G}l_3(K)$ an element of finite order n . The characteristic polynomial of γ over K has degree 3. All zeros of it are n -th unit roots $\zeta_n \in \mathbb{C}$. Over \mathbb{Q} the roots have degree not greater than 6. Denote by $\chi(x) \in \mathbb{Q}[x]$ a polynomial with zero ζ_n of degree ≤ 6 . We want to know, which unit roots can occur for $\gamma \in \Gamma(\sqrt{-3})$. We concentrate us first to prime numbers $n = p$ and to the field K of Eisenstein numbers. The minimal polynomial over \mathbb{Q} with zero ζ_p has degree $p - 1$. It divides $\chi(x)$, therefore $p - 1 \leq 6$. The only possibilities are $p = 2, 3, 5, 7$. The latter two primes can be excluded, because ζ_5, ζ_7 are not zeros of a polynomial of degree ≤ 3 over K .

If $n = p^k$ is a prime power, then the degree of the prime polynomial of ζ_{p^k} over \mathbb{Q} is equal to $(p - 1)p^{k-1}$. Over $\mathbb{Q}(\sqrt{-3})$ survive only the honest powers $2^2, 3^2$. In congruence subgroups we can further restrict the possible orders of finite elements. In [Apo], Lemma 7.1, we proved the elementary

Lemma 4.1 *Let \mathfrak{a} be an ideal in \mathcal{O} , $\gamma \in \Gamma(\mathfrak{a})$ of finite order n with n -th unit root ζ as eigenvalue. Then \mathfrak{a} divides the principal ideal $(1 - \zeta)$ in \mathcal{O}_L , where $L = K(\zeta)$.*

For the Eisenstein congruence subgroup $\Gamma(\sqrt{-3})$ we get for $\mathfrak{a} = (1 - \omega)$ the relation $(1 - \omega) \mid (1 - \zeta) \in \mathcal{O}_{K(\zeta)}$. The only possible prime order for ζ is $p = 3$. The prime powers $2^2, 3^2$ are excluded, but also $p = 2$. We notice the following

Corollary 4.2 *The only elements of finite order in $\Gamma(\sqrt{-3})$ have order 3. Each of them is conjugated to one of the diagonal elements $\text{diag}(\omega^i, \omega^j, \omega^k)$.*

We want to determine all reflections in $\Gamma(\sqrt{-3})$. Such a reflection σ must be of order 3 by Corollary 4.2 with a double and a single eigenvalue. Without loss of generality, we assume that 1 is the double eigenvalue and ω the other one (if not, take the inverse or/and multiply σ with ω or ω^2). The eigenplane is denoted by E . Its orthogonal complementary line in V is denoted by G . The integral part $\mathfrak{g} = \mathcal{O}^3 \cap G$ is spanned by a primitive (eigen) vector \mathfrak{g} with positive norm and sits on the line G orthogonal to E . We assert that its norm $n(\mathfrak{g}) = n(G)$ is equal to 1:

5 Elliptic elements

We want to prove that the Eisenstein quotient surface $\Gamma(\sqrt{-3}) \backslash \mathbb{B}$ is smooth and that the $\Gamma(\sqrt{-3})$ -reflection discs fill completely the ramification locus of the (locally finite) quotient morphism $\mathbb{B} \rightarrow \Gamma(\sqrt{-3}) \backslash \mathbb{B}$. For this purpose we must find all elliptic elements, points, of our Picard Eisenstein congruence lattice $\Gamma(\sqrt{-3})$ or on the surface $\Gamma(\sqrt{-3}) \backslash \mathbb{B}$, respectively.

Characterisation 5.1 of elliptic elements. *Let Δ be a Picard modular group of the 2-ball. An element $\delta \in \Delta$ is elliptic iff it has an isolated fixed point P on \mathbb{B} . This means that in a small open neighbourhood of P there is no other fixed point of δ . A point $P \in \mathbb{B}$ is a Δ -elliptic point iff the stationary group $\text{Stab}_\Delta(P) = \{\gamma \in \Delta; \gamma(P) = P\}$ contains an elliptic group element. (cp. Shimura [Sh], Ch.I)*

Remarks 5.2 *For each ball point P the stationary group $\text{Stab}_\Delta(P)$ is finite because Δ acts proper discontinuously on \mathbb{B} . An elliptic element $\delta \in \Delta$ has finite order, say n . Being not a reflection it has three different eigenvalues. Each of them is an n -th unit root. Any triple of eigenvectors of the different eigenvalues is an orthobasis of $\mathbb{C}^{2,1}$. Precisely one of these basis vectors, let's denote it by \mathfrak{c} , has negative norm. Its projection P on \mathbb{B} is the only ball fixed point of δ .*

Proposition 5.3 *The set of $\Gamma(\sqrt{-3})$ -elliptic ball points coincides with the set of -1 points on the Eisenstein Congruence surface. It's the same to say: with the set of intersection points of two $\Gamma(\sqrt{-3})$ -reflection discs on \mathbb{B} .*

Proof. The inclusion \subseteq has already been proved with Lemma 3.17: namely the -1 point O contains the elliptic element $\delta = \text{diag}(\omega, \omega^2, 1)$. For any -1 point P one finds a $\gamma \in \Gamma(\sqrt{-3})$ transporting O to P . Then $\gamma\delta\gamma^{-1}$ stabilizes P , has the same eigenvalues as δ , is therefore elliptic. The inverse inclusion has been proved in my book [EPD], Ch.I, 1.4.5, over two pages, elementary but a littlebit tricky.

6 Smoothness of the Eisenstein Congruence Surface (ECS)

6.1 Smoothness of the open surface $\Gamma(\sqrt{-3})\backslash\mathbb{B}$

Corollary 6.1 *The projective stabiliser group $\text{Stab}_{\Gamma(\sqrt{-3})}(p)$ of each $\Gamma(\sqrt{-3})$ -elliptic point $p \in \mathbb{B}$ is isomorphic to $Z_3 \times Z_3$. It is generated by two $\Gamma(\sqrt{-3})$ -reflections. Therefore the image point P on $\Gamma(\sqrt{-3})\backslash\mathbb{B}$ is smooth. Hence the whole (open) quotient surface is smooth.*

Proof. We saw, that p must be a ballpoint of norm -1 and that it is the intersection of precisely two reflection disc (Proposition 5.3). Because of the Γ -equivalence of all -1 -points we can assume that $p = O$. We visualized the local situation around O in Picture (1.2). The reflections $\begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$ generate obviously $K9 = Z_3^2 \cong \text{Stab}_{\Gamma(\sqrt{-3})}(O)$. The image point of O , and also of p on the Congruence Surface is locally isomorphic to its image of O on $\mathbb{C}^2/\text{Stab}_{\Gamma(\sqrt{-3})}(O)$. The latter image point is a smooth one by the following old result:

Proposition 6.2 (Chevalley [Ch]). *Let G be a finite subgroup of $\mathbb{G}l_n(\mathbb{C})$, $p: \mathbb{C}^n \rightarrow \mathbb{C}^n/G$ the quotient map. Then $p(O)$ is a regular point if and only if G is generated by reflections.*

□

We remember to the definitions for arbitrary \mathbb{B} -lattices Δ and Δ -discs \mathbb{D} to its *normalizer* and *centralizer* group (see the text before Diagram (4) in Section 1):

$$N_{\Delta}(\mathbb{D}) = \{\alpha \in \Delta; \alpha(\mathbb{D}) = \mathbb{D}\}, Z_{\Delta}(\mathbb{D}) = \{\alpha \in \Delta; \alpha|_{\mathbb{D}} = \text{id}|_{\mathbb{D}}\}. \quad (17)$$

The effectively on \mathbb{D} acting group is: $\Delta_{\mathbb{D}} = N_{\Delta}(\mathbb{D})/Z_{\Delta}(\mathbb{D})$.

