A Projector Based Decoupling of DAEs obtained from the Derivative Array

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Abstract

The solution vector of a differential-algebraic equation contains different types of components, depending on their properties. In this paper, we particularly present an orthogonal decoupling that, for higher-index DAEs, describes in which context these orthogonal components appear in the derivative array. In this sense, we characterize different types of so-called "higher-index" components with regard to the explicit and hidden constraints. As a consequence, for linear DAEs we obtain a straightforward possibility to determine a projected explicit ODE and compare it with the so-called inherent regular ODE related to the projector-based decoupling associated with the tractability matrix sequence. By several examples we illustrate the differences of these two projector-based approaches and discuss their relationship.

Keywords: DAE, differential-algebraic equation, consistent initial value, index, derivative array, projector based analysis, constraints, orthogonal decoupling, tractability, MNA

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1 Introduction

Higher-index differential-algebraic equations (DAEs) present explicit and hidden constraints that restrict the choice of consistent initial values. In fact, the dynamics can be characterized by lower-dimensional ODEs that might be not unique.

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Example 1. *Let us consider a well-understood higher-index example from [17], [20], [18]:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} x' + \begin{pmatrix} -\alpha & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} q_1 \\ q_{2,1} \\ q_{2,2} \\ q_{2,3} \\ q_{2,4} \end{pmatrix}.$$

The explicit constraint reads

$$x_5 = q_{2,4}$$

and the hidden constraints result to be

$$\begin{aligned} x_4 &= q_{2,3} - q_{2,4}', \\ x_3 &= q_{2,2} - (q_{2,3} - q_{2,4}')', \\ x_2 &= q_{2,1} - (q_{2,2} - (q_{2,3} - q_{2,4}')')'. \end{aligned}$$

Therefore, the degree of freedom d results to be one. To characterize the one-dimensional dynamics, there are different possibilities. On the one hand, the explicit scalar ODE

$$x_1' - \alpha x_1 = q_1 + q_{2,1} - q_{2,2}' + q_{2,3}'' - q_{2,4}''', \tag{1}$$

that depends on derivatives of the right-hand side q, could be considered. On the other hand, for

$$u_e := x_1 + x_3 - \alpha x_4 + \alpha^2 x_5$$

the explicit scalar ODE

$$u_e' - \alpha u_e = q_1 + q_{2,1} - \alpha q_{2,2} + \alpha^2 q_{2,3} - \alpha^3 q_{2,4}, \tag{2}$$

that does not depend on derivatives of q, could be considered. For the initialization, this means that if we consider (1)*, then an initial value is prescribed for $x_1(t_0)$. In contrast, if* (2) *is considered, then an initial value is prescribed for $u_e(t_0)$. In both cases, $x_2(t_0), \ldots, x_5(t_0)$ are determined by the explicit and hidden constraints and cannot be prescribed.*

While a general projector based characterization of ODEs associated to a DAE that do not involve derivatives (like (2)) can be found in [17] and the related work, such a general projector based description has not been developed so far for ODEs associated to a DAE with orthogonality properties like (1). Such ODEs will depend, in general, on derivatives of parts of the original DAE, i.e., parts of the so-called derivative array.

In this setting, our goal is to provide the framework of a projector based analysis of DAEs for approaches that are based on the consideration of the derivative array, whereas the associated ODE may depend on derivatives of the right-hand side, like (1).

In a first step, a new approach to compute consistent initial values for higher-index DAEs using the derivative array and a projector based approach was recently developed in [7], [9]. Starting from these results, in this paper we address an orthogonal decoupling of the solution vector and a corresponding decoupling of the equations of the DAE. In order to facilitate the readability, we start defining the basic concepts briefly again.

We consider DAEs of the form

$$f(x', x, t) = 0, \quad \text{where } f_{x'} \text{ is singular,} \tag{3}$$

assume that

$$\ker f_{x'}(x', x, t)$$

does not depend on (x', x) and that a continuously differentiable projector $Q = Q(t)$ onto $\ker f_{x'}$ exists. On the basis of the complementary projector $P(t) := I - Q(t)$ we can then reformulate the DAE as

$$f(x', x, t) = f(Px', x, t) = f((Px)' - P'x, x, t) = 0, \tag{4}$$

as already introduced in [15]. In this sense, we will use the notation:

- Px for the differentiated component,
- Qx for the undifferentiated component,

since, for the decoupling $x' = (Px)' + (Qx)'$, we can see that $(Px)' = \varphi_1(x, t)$ is implicitly given, cf. [7], [9]. In [7] we presented an orthogonal decoupling of Qx with regard to the explicit and hidden constraints.

In this article, we complete this approach by decoupling Px analogously, thus obtaining an orthogonal decoupling of the complete vector $x = Px + Qx$. The paper is organized as follows.

In Section 2 we summarize some definitions and the notations introduced in [7], [9]. Based on that, the orthogonal projectors used for the decoupling of x are defined in Section 3. In particular, the projector Π is defined, which is analyzed in more detail in Section 4.

Section 5 presents an extensive discussion of linear DAEs. For linear DAEs, Π turns out to deliver a description of an associated explicit ODE. We show and illustrate with examples the differences between the introduced orthogonal decoupling and the projector based decoupling associated with the tractability matrix sequence.

The computation of the projectors for the MNA is briefly presented in Section 6 in order to show that the new orthogonal decoupling is a direct generalization of a result presented already in [3].

In the Appendix, we provide some required results from linear algebra and analyze two illustrative classes of linear DAEs with constant coefficients.

2 Reinterpretation of the Differentiation Index

With the decoupling $x = Px + Qx$ in mind, we use the following definition of the differentiation index, which was introduced in [7], [9].

Definition 1. *The differentiation index is the smallest integer μ such that*

$$\begin{aligned} f(x',x,t) &= 0, \\ \frac{d}{dt} f(x',x,t) &= 0, \\ &\vdots \\ \frac{d^{\mu-1}}{dt^{\mu-1}} f(x',x,t) &= 0, \end{aligned}$$

uniquely determines Qx as a function of (Px,t).

Due to (4) there exists a function φ_1 such that, locally,

$$(Px)' = \varphi_1(x,t)$$

holds. If, according to Definition 1, there exists another function φ_2 such that

$$Qx = \varphi_2(Px,t),$$

then one further differentiation provides

$$(Qx)' = \varphi_3((Px)', Px, t) = \tilde{\varphi}_3(x,t).$$

Consequently, if μ is the differentiation index according to Definition 1, then the conventional differentiation index (see e.g. [2]) results to be μ as well.

In order to allow for the differentiations, we consider

$$F_j(x^{(j+1)}, x^{(j)}, \ldots, x', x, t) := \frac{d^j}{dt^j} f(x',x,t),$$

and define for $z_i \in \mathbb{R}^n$, $i = 0, \ldots, k$,

$$g^{[k]}(z_0, z_1, \ldots, z_k, t) := \begin{pmatrix} f(z_1, z_0, t) \\ F_1(z_2, z_1, z_0, t) \\ \vdots \\ F_{k-1}(z_k, \ldots, z_0, t) \end{pmatrix}, \tag{5}$$

which corresponds to the derivative array. To compute the index μ in this context, for $z_i \in \mathbb{R}^n$, $i = 0, \ldots, k$, we denote by

$$G^{[k]}_{(z_0)}(z_0, z_1, \ldots, z_k, t) \in \mathbb{R}^{nk \times n}$$

the Jacobian matrix of $g^{[k]}(z_0, z_1, \ldots, z_k, t)$ with respect to z_0, by

$$G^{[k]}_{(z_1,\ldots,z_k)}(z_0, z_1, \ldots, z_k, t) \in \mathbb{R}^{nk \times nk}$$

the Jacobian matrix of $g^{[k]}(z_0, z_1, \ldots, z_k, t)$ with respect to $(z_1, \ldots, z_k)$, and consider the matrices

$$\mathcal{B}^{[k]} := \begin{pmatrix} P & 0 \\ G^{[k]}_{(z_0)} & G^{[k]}_{(z_1,\ldots,z_k)} \end{pmatrix} \in \mathbb{R}^{n(k+1) \times n(k+1)}, \quad k = 1, \ldots.$$

According to [9], we check if the matrices $\mathcal{B}^{[k]}$ are 1-full with respect to the first n columns for $k = 1, 2, \ldots$, i.e., whether

$$\ker \mathcal{B}^{[k]} \subseteq \left\{ \begin{pmatrix} s_0 \\ s_1 \end{pmatrix} \ : \ s_0 \in \mathbb{R}^n,\ s_0 = 0,\ s_1 \in \mathbb{R}^{nk} \right\}. \tag{6}$$

We conclude that the index is μ if μ is the smallest integer for which $\mathcal{B}^{[\mu]}$ is 1-full and emphasize that g^μ consists of $f, F_1, \ldots, F_{\mu-1}$, such that no μ-th differentiation is needed.

