Singularities of the Robotic Arm DAE

Diana Estévez Schwarz
Beuth Hochschule für Technik, Berlin *

René Lamour, Roswitha März
Humboldt-University of Berlin†

August 22, 2019

Abstract

One of the benchmarks for higher-index DAEs is the so-called robotic arm, which results from a tracking problem in robotics. Testing this benchmark, we became aware of the singularities that appear and started to analyze them thoroughly. To our knowledge, there is no comprehensive description of these singularities in the DAE literature so far. For our analysis, we use different methodologies, which are elaborated in this article. This detailed inspection results from two different index concepts, namely the projector based analysis of the derivative array and the projector based DAE analysis. As a result, with both approaches we identify the same two types of singularities. One of them is obvious, but the other one is unexpected.

Keywords: Robotic Arm problem, singularity, differential-algebraic equation, DAE, projector based analysis, derivative array, differentiation index, tractability index

MSC-Classification: 34A09, 34C05, 65L05, 65L80, 93C85

1 Introduction

The diagnosis of singularities of differential-algebraic equations (DAEs) is necessary to evaluate the reliability of numerical results. Nevertheless, this aspect has not been considered sufficiently in practice up to now. Therefore, in the last couple of years we developed some tools that provided indications for numerical difficulties, in particular the code InitDAE, [7], [12]. The Taylor-coefficients computed with InitDAE can be used for an integration method described in [13]. Altogether, in this way we obtain detailed information while integrating DAEs.

Looking for an ambitious higher-index test example, we recalled the path control of a two-link, flexible joint, planar robotic arm from Campbell (1988) [3, 4], which based on a more general model by de Luca, Isidori (1987) [15].

*estevez@beuth-hochschule.de
†lamour@math.hu-berlin.de, maerz@math.hu-berlin.de

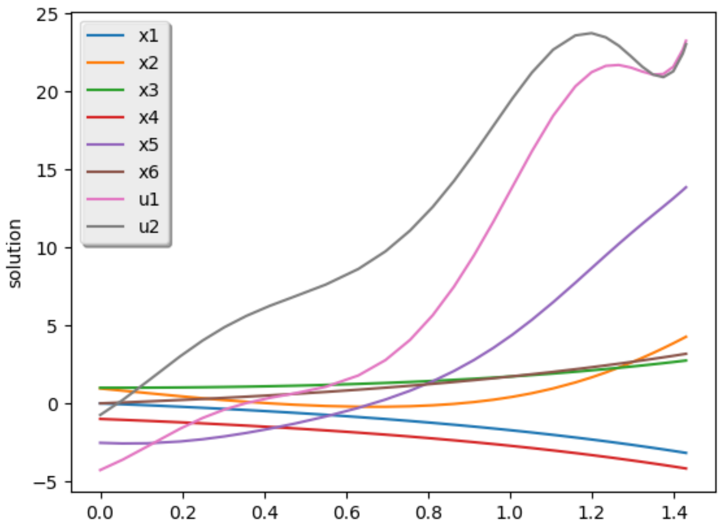

Figure 1: Numerical solution of Robotic Arm problem obtained with the Taylor methods described in [13]
.

To our surprise, first tests suggested the existence of various singularities. In particular, we wondered that integrating over the interval $[0, 2]$, InitDAE finds a singularity at $\sim$1.5 and the integration stops (see Fig. 1), whereas in [18] a successful integration up to 1.7 is reported. This motivated a deeper theoretical analysis of this particular higher-index DAE.

We aim at a comprehensive description of this famous test example and analyze the DAE in great detail from different points of view. The equations of the DAE are presented in Section 2. In Section 3 we investigate the properties of the DAE by direct consideration of the model equations. This direct analysis shows also a way to represent the solution of the Robotic Arm equations. From a more general point of view, we used two approaches to analyze the DAE:

- In Section 4 we apply the algorithm used in InitDAE, which is based on a projector based analysis of the derivative array.
- An admissible matrix function sequence and associated admissible projector functions in the context of the direct projector based DAE-analysis are developed in Section 5.