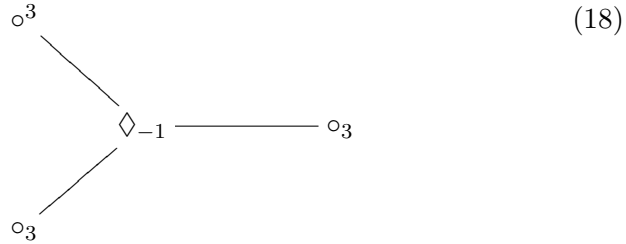
Proposition 6.3 *The image curve of any $\Gamma(\sqrt{-3})$ -reflection disc \mathbb{D} on the (open) Congruence Surface $\Gamma(\sqrt{-3})\backslash\mathbb{B}$ is smooth.*

Proof. Let $\sigma \in \Gamma' := \Gamma(\sqrt{-3})$ be a reflection and denote the corresponding reflection disc by $\mathbb{D} = \mathbb{D}_\sigma$. Knowing that the quotient curve $\Gamma'_\mathbb{D}\backslash\mathbb{D}$ is smooth we see that also $\Gamma'\backslash\mathbb{D} \subset \Gamma'\backslash\mathbb{B}$ has no singularities iff the Γ' -equivalence for points on \mathbb{D} is not stronger than the $N_{\Gamma'}(\mathbb{D})$ -equivalence. But this generally known for reflection discs (see [BSA], Lemma 4.5.2).

6.2 Smoothness at the cusps

We want not explain here the necessary notions and calculations for understanding things around cusps. We will shortly mention the things we used. The interested reader should consult my book [EPD] (for the ECS).

- Neat subgroup, toroidal compactifications, cusp bundle over elliptic curves, disc bundles over elliptic curves, their plugging in, smooth crossing fibres, cyclic quotients of elliptic cusp bundles, curve compactifications $\Delta\backslash\mathbb{B}$ around cusps.
- explicit calculations [EPD], yield the second graph type in [Ho83], Prop. 4.2):



This is a "dual graph" of four (irreducible) curves. The circles stand for reflection curves with (positive) indices announcing the branch order. The central diamond represents a compactifying curve with (negative) index, which means the selfintersection. The lines are interpreted as intersection points of the curves joint by them. So the cusp curve has three intersecting reflection curves. We know moreover that our ECS-cusp curves are rational.

Proposition 6.4 *All (four) cusp points of the ECS are non-singular. They are also regular points of each of (the six) reflection curves.*

Altogether with Propositions 6.1 and 6.3 we get

Theorem 6.5 *The BB-compactified Eisenstein Congruence Surface $\widehat{\Gamma(\sqrt{-3})\backslash\mathbb{B}}$ is smooth. All (six) (compact) reflection curves $\widehat{\Gamma(\sqrt{-3})\backslash\mathbb{D}}$ on it are smooth.*

6.3 Curve classification

Let \mathbb{D} be a $\Gamma(\sqrt{-3})$ -reflection disc on \mathbb{B} and C its (open) quotient curve $\Gamma(\sqrt{-3})\backslash\mathbb{D} \subset \Gamma(\sqrt{-3})\backslash\mathbb{B}$. The closure of the BB- compactified or curve compactified surface is denoted by \hat{C} or \bar{C} , respectively.

Exercise. Determine the Euler-Poincaré volume of a fundamental domain \mathcal{F} of the one-dimensional Eisenstein \mathbb{D} -lattice $\mathbb{P}\mathbb{U}((1, 1), \mathcal{O}) = \Gamma_{\mathbb{D}}$ and \mathcal{F}' of $\Gamma'_{\mathbb{D}} = \Gamma(\sqrt{-3})_{\mathbb{D}}$.

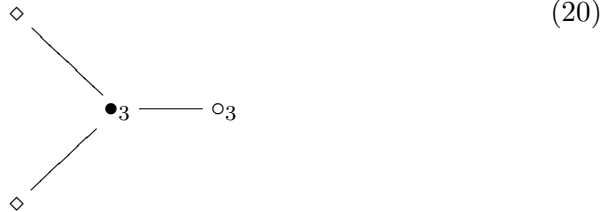
Answer (Feustel [Fe0]):

$$\int_{\mathcal{F}'} \gamma_1 = -2/3 \text{ with Euler form } \gamma_1 \text{ on } \mathbb{D} \quad (19)$$

Hint. Use a transfer of $\mathbb{U}((1, 1), \mathcal{O})$ to a $Sl_2(\mathbb{Z})$ -commensurable group acting on the upper half plane \mathbb{H} along a suitable biholomorphic fractional map $\mathbb{D} \xrightarrow{\sim} \mathbb{H}$. Relate it with the volume of a Sl_2 -fundamental domain on \mathbb{H} , which is equal to $-\frac{1}{6}$. For more details see [EPD], I.1.5.

Next we want to calculate Euler number $e(\bar{C})$ and signature $\sigma(\bar{C})$ of the $\Gamma(\sqrt{-3})\backslash\mathbb{B}$ - embedded compact reflection curve \bar{C} . A good geometric orientation is concentrated in the following

\bar{C} -Graph ([BSA], Figure in Example 4.7.6):



It's a "dual graph" again. The small circle and the diamonds stand for curves crossing \bar{C} represented by the central bullet. More precisely, the diamonds stand for cusp curves, the circle: for the reflection curve crossing the given one. The attached numbers 3 indicate the branch order (ramification index).

We engaged Euler number and selfintersection defining (1-dimensional) *orbital heights*: h_e the *Euler-height* and h_τ the *signature-height*. We only reduce the definition here to the cases, when at most two components of the irreducible branch curves are intersecting in any point of the curve compactified model of a Picard modular surface. I refer to the more general Definition 4.7.3 in my book [BSA]. Surface singularity data e_i, d_j can be simplified (see below) because our surface (ECS) is smooth. In [BSA] we wrote " e_f ", " τ_f " instead of $h_e, \frac{h_\tau}{3}$.

Definition 6.6 *The one-dimensional orbital Euler hight and orbital signature hight are defined as*

$$\begin{aligned} h_e(C) &= e(\overline{C}) - \sum_i \left(1 - \frac{1}{v_i d_i}\right) - \#\overline{C}_\infty \\ h_\tau(C) &= \frac{1}{v} \left[(\overline{C}^2) + \sum_i \frac{e_i}{d_i} + \sum_j \frac{e_j}{d_j} \right], \end{aligned} \tag{21}$$

where $e(\overline{C})$, (\overline{C}^2) are the Euler number respectively the selfintersection index of the reflection curve C . Singularity data are trivial: $e_i = 0, d_j = 1$. The branch order of C is denoted by v , while v_i is the branch order of the i -th curve crossing C .

The number of rational compactification curves crossing \overline{C} is denoted by $\#\overline{C}_\infty$. h_e is the algebraic expression for the Euler volume $\int_{\mathcal{F}'} \gamma_1$, and $h_\tau = \frac{1}{2} \cdot h_e$. The relations can be found in [BSA], (4.7.7). The latter one is known as *proportionality (Prop.1)*.

Now we can calculate with the help of (19), (??) and (21)

$$\begin{aligned} -\frac{2}{3} &= h_e(C) = e(\overline{C}) - \left(1 - \frac{1}{3 \cdot 1}\right) - 2, \\ -\frac{1}{3} &= h_\tau(C) = \frac{1}{3} \cdot \left[(\overline{C}^2) + 0 + 0 \right]. \end{aligned}$$

It follows that:

Lemma 6.7 *Any of the six reflection curves \overline{C} on the ECS have Euler number 2. So these are smooth rational curves. Moreover their selfintersection is equal to -1 .*

7 Surface classification

For brevity, e.g. at indices, we use following notations:

$$\Gamma = \mathbb{U}((2, 1), \mathcal{O}), \Gamma' = \Gamma(\sqrt{-3}), X = \Gamma \backslash \mathbb{B}, X' = \Gamma' \backslash \mathbb{B}$$

We defined in [BSA] two-dimensional *orbital hights* H_e, H_τ for any Picard modular surface. Let's pick out the easy variant of smooth curve compactified Picard modular surfaces and (smooth) branch curve on it without triple points. The irreducible compactification curves are assumed to be rational.

