# Coupled Electromagnetic Field and Electric Circuit Simulation: A Waveform Relaxation Benchmark

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**Abstract** We consider coupled dynamical systems, arising from electric circuit coupled electromagnetic devices. The resulting subsystems are an ordinary differential equation reflecting the spatially discretized electromagnetic field equations, see [15] and a differential-algebraic equation describing the equations of a lumped circuit obtained by the modified nodal analysis, e. g. see [11]. Notice that the systems dimension may easily reach millions of unknowns motivating the need of different methods such as the waveform relaxation method, see [12]. We discuss how to improve the convergence behavior in terms of Gauss-Seidel approach by changing the coupled system's formulation. Tests show, that an acceleration of magnitudes can be reached.

## 1 Introduction

As we consider coupled electric circuit and electromagnetic devices, as in previous works, e. g. [19], we first focus ourselves on the modeling and analysis of lumped circuits in Sec. 2 where we extend the decoupling theorem presented in [17] to fit in our framework. Next, in Sec. 3 we briefly introduce electromagnetic devices and their modeling as well as their spatial discretization before turning to the coupled modeling in Sec. 4, which mainly builds upon the approach presented in [2]. Here, we propose different formulations for the resulting coupled systems. With these

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preliminaries established, we then focus on the analysis of a waveform relaxation method, cf. [12], applied on one of these systems in Sec. 5. The subsequent section deals with numerical experiments where we observe an enormous impact of the coupling formulation onto the waveform relaxation method's convergence behavior. This work is concluded by Sec. 6 where we give an outlook on future work.

## 2 Electric Circuits

First we focus on so-called lumped circuits that rely on the assumption that the circuit elements do not interfere with each other due to their spatial distance, see [6]. We cover their modeling and the analysis of the resulting equations, especially with regards to their decoupling.

### *2.1 Modeling*

In the simulation of electrical circuits within the framework of industrial applications, the so-called modified nodal analysis (MNA) is a frequently employed approach, cf. [4], to which we give a brief introduction. This modeling philosophy has (like many others) an intimate relation to graph theory motivating a special consideration when it comes down to implementation, see e. g. [22].

We start with the observation that an electrical circuit can be interpreted topologically as an oriented, connected hypergraph $\mathscr{H} = (\mathscr{V}, \mathscr{E})$, consisting of a vertex set $\mathscr{V}$ and a hyperedge set $\mathscr{E}$, where a hyperedge is a non-empty ordered tuple of vertices in $\mathscr{V}$. The circuit nodes and elements are then identified with the vertices and hyperedges, respectively. A junction of two or more elements is called a node. The nodes that a circuit element is connected to, are referred to as the element's terminals and define the hyperedge's vertex tuple. Further, one of these terminals is set as the element's reference terminal and to each left terminal so-called branches are defined and named, which fixes the vertex tuple's order and hence yields the above-mentioned hypergraph's orientation.

To give an example, let the first vertex represent the reference terminal and successively, for every following vertex, enumerate the branches going from the reference terminal to the current terminal identified by the vertex. A three-terminal device represented by $E_i = (V_{i_1}, V_{i_2}, V_{i_3}) \in \mathscr{E}$ connects the nodes $i_1, i_2$, and $i_3$. The element has the reference terminal $i_1$ which is connected to the terminals $i_2$ and $i_3$ via the branches $(V_{i_1}, V_{i_2})$ and $(V_{i_1}, V_{i_3})$, respectively.

**Kirchhoff's laws** are the basis of circuit equations for almost every model approach [6], and read:

- Kirchhoff's current law: For any node and at any time the algebraic sum of all branch currents entering or leaving the node is zero.
- Kirchhoff's voltage law: For any loop and at any time the algebraic sum of branch voltages around the loop is zero.

Hence, the circuits unknowns are the currents and voltages on each branch and potentials at each node, acting on time.

Let $n \in \mathbb{N}$ be the number of circuit nodes and $m \in \mathbb{N}$ the amount of total branches. From the graph structure we deduce the *incidence matrix* $A_{\text{full}} \in \{1,-1,0\}^{n\times m}$ defined by

$$(A_{\text{full}})_{i,j} := \begin{cases} 1 & \text{if node } i \text{ is a non-reference terminal of branch } j \\ -1 & \text{if node } i \text{ is the reference terminal of branch } j \\ 0 & \text{else} \end{cases}$$

Let $\mathscr{I} := [t_0, T] \subset \mathbb{R}$ be some time interval. With $\mathbf{i}, \mathbf{v} : \mathscr{I} \to \mathbb{R}^m$ and $\mathbf{e}_{\text{full}} : \mathscr{I} \to \mathbb{R}^n$ being the vector-functions of all *branch currents, branch voltages* and *node potentials*, respectively, Kirchhoff's laws for current and voltage yield

$$A_{\text{full}}\mathbf{i} = 0\,, \qquad \mathbf{v} = A_{\text{full}}^{\top}\mathbf{e}_{\text{full}}\,.$$

As the rows in $A$ are linear dependent, i. e. they sum up to zero, we encounter redundancy which can lead to an ill defined problem with respect to unique solvability. Therefore, one row is eliminated which can be interpreted as choosing one node to be the circuit's reference node, usually the ground, and fix its potential, generally zero. By neglecting this node, we analogously obtain the *reduced incidence matrix* $A \in \{1,-1,0\}^{(n-1)\times m}$ with the left node potentials collected in $\mathbf{e} : \mathscr{I} \to \mathbb{R}^{(n-1)}$. For the Kirchhoff's circuit laws nothing changes, i. e.

$$A\mathbf{i} = 0\,, \qquad \mathbf{v} = A^{\top}\mathbf{e}\,. \tag{1}$$

**Constitutive element equations** contain/define relations between the branch's currents and voltages and thus complete the Kirchhoff's laws. Most of them can be categorized into the two classes of current and voltage controlling elements. For an arbitrary circuit element consider the quantity dissection $\mathbf{i} = (\mathbf{i}_{\text{elem}}, \mathbf{i}_{\text{compl}})$ and $\mathbf{v} = (\mathbf{v}_{\text{elem}}, \mathbf{v}_{\text{compl}})$ according to the quantities belonging to the element branches and the complementary ones. A circuit element is called current controlling if it has a constitutive equation which explicitly determines its branch currents, e. g.

$$\mathbf{i}_{\text{elem}} = f_{\text{elem}}\left(\tfrac{\mathrm{d}}{\mathrm{d}t} d_{\text{elem}}(\mathbf{i}_{\text{compl}}, \mathbf{v}, t), \mathbf{i}_{\text{compl}}, \mathbf{v}, t\right) \tag{2}$$

or voltage controlling if the voltages are explicitly determined, e. g.

$$\mathbf{v}_{\text{elem}} = f_{\text{elem}}\left(\tfrac{\mathrm{d}}{\mathrm{d}t} d_{\text{elem}}(\mathbf{i}, \mathbf{v}_{\text{compl}}, t), \mathbf{i}, \mathbf{v}_{\text{compl}}, t\right) , \tag{3}$$

for some functions $f_{\text{elem}}$ and $d_{\text{elem}}$. These constitutive equations do not cover all element types, but more than just the basic ones.

**Modified nodal analysis.** According to the above classification (2) and (3) sort the quantities and reduced incidence matrix such that $\mathbf{i} = (\mathbf{i}_{\text{curr}}, \mathbf{i}_{\text{vol}})$, $\mathbf{v} = (\mathbf{v}_{\text{curr}}, \mathbf{v}_{\text{vol}})$ and $A = \begin{bmatrix} A_{\text{curr}} & A_{\text{vol}} \end{bmatrix}$. Further, we collect all the current and voltage controlling constitutive equations into $f_{\text{curr}}$, $d_{\text{curr}}$ and $f_{\text{vol}}$, $d_{\text{vol}}$, respectively. Inserting $f_{\text{curr}}$ into KCL and replacing all the branch voltages in terms of potentials, we obtain from (1):

$$A_{\text{curr}} f_{\text{curr}}\left(\tfrac{\mathrm{d}}{\mathrm{d}t} d_{\text{curr}}(\mathbf{i}, A^{\top}\mathbf{e}, t), \mathbf{i}, A^{\top}\mathbf{e}, t\right) + A_{\text{vol}}\mathbf{i}_{\text{vol}} = 0 , \tag{4}$$

$$f_{\text{vol}}\left(\tfrac{\mathrm{d}}{\mathrm{d}t} d_{\text{vol}}(\mathbf{i}, A^{\top}\mathbf{e}, t)\mathbf{i}, A^{\top}\mathbf{e}, t\right) - A_{\text{vol}}^{\top}\mathbf{e} = 0 . \tag{5}$$

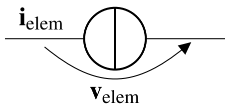
The system (4)-(5) is called the modified nodal analysis (MNA) and represents a system of differential-algebraic equations (DAE) which is usually completed to an initial value problem (IVP)

$$f\left(\tfrac{\mathrm{d}}{\mathrm{d}t} d(\mathbf{x}, t), \mathbf{x}, t\right) = 0 , \qquad \mathbf{x}(t_0) = x_0 , \tag{6}$$

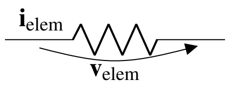
where $\mathbf{x} = (\mathbf{e}, \mathbf{i})$, while $x_0$ is some initial value for $t_0$ and the functions $f$ and $d$ deduced by (4)-(5).

**Considered Elements.** The first set of considered elements are the two-terminals

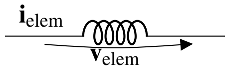
- current source: $\mathbf{i}_{\text{elem}} = i_s(t)$



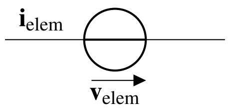

- resistor: $\mathbf{i}_{\text{elem}} = g(\mathbf{v}_{\text{elem}})$



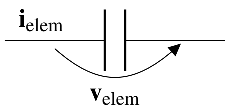

- inductor: $\mathbf{i}_{\text{elem}} = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\mathbf{v}_{\text{elem}})$




- voltage source: $\mathbf{v}_{\text{elem}} = v_s(t)$



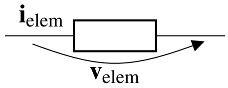

- capacitor: $\mathbf{v}_{\text{elem}} = \frac{\mathrm{d}}{\mathrm{d}t} q(\mathbf{i}_{\text{elem}})$




with scalar source functions $v_s : \mathscr{I} \to \mathbb{R}$ and $i_s : \mathscr{I} \to \mathbb{R}$ and characteristic functions $g, \phi : \mathbb{R} \times \mathscr{I} \to \mathbb{R}$ and $q : \mathbb{R} \times \mathscr{I} \to \mathbb{R}$ since each of them act on one branch only.