Both concepts are supported by certain constant-rank conditions. In both cases, the same singularities are indicated by corresponding rank drops.

Our numerical results are reported in Section 6. The paper closes with an investigation of a more general formulation for the Robotic Arm problem, in dependence of several parameters. The critical constellations for these parameters are described in Section 7.

2 Equations of the Robotic Arm

The problem we will consider is a semi-explicit DAE of dimension 8 with two constraints. The variables (x_1, x_2, x_3) are angular coordinates that describe the robot's configuration and (x_4, x_5, x_6) are their derivatives. Finally, the variables u_1 and u_2 are rotational torques and $(p_1(t), p_2(t))$ is the prescribed endpoint of the outer arm in Cartesian coordinates.

2.1 The Equations from the DAE-literature

First, in the Sections 2-6, we will analyze the equations in the form given in [4], i.e., the particular equations read in detail:

$$
\begin{aligned}
&x_1' - x_4 = 0,\\
&x_2' - x_5 = 0,\\
&x_3' - x_6 = 0,\\
&x_4' - 2c(x_3)(x_4 + x_6)^2 - x_4^2 d(x_3) - (2x_3 - x_2)(a(x_3) + 2b(x_3))\\
&\qquad - a(x_3)(u_1 - u_2) = 0,\\
&x_5' + 2c(x_3)(x_4 + x_6)^2 + x_4^2 d(x_3) - (2x_3 - x_2)(1 - 3a(x_3) - 2b(x_3))\\
&\qquad + a(x_3)(u_1 - u_2) - u_2 = 0\\
&x_6' + 2c(x_3)(x_4 + x_6)^2 + x_4^2 d(x_3) - (2x_3 - x_2)(a(x_3) - 9b(x_3))\\
&\qquad + (a(x_3) + b(x_3))(u_1 - u_2) + d(x_3)(x_4 + x_6)^2 + 2x_4^2 c(x_3) = 0,\\
&\cos x_1 + \cos(x_1 + x_3) - p_1(t) = 0,\\
&\sin x_1 + \sin(x_1 + x_3) - p_2(t) = 0,
\end{aligned} \tag{1}
$$

with

$$
p_1(t) = \cos(\mathrm{e}^t - 1) + \cos(t - 1), \quad p_2(t) = \sin(1 - \mathrm{e}^t) + \sin(1 - t), \tag{2}
$$

and

$$
\begin{aligned}
a(z) &= \frac{2}{2 - \cos^2 z}, & b(z) &= \frac{\cos z}{2 - \cos^2 z},\\
c(z) &= \frac{\sin z}{2 - \cos^2 z}, & d(z) &= \frac{\cos z \sin z}{2 - \cos^2 z}.
\end{aligned}
$$

This DAE is frequently used to illustrate DAE procedures (e.g., [1, 3, 4, 18, 10]). As will be confirmed below, at regular points its index is 5 and the degree of freedom is zero. Moreover, an explicit representation of the solution is given.

The structure of the Robotic Arm model is illustrated in Figure 2. It describes a two-link robotic arm with an elastic joint moving on a horizontal plane. In fact, x_1 corresponds to the rotation of the first link with respect to the base frame, x_2 to the rotation of the motor at the second joint and x_3 to the rotation of the second link with respect to the first link. For more details we refer to [15], [6]. Considerations on a more general formulation in dependence of several parameters can be found in Section 7.

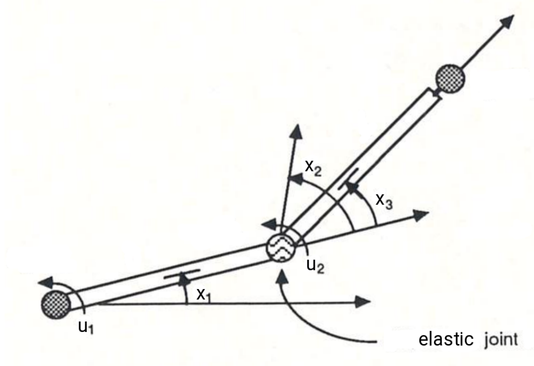

Figure 2: Drawing of the Robotic Arm problem. (Modification of a graphic from [6])