Definition 7.1 *The orbital Euler hight respectively signature hight look like this:*

$$\begin{aligned} H_e(X') &= E(\overline{X'}) - 2(\mathbf{1} - \mathbf{v}^{-1}) \cdot D \cdot \mathbf{t}_{\mathbf{v}^{-1}} - \frac{1}{2}(\mathbf{1} - \mathbf{v}^{-1}) \cdot S \cdot \mathbf{t}(\mathbf{1} - \mathbf{v}^{-1}) - t \\ H_\tau(X') &= \Sigma(\overline{X'}) - \frac{1}{3}(\mathbf{v} - \mathbf{v}^{-1}) \cdot D \cdot \mathbf{t}_{\mathbf{v}^{-1}} - \frac{1}{3}(T^2) \end{aligned} \tag{22}$$

Thereby T denotes the compactification divisor. Its number of components is denoted by t . D is the (diagonal) selfintersection matrix of branch curve components. S denotes the proper intersection matrix of the same curves, that means with selfintersections substituted by 0.

In the case of the Eisenstein Congruence Surface we have $t = 4$ by Corollary 3.3, and

$$D = \text{diag}(-1, -1, -1, -1, -1, -1)$$

by Lemma 6.7. With suitable numeration one gets for the six reflection curves on $\overline{X'}$:

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

because each component has only one intersecting partner, cp. Diagram 20 with Proposition 3.7 in the background.

The right hand sides in (22) can now be calculated:

$$\begin{aligned} E(\overline{X'}) &= \sum_{i=1}^6 \left(\left(1 - \frac{1}{3}\right) \cdot (-1) \cdot 1/3 \right) - 2 \cdot 4, -\frac{1}{2} \sum_{i=1}^6 \left(\left(1 - \frac{1}{3}\right) \cdot 1 \cdot \left(1 - \frac{1}{3}\right) \right) \\ \Sigma(\overline{X'}) &= \frac{1}{3} \sum_{i=1}^6 \left(\frac{8}{3} \cdot (-1) \cdot \frac{1}{3} \right) - \frac{1}{3} \cdot 4 \cdot (-1) \end{aligned} \tag{23}$$

The left hand sides of (22) are volumes of fundamental domains again:

$$\begin{aligned} &\int_{\mathcal{F}(\Gamma')} \gamma_2 \text{ with Euler form } \gamma_2 \text{ on } \mathbb{B} \\ &\int_{\mathcal{F}(\Gamma')} \tau_2 \text{ with signature form } \tau_2 \text{ on } \mathbb{B} \end{aligned}$$

For details we refer to [BSA] or [BHH]. There you find also the important Proportionality Theorem (Prop 2): [BSA],(4.9.1). Originally it comes from Hirzebruch's comparison of Chern forms of invariant metrics on symmetric domains, well-understandable summarized in [BHH]. Here we need the Proportionality Relation

$$\gamma_2 = 3\tau_2 \text{ consequently } H_e(X') = 3H_\tau(X').$$

Remarks 7.2 *The conclusion needs higher dimensional modern Riemann-Roch Theory in the sense of Grothendieck-Hirzebruch, see [BSA] with background in Hirzebruch's book [Hi].*

I determined the Euler volumes of full Picard modular groups at the second half of the 1970-s. They appear as L-series values, multiplied with an elementary number. A proof has been reproduced in my book [BSA],

Ch.V, 5A. Later, by means of functional equation they could be expressed in terms of (higher) Bernoulli numbers, see last formula in [Ho98]. Especially, we found

$$H_e(X') = \int_{\mathcal{F}(\Gamma')} \gamma_2 = 1/3 \quad , \quad H_\tau(X') = \int_{\mathcal{F}(\Gamma')} \tau_2 = 1/9,$$

see e.g. [BSA], proof of Lemma 5.2.2. We plug it into the left hand side of (22), calculate (23) to get

$$\frac{1}{3} = E(\overline{X'}) - \frac{20}{3} \quad , \quad \frac{1}{9} = \Sigma(\overline{X'}) + \frac{28}{9}.$$

It follows that $E(\overline{X'}) = 7$ and $\Sigma(\overline{X'}) = -3$.

Each of the four cusp lines on $\overline{X'}$ have selfintersection -1 . So, the birational morphism $\overline{X'} \rightarrow \widehat{X'}$ is the simultaneous blowing down of four exceptional curves to four (regular) cusp points. It induces classically the following change of Chern numbers:

$$E(\widehat{X'}) = E(\overline{X'}) - 4 = 3 \quad , \quad \Sigma(\widehat{X'}) = \Sigma(\overline{X'}) + 4 = 1.$$

Proposition 7.3 *The BB-compactification $\widehat{X'} = \Gamma(\sqrt{-3}) \backslash \mathbb{B}$ has the following Chern invariants:*

$$\text{Euler number } E(\widehat{X'}) = 3, \quad \text{Signatur } \Sigma(\widehat{X'}) = 1.$$

One derives from them two other Chern invariants, arithmetic genus and Selfintersection of the canonical class:

$$\chi = \frac{1}{4}(E + \Sigma) = 1 \quad , \quad (K^2) = 12\chi - E = 9.$$

Now we must remember to the following highlight theorem of the 1980-s:

Theorem 7.4 (Miyaoaka - Yau). *A smooth compact complex surface of general type is a ball quotient $\Delta \backslash \mathbb{B}$ (for a suitable cocompact ball lattice Δ) if and only if their Chern numbers (K^2) and E (Euler number) satisfy the relation*

$$(K^2) = 3E. \tag{24}$$

In this case there is no rational curve on the surface.

A simple resumé of Miyaoaka's and Yau's approach to the above theorem can be found in [BHH], Anhang B.2, F).

From classification theory of complex surfaces of Kodaira dimension < 2 one deduces (after careful check of the texts in [BPV] or [Shaf] around Chern invariants):

Proposition 7.5 *The only smooth compact complex algebraic surface, not of general type, satisfying condition (24) is the projective plane \mathbb{P}^2 .*

Corollary 7.6 *If there is a rational curve on the smooth compact complex surface satisfying Chern number relation (24), then it cannot be of general Type. It has to be isomorphic to the projective plane \mathbb{P}^2 .*

Main Theorem 7.7 *The Eisenstein Congruence Surface $\Gamma(\widehat{\sqrt{-3}})\backslash\mathbb{B}$ is the projective plane \mathbb{P}^2 . The four cusp points $K_i, i = 1..4$, are not collinear. The compactified branch locus of the quotient morphism $\Gamma(\sqrt{-3})\backslash\mathbb{B} \rightarrow \Gamma\backslash\mathbb{B}$ is the complete quadrilateral on \mathbb{P}^2 , visualized in Picture (1.2): Each component a line through two of the cusp points.*

Proof. We know by Proposition 7.3, that for our ECS the Chern relation (24) is satisfied. By Corollary 7.6 and Theorem 7.4 it cannot be a surface of general type. But then it follows from Proposition 7.5 that the BB-compactified Eisenstein Congruence Surface is the projective plane \mathbb{P}^2 .