For later investigation, we introduce another current controlling element, the multiterminal

- electromagnetic device: $\mathbf{i}_{\text{elem}} = \mathscr{E}[\mathbf{v}_{\text{elem}}, \dots]$




where the branch currents are obtained by some operator $\mathscr{E}$ acting on the branch voltages, their derivatives etc. The operator $\mathscr{E}$ is left to be defined, e. g. involving a solution operator for Maxwell's equation.

This leads us to the following assumption concerning the elements that are the subject of our investigation.

**Assumption 1.** *Let the electrical circuit consist of capacitors, resistors, inductors, voltage and current sources plus one electromagnetic device according to the previously defined models, inducing* $m_C, m_R, m_L, m_V, m_I$ *and* $m_{EM} \in \mathbb{N}$ *branches, respectively.*

The quantities and incidence matrix are sorted according to the order of the elements, i. e.

$$\mathbf{i} = (\mathbf{i}_C, \mathbf{i}_R, \mathbf{i}_L, \mathbf{i}_V, \mathbf{i}_I, \mathbf{i}_{EM}) \in \mathbb{R}^{m_C+m_R+m_L+m_V+m_I+m_{EM}} ,$$
$$A = \begin{bmatrix} A_C \ A_R \ A_L \ A_V \ A_I \ A_{EM} \end{bmatrix} \in \{1, -1, 0\}^{(n-1\times(m_C+m_R+m_L+m_V+m_I+m_{EM}))} .$$

With $q_C : \mathbb{R}^{m_C} \times \mathscr{I} \to \mathbb{R}^{m_C}, g_R : \mathbb{R}^{m_R} \times \mathscr{I} \to \mathbb{R}^{m_R}, \phi_L : \mathbb{R}^{m_L} \times \mathscr{I} \to \mathbb{R}^{m_L}, i_s : \mathscr{I} \to \mathbb{R}^{m_I}$ and $v_s : \mathscr{I} \to \mathbb{R}^{m_V}$ we describe the element type-wise resulting characteristic functions. Then the MNA (4)-(5) is of the form:

$$A_C \tfrac{\mathrm{d}}{\mathrm{d}t} q_C(A_C^\top \mathbf{e}) + A_R g_R(A_R^\top \mathbf{e}) + A_L \mathbf{i}_L + A_V \mathbf{i}_V + A_I i_s(t) + A_{EM}\mathscr{E} = 0 , \qquad (7)$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t} \phi_L(\mathbf{i}_L) - A_L^\top \mathbf{e} = 0 , \qquad (8)$$
$$v_s(t) - A_V^\top \mathbf{e} = 0 . \qquad (9)$$

## 2.2 *Analysis of the Electric Circuit System*

**Assumption 2** (Global Passivity)**.** *All resistances, inductances and capacitances in the electric circuit show a passive behavior, i. e.* $q_C, q_R$ *and* $\phi_L$ *are strongly monotone.*

**Assumption 3** (Cutsets and Loops)**.** *The electric circuit does neither have a cutset of current sources and electromagnetic devices nor a loop of voltage sources.*

Ass. 3 is met since these kind of circuits are generally forbidden and lead to nonspecific mathematical solutions.

**Lemma 1.** *If Assumption 3 is satisfied, then*

$$\ker A_V = \{0\}\,, \qquad \textit{and} \qquad \ker \begin{bmatrix} A_C\ A_V\ A_R\ A_L \end{bmatrix}^\top = \{0\}\,.$$

**Assumption 4.** *The resistive function $g_R$ and inductance function $\phi_L$ are globally Lipschitz continuous. The source functions $i_s$ and $v_s$ are sufficiently often continuously differentiable.*

**Theorem 5.** *If Assumptions 2, 3 and 4 are satisfied, then there exist globally Lipschitz continuous functions $f_0$, $f_1$, $f_2$ and an operator function $s(u)$ for each $t \in \mathscr{I}$ and a function $f_3 \in C^2(\mathscr{I})$ as well as a matrix $M_3$ and nonsingular matrices $M_1(\mathbf{y},\mathbf{z}_3)$ and $T = \begin{bmatrix} T_0\ T_1\ T_2\ T_3 \end{bmatrix}$, the latter one being composed matrix of blocks, such that the IVP* (6) *with an arbitrary given $u \in C^2(\mathscr{I})$ can be globally decoupled into an equivalent system of the form*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = f_0(\mathbf{y},\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,s_c(\mathscr{E}),s_i(t))\,, \qquad \mathbf{y}(t_0) = \mathbf{y}_0\,, \tag{10}$$

$$\mathbf{z}_1 = M_1(\mathbf{y},\mathbf{z}_3)\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_3 + f_1(\mathbf{y},\mathbf{z}_2,\mathbf{z}_3,s_c(\mathscr{E}),s_i(t))\,, \tag{11}$$

$$\mathbf{z}_2 = f_2(\mathbf{y},\mathbf{z}_3,s_c(\mathscr{E}),s_i(t),s_v(t))\,, \tag{12}$$

$$\mathbf{z}_3 = M_3 \begin{pmatrix} s_c(\mathscr{E}) + s_i(t) \\ s_v(t) \end{pmatrix}. \tag{13}$$

*For a given $C^1$ function $\mathscr{E}$ on $\mathscr{I}$, the function $\mathbf{x} \in C^1(\mathscr{I})$ solves the IVP* (6) *if and only if $\bar{\mathbf{x}}$ defined by $\mathbf{x} = T\bar{\mathbf{x}} = T_0\mathbf{y} + T_1\mathbf{z}_1 + T_2\mathbf{z}_2 + T_3\mathbf{z}_3$ is a $C^1$ solution of the IVP* (21)-(13) *on $\mathscr{I}$ with the uniquely defined initial value $y_0$ for which*

$$x_0 = T_0 y_0 + T_1\mathbf{z}_1(t_0) + T_2\mathbf{z}_2(t_0) + T_3\mathbf{z}_3(t_0)\,.$$

*Proof.* The proof mainly follows the one of the decoupling theorem in [17, ]. Here we use a different notation and extended it to the setting outlined by Assumptions 2, 3 and 4. For the inverses' existences and assumptions' influences we also refer to the latter reference. We use the Dissection-Index concept, introduced in [10], in order to split the DAE (6) into its different types of equations.

We start with some practical definitions:

$$s_c(\mathscr{E}) := A_{EM}\mathscr{E}\,, \qquad s_i(t) := A_I i_s(t)\,, \qquad s_v := v_s(t)\,.$$

Let $\{Q,P\}$ be co-kernel splitting pairs of the matrices $M$ as follows:

| $\{Q,P\}$ | $\{Q_c,P_c\}$ | $\{Q_v,P_v\}$ | $\{Q_r,P_r\}$ | $\{\bar{Q}_v,\bar{P}_v\}$ | $\{\bar{Q}_l,\bar{P}_l\}$ | $\{Q_e,P_e\}$ |
|---|---|---|---|---|---|---|
| $M$ | $A_C$ | $Q_c^\top A_V$ | $Q_v^\top Q_c^\top A_R$ | $A_V^\top Q_c P_v$ | $A_L^\top Q_c Q_v Q_r$ | $P_c^\top A_V \bar{Q}_v$ |

By definition, for each splitting pair $\{Q,P\}$ it is $[Q\ P]$ a nonsingular matrix, $\operatorname{im} Q = \ker M^\top$ and $\ker Q = \{0\}$. Justifying the unique split of $\mathbf{e}$, $\mathbf{i}_L$ and $\mathbf{i}_V$ as follows:

$$\mathbf{e} = Q_c[Q_v(Q_r\mathbf{z}_{1l} + P_r\mathbf{z}_{2r}) + P_v\mathbf{z}_{2v}] + P_c[Q_e\mathbf{y}_e + P_e\mathbf{z}_{3e}] \,, \tag{14}$$
$$\mathbf{i}_L = \bar{Q}_l\bar{\mathbf{y}}_l + \bar{P}_l\bar{\mathbf{z}}_{3l} \,, \tag{15}$$
$$\mathbf{i}_V = \bar{Q}_v\bar{\mathbf{z}}_{1v} + \bar{P}_v\bar{\mathbf{z}}_{2v} \,, \tag{16}$$

where we collect the new variables as $\mathbf{y} := (\mathbf{y}_e, \bar{\mathbf{y}}_l)$, $\mathbf{z}_1 := (\mathbf{z}_{1l}, \bar{\mathbf{z}}_{1v})$, $\mathbf{z}_2 := (\mathbf{z}_{2v}, \mathbf{z}_{2r}, \bar{\mathbf{z}}_{2v})$ and $\mathbf{z}_3 := (\mathbf{z}_{3e}, \bar{\mathbf{z}}_{3l})$. Exploiting the splitting pairs' properties we introduce

$$\begin{aligned}
g_R(\mathbf{z}_{2r}, \mathbf{z}_{2v}, \mathbf{y}_e, \mathbf{z}_{3e}) &:= g_R(A_R^\top(Q_c(Q_vP_r\mathbf{z}_{2r} + P_v\bar{\mathbf{z}}_{2v}) + P_c(Q_e\mathbf{y}_e + P_e\mathbf{z}_{3e})) = g_R(A_R^\top\mathbf{e}) \,, \\
\hat{q}_C(\mathbf{y}_e, \mathbf{z}_{3e}) &:= P_c^\top A_C q_C(A_C^\top P_c(Q_e\mathbf{y}_e + P_e\mathbf{z}_{3e}))A_C^\top P_c = P_c^\top A_C q_C(A_C^\top\mathbf{e})A_C^\top P_c \,, \\
\phi_L(\bar{\mathbf{y}}_l, \bar{\mathbf{z}}_{3l}) &:= \phi_L(\bar{Q}_l\bar{\mathbf{y}}_l + \bar{P}_l\bar{\mathbf{z}}_{3l}) = \phi_L(\mathbf{i}_L) \,.
\end{aligned}$$

Furthermore, we need some additional splitting pairs to split the equations. We choose co-kernel splitting pairs $\{W_c, V_c(\mathbf{y}_e, \mathbf{z}_{3e})\}$ and $\{\bar{W}_l, \bar{V}_l(\bar{\mathbf{y}}_l, \bar{\mathbf{z}}_{3l})\}$ of the matrices $P_c^\top A_V \bar{Q}_v$ and $A_L^\top Q_c Q_v Q_r$, respectively, with $W_c^\top \hat{q}_C(\mathbf{y}_e, \mathbf{z}_{3e})Q_e$ and $\bar{W}_l^\top \phi_L(\bar{\mathbf{y}}_l, \bar{\mathbf{z}}_{3l})\bar{Q}_l$ nonsingular and

$$\begin{aligned}
V_c^\top(\mathbf{y}_e, \mathbf{z}_{3e})\hat{q}_C(\mathbf{y}_e, \mathbf{z}_{3e})\begin{bmatrix}Q_e & P_e\end{bmatrix} &= \begin{bmatrix}0 & I\end{bmatrix} \,, \\
\bar{V}_l^\top(\bar{\mathbf{y}}_l, \bar{\mathbf{z}}_{3l})\phi_L(\bar{\mathbf{y}}_l, \bar{\mathbf{z}}_{3l})\begin{bmatrix}\bar{Q}_l & \bar{P}_l\end{bmatrix} &= \begin{bmatrix}0 & I\end{bmatrix} \,.
\end{aligned}$$

We derive equations of the form (21)-(13) in four steps, starting with (13) and finishing with (21):

1. Multiplying (7) by $Q_r^\top Q_v^\top Q_c^\top$ and (9) by $\bar{Q}_v^\top$ from the left yields:

$$\bar{\mathbf{z}}_{3l} = \bar{M}_{3l}(A_I i_s(t) + A_{EM}\mathscr{E}) \,, \quad \text{with } \bar{M}_{3l} := -(Q_r^\top Q_v^\top Q_c^\top A_L \bar{P}_l)^{-1} Q_r^\top Q_v^\top Q_c^\top \,, \tag{17}$$
$$\mathbf{z}_{3e} = M_{3e} v_s(t) \,, \qquad \text{with } M_{3e} := -(\bar{Q}_v^\top A_V^\top P_c P_e)^{-1}\bar{Q}_v^\top \,. \tag{18}$$

   Introducing $M_3 := \begin{bmatrix}\bar{M}_{3l} & \\ & M_{3e}\end{bmatrix}$ yields (13).