3 Defining Projectors with the Derivative Array

In order to characterize the different components we have a closer look onto the matrix

$$G^{[k]} = \begin{pmatrix} G^{[k]}_{(z_0)} & G^{[k]}_{(z_1,\ldots,z_k)} \end{pmatrix} =: \begin{pmatrix} G^{[k]}_L & G^{[k]}_R \end{pmatrix}, \tag{7}$$

where L and R stand for left- and right-hand side, respectively.

- To decouple the undifferentiated component Q for $k = 1, \ldots$, we consider a basis[1] $B^{[k]}_R$ along $\operatorname{im} G^{[k]}_R$ and define the projector T_k as the orthogonal projector onto
$$\ker \begin{pmatrix} P \\ B^{[k]}_R G^{[k]}_L \end{pmatrix} =: \operatorname{im} T_k.$$
Consequently, $T_k x$ corresponds to the part of the undifferentiated component Qx that, after $k-1$ differentiations, cannot yet be represented as a function of (Px, t). Note that, by definition, $T_k \neq 0$ for $k < \mu$ and $T_\mu = 0$, cf. [7].

- To characterize the different parts of the differentiated component Px, we further decouple $G^{[k]}$ in each step k and consider
$$\begin{pmatrix} Q & 0 & 0 \\ G^{[k]}_L P & G^{[k]}_L Q & G^{[k]}_R \end{pmatrix}.$$
With this decoupling from [7] in mind, we consider a basis[2] $B^{[k]}_{LQ-R}$ along
$$\operatorname{im} \begin{pmatrix} G^{[k]}_L Q & G^{[k]}_R \end{pmatrix}$$
and finally define the orthogonal projector V_k onto
$$\ker \begin{pmatrix} Q \\ B^{[k]}_{LQ-R} G^{[k]}_L \end{pmatrix} =: \operatorname{im} V_k.$$
Then $V_k x$ represents the part of the differentiated components Px that is not determined by the constraints resulting after $k-1$ differentiations.

[1] Instead of a basis, any matrix $W^{[k]}_R$ with $\ker W^{[k]}_R = \operatorname{im} G^{[k]}_R$ could be used in this context, especially a projector. According to our implementation in InitDAE, we consider a basis here.

[2] Again, instead of a basis, any matrix $W^{[k]}_{LQ-R}$ with $\ker W^{[k]}_{LQ-R} = \operatorname{im} G^{[k]}_{LQ-R}$ could be used in this context, analogously as for $B^{[k]}_R$.

By definition, the degree of freedom d is rank V_μ. In accordance with our previous work we define

$$\Pi := V_\mu.$$

Note that, by construction, we have $QV_k = 0$ for all k and, hence, $Z_k = (P - V_k)$ results to be a projector:

$$Z_k \cdot Z_k = (P - V_k)(P - V_k) = P - 2 \cdot PV_k + V_k = P - V_k = Z_k.$$

Consequently, $Z_k x$ describes the differentiated components that are determined by constraints resulting after $k-1$ differentiation and, in particular, $(P - \Pi)x = Z_\mu x$ the differentiated components that are determined by constraints after $\mu - 1$ differentiations.

According to Theorem 1 in [7], it holds

$$T_k = Q_0 T_k = T_k Q_0 = T_{k-1} T_k = T_k T_{k-1}, \tag{8}$$

and it can be proved analogously that

$$V_k = P_0 V_k = V_k P_0 = V_{k-1} V_k = V_k V_{k-1}. \tag{9}$$

Therefore, for $Z_k = (P - V_k)$, $U_k := (Q - T_k)$, $x = Px + Qx$ we can consider the decoupling

$$Px = PZ_1 x + V_1 Z_2 x + V_2 Z_3 x + \ldots + V_{\mu-2} Z_{\mu-1} x + \Pi x, \tag{10}$$
$$Qx = Q_0 U_1 x + T_1 U_2 x + T_2 U_3 x + \ldots + T_{\mu-2} U_{\mu-1} x + T_{\mu-1} x. \tag{11}$$

Example 2. *Let us consider the DAE resulting from the exothermic reactor model (cf. [21]), also described in [2]:*

$$\begin{aligned} C' &= K_1(C_0 - C) - R, \\ T' &= K_1(T_0 - T) + K_2 R - K_3(T - T_C), \\ 0 &= R - K_3 \mathrm{e}^{-\frac{\mathrm{K_4}}{\mathrm{T}}} \mathrm{C}, \\ 0 &= C - u, \end{aligned}$$

where K_1, K_2, K_3, K_4 are constants, C_0 and T_0 are the feed reactant concentration and feed temperature (assumed to be known functions). The variables C and T are the corresponding quantities in the product, $u(t)$ is an input function prescribing C, R is the reaction rate per unit volume, and T_C is the temperature of the cooling medium. The corresponding projectors can be found in Table 1. The index is three and since $\Pi = 0$, the degree of freedom is zero and no initial values can be prescribed in this case.

4 Properties of Π

To simplify the notation, we introduce matrices N and W fulfilling

$$N := B_R^{[\mu]} G_L^{[\mu]}, \quad \ker W = \operatorname{im} NQ, \tag{12}$$

	$x = (C, T, R, T_C)$
A	$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$G^{[1]}$	$T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ V_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$G^{[2]}$	$T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ V_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$G^{[3]}$	$T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ V_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: \Pi$

Table 1: Projectors associated with the derivative array analysis for the exothermic reactor model (Example 2).

where W can be an arbitrary matrix (e.g. an orthogonal basis or projector). Consequently, the orthogonal projector Π fulfills

$$\ker \begin{pmatrix} Q \\ WN \end{pmatrix} = \ker Q \cap \ker WN = \operatorname{im} \Pi.$$

According to the index definition, it further holds

$$\ker \begin{pmatrix} P \\ N \end{pmatrix} = \ker \begin{pmatrix} \Pi \\ N \end{pmatrix} = \{0\}.$$

Consequently, Lemma 2 from the Appendix A.1 implies that there exists a function φ_4 such that

$$(I - \Pi)x = \varphi_4(\Pi x, t). \tag{13}$$

In [8], [9] we have shown that, under suitable assumptions, the constraint optimization problem

$$\begin{aligned} \min \quad & \|P(z_0 - \alpha)\|_2 & (14) \\ \text{subject to} \quad & g^{[\mu]}(z_0, z_1, \ldots, z_\mu, t_0) = 0, & (15) \end{aligned}$$

turns out to compute consistent initial values fulfilling

$$\Pi(z_0 - \alpha) = 0.$$

In this sense, the consistent initialization computed by (14)-(15) corresponds to

$$z_0 = \Pi\alpha + \varphi_4(\Pi\alpha, t_0).$$

In the following, we pursue this idea for linear DAEs in order to obtain an associated regular ODE explicitly.