The six reflection discs visualized in Picture (1.2) cover our the branch curves along the ball quotient morphism. On any of the above discs \mathbb{D} ly, up $\Gamma(\sqrt{-3})$ -equivalence, precisely two cusps. Hence the compactified image curve \overline{C} goes through precisely two cusp curves of $\overline{X'}$. We see in Lemma 6.7 that the selfintersection index (\overline{C}^2) is equal to -1 . Each blowing down curve crossing \overline{C} increases the selfintersection by 1, therefore $(\hat{C}^2) = (\overline{C}^2) + 2 = 1$. But \hat{C} is a smooth rational curve on \mathbb{P}^2 . There are only two possibilities: It is a quadric or an embedded line on the plane. The first case can be excluded because selfintersection of a quadric is equal to 4. The Main Theorem is proved. □

Chapter II. Heegner Reihen Picardscher Modulflächen

8 Orbitale Invarianten

8.1 Orbital Surface Hights

\mathcal{F} denotes the category of open Picard modular surfaces, but with only finite morphisms. More precisely, we pick out only the finite morphisms of \mathcal{F} supported by inclusions $\Gamma' \subseteq \Gamma$, that means the quotient morphisms $\Gamma'\backslash\mathbb{B} \twoheadrightarrow \Gamma\backslash\mathbb{B}$. For any imaginary quadratic number field K we let \mathcal{F}_K be the complete subcategory of all Picard modular surfaces over fixed K . Conversely, we can consider \mathcal{F} as disjoint union of all \mathcal{F}_K .

We can also built the categories $\widehat{\mathcal{F}}, \overline{\mathcal{F}}$ of BB-compactified or curve-compactified objects and morphisms from \mathcal{F} . If we correspond to each object/morphism its compactification we get natural isomorphisms $\mathcal{F} \xrightarrow{\sim} \overline{\mathcal{F}} \xrightarrow{\sim} \widehat{\mathcal{F}}$.

Definition 8.1 An orbital invariant on \mathcal{F} in a \mathbb{Q} -Algebra R is a degree compatible map $H : \mathcal{F} \rightarrow R$, t.m. it holds that

$$H(Y) = [Y : X] \cdot H(X) = \deg(F) \cdot H(X)$$

for all (finite) morphisms $F : \Gamma' \backslash \mathbb{B} = Y \rightarrow X = \Gamma \backslash \mathbb{B}$ of \mathcal{F} .

In other words the condition describes a contravariant numerical functor $\deg : \mathcal{F} \rightarrow R$, t.m. with commutative diagrams

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & H(Y) \equiv H(\mathbb{P}\Gamma') \\ \downarrow F & & \bullet \uparrow \deg F = [\mathbb{P}\Gamma : \mathbb{P}\Gamma'] \uparrow \\ X & \xrightarrow{\quad} & H(X) \equiv H(\mathbb{P}\Gamma), \end{array} \quad (25)$$

where \bullet is nothing but the multiplication map (with $\deg F$ - factor). Sometimes it is convenient to express the factor in terms of index of the starting groups: If $Z(\Gamma)$ denotes the (finite cyclic) center of Γ and w its order, then we have the obvious relation

$$[\mathbb{P}\Gamma : \mathbb{P}\Gamma'] = [\Gamma/Z(\Gamma) : \Gamma'/Z(\Gamma')] = [\Gamma : \Gamma'] \cdot [Z(\Gamma') : Z(\Gamma)] = [\Gamma : \Gamma'] \cdot \frac{w'}{w} \quad (26)$$

Important is the the multiplicativity sitting in the categorial definition:

$$\deg(F \circ F') = \deg(F) \cdot \deg(F')$$

for all $F' : Z \rightarrow Y$ in \mathcal{F} .

It is easy to transfer the notion to the (naturally isomorphic) compactified categories $\widehat{\mathcal{F}}$ and $\overline{\mathcal{F}}$. This should be done in mind by the reader. We denote e.g. simultaneously orbital hights by H on each of our open or compactified categories. Distinctions $H, \widehat{H}, \overline{H}$ are not necessary. We also use sometimes $H(\Gamma) = H(\mathbb{P}\Gamma) = H(\Gamma \backslash \mathbb{B})$ as in the above diagram.

The background are Haar mesures. Denote by G the Lie group defining the ball \mathbb{B} and acting on it, say $G = \mathrm{SU}((2, 1), \mathbb{C})$. Take a G -invariant metric μ on \mathbb{B} (known as *Bergmann metric* in Differential Geometry) and let $d\mu$ be the corresponding volume form on \mathbb{B} . The volumes $\mu(\Gamma) := \int_{\mathcal{F}(\Gamma)} d\mu$ of fundamental domains of ball lattices Γ are finite (by definition of lattice), especially for Picard modular groups. Each such volume form defines an orbital invariant on $\widehat{\mathcal{F}}$ in \mathbb{R} . We can choose Chern forms, especially the Euler or signature form, see Section 7. Definition 7.1.

Example 8.2 The orbital Euler hight $H_e(X') = \int_{\mathcal{F}(\Gamma')} \gamma_2$ and also the orbital Signatur hight $H_\tau(X') = \int_{\mathcal{F}(\Gamma')} \tau_2$

have been essentially used for questions around surface classifications. They can be expressed in terms of Riemann-Roch and singularity theory. As important example we presented the hights of the Eisenstein Congruence Surface in this manner, see Remarks 7.2.

8.2 Picard modular curves

Let Γ be again a Picard modular surface, say over K . Consider a K -disc $\mathbb{D} \subset \mathbb{B}$. The quotient curve $D_\Gamma := \Gamma \backslash \mathbb{D} \subset \Gamma \backslash \mathbb{B}$ is called a *Picard Modular curve* (PMC). All of them form the *category \mathcal{D} of Picard modular curves*. The morphisms are the finite coverings $\Gamma' \backslash \mathbb{D} \rightarrow \Gamma \backslash \mathbb{D}$ induced by a surface covering $\Gamma' \backslash \mathbb{B} \rightarrow \Gamma \backslash \mathbb{B}$. We illustrate the situation in the following diagram:

$$\begin{array}{ccc}
 D_{\Gamma'} \equiv \Gamma' \backslash \mathbb{D} & \hookrightarrow & \Gamma' \backslash \mathbb{B} \\
 \downarrow f & & \downarrow F \\
 D_\Gamma \equiv \Gamma \backslash \mathbb{D} & \hookrightarrow & \Gamma \backslash \mathbb{B}
 \end{array} \quad (27)$$

The compactified extensions $\widehat{D}_{\Gamma'} \rightarrow \widehat{D}_\Gamma$ or $\overline{D}_{\Gamma'} \rightarrow \overline{D}_\Gamma$ restricting \widehat{F} or \overline{F} , respectively, form the categories $\widehat{\mathcal{D}}$ or $\overline{\mathcal{D}}$.

Definition 8.3 $p \in \mathbb{D}$ is called a Γ -cross point (of \mathbb{D}) iff there exists an element $\gamma \in \Gamma \setminus N_\Gamma(\mathbb{D})$ such that $p \in \mathbb{D} \cap \gamma \mathbb{D}$.

In other words: γ moves \mathbb{D} but not p . Or: The image point P of p along $\mathbb{D} \rightarrow D_\Gamma = \Gamma \backslash \mathbb{D}$ is a curve singularity of D_Γ . Or: The Γ -equivalence for points on \mathbb{D} is stronger than the $N_\Gamma(\mathbb{D})$ -equivalence.

Definition 8.4 A sublattice Γ' (of finite index) of Γ is called \mathbb{D} -neat if it is neat and has no Γ' -cross point.

In other words: for any $\gamma' \in \Gamma'$ it holds that $\gamma' \mathbb{D} = \mathbb{D}$ or $\mathbb{D} \cap \gamma' \mathbb{D} = \emptyset$. For a discussion we refer the reader to the original Definition 4.4.7 in my book [BSA] and the text around. There one finds the following

Facts 8.5 .

- For any K -disc $\mathbb{D} \subset \mathbb{B}$ and Picard modular group Γ (over K) there exists a \mathbb{D} -neat sublattice Γ' of Γ .
- Any Picard curve $D_\Gamma = \Gamma \backslash \mathbb{D}$ (as above) has only finitely many curve singularities.