2. Multiplying (9) by $\bar{P}_v^\top$ and (7) by $P_r^\top Q_v^\top Q_c^\top$ and $P_v^\top Q_c^\top$ from the left yields:

$$\begin{aligned}
\mathbf{z}_{2v} &= f_{2v}(\mathbf{y}_e, \mathbf{z}_{3e}, s_v(t)) := (\bar{P}_v^\top A_V^\top Q_c P_v)^{-1}\bar{P}_v^\top(-A_V^\top P_c(Q_e\mathbf{y}_e + P_e\mathbf{z}_{3e}) + v_s(t)) \,, \\
\mathbf{z}_{2r} &= f_{2r}(\mathbf{y}, \mathbf{z}_3, s_c(\mathscr{E}), s_i(t)) \,, \quad \text{with } f_{2r} \text{ satisfying} \\
&\qquad h_R(f_{2r}(\mathbf{y}, \mathbf{z}_3, s_c(\mathscr{E}), s_i(t)), \mathbf{y}, \mathbf{z}_3, s_c(\mathscr{E}), s_i(t)) = 0 \,, \\
\bar{\mathbf{z}}_{2v} &= \bar{f}_{2v}(\mathbf{y}, \mathbf{z}_3, s_c(\mathscr{E}), s_i(t)) := -(P_v^\top Q_c A_V \bar{P}_v)^{-1} P_v^\top Q_c^\top[A_R g_R(\mathbf{z}_{2r}, \bar{\mathbf{z}}_{2v}, \mathbf{y}_e, \mathbf{z}_{3e}) \\
&\qquad A_L(\bar{Q}_l\bar{\mathbf{y}}_l + \bar{P}_l\bar{\mathbf{z}}_{3l}) + A_V\bar{Q}_v\bar{\mathbf{z}}_{1v} + (A_I i_s(t) + A_{EM}\mathscr{E})] \,,
\end{aligned}$$

   where

$$\begin{aligned}
h_R(\mathbf{z}_{2r}, \mathbf{y}, \mathbf{z}_3, s_c(\mathscr{E}), s_i(t)) &:= P_r^\top Q_v^\top Q_c^\top[A_R g_R(\mathbf{z}_{2r}, f_{2v}(\mathbf{y}_e, \mathbf{z}_{3e}, t), \mathbf{y}_e, \mathbf{z}_{3e}) \\
&\qquad + A_L(\bar{Q}_l\bar{\mathbf{y}}_l + \bar{P}_l\bar{\mathbf{z}}_{3l}) + (A_I i_s(t) + A_{EM}\mathscr{E})] \,.
\end{aligned}$$

Introducing $f_2(\mathbf{y},\mathbf{z}_3,s_c(\mathscr{E}),s_i(t),s_v(t)) := \begin{pmatrix} f_{2v}(\mathbf{y}_e,\mathbf{z}_{3e},s_v(t)) \\ f_{2r}(\mathbf{y},\mathbf{z}_3,s_c(\mathscr{E}),s_i(t)) \\ \bar{f}_{2v}(\mathbf{y},\mathbf{z}_3,s_c(\mathscr{E}),s_i(t)) \end{pmatrix}$ gives (23).

3. Multiplying (8) by $\bar{V}_l^\top(\bar{\mathbf{y}}_l,\bar{\mathbf{z}}_{3l})$ and (7) by $V_c^\top(\mathbf{y}_e,\mathbf{z}_{3e})P_c^\top$ from the left yields:

$$\mathbf{z}_{1l} = M_{1l}(\bar{\mathbf{y}}_l,\bar{\mathbf{z}}_{3l})\tfrac{\mathrm{d}}{\mathrm{d}t}\bar{\mathbf{z}}_{3l} + f_{1l}(\mathbf{y},\mathbf{z}_2,\mathbf{z}_3)\ , \tag{19}$$

$$\bar{\mathbf{z}}_{1v} = -\bar{M}_{1v}(\mathbf{y}_e,\mathbf{z}_{3e})\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_{3e} + \bar{f}_{1v}(\mathbf{y},\mathbf{z}_2,\mathbf{z}_3,s_c(\mathscr{E}),s_i(t))\ . \tag{20}$$

with

$$\begin{aligned} M_{1l}(\bar{\mathbf{y}}_l,\bar{\mathbf{z}}_{3l}) :=& (\bar{V}_l^\top(\bar{\mathbf{y}}_l,\bar{\mathbf{z}}_{3l})\bar{A}_L^\top)^{-1}\ , \\ \bar{M}_{1v}(\mathbf{y}_e,\mathbf{z}_{3e}) :=& (V_c^\top(\mathbf{y}_e,\mathbf{z}_{3e})P_c^\top A_V\bar{Q}_v)^{-1}\ , \\ f_{1l}(\mathbf{y},\mathbf{z}_2,\mathbf{z}_3) :=& -M_{1l}(\bar{\mathbf{y}}_l,\bar{\mathbf{z}}_{3l})\bar{V}_l^\top(\bar{\mathbf{y}}_l,\bar{z}_{3l}) \\ & \cdot[A_L^\top(Q_c(Q_vP_r\mathbf{z}_{2r}+P_v\mathbf{z}_{2v})+P_c(Q_e\mathbf{y}_e+P_e\mathbf{z}_{3e}))]\ , \\ \bar{f}_{1v}(\mathbf{y},\mathbf{z}_2,\mathbf{z}_3,s_c(\mathscr{E}),s_i(t)) :=& -\bar{M}_{1v}(\mathbf{y}_e,\mathbf{z}_{3e})V_c^\top(\mathbf{y}_e,\mathbf{z}_{3e})P_c^\top \\ & \cdot[A_Rg_R(\mathbf{z}_{2r},\mathbf{z}_{2v},\mathbf{y}_e,\mathbf{z}_{3e})+A_L(\bar{Q}_l\bar{\mathbf{y}}_l+\bar{P}_l\bar{\mathbf{z}}_{3l}) \\ & +A_V\bar{P}_v\bar{\mathbf{z}}_{2v}+(A_Ii_s(t)+A_{EM}\mathscr{E})]\ . \end{aligned}$$

Introducing $M_1(\mathbf{y},\mathbf{z}_3) := \begin{bmatrix} 0 & M_{1l}(\bar{\mathbf{y}}_l,\bar{\mathbf{z}}_{3l}) \\ -\bar{M}_{1v}(\mathbf{y}_e,\mathbf{z}_{3e}) & 0 \end{bmatrix}$ and $f_1(\mathbf{y},\mathbf{z}_2,\mathbf{z}_3,s_c(\mathscr{E}),s_i(t)) := \begin{pmatrix} f_{1l}(\mathbf{y},\mathbf{z}_2,\mathbf{z}_3) \\ \bar{f}_{1v}(\mathbf{y},\mathbf{z}_2,\mathbf{z}_3,s_c(\mathscr{E}),s_i(t)) \end{pmatrix}$ gives (22).

4. Multiplying (8) by $\bar{W}_l^\top$ and (7) by $W_c^\top P_c^\top$ from the left and using (19)-(20) yields:

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t}\bar{\mathbf{y}}_l = \bar{f}_{0l}(\mathbf{y},\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3) := \\ &\quad(\bar{W}_l^\top\phi_L(\bar{\mathbf{y}}_l,\bar{\mathbf{z}}_{3l})\bar{Q}_l)^{-1}\bar{W}_l^\top\cdot[I-\phi_L(\bar{\mathbf{y}}_l,\bar{\mathbf{z}}_{3l})\bar{P}_l\bar{V}_l^\top(\bar{\mathbf{y}}_l,\bar{\mathbf{z}}_{3l})] \\ &\quad\cdot[A_L^\top(Q_c(Q_vP_r\mathbf{z}_{2r}+P_v\mathbf{z}_{2v})+P_c(Q_e\mathbf{y}_e+P_e\mathbf{z}_{3e}))+\bar{A}_L^\top\mathbf{z}_{1l}]\ , \\ &\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}_e = f_{0e}(\mathbf{y},\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,s_c(\mathscr{E}),s_i(t)) := \\ &\quad(W_c^\top\hat{q}_C(\mathbf{y}_e,\mathbf{z}_{3e})Q_e)^{-1}W_c^\top P_c^\top\cdot[I-\hat{q}_C(\mathbf{y}_e,\mathbf{z}_{3e})P_eV_c^\top(\mathbf{y}_e,\mathbf{z}_{3e})] \\ &\quad\cdot[A_L(\bar{Q}_l\bar{\mathbf{y}}_l+\bar{P}_l\bar{\mathbf{z}}_{3l})+A_V(\bar{Q}_v\bar{\mathbf{z}}_{1v}+\bar{P}_v\bar{\mathbf{z}}_{2v}) \\ &\quad+A_Rg_R(\mathbf{z}_{2r},\mathbf{z}_{2v},\mathbf{y}_e,\mathbf{z}_{3e})+(A_Ii_s(t)+A_{EM}\mathscr{E})]\ . \end{aligned}$$

Introducing $f_0(\mathbf{y},\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,s_c(\mathscr{E}),s_i(t)) := \begin{pmatrix} f_{0e}(\mathbf{y},\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,s_c(\mathscr{E}),s_i(t)) \\ \bar{f}_{0l}(\mathbf{y},\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3) \end{pmatrix}$ gives (21).