5 Linear DAEs

In this section we consider linear DAEs with constant or time-dependent coefficient matrices of the form

$$A(t)x' + B(t)x \quad = \quad q(t), \tag{16}$$

which are regular on an open finite interval $\mathcal{I}$ according to the definition introduced in [7]. Recall that this regularity assumption also excludes so-called harmless critical points like the one described in Example 2.71 from [17], since we assume that all the projectors considered have constant rank. With the notation from (12), the explicit and hidden constraints can then be described in terms of

$$N(t)x = s(t) := B_R^{[\mu]}(t) \begin{pmatrix} q(t) \\ q'(t) \\ \vdots \\ q^{(\mu-1)}(t) \end{pmatrix}. \tag{17}$$

Recall further that for linear DAEs

$$W(t)N(t)x = W(t)N(t)P(t)x = W(t)s(t)$$

represents the constraints that restrict Px. For simplicity, we will drop the argument t in the following.

For our purposes, we basically consider the orthogonal splittings

$$P \quad = \quad P\Pi + P(I-\Pi) = \Pi + (P-\Pi), \qquad I = \Pi + (I-\Pi)$$

and assume that all pseudo-inverses used below can be defined pointwise. For a detailed discussion on the existence of time-dependent pseudo-inverses in an analogous context we refer to [17], [19].

Note that there are some relations between Π and the projector $\Pi_{\mu-1}$ from [17]. In fact, by definition, for index-2 DAEs $I - \Pi$ results to be the orthogonal projector along $\operatorname{im} \Pi_1 = \ker(I - \Pi_1)$, i.e., $\ker(I - \Pi) = \ker(I - \Pi_1)$.

In the higher-index case the relationship between Π and $\Pi_{\mu-1}$ seems to be more complex. For a better appraisal, we start comparing the definitions of explicit ODEs related to a DAE that result from the different concepts.

5.1 On explicit ODEs associated with a DAE

In the literature, there are several explicit ODEs that are associated with a DAEs, in particular:

- The completion ODE, or underlying ODE, is an explicit ODE for the complete vector x that is associated with the differential index concept. It can be extracted from the derivative array (cf., e.g., [2], [16] and the references therein) and depends on the derivatives of q up to the order μ:

  $$x' = \varphi_c(x, q, q', \dots, q^{(\mu)}),$$

  for a suitable function φ_c.

- The inherent explicit regular ODE (IERODE) is closely related to the tractability index concept. It is formulated for $u_i := \Pi_{\mu-1}x \in \mathbb{R}^n$, where $\Pi_{\mu-1}$ is a suitably defined projector fulfilling rank $\Pi_{\mu-1} = d$. It lives in $\mathbb{R}^n$, $n \geq d$ and is unique in the scope of fine decoupling (see [17], [20] and the references therein). The projector $\Pi_{\mu-1}$ is precisely chosen such that the IERODE does not depend on derivatives of q, i.e.,

$$(\Pi_{\mu-1}x)' = \varphi_i(\Pi_{\mu-1}x, q), \quad \text{or,} \quad u_i' = \varphi_i(u_i, q),$$

  for $u_i \in \mathbb{R}^n$ and a suitable function φ_i.

- An essential underlying ODEs (EUODEs) has minimal size d (cf. [1], [20] and the references therein). There may be several EUODEs living in a transformed space with dimension d. EUODEs are also free of derivatives of q and can be considered a condensed IERODE, cf. [20]. We will represent EUODEs in terms of

$$u_e' = \varphi_e(u_e, q)$$

  for $u_e \in \mathbb{R}^d$ and a suitable function φ_e.

In this section, we consider a closely related definition of explicit ODEs:

- A projected explicit ODE (PEODE) of a DAE is an explicit ODE formulated for $u_p := \Pi_p x \in \mathbb{R}^n$ for a projector Π_p. A PEODE lives in $\mathbb{R}^n$, $n \geq \operatorname{rank} \Pi_p \geq d$ and may depend on derivatives of q up to the order μ:

$$(\Pi_p x)' = \varphi_p(\Pi_p x, q, q', \ldots, q^{(\mu)}),$$

  or

$$u_p' = \varphi_p(u_p, q, q', \ldots, q^{(\mu)}),$$

  for $u_p \in \mathbb{R}^n$ and a suitable function φ_p.

- Essential projected ODEs (EPEODEs) are corresponding condensed PEODEs with minimal size rank Π_p. They can also depend on derivatives of q in general. We will represent EPEODEs in terms of

$$u_{ep}' = \varphi_{ep}(u_{ep}, q, q', \ldots, q^{(\mu)})$$

  for $u_{ep} \in \mathbb{R}^{\operatorname{rank} \Pi_p}$ and a suitable function φ_{ep}.

Note that in this sense, completion ODEs are PEODEs for $\Pi_p = I$ and therefore also EPEODEs. Moreover, IERODEs are PEODEs for $\Pi_p = \Pi_{\mu-1}$, while EUODEs are the corresponding EPEODEs of dimension d.

The following Lemma generalizes Lemma 2.27 from [17] for PEODEs:

Lemma 1. *Let Π_p be a projector with $\Pi_p \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)$ and $u \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)$ be a solution of an ODE of the form*

$$u' - \Pi_p' u + \Pi_p C(t) u = \Pi_p c(t) \tag{18}$$

for suitable $C(t), c(t)$. Then the subspace $\operatorname{im} \Pi_p$ *is an invariant subspace for the ODE* (18)*, i.e., the following assertion is valid for the solutions $u \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)$:*

$$u(t_*) \in \operatorname{im} \Pi_p(t_*), \text{ with a certain } t_* \in \mathcal{I} \Leftrightarrow u(t) \in \operatorname{im} \Pi_p(t) \text{ for all } t \in \mathcal{I}.$$

Proof. This proof follows the steps of Lemma 2.27 in [17], which traces back to [15]. Let $\bar{u} \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)$ denote the unique solution of

$$\begin{aligned} \bar{u}' - \Pi_p'(t)\bar{u} + \Pi_p(t)C(t)\bar{u} &= \Pi_p(t)c(t), & (19) \\ \bar{u}(t_*) &= \Pi_p(t_*)\alpha & (20) \end{aligned}$$

for an arbitrary $\alpha \in \mathbb{R}^n$. If we multiply (19) and (20) by $(I - \Pi_p(t))$ and $(I - \Pi_p(t_*))$, respectively, then we obtain

$$\begin{aligned} (I - \Pi_p(t))\bar{u}' - (I - \Pi_p(t))\Pi_p'(t)\bar{u} &= 0, \\ (I - \Pi_p(t_*))\bar{u}(t_*) &= 0. \end{aligned}$$

For the function $\bar{v} := (I - \Pi_p)\bar{u} \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)$ with

$$\bar{v}' = (I - \Pi_p)'\bar{u} + (I - \Pi_p)\Pi_p'\bar{u}'$$

then

$$0 = \bar{v}' - (I - \Pi_p)'\bar{u} - \underbrace{(I - \Pi_p)\Pi_p'}_{-(I-\Pi_p)'\Pi_p}\bar{u} = \bar{v}' - (I - \Pi_p)'(I - \Pi_p)\bar{u}$$

and, therefore, $\bar{v}' - (I - \Pi_p)'\bar{v} = 0$ and $\bar{v}(t_*) = 0$ hold. Consequently, $\bar{v}$ vanishes identically, implying $\bar{u} = \Pi_p u(t)$. □

In the following, for a given DAE we will consider a particular type of projected explicit ODE, choosing $\Pi_p = \Pi$.