2.2 Structural Properties

In order to present a more intelligible representation of some structural properties, we reformulate the equations with new variables $x_7 := u_1 - u_2$, $x_8 = u_2$, as already done in [1], and discuss the corresponding DAE, which will be considered in the form

$$\left(\begin{pmatrix} I_6 & \\ & 0_2 \end{pmatrix} x\right)' + b(x,t) = 0. \tag{3}$$

For this notation, we have then

$$\begin{aligned}
b_1(x,t) &= -x_4,\\
b_2(x,t) &= -x_5,\\
b_3(x,t) &= -x_6,\\
b_4(x,t) &= -2c(x_3)(x_4+x_6)^2 - x_4^2 d(x_3) - (2x_3 - x_2)(a(x_3) + 2b(x_3)) - a(x_3)x_7,\\
b_5(x,t) &= +2c(x_3)(x_4+x_6)^2 + x_4^2 d(x_3) - (2x_3 - x_2)(1 - 3a(x_3) - 2b(x_3))\\
&\qquad + a(x_3)x_7 - x_8,\\
b_6(x,t) &= +2c(x_3)(x_4+x_6)^2 + x_4^2 d(x_3) - (2x_3 - x_2)(a(x_3) - 9b(x_3))\\
&\qquad + (a(x_3) + b(x_3))x_7 + d(x_3)(x_4+x_6)^2 + 2x_4^2 c(x_3),\\
b_7(x,t) &= \cos x_1 + \cos(x_1 + x_3) - p_1(t),\\
b_8(x,t) &= \sin x_1 + \sin(x_1 + x_3) - p_2(t),
\end{aligned}$$

resulting in $x \in \mathbb{R}^8$. The function $b : \mathbb{R}^8 \times \mathbb{R} \to \mathbb{R}^8$ is continuously differentiable and the partial Jacobian matrix $b_x(x,t)$ reads

$$b_x(x,t) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & b_{42} & b_{43} & b_{44} & 0 & b_{46} & b_{47} & 0 \\ 0 & b_{52} & b_{53} & b_{54} & 0 & b_{56} & b_{57} & -1 \\ 0 & b_{62} & b_{63} & b_{64} & 0 & b_{66} & b_{67} & 0 \\ b_{71} & 0 & b_{73} & 0 & 0 & 0 & 0 & 0 \\ b_{81} & 0 & b_{83} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for entries $b_{ik} = \frac{\partial b_i}{\partial x_k}$ that are again smooth functions of x. In particular, $b_{42}, b_{47}, b_{57}, b_{62}, b_{67}$ depend only on x_3, and $b_{71}, b_{73}, b_{81}, b_{83}$ depend only on x_1, x_3. We drop the arguments in the majority of cases.

Since the particular form of several coefficients b_{ik} does not matter, we present only those coefficients which will actually play a role later on:

$$\begin{aligned} b_{42} &= a(x_3) + 2b(x_3) = \frac{2 + 2\cos x_3}{2 - \cos^2 x_3}, \\ b_{47} &= -a(x_3) = -\frac{2}{2 - \cos^2 x_3}, \\ b_{62} &= a(x_3) - 9b(x_3) = \frac{2 - 9\cos x_3}{2 - \cos^2 x_3}, \\ b_{67} &= a(x_3) + b(x_3) = \frac{2 + \cos x_3}{2 - \cos^2 x_3}, \\ b_{71} &= -\sin x_1 - \sin(x_1 + x_3), \\ b_{73} &= -\sin(x_1 + x_3), \\ b_{81} &= \cos x_1 + \cos(x_1 + x_3), \\ b_{83} &= \cos(x_1 + x_3). \end{aligned}$$

We also want to emphasize some special relations.

Lemma 1. (a) *The function b_{67} is smooth and has no zeros. It depends on x_3 only.*

(b) *The functions*

$$p := \frac{b_{47}}{b_{67}} \quad \text{and} \quad r := \frac{1}{b_{67}}$$

are smooth and depend on x_3 only. They have no zeros.