Regarding (14)-(16), the initial condition $\mathbf{y}_0$ and the nonsingular transformation matrix $T = \begin{bmatrix} T_0 & T_1 & T_2 & T_3 \end{bmatrix}$ are given by

$$T_0 = \begin{bmatrix} P_cQ_e & 0 \\ 0 & \bar{Q}_l \\ 0 & 0 \end{bmatrix}, T_1 = \begin{bmatrix} Q_cQ_vQ_r & 0 \\ 0 & 0 \\ 0 & \bar{Q}_v \end{bmatrix}, T_2 = \begin{bmatrix} Q_cP_v & Q_cQ_vP_r & 0 \\ 0 & 0 & \\ 0 & 0 & \bar{P}_v \end{bmatrix}, T_3 = \begin{bmatrix} P_cP_e & 0 \\ 0 & \bar{P}_l \\ 0 & 0 \end{bmatrix}$$

with the unique decomposition $\mathbf{x}(t_0) = x_0 = T_0\mathbf{y}_0 + T_1\mathbf{z}_{10} + T_2\mathbf{z}_{20} + T_3\mathbf{z}_{30}$. Finally, we can conclude that the system (7)-(9) is equivalent to the system (21)-(13) as the matrices with which we multiplied through steps 1-4 form a nonsingular one by construction. □

**Assumption 6.** *The EM device is not a subset of an LI-cutset.*

**Lemma 2.** *With Ass. 6 to hold, the decoupled system* (21)-(13) *can be equivalently transformed into a system of the form*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = \hat{f}_0(\mathbf{y}, \mathbf{z}_1, \mathbf{z}_2, s_c(\mathscr{E}), s_i(t), s_v(t)), \qquad \mathbf{y}(t_0) = \mathbf{y}_0, \tag{21}$$

$$\mathbf{z}_1 = \hat{M}_1(\mathbf{y}, s_i(t), s_v(t))\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} s_i(t) \\ s_v(t) \end{pmatrix} + \hat{f}_1(\mathbf{y}, \mathbf{z}_2, s_c(\mathscr{E}), s_i(t), s_v(t)), \tag{22}$$

$$\mathbf{z}_2 = \hat{f}_2(\mathbf{y}, s_c(\mathscr{E}), s_i(t), s_v(t)). \tag{23}$$

*With transformation matrices $\hat{T}_0, \hat{T}_1$ and $\hat{T}_2$.*

*Proof.* Due to Ass. 6 we deduce that $Q_r^\top Q_v^\top Q_c^\top A_{EM} = 0$ by the co-kernel splitting pair properties. Hence, $\bar{\mathbf{z}}_{3l} = \bar{M}_{3l}A_I i_s(t)$ in (17) which makes $\mathbf{z}_3$ dependent on $t$ only, to be more precise on $s_i(t)$. Now define new functions $\hat{f}_0, \hat{f}_1, \hat{f}_2$ and $\hat{M}_1$ by inserting $\mathbf{z}_3$ and incorporating $M_3$. □

## 3 Electromagnetic Devices

As of the high importance of these devices in everyday life, on chips become smaller in size whereby the operating frequency increases. In order to serve this trend, industry persuades the development of new models and simulation techniques to enhance these devices while saving expenditures by virtue of laboratory testing etc. These models require to cover all the physical phenomena which no longer can be ignored, e. g. cross-talking and skin effect. Hence, modern models make use of the full set of Maxwell's equations, since they are believed to govern all large-scaled electromagnetic phenomena, cf. [21]. Depending on the particular application and the numerics, various equivalent formulations have been developed of which the so-called potential formulation is widely used.

## 3.1 Modeling

As we intend to incorporate *electromagnetic devices* (EM), such as incorporated circuits or other on-chips, into an electric circuit, we first have to provide a suitable EM model.

**Maxwell's equations** as published by James Clerk Maxwell's in 1865 [14] are a set of equations unifying the electric and magnetic field theory into classical electromagnetism. Given in their modern macroscopic differential vector-valued formulation using the SI-unit convention, *Maxwell's equations* (ME) consist of four first order partial-differential equations:

Gauss's law (GL) $$\nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t) \,, \tag{24}$$

Gauss's law for magnetism (GLM) $$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0 \,, \tag{25}$$

Maxwell-Faraday's law (MF) $$\nabla \times \mathbf{E}(\mathbf{r},t) = -\tfrac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) \,, \tag{26}$$

Maxwell-Ampère's law (MA) $$\nabla \times \mathbf{H}(\mathbf{r},t) = \mathbf{J}(\mathbf{r},t) + \tfrac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) \,, \tag{27}$$

They describe the behavior of four vector-valued functions of space $\Omega \subseteq \mathbb{R}^3$ and time $\mathscr{I} \subseteq \mathbb{R}$. The quantities are the *electric* and *magnetic induction* $\mathbf{D},\mathbf{B} : \Omega \times \mathscr{I} \to \mathbb{R}^3$ and the *electric* and *magnetic field* $\mathbf{E},\mathbf{H} : \Omega \times \mathscr{I} \to \mathbb{R}^3$ depending on the *distribution of charge* and *conduction current density* given by $\rho,\mathbf{J} : \Omega \times \mathscr{I} \to \mathbb{R}^3$.

**Constitutive equations** Similar to the constitutive element equations of electrical circuits, we need additional quantity relations in order to make the system (24)-(27) determinate, cf. [9]. The idea is to deduce $\mathbf{D},\mathbf{H}$ and $\mathbf{J}$ from $\mathbf{E}$ and $\mathbf{B}$, by making use of empirical observations when material comes into play.

**Assumption 7** (Linear and Inhomogeneous Media)**.** *The constitutive equations read:*

$$\mathbf{D}(\mathbf{r},t) = \varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r},t) \,, \quad \mathbf{H}(\mathbf{r},t) = \nu(\mathbf{r})\mathbf{B}(\mathbf{r},t) \,, \quad \mathbf{J}(\mathbf{r},t) = \sigma(\mathbf{r})\mathbf{E}(\mathbf{r},t) \,, \tag{28}$$

*with permittivity $\varepsilon$, reluctivity (or inverse permeability) $\nu = \mu^{-1}$ and conductivity $\sigma$ being rank-2 tensors $\Omega \to \mathbb{R}^{3\times 3}$.*

As we typically consider linear inhomogeneous media which are neither ferromagnetic nor ferroelectric, we let Ass. 7 hold.

**Boundary conditions** are of importance when solving ME, e. g. during simulations. As we intend to spatially discretize ME we would lack resources if we would not restrict the domain $\Omega$ to some bounded area of interest.

**Assumption 8** (Lipschitz Domain)**.** *The spatial domain $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain, i. e. open, bounded, connected and the boundary $\partial\Omega$ is a Lipschitz boundary.*

Let Ass. 8 hold. In order to keep ME (24)-(27) determinate we need to introduce boundary conditions. A typical choice is to assume $\Omega$ being surrounded by a *perfectly electric conducting* (PEC) medium, see e. g. [1, 7, 3, 18]:
**Assumption 9** (PEC Boundary)**.** *The tangential component of the electric field at the boundary vanishes, i. e. with* **n** *being the outer unit normal*

$$\mathbf{E} \times \mathbf{n} = 0, \qquad \text{on } \partial\Omega \times \mathscr{I}.$$

## $\mathbf{A} - \varphi$ Formulation

Instead of solving ME in their classical form, one typically makes use of alternative formulations. Here, we focus on the widely-spread $\mathbf{A} - \varphi$ *formulation*, see e. g. [16, 23], which formulates ME in terms of so-called potentials.

First, we observe that the electric field **E** and the magnetic induction **B** can be expressed as

$$\mathbf{E} = \nabla\varphi + \tfrac{\partial}{\partial t}\mathbf{A}\,, \qquad \mathbf{B} = \nabla \times \mathbf{A}\,, \tag{29}$$

for some $\mathbf{A} : \Omega \times \mathscr{I} \to \mathbb{R}^3$ and $\varphi : \Omega \times \mathscr{I} \to \mathbb{R}$ which we call *magnetic vector potential* and *electric scalar potential*, respectively. With (29) the homogeneous ME, i. e. GLM (25) and MF (26), are implicitly satisfied, which originally motivated this approach. What is left to solve are the GL (24) and MA (27) which, after incorporating the constitutive equations (28), yield the so-called called $\mathbf{A} - \varphi$ *formulation*:

$$-\nabla \cdot \left[\varepsilon\left(\nabla\varphi + \tfrac{\partial}{\partial t}\mathbf{A}\right)\right] = \rho\,, \tag{30}$$

$$\nabla \times (\nu\nabla \times \mathbf{A}) + \tfrac{\partial}{\partial t}\left[\varepsilon\left(\nabla\varphi + \tfrac{\partial}{\partial t}\mathbf{A}\right)\right] + \sigma\left(\nabla\varphi + \tfrac{\partial}{\partial t}\mathbf{A}\right) = 0\,. \tag{31}$$

**Gauging** As with the reduction of equations comes ambiguity, i. e. system (30)-(31) is not uniquely solvable, we follow the *grad-div regularization* approach from [5] and introduce the generalized *gauge condition*

$$\vartheta\varepsilon\nabla\tfrac{\partial}{\partial t}\varphi + \zeta\nabla\left[\xi\nabla \cdot (\zeta\mathbf{A})\right] = 0\,, \tag{32}$$

for some material tensors $\zeta \in \mathbb{R}^{3\times3}$ and $\xi \in \mathbb{R}^{3\times3}$, see [2].
**Assumption 10.** *We fix the $A - \varphi$ formulation's ambiguity by choosing a grad-type Lorenz gauge condition, i. e.* (32) *with* $\vartheta = 1$.

**Boundary conditions** A possible choice for the $A - \varphi$ formulation's boundary conditions can be deduced from Ass. 9 as follows, see e. g. [8]:
**Assumption 11** (Boundary Conditions for $A - \varphi$ formulation)**.** *We assume PEC medium at artificial boundaries:*

$$\mathbf{A} \times \mathbf{n} = 0\,, \qquad on\ \partial\Omega \times \mathscr{I}\,, \tag{33}$$
$$\nabla\varphi \times \mathbf{n} = 0\,, \qquad on\ \partial\Omega \times \mathscr{I}\,. \tag{34}$$

As proposed in [2] we end up with the following set of partial differential equations in order to solve an EM problem:

$$\varepsilon\nabla\tfrac{\partial}{\partial t}\varphi + \zeta\nabla\left[\xi\nabla\cdot(\zeta\mathbf{A})\right] = 0\,, \tag{35}$$
$$\nabla\times(\nu\nabla\times\mathbf{A}) + \tfrac{\partial}{\partial t}\left[\varepsilon\left(\nabla\varphi + \Pi\right)\right] + \sigma\left(\nabla\varphi + \tfrac{\partial}{\partial t}\mathbf{A}\right) = 0\,, \tag{36}$$
$$\tfrac{\partial}{\partial t}\mathbf{A} - \Pi = 0\,, \tag{37}$$

plus the boundary conditions (33)-(34), where $\Pi : \Omega\times\mathscr{I}\to\mathbb{R}^3$ is a *quasi-canonical momentum* in order to avoid second order derivatives.