These particular PEODEs are specially relevant for the analysis of the Taylor series method discussed in [10]. Since automatic differentiation is used there, the higher order derivatives can perfectly be handled for sufficiently smooth DAEs. This is a fundamental difference to other integration schemes, which require a special treatment of these derivatives in general.

5.2 A closer look at the constraints

With the results from Appendix A.1, the constraints can be split into different parts with regard to $P(t)$ and $\Pi(t)$.

- On the one hand, we consider the constraints for Px

  $$WNx = Ws, \tag{21}$$

  which lead to

  $$(P - \Pi)x = (WN)^+(WN)x = (WN)^+Ws. \tag{22}$$

- On the other hand, we reformulate (17), obtaining

  $$N(I - \Pi)x = s - N\Pi x.$$

  According to Corollary 2 from the Appendix A.1, the multiplication by $(N(I - \Pi))^+$ provides the representation

  $$(I - \Pi)x = (N(I - \Pi))^+ (s - N\Pi x). \tag{23}$$

  Note that this particularly yields

  $$Qx = Q\,(N(I - \Pi))^+ (s - N\Pi x).$$

By definition, Πx is missing. On that account, we deduce a projected explicit ODE for Πx in the following.

5.3 Obtaining a projected explicit ODE for $u = \Pi x$

Now we show how to obtain a projected explicit ODE (PEODE) for Πx in four steps.

(i) Reformulation of the derivative with the projector P
If, for $r = \operatorname{rank} A$ and the SVD $A = U\operatorname{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)V^T$, we define the nonsingular matrix $\hat{A}$ by

$$\hat{A} = V\operatorname{diag}(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_r}, 1, \ldots, 1)U^T,$$

then the property $\hat{A}A = P$ is given. Moreover, the multiplication of (16) by $\hat{A}(t)$ leads to the DAE

$$(Px)' + B_{(i)}x \quad = \quad q_{(i)} \tag{24}$$

for

$$B_{(i)} := \hat{A}B + P', \quad q_{(i)} := \hat{A}q.$$

(ii) Reformulation of the derivative with the projector Π

If we use equation (22) and the splitting

$$(Px)' = (\Pi x)' + ((P - \Pi)x)',$$

then equation (24) leads to

$$(\Pi x)' + B_{(i)}x \quad = \quad q_{(ii)} \tag{25}$$

for

$$q_{(ii)} = q_{(i)} - ((WN)^+ Ws)'.$$

(iii) Formulation of an ODE in terms of Πx
With equation (23), in equation (25) we consider the splitting

$$B_{(i)}x = B_{(i)}(\Pi x + \underbrace{(N(I - \Pi))^+ (s - N\Pi x)}_{=(I-\Pi)x}).$$

Consequently, for

$$\begin{aligned} B_{(iii)} &:= B_{(i)}(I - (N(I - \Pi))^+ N)\Pi, \\ q_{(iii)} &:= q_{(ii)} - B_{(i)}(N(I - \Pi))^+ s, \end{aligned}$$

we obtain the ODE

$$(\Pi x)' + B_{(iii)}(\Pi x) \quad = \quad q_{(iii)}. \tag{26}$$

(iv) Formulation of an invariant ODE for $u = \Pi x$
If we finally multiply (26) by Π and use $(\Pi x)' = (\Pi\Pi x)' = \Pi'(\Pi x) + \Pi(\Pi x)'$, the projected explicit ODE (PEODE)

$$(\Pi x)' - \Pi'(\Pi x) + \Pi C(t)(\Pi x) = \Pi c(t) \tag{27}$$

results for

$$C(t) = B_{(iii)} = (\hat{A}B + P')(I - (N(I-\Pi))^{+}N)\Pi \tag{28}$$

$$c(t) = q_{(iii)} = \hat{A}q - ((WN)^{+}Ws)' - (\hat{A}B + P')(N(I-\Pi))^{+}s \tag{29}$$

in the invariant subspace $\operatorname{im}\Pi$, cf. Lemma 1.

Summarizing, we have proved the following result:

Theorem 1. *Let the DAE* (16) *be regular with index* μ *such that the constraints can be described by* (17) *and the used pseudo-inverses exist. Then a solution* $x = \Pi x + (I - \Pi)x$ *of the DAE can be determined*

- *considering an initial value problem for the ODE* (27) *in the invariant subspace* $\operatorname{im}\Pi$ *in order to obtain* Πx*, and*
- *computing* $(I - \Pi)x$ *afterwards according to* (23)*.*

Remark 1. *In general we allow for* (21) *that*

$$WNx = Ws = \phi(q, \ldots, q^{(\mu-1)})$$

and, therefore,

$$c(t) = \hat{c}(t, q, q', \ldots, q^{(\mu)}),$$

such that φ_p *and* φ_{ep} *may depend on derivatives of* q *up to order* μ*. However, for the classes of DAEs inspected rigorously in [6], [7] we obtained* $V_\mu = V_{\mu-1}$*, consequently* $Z_\mu = Z_{\mu-1}$*, and therefore*

$$WNx = Ws = \phi(q, \ldots, q^{(\mu-2)})$$

and

$$c(t) = \hat{c}(t, q, q', \ldots, q^{(\mu-1)}).$$

This holds particularly for properly stated linear DAEs of index $\mu \leq 2$ *and linear DAEs with constant coefficient matrices with an arbitrary index. Consequently, for these classes of DAEs,* φ_p *and* φ_{ep} *depend on derivatives of* q *up to order* $\mu - 1$*.*

5.4 Illustrative Examples

Example 3. *We start illustrating our approach with a small index-2 example, which is slightly more general than the one discussed in [7].*

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' + \underbrace{\begin{pmatrix} 1 & 0 & a \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}}_{B} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{30}$$

	$x = (x_1, x_2, x_3)$
A	$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$G^{[1]}$	$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ V_1 = \frac{1}{5} \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$G^{[2]}$	$T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ V_2 = \frac{1}{5} \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: \Pi$

Table 2: Projectors associated with the derivative array analysis for Example 3.

for functions $q_1(t), q_2(t), q_3(t)$ and a parameter a. According to the analysis shown in Table 2, the differentiation index is 2 and the constraints can be described by

$$\underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{=:N} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} q_3 \\ q_2 - q_3' \end{pmatrix}}_{=:s}.$$

Consequently,

$$\begin{aligned}
NQ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad WN = WNP = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix}, \\
(WN)^+ WN &= \frac{1}{5} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (P - \Pi), \\
N(I - \Pi) &= \begin{pmatrix} 1 & 2 & 0 \\ \frac{3}{5} & \frac{6}{5} & 1 \end{pmatrix}, \quad (N(I - \Pi))^+ = \begin{pmatrix} \frac{1}{5} & 0 \\ \frac{2}{5} & 0 \\ -\frac{3}{5} & 1 \end{pmatrix}, \\
(N(I - \Pi))^+ N(I - \Pi) &= \begin{pmatrix} \frac{1}{5} & \frac{2}{5} & 0 \\ \frac{2}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} = (I - \Pi), \\
(I - (N(I - \Pi))^+ N) &= \frac{1}{5} \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ -2 & 1 & 0 \end{pmatrix}.
\end{aligned}$$

With

$$\hat{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{A}B = \begin{pmatrix} 1 & -1 & 2a - 1 \\ 0 & 1 & 1 - a \\ 1 & 2 & 0 \end{pmatrix}$$

the PEODE described by equation (27) *reads*

$$\left(\frac{1}{5} \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x \right)' + (2 - a) \frac{1}{5} \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

for

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 2q_1 - 2aq_2 - \frac{2}{5}(1-3a)\,q_3 \\ -q_1 + aq_2 + \frac{1}{5}(1-3a)\,q_3 \\ 0 \end{pmatrix} + \begin{pmatrix} (2a - \frac{6}{5}) \\ -(\frac{3}{5} - a) \\ 0 \end{pmatrix} q_3'.$$