We set shortly $v = |Z_\Gamma(\mathbb{D})|$. It's nothing else but the branch order of the covering (reflection order, ramification index) of the covering F in Diagram (27) at the quotient curve D_Γ . The degree of the curve covering $D_{\Gamma'}/D_\Gamma$ can be expressed as

$$[D_{\Gamma'} : D_\Gamma] = [\Gamma_\mathbb{D} : \Gamma'_\mathbb{D}] = \frac{[N_\Gamma(\mathbb{D}) : N_{\Gamma'}(\mathbb{D})]}{[Z_\Gamma(\mathbb{D}) : Z_{\Gamma'}(\mathbb{D})]} = \frac{[N_\Gamma(\mathbb{D}) : N_{\Gamma'}(\mathbb{D})]}{v : v'}$$

with obvious notation $v' = |Z_{\Gamma'}(\mathbb{D})|$.

9 Orbital Curve Categories

For Riemann-Roch calculations around curves D_Γ we need the embedding of the compact curve $\overline{D_\Gamma}$ in an open neighbourhood $U \subseteq \overline{\Gamma \backslash \mathbb{B}}$. Namely, for orbital invariants we need data of surface and curve singularities living on $\overline{D_\Gamma}$, moreover the branch order of the curve with respect to \mathbb{D} and Γ . Instead of smaller neighbourhoods - as used in earlier papers, e.g. in [Ho02] - it suffices to take only the big one $\overline{U} = \overline{\Gamma \backslash \mathbb{B}}$. So we thicken our open curves to \mathbf{D}_Γ , which is nothing else but the curve together with its embedding:

$$\mathbf{D}_\Gamma : \quad D_\Gamma \hookrightarrow \Gamma \backslash \mathbb{B}$$

Definition 9.1 *The thickened curve \mathbf{D}_Γ is called an orbital Picard modular curve (on $\Gamma \backslash \mathbb{B}$).*

An orbital covering is an embedding pair $\mathbf{f} = (f, F)$ as in Diagram (27). More simply, it is an embedding diagram

$$\begin{array}{ccc} D_{\Gamma'} & \hookrightarrow & \Gamma' \backslash \mathbb{B} \\ f \downarrow & \mathbf{f} & \downarrow F \\ D_\Gamma & \hookrightarrow & \Gamma \backslash \mathbb{B} \end{array} \quad (28)$$

On this way we built the category \mathfrak{D} of open orbital Picard curves together with orbital coverings as defined above as (only) morphisms. Via compactifications we get in obvious manner the orbital categories $\widehat{\mathfrak{D}}$ and $\overline{\mathfrak{D}}$. If we fix the imaginary quadratic field K , then then we get complete subcategories $\mathfrak{D}_K, \widehat{\mathfrak{D}}_K, \overline{\mathfrak{D}}_K$ of orbital Picard modular curves over K .

The diagram

$$\begin{array}{ccc} D_{\Gamma''} & \hookrightarrow & \Gamma'' \backslash \mathbb{B} \\ g \downarrow & \mathbf{g} & \downarrow G \\ D_{\Gamma'} & \hookrightarrow & \Gamma' \backslash \mathbb{B} \\ f \downarrow & \mathbf{f} & \downarrow F \\ D_\Gamma & \hookrightarrow & \Gamma \backslash \mathbb{B} \end{array} \quad (29)$$

describes the multiplicativity of orbital morphisms in full details:

$$\mathbf{f} \circ \mathbf{g} = (f, F) \circ (g, G) = (f \circ g, F \circ G) = \mathbf{fg}.$$

9.1 Orbital invariants for orbital curves

Definition 9.2 *A (rational) orbital invariant*

$$\mathbf{h} : \mathfrak{D} \longrightarrow \mathbb{Q}$$

is a multiplicative numerical functor on \mathfrak{D} satisfying

$$\mathbf{h}(\mathbf{D}) = [\mathbf{D} : \mathbf{C}] \cdot \mathbf{h}(\mathbf{C}) \text{ where } [\mathbf{D} : \mathbf{C}] = \frac{[D : C]}{w : v} = \frac{[\Gamma_{\mathbb{D}} : \Gamma'_{\mathbb{D}}]}{w : v}$$

for orbital curve coverings $\mathbf{D} \twoheadrightarrow \mathbf{C}$ in \mathfrak{D} with branch orders $v = v_{\mathbf{C}}$ and $w = v_{\mathbf{D}}$. The numbers v, w are also called (orbital) weights.

The single value $\mathbf{h}(\mathbf{C})$ is called the orbital hight (w.r.t. \mathbf{h}) of the orbital curve $\mathbf{C} = \mathbf{D}_{\Gamma}$.

Example 9.3 Take a curve-degree compatible numerical hight functor $h : \mathcal{D} \rightarrow \mathbb{Q}$. This means that for all covering $f : D_{\Gamma'} \twoheadrightarrow D_{\Gamma}$ in \mathcal{D} it holds that

$$h(D_{\Gamma'}) = [D_{\Gamma'} : D_{\Gamma}] \cdot h(D_{\Gamma}).$$

We orbitalize it to the obital invariant $\mathbf{h} : \mathfrak{D} \rightarrow \mathbb{Q}$ setting

$$\mathbf{h}(\mathbf{D}) = \frac{1}{w} h(D).$$

It is easy to check that (with notations of Definition 9.2)

$$\mathbf{h}(\mathbf{D}) = \frac{1}{w} h(D) = \frac{[D : C] \cdot h(C)}{w} = \frac{[D : C] \cdot v \cdot \mathbf{h}(\mathbf{C})}{w} = [\mathbf{D} : \mathbf{C}] \cdot \mathbf{h}(\mathbf{C}).$$

So we orbitalize the Euler and signature hights in Definition 6.6 to *orbital Euler* respective *signature invariant* on \mathfrak{D} . The second formula there shows that the branch orders are very necessary. This is the reason for the change from simple hights to (fat) orbital invariants. The branch orders will be most important in the next section preparing Heegner series.

10 Orbital Intersection Products

It becomes convenient to work in the category $\widehat{\mathfrak{D}}$ of BB-compactified orbital Picard curves. We restrict ourselves to a fixed imaginary quadratic number field K . So we work only with Picard modular groups Γ commensurable with the full one $\mathbb{U}((2, 1), \mathcal{O}_K)$ and the compactified quotient surfaces $\widehat{X}_{\Gamma} = \widehat{\Gamma \backslash \mathbb{B}}$. Each K -disc \mathbb{D} defines an (embedded) Orbital Picard Modular Curve $\widehat{\mathbf{D}}_{\Gamma} : \widehat{D}_{\Gamma} \hookrightarrow \widehat{X}_{\Gamma}$ in $\widehat{\mathfrak{D}}_K$, cp. Definition 9.1.

We orbitalize first the well-known divisor groups

$$\text{Div } S = \bigoplus_{C \subset S} \mathbb{Q} \cdot C$$

on compact algebraic surfaces S (where C runs through all irreducible compact algebraic curves on S). There is a nice intersection product on complex

compact normal algebraic surfaces for the generating curves C . The definition goes back to Mumford, (later Fulton). It can be \mathbb{Q} -linearly extended to a \mathbb{Q} -bilinear form

$$(\cdot) : \text{Div } S \times \text{Div } S \longrightarrow \mathbb{Q}.$$

For orbitalization we fix Γ and define on $\widehat{X} = \widehat{X}_\Gamma$ as above the formal \mathbb{Q} -vector space

$$\mathbf{Div } \widehat{X} = \bigoplus_{\widehat{D}_\Gamma \subset \widehat{X}_\Gamma} \mathbb{Q} \cdot \widehat{\mathbf{D}}_\Gamma$$

generated by the OPM curves $\widehat{\mathbf{D}}_\Gamma$ on \widehat{X}_Γ , see Definition 9.1 again.