#### 3.1.1 Spatial Discretization

The EM system (33)-(37) needs to be spatially discretized in order to apply time integration schemes. For this purpose we make use of the *Finite-Integration-Technique* (FIT), originally introduced 1977 by Thomas Weiland [25]. In the following we are not going into the details of FIT, but refer to the aforementioned reference for further reading.

From the FIT we obtain two struggling meshes defined on $\Omega$ which are dual to each other. Let $n_{\mathscr{P}}, n_{\mathscr{L}}, n_{\mathscr{F}}$ and $n_{\mathscr{V}} \in \mathbb{N}$ be the number of geometrical objects, i. e. *mesh points*, *links*, *faces* and *volumes*, respectively, of the *primal mesh*. According to the mesh's duality this are the amounts of *dual mesh volumes*, *facets*, *links* and *points*, respectively, of the *dual mesh*.

Motivated by the boundary conditions 11 we deduce that the discrete vector potential tangential to the boundary vanishes and that the scalar potential at the boundary is constant in space, i. e. described by some component identical $\phi_\Gamma : \mathscr{I}\to\mathbb{R}^{n_{\mathscr{P}}-n_{\mathscr{P}}^{\mathrm{int}}}$. Incorporating these conditions leads us to consider mainly the $n_{\mathscr{P}}^{\mathrm{int}}\in\mathbb{N}$ mesh points and $n_{\mathscr{L}}^{\mathrm{int}}\in\mathbb{N}$ links which do not belong to the boundary $\partial\Omega$.

For the potential $\mathbf{A}$ and $\varphi$ we obtain as spatially discretized versions the *discrete vector potential* $\mathbf{a} : \mathscr{I}\to\mathbb{R}^{n_{\mathscr{L}}^{\mathrm{int}}}$ and the *discrete scalar potential* $\phi : \mathscr{I}\to\mathbb{R}^{n_{\mathscr{P}}^{\mathrm{int}}}$ as integral quantities on the internal mesh links and points, respectively. On the meshes we introduce discrete operators for the gradient, divergence and curl as matrices, where the dual ones are notated with a tilde above:

| | |
|---|---|
| primal gradient matrix: | $G \in \{1,-1,0\}^{n_{\mathscr{L}}^{\text{int}} \times n_{\mathscr{P}}^{\text{int}}}$ , |
| dual divergence matrix | $\tilde{S} \in \{1,-1,0\}^{n_{\mathscr{P}}^{\text{int}} \times n_{\mathscr{L}}^{\text{int}}}$ , |
| primal curl matrix: | $C \in \{1,-1,0\}^{n_{\mathscr{F}} \times n_{\mathscr{L}}^{\text{int}}}$ , |
| dual curl matrix: | $\tilde{C} \in \{1,-1,0\}^{n_{\mathscr{L}}^{\text{int}} \times n_{\mathscr{F}}}$ . |

The constitutive material parameters turn into diagonal matrices $M_\varepsilon \in \mathbb{R}^{n_{\mathscr{L}}^{\text{int}} \times n_{\mathscr{L}}^{\text{int}}}$, $M_\nu \in \mathbb{R}^{n_{\mathscr{L}}^{\text{int}} \times n_{\mathscr{L}}^{\text{int}}}$ and $M_\sigma \in \mathbb{R}^{n_{\mathscr{F}} \times n_{\mathscr{F}}}$ and are connecting the primal and dual meshes fields. The parameters' modeling opens a whole new field of investigation, see e. g. [24]. From the gauging we obtain further material matrices $M_\zeta \in \mathbb{R}^{n_{\mathscr{L}}^{\text{int}} \times n_{\mathscr{L}}^{\text{int}}}$ and $M_\xi \in \mathbb{R}^{n_{\mathscr{L}}^{\text{int}} \times n_{\mathscr{L}}^{\text{int}}}$. As the boundary's discrete scalar potentials will be put in the role of interacting with the outer environment, such as lumped circuits, they still need to be considered in the ME. Considering them again results in an additional operator $G_\Gamma \in \mathbb{R}^{n_{\mathscr{L}}^{\text{int}} \times (n_{\mathscr{P}} - n_{\mathscr{P}}^{\text{int}})}$ extending the gradient operator.

Finally, the discrete pendant of the $\mathbf{A} - \varphi$ formulation with Lorenz gauge (35)-(37) incorporating the boundary conditions (33)-(34) reads

$$\tilde{S} M_\varepsilon G \tfrac{\mathrm{d}}{\mathrm{d}t} \phi + \tilde{S} M_\zeta G M_\xi \tilde{S} M_\zeta \mathbf{a} = 0 \,, \quad (38)$$

$$\tilde{C} M_\nu C \mathbf{a} + \tfrac{\mathrm{d}}{\mathrm{d}t} \left( M_\varepsilon \left( G\phi + G_\Gamma \phi_\Gamma(t) + \pi \right) \right) + M_\sigma \left( G\phi + G_\Gamma \phi_\Gamma(t) + \tfrac{\mathrm{d}}{\mathrm{d}t} \mathbf{a} \right) = 0 \,, \quad (39)$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \mathbf{a} - \pi = 0 \,, \quad (40)$$

again with the *discrete quasi-canonical momentum* $\pi : \mathscr{I} \to \mathbb{R}^{n_{\mathscr{L}}^{\text{int}}}$. System (38)-(40) is referred to as *Maxwell's grid equations* (MGE) in this article. In addition, we introduce $u_0$ as an initial value for $t_0$ such that there is a solution complying to $\mathbf{u}(t_0) = u_0$. According to [2] MGE, the above met assumptions form an ordinary differential equation system. With $\mathbf{u} = (\phi, \mathbf{a}, \pi) \in \mathbb{R}^{n_U}$ for $n_U = n_{\mathscr{P}}^{\text{int}} + n_{\mathscr{L}}^{\text{int}} + n_{\mathscr{L}}^{\text{int}}$ and

$$M_{EM} := \begin{bmatrix} \tilde{S} M_\varepsilon G & 0 & 0 \\ M_\varepsilon G & M_\sigma & M_\varepsilon \\ 0 & I & 0 \end{bmatrix} , \quad B_{EM} := \begin{bmatrix} 0 & \tilde{S} M_\zeta G M_\xi \tilde{S} M_\zeta & 0 \\ M_\sigma G & \tilde{C} M_\nu C & 0 \\ 0 & 0 & I \end{bmatrix} ,$$

$$b_{EM}(\mathbf{u}) := M_{EM}^{-1} B_{EM} \mathbf{u} \,, \qquad c(t) := -M_{EM}^{-1} \begin{pmatrix} 0 \\ \tfrac{\mathrm{d}}{\mathrm{d}t} \left( M_\varepsilon G_\Gamma \phi_\Gamma(t) \right) + M_\sigma G_\Gamma \phi_\Gamma(t) \\ 0 \end{pmatrix} ,$$

we know that $M_{EM}^{-1}$ exists and the equations (38)-(40) with $u_0$ can be expressed as an IVP

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \mathbf{u} + b_{EM}(\mathbf{u}) = c(t) \,, \qquad \mathbf{u}(t_0) = u_0 \,. \quad (41)$$

# 4 Coupled Electric Circuits and Electromagnetic Devices

In this section wen want to provide a coupled model for electric circuits, incorporating EM devices, cf. Fig. 1, that follows the approach in [2] and [20].

First, we have to introduce the EM device as a circuit element that complies to a constitutive equation, i, e. that satisfies either (2) or (3). Second, we try to interpret it in terms of the considered elements only. Finally, we will be able to write down the complete coupled system.

## 4.1 Modeling

Let the EM models' boundary $\partial\Omega$ fall into $m_{EM}+1$ disjoint nonempty parts $\Gamma_j \subset \partial\Omega$, for $0 \leq j \leq m_{EM}$. It holds $\partial\Omega = \bigcup_{1\leq j\leq m_{EM}} \Gamma_j \cup \Gamma_0$. Further, each part is assumed to be connected to exactly one node of the electric circuit which makes the EM device an $(m_{EM}+1)$-terminal element that has $m_{EM}$ branches, according to Sec. 2.1.

By convention, the 0-th terminal is attached to the ground and can be considered as the element's reference terminal. The boundary parts can now be mapped to the circuit nodes by the devices incidence matrix $A_{EM} \in \{1,-1,0\}^{n-1\times m_{EM}}$ with $n$ being the total number of circuits.

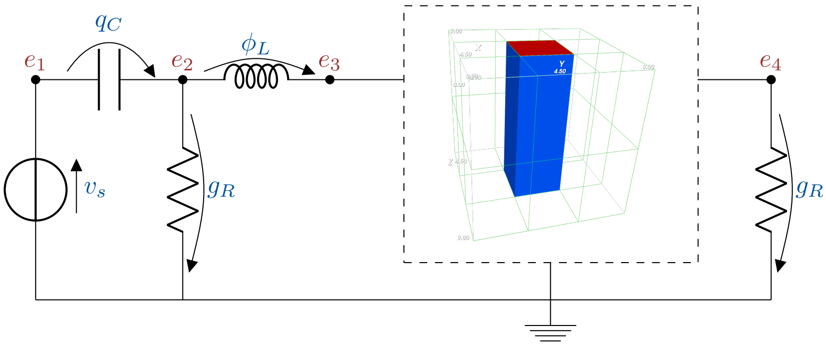



**Fig. 1** Electric circuit incorporating an EM device

Coupling branch voltages: $\mathbf{v}_{EM}$

In order to archive an excitation of the EM fields via the boundary, a possible choice would be to vary the Dirichlet boundary conditions, e. g. $\phi_\Gamma$, in a reasonable way.

First, we select the primal mesh points belonging to the $j$-th terminal, for $0 \leq j \leq m_{EM}$, by making use of the matrix

$$\Lambda = (\lambda_{ij})_{i \in \mathscr{N}_{\mathscr{P}}^{bound}, 1 \leq j \leq m_{EM}}, \qquad \lambda_{ij} = \begin{cases} 1 & \text{if } P(i) \subset \Gamma_j \\ 0 & \text{else} \end{cases}.$$

Next, we denote the EM device's branch voltages by $\mathbf{v}_{EM} : \mathscr{I} \to \mathbb{R}^{m_{EM}}$. Since they can be considered as potentials with respect to the ground as well, we map these quantities directly to the adjacent scalar potential, which yields

$$\phi_\Gamma = \Lambda \mathbf{v}_{EM} . \tag{42}$$

Note that with (42) the scalar potential at the reference terminal's boundary part is excited with 0 potential.