Hence, an EPEODE can be formulated for $u_{ep} := 2x_1 - x_2$*:*

$$u_{ep}' + (2-a)u_{ep} = -5r_2. \tag{31}$$

Once this ODE is solved, the solution of the original DAE can be computed using

$$(I - \Pi)x = \frac{1}{5}\begin{pmatrix} x_1 + 2\,x_2 \\ 2\,x_1 + 4\,x_2 \\ 5x_3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} q_3 \\ 2q_3 \\ q_2 - q_3' - 3q_3 - (2x_1 - x_2) \end{pmatrix} = \varphi_4(\Pi x, t).$$

For this example, the matrices defined in [17], page 23 ff., which are part of the tractability matrix sequence, read:

$$G_2 = \begin{pmatrix} 2\,a & 4\,a-1 & a \\ a+1 & 2\,a+2 & 1 \\ 1 & 2 & 0 \end{pmatrix}, \quad G_2^{-1} = \begin{pmatrix} 2 & -2\,a & 2\,a^2 - 2\,a + 1 \\ -1 & a & a - a^2 \\ 0 & 1 & -a-1 \end{pmatrix},$$

$$\Pi_1 = \begin{pmatrix} 2-2\,a & 2-4\,a & 0 \\ a-1 & 2\,a-1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consequently, for $u = \Pi_1 x$ *the IERODE reads:*

$$u_i' + \begin{pmatrix} 2\,a^2 - 6\,a + 4 & 4\,a^2 - 10\,a + 4 & 0 \\ -a^2 + 3\,a - 2 & -2\,a^2 + 5\,a - 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} u_i = \begin{pmatrix} 2 & -2\,a & 2\,a^2 - 2\,a + \frac{4}{5} \\ -1 & a & -a^2 + a - \frac{2}{5} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

Since it holds for $u_e = (a-1)x_1 + (2a-1)x_2$ *that*

$$\Pi_1 x = u_i = \begin{pmatrix} -2u_e \\ u_e \end{pmatrix},$$

it suffices to consider the EUODE

$$u_e' + (-a^2 + 3\,a - 2)(-2u_e) + (-2\,a^2 + 5\,a - 2)u_e = -q_1 + aq_2 + (-a^2 + a - \frac{2}{5})q_3,$$

i.e.,

$$u_e' + (2-a)u_e = -q_1 + aq_2 + (-a^2 + a - \frac{2}{5})q_3.$$

Note that in contrast to (31)*, this ODE does not depend on derivatives of the right-hand side.*

Example 4. *Let us consider again Example 1. Since* $T_3 \neq 0$ *and* $T_4 = 0$*, the index is 4 and we obtain* $\Pi = V_4$*, cf. Table 3. Consequently, the associated EPEODE we obtain coincides with the one discussed in [20], [18]:*

$$x_1' - \alpha x_1 = q_1 + q_{2,1} - (q_{2,2} - (q_{2,4} - q_{2,4}')')'.$$

	$x = (x_1, x_2, x_3, x_4, x_5)$
A	$Q = \begin{pmatrix} 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \; P = \begin{pmatrix} 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$
$G^{[1]}$	$T_1 = \begin{pmatrix} 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \; V_1 = \begin{pmatrix} 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0 \end{pmatrix}$
$G^{[2]}$	$T_2 = \begin{pmatrix} 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \; V_2 = \begin{pmatrix} 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$
$G^{[3]}$	$T_3 = \begin{pmatrix} 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \; V_3 = \begin{pmatrix} 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$
$G^{[4]}$	$T_4 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \; V_4 = \begin{pmatrix} 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} =: \Pi$

Table 3: Projectors associated with the derivative array analysis for Example 4.

In contrast, according to [20], [18], with

$$G_4 = \begin{pmatrix} 1 & -1 & \alpha & -\alpha^2 & \alpha^3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Pi_3 := \begin{pmatrix} 1 & 0 & 1 & -\alpha & \alpha^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

the EUODE (without derivatives of q) results to be

$$u_e' - \alpha u_e = q_1 + q_{2,1} - \alpha q_{2,2} + \alpha^2 q_{2,3} - \alpha^3 q_{2,4}$$

for

$$u_e = x_1 + x_3 - \alpha x_4 + \alpha^2 x_5.$$

Remark 2. *Observe that, as expected, in Examples 3 and 4 the spectra of the EUODE and the EPEODE coincide. This has to be given due to stability reasons.*

The discussion of a more general class of linear DAEs with constant coefficients that includes Example 4 can be found in the Appendix A.2, see Example 5.

6 Modified Nodal Analysis (MNA)

For the equations resulting in circuit simulation with the conventional MNA, Lemma 4 permits an easy interpretation of the representation of Π described already in [3] and the projector PQ_1 given in [13].

Using the same notation as in [13], [3], the conventional MNA for circuits without controlled sources leads to equations of the form

$$\begin{aligned} A_C C(A_C^T e, t) A_C e' + A_R r(A_R^T e, t) + A_L j_L + A_V j_V + A_I i(t) &= 0, \\ L(j_L, t) j_L' - A_L^T e &= 0, \\ A_V^T - v(t) &= 0, \end{aligned}$$

for incidence matrices A_C, A_R, A_V, A_L, A_I, suitable given functions C, L, r, v, i, and the unknown functions (e, j_L, j_V). If we suppose that $C(A_C^T e, t)$, $L(j_L, t)$ and $G(u,t) := \frac{\partial r(u,t)}{\partial u}$ are positive definite, in [3] it was shown that the projector Π is constant and depends only on the topological properties of the network.

For the description, we merely require projectors with

$$\text{im } Q_C = \ker A_C^T, \quad \text{im } Q_{CRV} = \ker(A_C A_R A_V)^T, \quad \text{im } \bar{Q}_{V-C} = \text{im } A_V^T Q_C.$$

Analogously to [4] we define

$$Q := \begin{pmatrix} Q_C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}, \quad T = T_1 := \begin{pmatrix} Q_{CRV} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \bar{Q}_{V-C} \end{pmatrix},$$

but assume now that these projectors are orthogonal. Due to the symmetry of the equations we can further define

$$\begin{aligned} \bar{H}_1 &:= \begin{pmatrix} A_C A_C^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + (I - Q) \\ H_1(A_C^T e, j_L, t) &:= \begin{pmatrix} A_C C(A_C^T e, t) A_C^T & 0 & 0 \\ 0 & L(j_L, t) & 0 \\ 0 & 0 & 0 \end{pmatrix} + (I - Q) \\ WN &:= \begin{pmatrix} 0 & Q_{CRV}^T A_L & 0 \\ 0 & 0 & 0 \\ \bar{Q}_{V-C}^T A_V^T & 0 & 0 \end{pmatrix}, \\ \bar{H}_2 &:= (WN)(WN)^T + (I - T) \\ &= \begin{pmatrix} Q_{CRV}^T A_L A_L^T Q_{CRV} + P_{CRV} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \bar{Q}_{V-C}^T A_V^T A_V \bar{Q}_{V-C} + \bar{P}_{V-C} \end{pmatrix}, \\ H_2(A_C^T e, j_L, t) &:= (WN) H_1^{-1}(A_C^T e, j_L, t)(WN)^T + (I - T). \end{aligned}$$

By construction, these matrices are nonsingular and $\bar{H}_2$ is symmetric such that the projector Π described already in [3] results to be the orthogonal projector Π, since

$$\begin{aligned} (WN)^+(WN) &= (WN)^T \bar{H}_2^{-1}(WN) \\ &= \begin{pmatrix} A_V \bar{Q}_{V-C} (\ldots)^{-1} \bar{Q}_{V-C} A_V^T & 0 & 0 \\ 0 & A_L^T Q_{CRV} (\ldots)^{-1} Q_{CRV}^T A_L & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

and therefore the orthogonal projector

$$\Pi = P - (WN)^T \bar{H}_2^{-1}(WN)$$

results to be constant. In contrast, in [13] it was shown that

$$\Pi_1(A_C^T e, j_L, t) = P - \left(H_1(A_C^T e, j_L, t)\right)^{-1} (WN)^T \left(H_2(A_C^T e, j_L, t)\right)^{-1} (WN).$$

This projector is neither orthogonal nor constant. However, by construction it holds that $\ker \Pi = \ker \Pi_1$, cf. Lemma 4.