For two OPM curves $\widehat{\mathbf{C}}, \widehat{\mathbf{D}}$ of orbital weight w respectively v on \widehat{X} we define the *orbital intersection*

$$(\widehat{\mathbf{C}} \cdot \widehat{\mathbf{D}}) = \frac{(\widehat{C} \cdot \widehat{D})}{v \cdot w}$$

It can be \mathbb{Q} -bilinearly extended to the *orbital intersection* on the OPM divisor group:

$$\mathbf{Div } \widehat{X} \times \mathbf{Div } \widehat{X} \longrightarrow \mathbb{Q}$$

For any neat Picard modular congruence subgroup $\Gamma_0 = \Gamma(\mathfrak{a}) \subseteq \Gamma$ the curve intersections of curves on the (smooth) compactified model $\overline{\Gamma_0 \backslash \mathbb{B}}$ have been also well-understood by Cogdell [Cog]. Locally, everything is clear. We use finite coverings to push down intersection products from neat surface models to arbitrary OPM surfaces. It's a longer procedure through Riemann-Roch, curve and surface singularities to realize this way in terms of geometric local and global Galois-Theory. For details we refer to [Ho02] and [Ho07].

Important was the construction of direct and inverse images along finite coverings $F : \widehat{Y} \rightarrow \widehat{X}$ of PM surfaces

$$\mathbf{F}_\# : \mathbf{Div } \widehat{Y} \longrightarrow \mathbf{Div } \widehat{X}, \quad \mathbf{F}^\# : \mathbf{Div } \widehat{X} \longrightarrow \mathbf{Div } \widehat{Y}$$

We proved the *Orbital Projection Formula*

$$(\mathbf{F}_\# \mathbf{B} \cdot \mathbf{A}) = (\mathbf{B} \cdot \mathbf{F}^\# \mathbf{A}),$$

where $\mathbf{A} \in \mathbf{Div } \widehat{X}$, $\mathbf{B} \in \mathbf{Div } \widehat{Y}$ are cycles on the PM surfaces \widehat{X} or \widehat{Y} , respectively.

This is the escalator we need for shifting intersection products from neat PM surfaces to non-neats and vice versa along finite coverings: From well-understood to less-understood intersection products.

11 Orbital Heegner Functionals

We consider (rational) functionals

$$\mathbf{f}_{\hat{X}} : \mathbf{Div} \hat{X} \longrightarrow \mathbb{Q}$$

on orbital divisor groups on a PM surface \hat{X} .

Definition 11.1 *We call a set $\check{\mathbf{f}} = \{\mathbf{f}_{\hat{X}}; \hat{X} \in \hat{\mathcal{D}}_K\}$ an orbital functional on $\hat{\mathcal{D}}_K$ iff it is compatible with orbital direct images along finite coverings $F : \hat{Y} \rightarrow \hat{X}$. This means that $\mathbf{f}_{\hat{X}} \circ F_{\#} = \mathbf{f}_{\hat{Y}}$ holds for all finite orbital coverings \mathbf{F} in $\hat{\mathcal{D}}_K$.*

We repeat the definition of the orbital Heegner functional presented first in [Ho02], 3.4. First remember to norms $n(\mathbb{D}) \in \mathbb{N}_+$ of K -discs \mathbb{D} . You find it in Subsection 3.3, Definition 3.6, only for $K = \mathbb{Q}(\sqrt{-3})$. But it works also in all cases, when \mathcal{O}_K is an principal domain: \mathbb{D} is the ortho-disc of a primitive \mathcal{O} -vector \mathbf{a} and $n(\mathbb{D}) = n(\mathbf{a})$. We set also for the image curve \mathcal{D}_Γ on $X = X_\Gamma = \Gamma \backslash \mathbb{B}$ (also for smooth and other compactified models)

$$n(\hat{\mathcal{D}}_\Gamma) = n(\mathcal{D}_\Gamma) = n(\Gamma \backslash \mathbb{D}) = n(\mathbb{D}).$$

Definition 11.2 *For $N \in \mathbb{N}_+$ we call the reduced Weil-divisor*

$$H_N = H_N(\hat{X}_\Gamma) = \sum_{n(\hat{\mathcal{D}}_\Gamma)=N} \hat{\mathcal{D}}_\Gamma \quad (30)$$

the Heegner divisor of norm N on \hat{X}_Γ , (set 0, if the sum is void). Taking the orbitalized curves in the sum, we get the orbital Heegner divisors $\mathbf{H}_N = \mathbf{H}_N(\hat{X}_\Gamma)$.

Moreover we introduce on \hat{X}_Γ the orbital Heegner functionals

$$\mathbf{h}_N : \mathbf{Div} \hat{X} \longrightarrow \mathbb{Q}, \hat{\mathbf{C}} \mapsto \langle \hat{\mathbf{C}}, \mathbf{H}_N \rangle$$

We construct from them the formal series (with independent variable q)

$$\mathbf{H}(q) = \sum_{N=1}^{\infty} \mathbf{h}^N \cdot q^N$$

We apply now simultaneously the functionals to an irreducible orbital curve $\hat{\mathbf{C}}$ on \hat{X} to get the formal power series

$$\mathbf{H}_{\hat{\mathbf{C}}}(q) = \sum_{N=1}^{\infty} \mathbf{h}_N(\hat{\mathbf{C}}) \cdot q^N \in \mathbb{Q}[[q]] \quad (31)$$

It is called the *formal orbital Heegner series* for $\widehat{\mathbf{C}}$. In [Ho02] between Definitions 3.5 and 3.6 we included also \mathbf{h}_0 , but it doesn't play any role in the actual article. It is componentwise clear that

$$\mathbf{H}_{\widehat{\mathbf{D}}}(q) = [\widehat{\mathbf{D}} : \widehat{\mathbf{C}}] \cdot \mathbf{H}_{\widehat{\mathbf{C}}}(q) \quad (32)$$

for any orbital curve covering $\widehat{\mathbf{D}}$ of $\widehat{\mathbf{C}}$, see also [Ho02], (11).

We fill our formal series with more life substituting q by $e^{2\pi i\tau}$, $\tau \in \mathbb{H}$, the upper half plane.

Definition 11.3 *The series*

$$\text{Heeg}_{\widehat{\mathbf{C}}}(\tau) = h_0 + \sum_{N=1}^{\infty} \mathbf{h}_N(\widehat{\mathbf{C}}) \cdot q^{2\pi i N \tau}$$

(with h_0 uniquely defined, see Remark 11.6, below) is called the orbital Heegner series of the orbital Picard curve $\widehat{\mathbf{C}}$.

11.1 Modular forms of Nebentypus

Remember the classical congruence subgroups ($m \in \mathbb{N}_+$)

$$\Gamma_0(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sl}_2(\mathbb{Z}); c \equiv 0 \pmod{m} \right\}$$

of the modular group $\text{Sl}_2(\mathbb{Z})$ acting on the upper half plane \mathbb{H} . Let, moreover,

$$\chi = \chi_K : \mathbb{Z} \rightarrow \{0, \pm 1\}, \quad h \mapsto \left(\frac{-D_{K/\mathbb{Q}}}{h} \right) \text{ (Legendre symbol)}$$

be the Dirichlet character of the imaginary quadratic number field K . It factorizes through the residue ring modulo the discriminant $D_{K/\mathbb{Q}}$.

Definition 11.4 *A holomorphic function $f(\tau)$ on \mathbb{H} is called modular form of weight $k \in \mathbb{N}$, level $N \in \mathbb{N}$ and Nebentypus χ , iff it satisfies the following functional equations:*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \chi(d)^k f(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N), \quad (33)$$

(and f has meromorphic extensions to the cusps). If it is 0 at the cusps, we call it a *cusp form*. The space of modular forms of weight k , level N and Dirichlet character χ will be denoted by $M_k(N, \chi)$. It is a finite-dimensional \mathbb{C} -vector space.