Coupling branch currents: $\mathbf{i}_{EM}$

Let $\mathbf{J}_t : \Omega \times \mathscr{I} \to \mathbb{R}^3$ be the model for total current density flowing through a point in space at a certain time. The current through an arbitrary surface $\Gamma$, with unit normal $\mathbf{n}$, is then obtained by $\int_\Gamma \mathbf{J}_t \cdot \mathbf{n}\, \mathrm{d}\mathbf{s}$.

**Assumption 12.** *The total current density is given by the sum of displacement and conduction current density:* $\mathbf{J}_t := \frac{\partial}{\partial t}\mathbf{D} + \mathbf{J}$*, as it is implemented in* `devEM`*, see [13].*

With Ass. 12 the current through $\Gamma$ expressed in potentials reads

$$\int_\Gamma -\tfrac{\partial}{\partial t}\left[\varepsilon\left(\nabla\varphi + \Pi\right)\right] - \sigma\left(\nabla\varphi + \tfrac{\partial}{\partial t}\mathbf{A}\right)\cdot \mathbf{n}\,\mathrm{d}\mathbf{s} = \int_\Gamma \nabla \times \left(\nu \nabla \times \mathbf{A}\right)\cdot \mathbf{n}\,\mathrm{d}\mathbf{s}\,,$$

according to MA. From the latter variant we observe that the cumulated sum through all terminals, i. e. the whole boundary $\partial\Omega$, equals zero:

$$\int_{\partial\Omega} \nabla \times \left(\nu \nabla \times \mathbf{A}\right)\cdot \mathbf{n}\,\mathrm{d}\mathbf{s} = \int_\Omega \nabla \cdot \nabla \times \left(\nu \nabla \times \mathbf{A}\right)\mathrm{d}\mathbf{r} = 0\,. \tag{43}$$

From (43) it becomes clear that the model assumption Ass. 12 is compatible to Kirchhoff's current law. Hence, we can omit one current quantity, here the reference terminal's current, and define the EM device's branch currents $\mathbf{i}_{EM} : \mathscr{I} \to \mathbb{R}^{m_{EM}}$ for the left terminals.

As the discrete variables for both $\mathbf{J}$ and $\mathbf{D}$ originated on dual mesh facets, we introduce a mapping and selection to the boundary's primal mesh points by $-G_\Gamma^\top$. Then the branch currents in terms of discrete potentials read

$$\mathbf{i}_{EM} = \Lambda^\top G_\Gamma^\top \left[\tfrac{\mathrm{d}}{\mathrm{d}t}\left(M_\varepsilon \left(G\phi + G_\Gamma \phi_\Gamma(t) + \pi\right)\right) + M_\sigma \left(G\phi + G_\Gamma \phi_\Gamma(t) + \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}\right)\right] \tag{44}$$

or analogously, according to MA (39) in MGE,

$$\mathbf{i}_{EM} = -\Lambda^\top G_\Gamma^\top \tilde{C} M_\nu C \mathbf{a}\ . \tag{45}$$

Element description

In the following, we derive all the functions and operators that are necessary in order to incorporate the EM device as a current controlling element in the sense of Sec. 2.1.

Since the MNA approach substitutes branch voltages by node potentials $\mathbf{e}$, we replace $\mathbf{v}_{EM}$ in (42) by $A_{EM}^\top \mathbf{e}$ and obtain

$$\phi_\Gamma = \Lambda A_{EM}^\top \mathbf{e}\ . \tag{46}$$

Defining

$$d_{EM}(\mathbf{x}) := M_\varepsilon G_\Gamma \Lambda A_{EM}^\top \mathbf{e}$$

and

$$c_{EM}(\mathbf{w}_{EM}, \mathbf{x}) := -M_{EM}^{-1} \begin{pmatrix} 0 \\ \mathbf{w}_{EM} + M_\sigma G_\Gamma \Lambda A_{EM}^\top \mathbf{e} \\ 0 \end{pmatrix}\ ,$$

incorporating the branch voltage coupling equation (46) into (41) yields

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} + b_{EM}(\mathbf{u}) = c_{EM}\left(\tfrac{\mathrm{d}}{\mathrm{d}t} d_{EM}(\mathbf{x}), \mathbf{x}\right)\ , \qquad \mathbf{u}(t_0) = u_0\ , \tag{47}$$

Similarly, we incorporate (46) into (44) and obtain

$$\mathbf{i}_{EM} = f_{EM}\left(\tfrac{\mathrm{d}}{\mathrm{d}t} d_{EM}(\mathbf{x}), \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}, \mathbf{x}, \mathbf{u}\right)\ , \tag{48}$$

for

$$\begin{aligned} &f_{EM}\left(\mathbf{w}_{EM}, \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}, \mathbf{x}, \mathbf{u}\right) := \\ &\Lambda^\top G_\Gamma^\top \left[M_\varepsilon \left(G \tfrac{\mathrm{d}}{\mathrm{d}t}\phi + \tfrac{\mathrm{d}}{\mathrm{d}t}\pi\right) + \mathbf{w}_{EM} + M_\sigma \left(G\phi + G_\Gamma \Lambda A_{EM}^\top \mathbf{e} + \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}\right)\right]\ . \end{aligned}$$

Note that shifting the differentiation operator in front of each component of $\mathbf{u}$ does not require any further smoothness as we need them anyway and bypass matrix vector calculations only. According to (48), this EM model fits the EM device in

Sec. 2.1 for some suitable operator $\mathscr{E}$. As it explicitly determines the EM device's branch currents, the EM device can be considered as a current controlling element. Therefore, its constitutive element equation should be incorporated into KCL to follow the regime of the MNA. By splitting up the summands in $f_{EM}$, we obtain new functions

$$\begin{aligned}
q_{EM}(A_{EM}^\top \mathbf{e}) &:= \Lambda^\top G_\Gamma^\top M_\varepsilon G_\Gamma \Lambda A_{EM}^\top \mathbf{e}\,,\\
g_{EM}(A_{EM}^\top \mathbf{e}) &:= \Lambda^\top G_\Gamma^\top M_\sigma G_\Gamma \Lambda A_{EM}^\top \mathbf{e}\,,\\
i_{EM}(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u},\mathbf{u}) &:= \Lambda^\top G_\Gamma^\top \left[M_\varepsilon \left(G\tfrac{\mathrm{d}}{\mathrm{d}t}\phi + \tfrac{\mathrm{d}}{\mathrm{d}t}\pi\right) + M_\sigma \left(G\phi + \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}\right)\right]\,.
\end{aligned}$$

such that (48) is equivalent to

$$\mathbf{i}_{EM} = \tfrac{\mathrm{d}}{\mathrm{d}t} q_{EM}(A_{EM}^\top \mathbf{e}) + g_{EM}(A_{EM}^\top \mathbf{e}) + i_{EM}(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u},\mathbf{u})\,. \tag{49}$$

*Remark 1.* $\Lambda^\top G_\Gamma^\top M_\varepsilon G_\Gamma \Lambda$ and $\Lambda^\top G_\Gamma^\top M_\sigma G_\Gamma \Lambda$ are diagonal.

According to Rem. 1, the derivative operator can be pulled out of the first summand without requiring further smoothness. Moreover, the constitutive element equation (49) allows us to interpret the EM device as a composition of parallel capacitor, resistor and controlled current source alongside each branch, linking the ground to terminal; cf. Fig. 2. With this new interpretation we can deduce statements about the coupling which reason certain behavior during simulation processes.

On the other hand, with the alternative version of the total current model (45), that complies the one in [2], we obtain

$$\mathbf{i}_{EM} = \hat{f}_{EM}(\mathbf{u}) := -\Lambda^\top G_\Gamma^\top \tilde{C} M_\nu C \mathbf{a}\,. \tag{50}$$

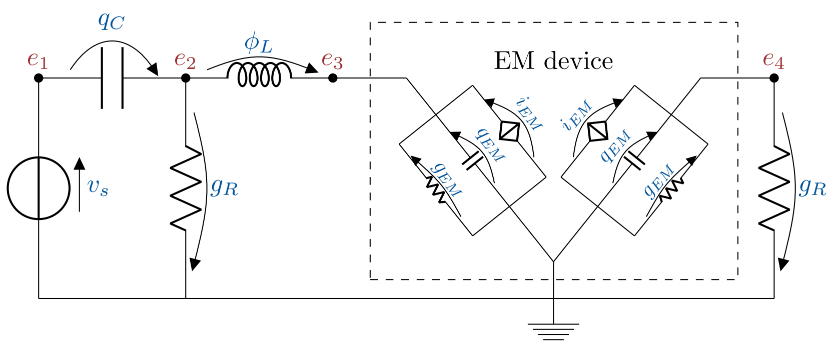



**Fig. 2** Electric circuit incorporating an EM device whose constitutive element equations are interpreted as a composition of capacitors, resistors and controlled sources along each branch.

#### 4.1.1 Systems

On the basis of the above preliminaries, we now provide a few categories of system formulations for the coupled problem that arise either due to implementation purposes, performance reasons or analysis results.

`devEM` version

As the `devEM` solver's interface takes control of the current coupling (48) it used to be convenient not to incorporate (48) into KCL, cf. [13, 19]. Hence, the coupled system that is solved arises from (7)-(9), (47), and (48) and is of the form

$$f_{MNA}(\tfrac{\mathrm{d}}{\mathrm{d}t} d_{MNA}(\mathbf{x}), \mathbf{x}, t) = c_{MNA}(\mathbf{y}) \,, \tag{51}$$

$$f_{EM}(\tfrac{\mathrm{d}}{\mathrm{d}t} d_{EM}(\mathbf{x}), \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}, \mathbf{x}, \mathbf{u}) - \mathbf{y} = 0 \,, \tag{52}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} + b_{EM}(\mathbf{u}) = c_{EM}(\tfrac{\mathrm{d}}{\mathrm{d}t} d_{EM}(\mathbf{x}), \mathbf{x}) \,, \tag{53}$$

with $\mathbf{y} = \mathbf{i}_{EM}$. The functions involved, with $\mathbf{w} = (\mathbf{w}_C, \mathbf{w}_L, \mathbf{w}_{EM})$, read

$$f_{MNA}(\mathbf{w}, \mathbf{x}, t) := \begin{pmatrix} A_C \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{w}_C + A_R g_R(A_R^\top \mathbf{e}) + A_L \mathbf{i}_L + A_V \mathbf{i}_V + A_I i_s(t) \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{w}_L - A_L^\top \mathbf{e} \\ v_s(t) - A_V^\top \mathbf{e} \end{pmatrix} ,$$

$$d_{MNA}(\mathbf{x}) := \begin{pmatrix} q_C(A_C^\top \mathbf{e}) \\ \phi_L(\mathbf{i}_L) \end{pmatrix} , \quad c_{MNA}(\mathbf{y}) := \begin{pmatrix} -A_{EM}^\top \mathbf{y} \\ 0 \\ 0 \end{pmatrix} .$$