7 Summary

In the present paper, we developed a new decoupling of DAEs that was obtained with orthogonal projectors and the derivative array.

The discussed projectors characterize the dependence of the different components on derivatives of the right-hand side. Moreover, they turned out to be constant for several examples from applications. Consequently, the components can be described easily and the verification of beneficial structural properties in the equations becomes simple. In fact, often higher-index components appear only linearly.

The presented decoupling of linear DAEs provides a projected explicit ODE (PEODE) that is described in terms of a specific orthogonal projector. The consideration of this particular PEODE permits a better understanding of projected integration methods, in particular the Taylor series method described in [10]. Further research will be conducted in this field [11].

The approach was applied to several examples, in particular to the equations from the exothermic reactor model discussed in [21], the MNA equations and DAEs in Kronecker canonical form. An application to the well-known index-5 DAE of the robotic arm can be found in the article [12] in this volume. Altogether, we illustrated that the introduced decoupling presents a valuable tool to analyze the structure of DAEs from various fields of applications. The algorithms for the computation were implemented in Python and are available online, cf. [5].

A Appendix

A.1 Linear Algebra Toolbox

In this part of the appendix, we summarize some results concerning the relationship of (orthogonal) projectors and constraints.

Lemma 2. *[7] Consider a pair of projectors $P, Q \in \mathbb{R}^{n\times n}$, $P = I - Q$.*

1. *For a matrix $N \in \mathbb{R}^{m\times n}$ and a vector $b \in \operatorname{im} N$, the linear system of equations*

$$Nz = b$$

uniquely determines Qz as a linear function of Pz and b iff

$$\ker \begin{pmatrix} P \\ N \end{pmatrix} = \{0\}. \tag{32}$$

2. *For $G_L \in \mathbb{R}^{m_G \times n}$, $G_R \in \mathbb{R}^{m_G \times p}$, a projector W_R along* $\operatorname{im} G_R$*, and for* $b \in \operatorname{im}(G_L\ G_R)$*, the linear system of equations*

$$\begin{pmatrix} G_L & G_R \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = b, \ \ z_1 \in \mathbb{R}^n, z_2 \in \mathbb{R}^p$$

uniquely determines Qz_1 as a linear function of Pz_1 and b iff, for $N := W_R G_L$,

$$\ker \begin{pmatrix} P \\ N \end{pmatrix} = \{0\}. \tag{33}$$

A proof can be found in [7] (Lemma 1).

Theorem 2. *[8] Suppose that an arbitrary matrix $N \in \mathbb{R}^{m \times n}$ and complementary projectors Q, $P := I - Q \in \mathbb{R}^{n \times n}$ fulfilling*

$$\ker \begin{pmatrix} P \\ N \end{pmatrix} = \{0\}$$

are given, and that W is an arbitrary matrix with the property $\ker W = \operatorname{im} NQ$ *such that $WN = WNP$. Then all projectors Π onto*

$$\ker \begin{pmatrix} Q \\ WN \end{pmatrix} = \ker Q \cap \ker WN$$

fulfill

$$\ker \begin{pmatrix} \Pi \\ N \end{pmatrix} = \{0\}.$$

A proof that is based on the SVD can be found in [8], cf. Theorem 3.

Lemma 3. *Consider an arbitrary matrix $N \in \mathbb{R}^{m \times n}$ and a pair of complementary orthogonal projectors Q, $P := I - Q \in \mathbb{R}^{n \times n}$. Then it holds*

$$\begin{pmatrix} P \\ NQ \end{pmatrix}^+ = \begin{pmatrix} P & (NQ)^+ \end{pmatrix}.$$

Proof. For $r := \operatorname{rank}(NQ)$, the singular value decomposition $NQ = U\Sigma V^T$ leads to

$$NQ = NQ \cdot Q = U\Sigma V^T \cdot Q = U\Sigma \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} V^T Q = U\Sigma \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} V^T.$$

Hence,

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} V^T = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} V^T Q \quad \text{and} \quad V \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = QV \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$(NQ)^+ = V\Sigma^+ U^T = Q \cdot V\Sigma^+ U^T = Q(NQ)^+,$$

such that

$$P \cdot (NQ)^+ = 0, \quad \left((NQ)^+\right)^T \cdot P = 0. \tag{34}$$

With the properties (34), the four Moore-Penrose conditions for

$$A := \begin{pmatrix} P \\ NQ \end{pmatrix}$$

can be verified easily:

1.
$$AA^+A = \begin{pmatrix} P \\ NQ(NQ)^+NQ \end{pmatrix} = A.$$

2.
$$A^+AA^+ = \begin{pmatrix} P & (NQ)^+NQ(NQ)^+ \end{pmatrix} = A^+.$$

3.
$$AA^+ = \begin{pmatrix} P & 0 \\ 0 & (NQ)(NQ)^+ \end{pmatrix} = (AA^+)^T.$$

4.
$$A^+A = P + (NQ)^+(NQ) = P^T + ((NQ)^+(NQ))^T = (A^+A)^T.$$

□

Corollary 1. *If, additionally to the assumptions of Lemma 3, the property*

$$\ker \begin{pmatrix} P \\ N \end{pmatrix} = \{0\}$$

is given, then

$$(NQ)^+NQ = Q$$

holds.

Proof. From

$$\{0\} = \ker \begin{pmatrix} P \\ N \end{pmatrix} = \ker \begin{pmatrix} P \\ NQ \end{pmatrix}$$

it follows that, in the proof of Lemma 3, we have

$$I = A^+A = P + (NQ)^+(NQ)$$

such that $(NQ)^+(NQ) = Q$ must hold. □

Corollary 2. *If the assumptions of Corollary 1 are given and we consider an arbitrary matrix* W *with the property* $\ker W = \operatorname{im} NQ$*, then, for the orthogonal projector* Π *fulfilling*

$$\ker \begin{pmatrix} Q \\ WN \end{pmatrix} = \ker Q \cap \ker WN = \operatorname{im} \Pi,$$

we have

$$I - \Pi = (N(I-\Pi))^+N(I-\Pi) = (I-\Pi)(N(I-\Pi))^+N(I-\Pi) \tag{35}$$

and

$$I - \Pi = Q + (WN)^+(WN), \tag{36}$$

where the latter representation implies

$$P - \Pi = (WN)^+(WN).$$

Proof. Since

$$\ker \begin{pmatrix} P \\ N \end{pmatrix} = \ker \begin{pmatrix} \Pi \\ N \end{pmatrix} = \ker \begin{pmatrix} \Pi \\ N(I-\Pi) \end{pmatrix} = \{0\}\,,$$

property (35) follows directly from Corollary 1. Moreover, by the definition of Π, Lemma 1 implies

$$I - \Pi = \begin{pmatrix} Q \\ WN \end{pmatrix}^{+} \begin{pmatrix} Q \\ WN \end{pmatrix} = Q + (WN)^{+}(WN).$$

□

Let us now focus on some relationships used in Section 6.