In my paper [Ho07], 7.4 you find the following

Theorem 11.5 *With suitable $h_0 = h_0(\widehat{\mathbf{C}})$ the Heegner series $\text{Heeg}_{\widehat{\mathbf{C}}}(\tau)$ of a Picard curve $\widehat{\mathbf{C}}$ of the field K is a modular form belonging to $M_k(m, \chi_K)$.*

Remark 11.6 Since a non-zero constant cannot satisfy (31), it does not belong to $M_k(m, \chi_K)$. Therefore the constant h_0 is uniquely determined by the modular property of the series. It can be expressed in terms of the orbital Euler number of $\widehat{\mathbf{C}}$ (Cogdell, [Cog]).

Remark 11.7 The above theorem was first proved by Cogdell in [Cog] for neat congruence subgroups. The extension of the result to all Picard modular groups one can find in my papers [Ho02] and [Ho07].

12 Hecke's explicit Construction of Modular Forms of Nebentypus

Hecke's notation of the space of *modular forms* of Γ of *weight* k , where Γ is a subgroup of finite index of $\subseteq \mathrm{Sl}_2(\mathbb{Z})$, is:

$$M_k(\Gamma) = \left\{ f \in \mathrm{Hol}(\mathbb{H}); f \left(\frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k f(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \right\}. \quad (34)$$

Keep in mind the meromorphic condition at cusps, after Definition 11.4 for all modular forms considered in this paper.

Well-defined in many textbooks about modular forms is the congruence subgroup

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \pmod{N} \right\} \subseteq \mathrm{Sl}_2(\mathbb{Z})$$

Obviously, we have the exact sequence of groups:

$$1 \longrightarrow \Gamma_1(N) \longrightarrow \Gamma_0(N) \longrightarrow (\mathbb{Z}/N\mathbb{Z})^* \longrightarrow 1$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d$$

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z})$ Hecke uses the operation (from right) $f|_{\gamma_k} := (c\tau + d)^{-k} f(\gamma(\tau))$, see also [Sh], § 2.1. Then the Definition in (34) is extended to modular forms of *Nebentypus* χ .

$$M_k(N, \chi) = \left\{ f \in M_k(\Gamma_1(N)); f|_{\gamma_k} = \chi(d) \cdot f, \forall \gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z}) \right\} \quad (35)$$

It's easy to verify that the functional conditions in (33) and (34) are the same. Helpful are the following facts:

- For odd primes $N = p$ and $k \equiv \frac{p-1}{2}$ is $\chi = \left(\frac{\bullet}{p} \right)$ the quadratic residue modulo p ([He59], p. 810). Especially $M_3(3, \chi) = M_3(3, \left(\frac{\bullet}{3} \right))$.
- $\left(\frac{\bullet}{3} \right) = \left(\frac{3}{\bullet} \right) = \left(\frac{-3}{\bullet} \right)$ (for MAPLE-application "jacobi(-3, n)")
- In $M_3(3)$ there exists no (non-zero) cusp form of Nebentypus $\left(\frac{3}{\bullet} \right)$, see [He59], Prop.10, p.817.

- There are precisely two linearly independent Eisenstein series in $M_3(3)$ of Nebentypus $\left(\frac{3}{\bullet}\right)$, see [He59], Prop.12, p.818 (with $q = e^{2\pi i\tau}$ according to (20), p.811)

$$E_1(\tau) = \sum_{n=1}^{\infty} c_1(n)q^n \quad , \quad c_1(n) = \sum_{0 < d|n} d^2 \chi(n/d)$$

$$E_2(\tau) = A + \sum_{n=1}^{\infty} c_2(n)q^n \quad , \quad c_2(n) = \sum_{0 < d|n} d^2 \chi(d)$$

(We don't need Hecke's explicit determination of $A = -\frac{3^{5/2}}{(2\pi)^3} \cdot 2 \cdot \sum_{n=1}^{\infty} \chi(n)n^{-3}$).

$$E_1 = \quad q + 3q^2 + 9q^3 + O(q^4)$$

$$E_2 = A + \quad q - 3q^2 + q^3 + O(q^4)$$

13 Heegner series for Picard-curves of Eisenstein type

Let Γ be a Picard modular group of the field $K = \mathbb{Q}(\sqrt{-3})$ of Eisenstein numbers and \widehat{X}_Γ the BB-compactified quotient surface $\Gamma \backslash \mathbb{B}$. For simplicity we assume that Γ is a sublattice of $\Gamma(\sqrt{-3})$. Take moreover one of the discs \mathbb{D} of norm 1, e.g. the orthogonal disc of $\mathfrak{d}_{2,4} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in Definition 2.4 visualized in Picture (1.2) joining the cusps -1 and $+1$. The embedded quotient curve of \mathbb{D} on \widehat{X}_Γ is denoted by $\widehat{\mathbf{D}}$, that on $\mathbb{P}^2 = X_{\Gamma(\sqrt{-3})}$ by $\widehat{\mathbf{T}}$.

According to Definition 11.3 we have to calculate the first coefficients $\mathbf{h}^{\mathbf{N}}(\widehat{\mathbf{T}}) = \langle \widehat{\mathbf{T}}, \mathbf{H}_{\mathbf{N}} \rangle$ of the Heegner series. Since $\widehat{\mathbf{T}}$ is a line on the projective plane \mathbb{P}^2 its intersection with any plane curve B coincides with the degree $\deg B$. Moreover we know the weight $v(\widehat{\mathbf{T}}) = 3$ (branch index). So

$$\langle \widehat{\mathbf{T}}, \widehat{\mathbf{C}} \rangle = \frac{1}{3} \cdot \deg(\widehat{\mathbf{C}}). \quad (36)$$

We know all curves of norm 1, namely: $\widehat{\mathbf{C}}_j, j = 1..6$. These are the lines of the quadrilateral visualized in Picture (2.2). They are also the components of branch locus of the covering $\widehat{\mathbb{B}} \rightarrow \widehat{X}_{\Gamma(\sqrt{-3})}$. For any irreducible plane curve $\widehat{\mathbf{C}}$ one gets

$$\langle \widehat{\mathbf{T}}, \widehat{\mathbf{C}} \rangle = \frac{1}{3} \cdot \frac{\deg(\widehat{\mathbf{C}})}{v_{\widehat{\mathbf{C}}}} = \begin{cases} \frac{1}{9}, & \text{if } \widehat{\mathbf{C}} = \widehat{\mathbf{C}}_j, j = 1..6 \\ \frac{1}{3} \cdot \deg(\widehat{\mathbf{C}}), & \text{else} \end{cases}$$

So

$$\begin{aligned} \langle \widehat{\mathbf{T}}, \mathbf{H}_1 \rangle &= \langle \widehat{\mathbf{T}}, \mathbf{C}_1 \rangle + \dots + \langle \widehat{\mathbf{T}}, \mathbf{C}_6 \rangle = \frac{2}{3} \\ \langle \widehat{\mathbf{T}}, \mathbf{H}_2 \rangle &= \frac{1}{3} \cdot \deg(H_2) \end{aligned} \quad (37)$$

because H_2 has no component of the branch locus. To get the Heegner series as explicit linear combination of Hecke's Eisenstein series $Heeg_{\widehat{\mathbf{T}}}(\tau)$ as linear combination of Hecke's Eisenstein series E_1, E_2 (see end of Section 12) we must know:

- A) $\dim M_3(3, \chi) = 2$;
- B) H_2 is the sum of three lines (of weight 1).