(c) *The matrix function*

$$\mathcal{M}(x_3) := \begin{pmatrix} b_{42} & b_{47} \\ b_{62} & b_{67} \end{pmatrix} = \begin{pmatrix} a(x_3) + 2b(x_3) & -a(x_3) \\ a(x_3) - 9b(x_3) & a(x_3) + b(x_3) \end{pmatrix} \tag{4}$$

has smooth entries depending on x_3 only. $\mathcal{M}(x_3)$ is nonsingular precisely if

$$b_{42}(x_3) - p(x_3) b_{62}(x_3) \neq 0.$$

(d) *For $z_\star = 3 - \sqrt{5}$ it holds that $\cos x_3 = z_\star$ implies $b_{42}(x_3) - p(x_3)b_{62}(x_3) = 0$ and vice versa.*

(e) *The function*

$$\mathcal{S}(x_3) = \frac{1}{b_{42}(x_3) - p(x_3)b_{62}(x_3)}, \quad x_3 \in \operatorname{dom}\mathcal{S} = \{\tau \in \mathbb{R} : \cos\tau \neq 3 - \sqrt{5}\},$$

is smooth on its definition domain, and so is

$$\mathcal{M}^{-1} = \begin{pmatrix} \mathcal{S} & -p\mathcal{S} \\ -rb_{62}\mathcal{S} & r + rpb_{62}\mathcal{S} \end{pmatrix}.$$

(f) *The matrix function*

$$\mathcal{N}(x_1, x_3) = \begin{pmatrix} b_{71} & b_{73} \\ b_{81} & b_{83} \end{pmatrix} = \begin{pmatrix} -\sin x_1 - \sin(x_1 + x_3) & -\sin(x_1 + x_3) \\ \cos x_1 + \cos(x_1 + x_3) & \cos(x_1 + x_3) \end{pmatrix} \tag{5}$$

depends only on x_1 *and* x_3*.* $\mathcal{N}(x_1, x_3)$ *is nonsingular precisely if* $\det \mathcal{N}(x_1, x_3) = \sin x_3 \neq 0$.

Proof. Assertion (d): By definition one has

$$b_{42}(x_3) - p(x_3)b_{62}(x_3) = \frac{2}{2 - \cos^2 x_3} \, \frac{4 + \cos^2 x_3 - 6\cos x_3}{2 + \cos x_3}.$$

This expression becomes zero exactly if $4 + \cos^2 x_3 - 6\cos x_3 = 0$. Next, $z_\star = 3 - \sqrt{5}$ is the only zero of the polynomial $4 + z^2 - 6z$, which belongs to the interval $[-1, 1]$. This proves the assertion.

The other assertions are now evident. $\square$

A corresponding generalization can be found in Section 7.

3 Inspection by Hand Method

For readability, in this section, we illustrate how the properties we will characterize in general terms in the following sections can also be appreciated by an intuitive analysis of this particular DAE. However, we want to emphasize that, in general, such a direct manual analysis can only be conducted if structural information is given a priori.

Rearranging the equations (similar as in [4]) we obtain

$$\begin{pmatrix} x_1' \\ x_3' \end{pmatrix} - \begin{pmatrix} x_4 \\ x_6 \end{pmatrix} = 0, \tag{6}$$

$$\begin{aligned} \begin{pmatrix} x_4' \\ x_6' \end{pmatrix} + \begin{pmatrix} -2c(x_3)(x_4 + x_6)^2 - x_4^2 d(x_3) - 2x_3(a(x_3) + 2b(x_3)) \\ 2c(x_3)(x_4 + x_6)^2 + x_4^2 d(x_3) - 2x_3(a(x_3) \ \ldots \\ -9b(x_3) + d(x_3)(x_4 + x_6)^2 + 2x_4^2 c(x_3)) \end{pmatrix} & \\ + \begin{pmatrix} a(x_3) + 2b(x_3) & -a(x_3) \\ a(x_3) - 9b(x_3) & a(x_3) + b(x_3) \end{pmatrix} \begin{pmatrix} x_2 \\ x_7 \end{pmatrix} &= 0, \end{aligned} \tag{7}$$