Lemma 4. *1. If A is an arbitrary matrix, Q is the orthogonal projector onto $\ker A$, then, for any positive definite matrix C, the matrix*

$$H_1 := A^T C A + Q$$

is nonsingular and positive definite.

2. We assume further that N is a matrix fulfilling

$$\ker \begin{pmatrix} P \\ N \end{pmatrix} = \{0\}\,,$$

W is a matrix with $\ker W = \operatorname{im} NQ$, and

$$P - \Pi = (WN)^{+}(WN).$$

Let further $\tilde{Q}$ be an orthogonal projector onto $\ker(WN)^T$. Then the matrix

$$H_2 = (WN)H_1^{-1}(WN)^T + \tilde{Q}$$

is nonsingular and positive definite.

3. Under these assumptions, the matrix

$$\Psi := H_1^{-1}(WN)^T H_2^{-1}(WN)$$

is a projector fulfilling $\Psi = \Psi \cdot P$ and

$$\Psi \cdot (P - \Pi) = \Psi, \quad (P - \Pi) \cdot \Psi = (P - \Pi),$$

i.e., $\ker \Psi = \ker(P - \Pi)$ and therefore $\Psi^{+}\Psi = (P - \Pi)$.

4. Finally, the above equations lead to

$$Q + \Psi^{+}\Psi = I - \Pi$$

and

$$(WN)\Psi = WN.$$

Proof. 1. A slightly weaker form of this lemma was proved in [13] for a specific application. For completeness, we give a general proof here. Let z be an element of $\ker H$. Then we have

$$(A^T CA + Q)z = 0.$$

If we multiply this equation by Q, it results that $Qz = 0$. Hence,

$$A^T CAz = 0$$

holds. From the positive definiteness of C it follows that $Az = 0$, and therefore $Pz = 0$. Finally, the positive definiteness of H_1 follows from

$$H_1 = \begin{pmatrix} A^T & Q \end{pmatrix} \underbrace{\begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}}_{\text{positive definite}} \begin{pmatrix} A \\ Q \end{pmatrix} \quad \text{and} \quad \ker \begin{pmatrix} A \\ Q \end{pmatrix} = \ker A \cap \ker Q = \{0\}.$$

2. The second assertion results directly for $A = (WN)^T$, $C = H_1^{-1}$.

3. We focus now on the properties of Ψ:

   (a) Let us first show that Ψ is a projector using $\tilde{P} := I - \tilde{Q}$

$$\begin{aligned} \Psi \cdot \Psi &= H_1^{-1}(WN)^T H_2^{-1} \underbrace{(WN) \cdot H_1^{-1}(WN)^T}_{\tilde{P}H_2 = H_2\tilde{P}} H_2^{-1}(WN) \\ &= H_1^{-1}(WN)^T H_2^{-1}(WN) = \Psi. \end{aligned}$$

   (b) We finally show

$$\begin{aligned} \Psi \cdot (P - \Pi) &= H_1^{-1}(WN)^T H_2^{-1}(WN) \cdot (WN)^+(WN) \\ &= H_1^{-1}(WN)^T H_2^{-1}(WN) = \Psi, \\ (P - \Pi) \cdot \Psi &= (WN)^+ \underbrace{(WN) \cdot H_1^{-1}(WN)^T}_{=\tilde{P}H_2} H_2^{-1}(WN) \\ &= (WN)^+(WN) = (P - \Pi). \end{aligned}$$

4. The last assertions follow directly form the above representation.

□

With the notation of Lemma 4 and

$$\Pi_\Psi := P - \Psi$$

we obtain the relations

$$\Pi_\Psi \Pi = \Pi, \quad \Pi\, \Pi_\Psi = \Pi_\Psi.$$

Note that in Section 6 we have shown that, for the considered index-2 DAEs, the projector Π_1 of the tractability index results to be a projector Π_Ψ with these properties.

A.2 Examples for Linear DAEs

To facilitate the understanding of our approach, we show the differences between

- the introduced orthogonal decoupling, leading to an PEODE that involves derivatives of the right-hand side, and
- a decoupling leading to an IERODE that precisely does not involve any derivatives of the right-hand side

for the Kronecker Canonical Form and a slightly more general class of DAEs, which particularly includes Example 1, 4.

A.2.1 Kronecker canonical form (KCF)

A linear differential-algebraic equation with constant coefficients and regular matrix pair can be transformed by a premultiplication of a nonsingular matrix and a linear coordinate change into a DAE in Kronecker canonical form (KCF), i.e., a DAE of the form

$$\begin{pmatrix} I_{n_1} & 0 \\ 0 & \mathcal{N} \end{pmatrix} x' + \begin{pmatrix} \mathcal{W} & 0 \\ 0 & I_{n_2} \end{pmatrix} x = q(t) \tag{37}$$

for $x(t) \in \mathbb{R}^n$, an arbitrary $\mathcal{W} \in \mathbb{R}^{n_1 \times n_1}$, a nilpotent matrix $\mathcal{N} \in \mathbb{R}^{n_2 \times n_2}$ with nilpotency-index μ, i.e., $\mathcal{N}^{\mu-1} \neq 0$, $\mathcal{N}^{\mu} = 0$, $n = n_1 + n_2$, and identity matrices $I_{n_1} \in \mathbb{R}^{n_1 \times n_1}$ and $I_{n_2} \in \mathbb{R}^{n_2 \times n_2}$, cf. [14]. Rewriting the equations as

$$x_1' + \mathcal{W} x_1 = q_1(t), \tag{38}$$

$$\mathcal{N} x_2' + x_2 = q_2(t), \tag{39}$$

for $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$, equation (38) corresponds to the inherent ODE and, by a recursive approach, the so-called pure DAE corresponding to equation (39) leads to the constraints

$$x_2 = q_2(t) - \mathcal{N} x_2' = q_2(t) - \mathcal{N}(q_2'(t) - \mathcal{N} x_2') = \cdots = \sum_{j=0}^{\mu-1} (-1)^j \mathcal{N}^j q_2^{(j)}(t).$$

A.2.2 Π for DAEs in KCF

We consider $Q_{\mathcal{N}} := I - \mathcal{N}^+ \mathcal{N}$, $P_{\mathcal{N}} = I - Q_{\mathcal{N}}$ and obtain the projectors

$$Q = \begin{pmatrix} 0 & \\ & Q_{\mathcal{N}} \end{pmatrix}, \quad P = \begin{pmatrix} I & \\ & P_{\mathcal{N}} \end{pmatrix}.$$

The Jacobian matrix (7) of the derivative array reads

$$G^{[k]} = \begin{pmatrix} \mathcal{W} & & I & & & & & & \\ & I & & \mathcal{N} & & & & & \\ & & \mathcal{W} & & I & & & & \\ & & & I & & \mathcal{N} & & & \\ & & & & \ddots & & \ddots & & \\ & & & & & & \mathcal{W} & & I \\ & & & & & & & I & & \mathcal{N} \end{pmatrix} = \begin{pmatrix} G_L^{[k]} & G_R^{[k]} \end{pmatrix}.$$

For index μ DAEs, i.e., $\mathcal{N}^\mu = 0$, a basis $B_R^{[\mu]}$ with $\ker B_R^{[\mu]} = \operatorname{im} G_R^{[\mu]}$ is given by

$$B_R^{[\mu]} = \begin{pmatrix} 0 & I & 0 & -\mathcal{N} & 0 & \mathcal{N}^2 & 0 & -\mathcal{N}^3 & \cdots & (-1)^{\mu-1}\mathcal{N}^{\mu-1} \end{pmatrix}.$$

Therefore, according to (12), $N := B_R^{[\mu]} G_L^{[\mu]} = \begin{pmatrix} 0 & I \end{pmatrix}$ and

$$\Pi = \begin{pmatrix} I & \\ & 0 \end{pmatrix}, \quad B(I - (N(I-\Pi))^+ N)\Pi = B\Pi = \begin{pmatrix} \mathcal{W} & \\ & 0 \end{pmatrix},$$

as expected. The corresponding projectors for the tractability index concept can be found in Section 1.2.6 from [17]. In this particular case, Π and $\Pi_{\mu-1}$ coincide and the PEODE for Πx is the IERODE as well.