Incorporated version

In order to be consistent with the MNA approach (4)-(5), we incorporate the current coupling equation into KCL of (7)-(9). Here, we make use of the previous observation (49) and group the resistance, capacitance and controlled current source like constitutive equations by introducing

$$A_{\mathscr{C}} := \begin{bmatrix} A_C \; A_{EM} \end{bmatrix} , \quad A_{\mathscr{R}} := \begin{bmatrix} A_R \; A_{EM} \end{bmatrix} , \quad q_{\mathscr{C}} := \begin{pmatrix} q_C \\ q_{EM} \end{pmatrix} , \quad g_{\mathscr{R}} := \begin{pmatrix} g_R \\ g_{EM} \end{pmatrix} .$$

Together with the EM system (47) we obtain

$$f_{MNA2}(\tfrac{\mathrm{d}}{\mathrm{d}t} d_{MNA2}(\mathbf{x}), \mathbf{x}, t) = c_{MNA2}(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}, \mathbf{u}) \,, \tag{54}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} + b_{EM}(\mathbf{u}) = c_{EM}(\tfrac{\mathrm{d}}{\mathrm{d}t} d_{EM}(\mathbf{x}), \mathbf{x}) \,, \tag{55}$$

for

$$f_{MNA2}(\mathbf{w},\mathbf{x},t) := \begin{pmatrix} A_{\mathscr{C}} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{w}_C + A_{\mathscr{R}} g_{\mathscr{R}}(A_{\mathscr{R}}^\top \mathbf{e}) + A_L \mathbf{i}_L + A_V \mathbf{i}_V + A_I i_s(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{w}_L - A_L^\top \mathbf{e} \\ v_s(t) - A_V^\top \mathbf{e} \end{pmatrix} ,$$

$$d_{MNA2}(\mathbf{x}) := \begin{pmatrix} q_{\mathscr{C}}(A_{\mathscr{C}}^\top \mathbf{e}) \\ \phi_L(\mathbf{i}_L) \end{pmatrix} , \quad c_{MNA2}(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u},\mathbf{u}) := \begin{pmatrix} -A_{EM}^\top i_{EM}(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u},\mathbf{u}) \\ 0 \\ 0 \end{pmatrix} .$$

Alternative version

Using (50) instead of (49) as the current coupling equation, we obtain:

$$f_{MNA}(\tfrac{\mathrm{d}}{\mathrm{d}t} d_{MNA}(\mathbf{x}),\mathbf{x},t) = c_{MNA3}(\mathbf{u}) , \tag{56}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} + b_{EM}(\mathbf{u}) = c_{EM}(\tfrac{\mathrm{d}}{\mathrm{d}t} d_{EM}(\mathbf{x}),\mathbf{x}) , \tag{57}$$

with

$$c_{MNA3}(\mathbf{u}) := \left(\Lambda^\top G_\Gamma^\top \tilde{C} M_\nu C \mathbf{a}\right) .$$

*Remark 2.* The systems (54)-(55) and (56)-(56) are analytically equivalent due to the discrete MA equation (39).

## 5 Waveform Relaxation Method

While the coupled systems in Sec. 4.1.1 can be solved numerically by various time integration methods, e. g. Runge-Kutta or the backward differentiation formula, it is also possible to solve only the subsystems. The latter can be realized for example in a Gauss-Seidel method style, cf. [12].

### *5.1 Gauss-Seidel Method*

When talking about the Gauss-Seidel method in this article, we mean the following procedure: Solve the subsystems, successively replace the solutions with newer ones and then start over the iteration. As we want to take advantage of the numerical treatment of ODEs, the EM equations (47) of the coupled systems will always be considered as one subsystem.

Let $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}^{(0)}$ and $\mathbf{x}^{(0)}$ be functions on $\mathscr{I}$ for the initial guess, e. g. obtained by extrapolation of the initial value $x_0$.

Applying the Gauss-Seidel iteration scheme on the systems (51)-(53), (54)-(55) and (56)-(57), respectively, we obtain the evaluation procedures

- for the `devEM` version:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k)} + b_{EM}(\mathbf{u}^{(k)}) = c_{EM}(\tfrac{\mathrm{d}}{\mathrm{d}t}d_{EM}(\mathbf{x}^{(k-1)}), \mathbf{x}^{(k-1)}) \,, \tag{58}$$

$$f_{MNA}(\tfrac{\mathrm{d}}{\mathrm{d}t}d_{MNA}(\mathbf{x}^{(k)}), \mathbf{x}^{(k)}, t) = c_{MNA}(\mathbf{y}^{(k)}) \,, \tag{59}$$

$$f_{EM}(\tfrac{\mathrm{d}}{\mathrm{d}t}d_{EM}(\mathbf{x}^{(k)}), \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k)}, \mathbf{x}^{(k)}, \mathbf{u}^{(k)}) - \mathbf{y}^{(k)} = 0 \,. \tag{60}$$

- for the incorporated version:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k)} + b_{EM}(\mathbf{u}^{(k)}) = c_{EM}(\tfrac{\mathrm{d}}{\mathrm{d}t}d_{EM}(\mathbf{x}^{(k-1)}), \mathbf{x}^{(k-1)}) \,, \tag{61}$$

$$f_{MNA2}(\tfrac{\mathrm{d}}{\mathrm{d}t}d_{MNA2}(\mathbf{x}^{(k)}), \mathbf{x}^{(k)}, t) = c_{MNA2}(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k)}, \mathbf{u}^{(k)}) \,. \tag{62}$$

- for the alternative version:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k)} + b_{EM}(\mathbf{u}^{(k)}) = c_{EM}(\tfrac{\mathrm{d}}{\mathrm{d}t}d_{EM}(\mathbf{x}^{(k-1)}), \mathbf{x}^{(k-1)}) \,, \tag{63}$$

$$f_{MNA}(\tfrac{\mathrm{d}}{\mathrm{d}t}d_{MNA}(\mathbf{x}^{(k)}), \mathbf{x}^{(k)}, t) = c_{MNA3}(\mathbf{u}^{(k)}) \,. \tag{64}$$

Here $k \in \mathbb{N}$ is the iteration parameter and all these equations have to comply to the initial values, i. e. $\mathbf{u}^{(k)}(t_0) = u_0$, $\mathbf{x}^{(k)}(t_0) = x_0^{(k)}$ as well as the auxiliary vector-function $\mathbf{y}^{(k)}(t_0) = y_0^{(k)}$, whereby the DAE variables may vary in each iteration and hence are decorated by ${}^{(k)}$.

## *5.2 Convergence Analysis*

In this section we focus on the Gauss-Seidel method's convergence for the alternative version of the coupled electric circuit EM device system (63)-(64).

Note that the decoupling of the subsystems is an invariant of the iteration parameter. Decoupling (64) by making use of Theorem 5 yields for $s_c(\mathscr{E}) = s_c(\mathbf{u}^{(k)}) := A_{EM}c_{MNA3}(\mathbf{u}^{(k)})$ and dropping the argument $t$ for simplicity:

$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k)} &= f_0(\mathbf{y}^{(k)},\mathbf{z}_1^{(k)},\mathbf{z}_2^{(k)},\mathbf{z}_3^{(k)},s_c(\mathbf{u}^{(k)}),s_i)\ , \qquad \mathbf{y}^{(k)}(t_0)=\mathbf{y}_0^{(k)}\ ,\\
\mathbf{z}_1^{(k)} &= M_1(\mathbf{y}^{(k)},\mathbf{z}_3^{(k)})\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_3^{(k)} + f_1(\mathbf{y}^{(k)},\mathbf{z}_2^{(k)},\mathbf{z}_3^{(k)},s_c(\mathbf{u}^{(k)}),s_i)\ ,\\
\mathbf{z}_2^{(k)} &= f_2(\mathbf{y}^{(k)},\mathbf{z}_3^{(k)},s_c(\mathbf{u}^{(k)}),s_i,s_v)\ ,\\
\mathbf{z}_3^{(k)} &= M_3\begin{pmatrix} s_i\\ s_v\end{pmatrix}.
\end{aligned}
$$

Under the prerequisite of Ass. 6, we obtain from Lemma 2

$$
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k)} = \hat{f}_0(\mathbf{y}^{(k)},\mathbf{z}_1^{(k)},\mathbf{z}_2^{(k)},s_c(\mathbf{u}^{(k)}),s_i,s_v)\ , \qquad \mathbf{y}^{(k)}(t_0)=\mathbf{y}_0^{(k)}\ , \tag{65}
$$

$$
\mathbf{z}_1^{(k)} = \hat{M}_1(\mathbf{y}^{(k)},s_i,s_v)\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} s_i\\ s_v\end{pmatrix} + \hat{f}_1(\mathbf{y}^{(k)},\mathbf{z}_2^{(k)},s_c(\mathbf{u}^{(k)}),s_i,s_v)\ , \tag{66}
$$

$$
\mathbf{z}_2^{(k)} = \hat{f}_2(\mathbf{y}^{(k)},s_c(\mathbf{u}^{(k)}),s_i,s_v)\ . \tag{67}
$$

Then, for (63) we obtain

$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k)} = -\,b_{EM}(\mathbf{u}^{(k)}) + c_{EM}\big(&\hat{T}_0\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k-1)} + \hat{T}_1\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_1^{(k-1)} + \hat{T}_2\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_2^{(k-1)},\\
&\hat{T}_0\mathbf{y}^{(k-1)} + \hat{T}_1\mathbf{z}_1^{(k-1)} + \hat{T}_2\mathbf{z}_2^{(k-1)}\big)\ .
\end{aligned} \tag{68}
$$