From B) and (37) it follows that

$$\langle \widehat{\mathbf{T}}, \mathbf{H}_2 \rangle = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$

Now we have to solve

$$x \cdot E_1 + y \cdot E_2 = \dots + (x + y)q + (3x - 3y)q^2 + \dots = \dots + \frac{2}{3}q + \frac{3}{3}q^2 + \dots$$

With the unique solution $x = \frac{3}{2}, y = \frac{1}{2}$ we receive

$$\begin{aligned} 3 \cdot Heeg_{\widehat{\mathbf{T}}}(\tau) &=: Heeg_{\widehat{\mathbf{T}}}(\tau) = \\ &= \frac{3}{2} \cdot E_1 + \frac{1}{2} \cdot E_2 = \dots + 2q + 3q^2 + 14q^3 + \dots + \langle \widehat{\mathbf{T}}, \mathbf{H}_N \rangle q^N + \dots \\ &= \dots + \langle \widehat{\mathbf{T}}, \mathbf{H}_1 \rangle q + \langle \widehat{\mathbf{T}}, H_2 \rangle q^2 + \langle \widehat{\mathbf{T}}, \mathbf{H}_3 \rangle q^3 + \dots \end{aligned}$$

With the explicit knowledge of Hecke's two basic Eisenstein series E_1, E_2 (end of Section 12) we can determine precisely each q^N -coefficient ($N > 0$) of the Heegner series. for the first 8 we calculated: ⁵

$$Heeg_{\widehat{\mathbf{T}}}(\tau) = a + 2q + 3q^2 + 14q^3 + 26q^4 + 24q^5 + 39q^6 + 100q^7 + 51q^8 + \dots \quad (38)$$

Theorem 13.1 *The Heegner series of the Picard line $\widehat{\mathbf{T}} \hookrightarrow \mathbb{P}^2$ (of norm 1) is*

$$Heeg_{\widehat{\mathbf{T}}}(\tau) = a + \sum_{n=1}^{\infty} \left\{ \sum_{0 < d|n} d^2 \cdot \left[\left(\frac{d}{3} \right) + \left(\frac{n/d}{3} \right) \right] \right\} \cdot q^n \quad (39)$$

We have to prove the items A) and B). We start with the

Proof of B). The dimension of the vector space $M_3(\Gamma_0(3), \chi)$ will be calculated with a formula of Cohen/Oesterlé [C-O].

Proposition 13.2 ([C-O]) *For $k \in \mathbb{Z}$ it holds that*

$$\begin{aligned} \dim S_k(\Gamma_0(N, \chi)) - \dim M_{2-k}(\Gamma_0(N, \chi)) &= \\ &= \frac{k-1}{12} N \prod_{p|N} \left(1 + \frac{1}{p}\right) - \frac{1}{2} \prod_{p|N} \lambda(r_p, s_p, p) \\ &+ \varepsilon_k \cdot \sum_{\substack{x \pmod N \\ x^2 \equiv -1 \pmod N}} \chi(x) + \mu_k \cdot \sum_{\substack{x \pmod N \\ x^2 + x \equiv -1 \pmod N}} \chi(x) \end{aligned}$$

⁵We determined the constant term with help of Hecke's article [He40] §3: $a = -1/18$

Thereby is r_p the exponent of p in the prime decomposition of N , and s_p is the p -exponent of the f uhrer $f(\chi)$ of the character χ . Moreover the authors of [C-O] defined

$$\lambda(r_p, s_p, p) = \begin{cases} p^{r'} + p^{r'-1}, & \text{if } 2s_p \leq r_p = 2r' \\ 2p^{r'}, & \text{if } 2s_p \leq r_p = 2r' + 1 \\ 2p^{r_p - s_p}, & \text{if } 2s_p > r_p \end{cases}$$

$$\varepsilon_k = \begin{cases} 0, & \text{if } k \text{ odd} \\ \frac{-1}{4}, & \text{if } 2s_p \leq r_p = 2r' + 1 \\ \frac{1}{4}, & \text{if } 2s_p > r_p \end{cases}$$

$$\mu_k = \begin{cases} 0, & \text{if } k \equiv 1 \pmod{3} \\ \frac{-1}{3}, & \text{if } k \equiv -1 \pmod{3} \\ \frac{1}{3}, & \text{if } k \equiv 0 \pmod{3} \end{cases}$$

Corollary 13.3 *It follows that $\dim M_3(\Gamma_0(3), \chi_3) = 2$.*

Proof of A)⁶. We have to set $N = 3, k = -1, \chi = \chi_3$. Then:

$$f(\chi_3) = 3, r_3 = 1, s_3 = 1, \varepsilon_{-1} = 0, \mu_{-1} = \frac{-1}{3} \text{ and } \dim S_{-1}(\Gamma_0(3), \chi_3) = 0,$$

hence

$$0 - \dim M_3(\Gamma_0(3), \chi_3) = \frac{-2}{12} \cdot 3 \cdot \left(1 + \frac{1}{3}\right) - \frac{1}{2} \lambda(1, 1, 3) + 0 - \frac{1}{3} \cdot \sum_{\substack{x \pmod{3} \\ x^2 + x \equiv -1 \pmod{3}}} \chi_3(x);$$

finally: $-\dim M_3(\Gamma_0(3), \chi_3) = \frac{-2}{3} - \frac{1}{2} \cdot 2 - \frac{1}{3} \chi_3(1) = -2.$

□

Proof of B). It is easy to see, that in $\mathbb{F}^{2,1}$ the only vectors of \mathbb{F} -norm -1 are

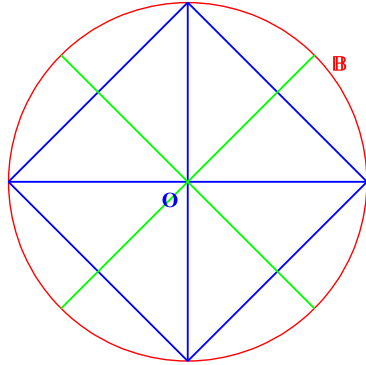
$$\pm \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Preimages in $\mathcal{O}^{2,1}$ of norm 2 along the residue map are

$$\pm \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} \sqrt{-3} \\ 0 \\ 1 \end{pmatrix} \quad (40)$$

All other norm 2 vectors in $\mathcal{O}^{2,1}$ are $\Gamma(\sqrt{-3})$ -congruent to one of them. We can draw two of the three orthodiscs of the above vectors (green):

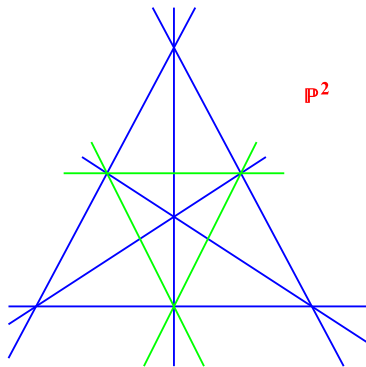
⁶The proof (including reference [C-O]) was found by my Ex-Doctorand Christian Sch on



(41)

The (green) diagonal disc \mathbb{D} is a Γ -reflection disc, pointwise fixed by the reflection $\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The quotient curves of the three orthodiscs of the vectors (40) on the Eisenstein Congruence Surface \mathbb{P}^2 are all three visible. They must be S_4 -reflection lines on \mathbb{P}^2 . For instance σ goes down to an effectively on \mathbb{P}^2 acting element of order 2 leaving pointwise fixed the (green) image line L of \mathbb{D} . It goes through two of the double points of the quadrilateral (look back to Picture (1.2) and also to Proposition 3.5 at the end of Subsection 3.2):



(42)

Altogether: The Heegner divisor H_2 has precisely 3 curves on \mathbb{P}^2 . They do not belong to the $\Gamma(\sqrt{-3})$ -branch divisor on \mathbb{P}^2 . Therefore

$$H_2 = L_1 + L_2 + L_3.$$

□

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