A.2.3 ODEs for slightly more general DAEs

Consider the DAE

$$\begin{pmatrix} I_{n_1} & 0 \\ 0 & \mathcal{N} \end{pmatrix} x' + \begin{pmatrix} \mathcal{W}_1 & \mathcal{W}_2 \\ 0 & I_{n_2} \end{pmatrix} x = q(t). \tag{40}$$

Analogously as above, we obtain

$$x_2 = \sum_{j=0}^{\mu-1} (-1)^j \mathcal{N}^j q_2^{(j)}(t).$$

Obtaining the PEODE for Πx corresponds to substituting this into the first block of equations, i.e.,

$$x_1' + \mathcal{W}_1 x_1 = -\mathcal{W}_2 \sum_{j=0}^{\mu-1} (-1)^j \mathcal{N}^j q_2^{(j)}(t) + q.$$

In fact, it holds

$$G^{[k]} = \begin{pmatrix} \mathcal{W}_1 & \mathcal{W}_2 & I & & & & & & \\ & I & & \mathcal{N} & & & & & \\ & & \mathcal{W}_1 & \mathcal{W}_2 & I & & & & \\ & & & I & & \mathcal{N} & & & \\ & & & & \ddots & & \ddots & & \\ & & & & & \mathcal{W}_1 & \mathcal{W}_2 & I & \\ & & & & & & I & & \mathcal{N} \end{pmatrix} = \begin{pmatrix} G_L^{[k]} & G_R^{[k]} \end{pmatrix}.$$

Therefore, for the index μ DAEs, a basis $B_R^{[\mu]}$ with $\ker B_R^{[\mu]} = \operatorname{im} G_R^{[\mu]}$ is given again by

$$B_R^{[\mu]} = \begin{pmatrix} 0 & I & 0 & -\mathcal{N} & 0 & \mathcal{N}^2 & 0 & -\mathcal{N}^3 & \cdots & (-1)^{\mu-1}\mathcal{N}^{\mu-1} \end{pmatrix}.$$

Consequently, N, Π and $B\Pi$ are the same as above for DAEs in KCF, as expected. However, the projectors related to the tractability index concept, are different, since the PEODE for Πx is not an IERODE.

For illustrative reasons, we show how the IERODE can be obtained for this particular class of DAEs without the tractability index sequence. We start noticing that we can substitute

$$x_2 = -\mathcal{N}x_2' + q_2$$

into the first block of equations, which leads to

$$\begin{pmatrix} I_{n_1} & -\mathcal{W}_2\mathcal{N} \\ 0 & \mathcal{N} \end{pmatrix} x' + \begin{pmatrix} \mathcal{W}_1 & 0 \\ 0 & I_{n_2} \end{pmatrix} x = \begin{pmatrix} q_1 - \mathcal{W}_2 q_2 \\ q_2 \end{pmatrix}.$$

This corresponds to a multiplication from the left-hand side by

$$\begin{pmatrix} I_{n_1} & -\mathcal{W}_2 \\ 0 & I_{n_2} \end{pmatrix}.$$

If we now define x_{1p} as follows

$$x = \begin{pmatrix} I & \mathcal{W}_2\mathcal{N} \\ 0 & I \end{pmatrix} \underbrace{\begin{pmatrix} I & -\mathcal{W}_2\mathcal{N} \\ 0 & I \end{pmatrix} x}_{=:x_{p1}} = \begin{pmatrix} I & \mathcal{W}_2\mathcal{N} \\ 0 & I \end{pmatrix} x_{p1},$$

then we obtain

$$\begin{pmatrix} I_{n_1} & -\mathcal{W}_2\mathcal{N} \\ 0 & \mathcal{N} \end{pmatrix} x' = \begin{pmatrix} I_{n_1} & 0 \\ 0 & \mathcal{N} \end{pmatrix} (x_{p1})'$$

and thus

$$\begin{pmatrix} I_{n_1} & 0 \\ 0 & \mathcal{N} \end{pmatrix} (x_{p1})' + \begin{pmatrix} \mathcal{W}_1 & \mathcal{W}_1\mathcal{W}_2\mathcal{N} \\ 0 & I_{n_2} \end{pmatrix} x_{1p} = \begin{pmatrix} q_1 - \mathcal{W}_2 q_2 \\ q_2 \end{pmatrix}.$$

This procedure can be repeated if we multiply from the left-hand side by

$$\begin{pmatrix} I_{n_1} & -\mathcal{W}_1\mathcal{W}_2\mathcal{N} \\ 0 & I_{n_2} \end{pmatrix}$$

to obtain

$$\begin{pmatrix} I_{n_1} & -\mathcal{W}_1\mathcal{W}_2\mathcal{N}^2 \\ 0 & \mathcal{N} \end{pmatrix} (x_{1p})' + \begin{pmatrix} \mathcal{W}_1 & 0 \\ 0 & I_{n_2} \end{pmatrix} x_{1p} = \begin{pmatrix} q_1 - \mathcal{W}_2 q_2 - \mathcal{W}_1\mathcal{W}_2\mathcal{N} q_2 \\ q_2 \end{pmatrix}.$$

If we repeat this analogously until the nilpotency index is reached, then we obtain

$$\begin{pmatrix} I_{n_1} & 0 \\ 0 & \mathcal{N} \end{pmatrix} (x_{p(\mu-1)})' + \begin{pmatrix} \mathcal{W}_1 & 0 \\ 0 & I_{n_2} \end{pmatrix} x_{p(\mu-1)} = \begin{pmatrix} q_1 - \sum_{j=0}^{\mu-1} (\mathcal{W}_1)^j \mathcal{W}_2\mathcal{N}^j q_2 \\ q_2 \end{pmatrix}$$

for

$$x_{p(\mu-1)} = \prod_{j=1}^{\mu-1} \begin{pmatrix} I_{n_1} & \mathcal{W}_1^{j-1}\mathcal{W}_2\mathcal{N}^j \\ 0 & I_{n_2} \end{pmatrix} x = \begin{pmatrix} I_{n_1} & \sum_{j=1}^{\mu-1} \mathcal{W}_1^{j-1}\mathcal{W}_2\mathcal{N}^j \\ 0 & I_{n_2} \end{pmatrix} x.$$

Example 5. *For the Examples 1 and 4 this means*

$$\mathcal{W}_1 = (-\alpha), \quad \mathcal{W}_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_{2,1} \\ q_{2,2} \\ q_{2,3} \\ q_{2,4} \end{pmatrix},$$

and therefore it holds

$$\begin{pmatrix} 1 & \sum_{j=1}^{\mu-1} \mathcal{W}_1^{j-1} \mathcal{W}_2 \mathcal{N}^j \end{pmatrix} x = \begin{pmatrix} 1 & 0 & 1 & -\alpha & \alpha^2 \end{pmatrix} x = x_1 + x_3 - \alpha x_4 + \alpha^2 x_5,$$

$$-\sum_{j=0}^{\mu-1} (\mathcal{W}_1)^j \mathcal{W}_2 \mathcal{N}^j q_2 = \begin{pmatrix} 1 & -\alpha & \alpha^2 & -\alpha^3 \end{pmatrix} q_2 = q_{2,1} - \alpha q_{2,2} + \alpha^2 q_{2,3} - \alpha^3 q_{2,4}.$$

Consequently, we obtain the IERODE and EUODE that are not a PEODE or EPEODE for Πx*, respectively.*

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