After inserting the right hand sides of

$$
\begin{aligned}
\mathbf{z}_1^{(k-1)} =& \hat{M}_1(\mathbf{y}^{(k-1)},s_i,s_v)\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} s_i \\ s_v \end{pmatrix} + \hat{f}_1(\mathbf{y}^{(k-1)},\mathbf{z}_2^{(k-1)},s_c(\mathbf{u}^{(k-1)}),s_i,s_v)\ , \\
\mathbf{z}_2^{(k-1)} =& \hat{f}_2(\mathbf{y}^{(k-1)},s_c(\mathbf{u}^{(k-1)}),s_i,s_v)\ , \\
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k-1)} =& \hat{f}_0(\mathbf{y}^{(k-1)},\mathbf{z}_1^{(k-1)},\mathbf{z}_2^{(k-1)},s_c(\mathbf{u}^{(k-1)}),s_i,s_v)\ , \\
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_1^{(k-1)} =& \hat{M}_1(\mathbf{y}^{(k-1)},s_i,s_v)\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} s_i \\ s_v \end{pmatrix} + \Big[\tfrac{\partial}{\partial \mathbf{y}^{(k-1)}}\hat{M}_1(\mathbf{y}^{(k-1)},s_i,s_v)\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k-1)} \\
&+\tfrac{\partial}{\partial s_i}\hat{M}_1(\mathbf{y}^{(k-1)},s_i,s_v)\tfrac{\mathrm{d}}{\mathrm{d}t}s_i + \tfrac{\partial}{\partial s_v}\hat{M}_1(\mathbf{y}^{(k-1)},s_i,s_v)\tfrac{\mathrm{d}}{\mathrm{d}t}s_v\Big]\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} s_i \\ s_v \end{pmatrix} \\
&+\tfrac{\partial}{\partial \mathbf{y}^{(k-1)}}\hat{f}_1(\mathbf{y}^{(k-1)},\mathbf{z}_2^{(k-1)}s_i,s_v)\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k-1)} \\
&+\tfrac{\partial}{\partial \mathbf{z}_2^{(k-1)}}\hat{f}_1(\mathbf{y}^{(k-1)},\mathbf{z}_2^{(k-1)}s_i,s_v)\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_2^{(k-1)} \\
&+\tfrac{\partial}{\partial s_i}\hat{f}_1(\mathbf{y}^{(k-1)},s_i,s_v)\tfrac{\mathrm{d}}{\mathrm{d}t}s_i + \tfrac{\partial}{\partial s_v}\hat{f}_1(\mathbf{y}^{(k-1)},s_i,s_v)\tfrac{\mathrm{d}}{\mathrm{d}t}s_v\ , \\
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_2^{(k-1)} =& \tfrac{\partial}{\partial \mathbf{y}^{(k-1)}}\hat{f}_2(\mathbf{y}^{(k-1)},s_c(\mathbf{u}^{(k-1)}),s_i,s_v)\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k-1)} \\
&+\tfrac{\partial}{\partial s_c^{(k-1)}}\hat{f}_2(\mathbf{y}^{(k-1)},s_c(\mathbf{u}^{(k-1)}),s_i,s_v)\tfrac{\partial}{\partial \mathbf{u}^{(k-1)}}s_c(\mathbf{u}^{(k-1)})\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k-1)} \\
&+\tfrac{\partial}{\partial s_i}\hat{f}_2(\mathbf{y}^{(k-1)},s_c(\mathbf{u}^{(k-1)}),s_i,s_v)\tfrac{\mathrm{d}}{\mathrm{d}t}s_i \\
&+\tfrac{\partial}{\partial s_v}\hat{f}_2(\mathbf{y}^{(k-1)},s_c(\mathbf{u}^{(k-1)}),s_i,s_v)\tfrac{\mathrm{d}}{\mathrm{d}t}s_v\ ,
\end{aligned}
$$

into (68), we obtain an expression in terms of $\mathbf{u}^{(k)},\mathbf{y}^{(k-1)},\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k-1)},\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k-1)}$ and $t$ allowing us to define a function $\theta_1$ such that the EM subsystem reads

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k)} = \theta_1(\mathbf{u}^{(k)},\mathbf{y}^{(k-1)},\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k-1)},\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k-1)},t)\ .$$

Further, from (65) we obtain

$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k)} =& \hat{f}_0(\mathbf{y}^{(k)},M(\mathbf{y}^{(k)},\hat{f}_2(\mathbf{y}^{(k)},s_c(\mathbf{u}^{(k)}),s_i,s_v)\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} s_i \\ s_v \end{pmatrix} + \hat{f}_1(\mathbf{y}^{(k)},s_i,s_v), \\
&\hat{f}_2(\mathbf{y}^{(k)},s_c(\mathbf{u}^{(k)}),s_i,s_v)) =: \theta_2(\mathbf{u}^{(k)},\mathbf{y}^{(k-1)},\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k-1)},\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k-1)},t)\ .
\end{aligned}
$$

The overall inherent ODE system of the alternative coupled system with applied Gauss-Seidel method (63)-(64) reads

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k)} = \theta_1(\mathbf{u}^{(k)},\mathbf{y}^{(k-1)},\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k-1)},\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k-1)},t)\ , \tag{69}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k)} = \theta_2(\mathbf{u}^{(k)},\mathbf{y}^{(k-1)},\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k-1)},\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k-1)},t)\ . \tag{70}$$

According to [17] we introduce the convergence matrix for (69)-(70) as

$$\bar{M} := \begin{pmatrix} \frac{\partial\theta_1}{\partial\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k-1)}} & \frac{\partial\theta_1}{\partial\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k-1)}} \\ \frac{\partial\theta_2}{\partial\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k-1)}} & \frac{\partial\theta_2}{\partial\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k-1)}} \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_1}{\partial\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k-1)}} & \frac{\partial\theta_1}{\partial\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}^{(k-1)}} \\ 0 & 0 \end{pmatrix}$$

and observe that the spectral radius $\rho(\bar{M}) = \rho\left(\frac{\partial \theta_1}{\partial \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{(k-1)}}\right)$. From [17] we know that the Gauss-Seidel method converges if the latter expression is less than 1.

### *5.3 Benchmark*

In this section we provide some numerical results according to the waveform relaxation method introduced in Sec. 5.

As a coupled electric circuit and EM device coupled system, we chose the bandpass filter in Fig. 1. The bandpass filter was solved using the original formulation (51)-(53) and the incorporated split formulation (54)-(55). Formulations of the alternative type (56)-(57) are not accessible to us. For both systems we applied the Gauss-Seidel method and compared the resulting monolithic to each iteration's solution. As an error measure we picked the maximal error over each component which itself is defined by the maximal offset for the whole time-interval $\mathscr{I}$. For the original system formulation we obtained convergence to machine precision after approximately 5000 iterations as shown in Fig. 3. In contrast, using Gauss-Seidel method in the incorporated split system (61)-(62) we achieved convergence after 4 iteration steps, see Fig. 4.

## 6 Conclusion and Outlook

So far, the coupled electric circuit and EM device problem was simulated by making use of system formulation (51)-(53), as the solver `PyCEM`, used in [19], interfaces the industrial EM solver `devEM`. Tests have shown that incorporating the EM device's branch current into KCL of the circuit equations, as done in (61)-(62), leads to a convergence acceleration by several orders of magnitudes. To explain this behavior, we made the coupled systems the subject of a waveform relaxation analysis whose techniques we adopted from [17]. As we are in an early stage of the analysis, we switched to the less sophisticated coupled system (56)-(57), which itself can be considered equivalent to (54)-(55). Concerning the theory, we then were able to provide convergence analysis for the Gauss-Seidel method applied to the alternative formulation (63)-(64). As a matter of fact, the alternative system (63)-(64) and the split system (61)-(62) are no more equivalent to each other when applying iteration schemes. But in return, the structural properties, revealed by the split formulation (54)-(55), might be useful to make statements about the convergence behavior. Moreover, they motivate adopting the tool's interfaces in order to cover more than just system formulations (51)-(53).

A goal for future work is to analyze the split system formulation itself in terms of decoupling and convergence. Further, the range of waveform relaxation schemes should be expanded and applied to multiple EM devices.

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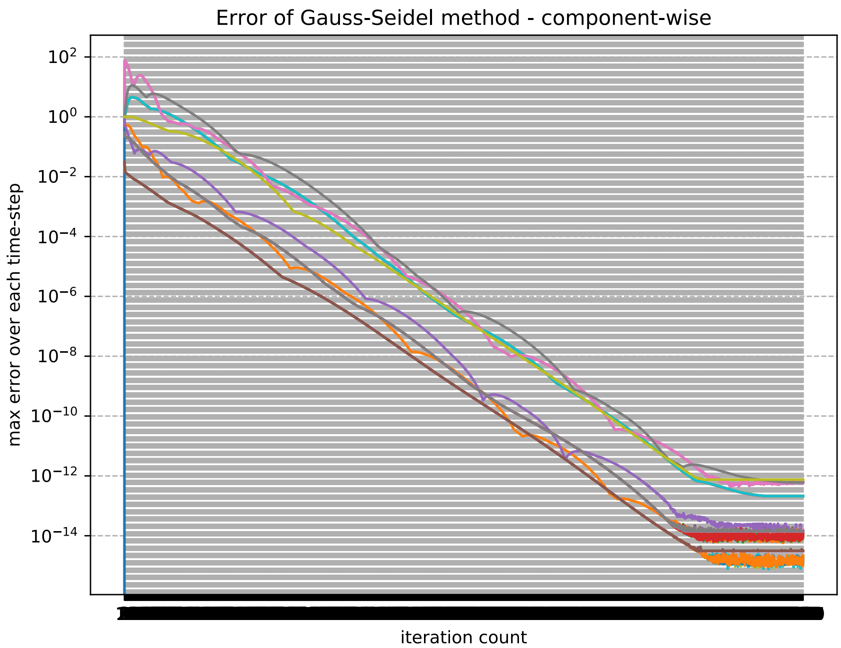


**Fig. 3** Solving the bandpass filter in Fig. 3 using (58)-(60). The plot shows error for each iteration's error.

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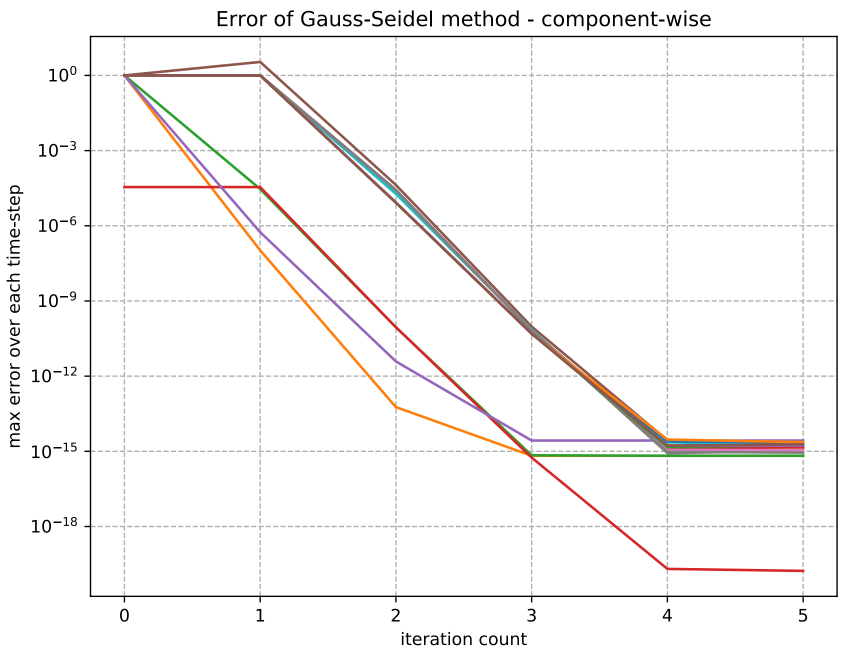


**Fig. 4** Solving the bandpass filter in Fig. 3 using (61)-(62). The plot shows error for each iteration's error.

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