$$x_2' - x_5 = 0, \tag{8}$$

$$\begin{aligned} x_5' + 2c(x_3)(x_4 + x_6)^2 + x_4^2 d(x_3) - (2x_3 - x_2)(1 - 3a(x_3) & \\ -2b(x_3)) + a(x_3)x_7 - x_8 &= 0, \end{aligned} \tag{9}$$

$$\cos x_1 + \cos(x_1 + x_3) - p_1(t) = 0, \tag{10}$$

$$\sin x_1 + \sin(x_1 + x_3) - p_2(t) = 0. \tag{11}$$

In this form, we can recognize that

1. x_1 and x_3 are uniquely determined by (10) - (11) whenever the Jacobian matrix with respect to x_1 and x_3, i.e., the matrix $\mathcal{N}$ from Lemma 1, is nonsingular, i.e., if $\det \mathcal{N} = \sin x_3 \neq 0$, leading to explicit expressions for x_1, x_3. This implies that we cannot prescribe initial values for x_1 and x_3. For the particular choice of p_1, p_2, cf. (2), we obtain indeed
$$\begin{pmatrix} x_1(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 - e^t \\ e^t - t \end{pmatrix}.$$
At points $t_\star$ with $x_3(t_\star) = k\pi, k \in \mathbb{Z}$ singularities are indicated.

2. By differentiation in this particular case we further obtain
$$\begin{pmatrix} x_1'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} x_4(t) \\ x_6(t) \end{pmatrix} = \begin{pmatrix} -e^t \\ e^t - 1 \end{pmatrix},$$
such that it becomes clear that for x_4 and x_6 we cannot prescribe initial values either.

3. Differentiating this latter expression and inserting it into (7) delivers expressions for (x_2, x_7) everywhere where the matrix $\mathcal{M}$ is nonsingular, cf. Lemma 1. This implies that singularities appear if
$$a(x_3)^2 - 3a(x_3) \cdot b(x_3) + b(x_3)^2 = 0,$$
i.e., if $\cos x_3 = 3 - \sqrt{5}$. We cannot prescribe initial values for x_2 and x_7.

4. Differentiating the expression obtained for x_2 (8) provides an expression for x_5, such that it becomes clear that we cannot prescribe a single initial value and the degree of freedom of the Robotic Arm DAE is zero.

5. A final differentiation of x_5 provides, with (9), an expression for x_8.

Consequently, we have an explicit representation of the solution of (1). This was also described in [1], but without mentioning the possible singularities.
A Python code of the solution can be found in [11].

The number of differentiations indicates that, at regular points, the classical differentiation index is 5.

4 Projector Based Derivative Array Procedures

In this section, we show how the above steps are described in terms of the approach presented in [8] and [9].

The Robotic Arm problem is a semi-explicit DAE
$$f(x', x, t) := f((Px)', x, t) = (Px)' + b(x(t), t) = 0,$$
with the constant projectors P and for later use $Q := I - P$
$$P = \begin{pmatrix} I_6 & \\ & 0_2 \end{pmatrix}, \quad Q = \begin{pmatrix} 0_6 & \\ & I_2 \end{pmatrix}.$$

This means that $x_1, \dots, x_6$ (the Px-component) is the differentiated component, while x_7 and x_8 (or, in the original formulation, u_1 and u_2) (the Qx-component) is the undifferentiated component.
For $z_i \in \mathbb{R}^8$, $j = 0, \dots, k$, we define

$$F_j(x^{(j+1)}, x^{(j)}, \dots, \dot{x}, x, t) := \frac{\mathrm{d}^j}{\mathrm{d}t^j} f(\dot{x}, x, t)$$

and

$$g^{[k]}(z_0, z_1, \dots, z_k, t) := \begin{pmatrix} F_0(z_1, z_0, t) \\ F_1(z_2, z_1, z_0, t) \\ \vdots \\ F_{k-1}(z_k, \dots, z_0, t) \end{pmatrix}.$$

Furthermore, by

$$G^{[k]}(z_0, z_1, \dots, z_k, t) \in \mathbb{R}^{8k \times 8(k+1)}$$

we denote the Jacobian matrix of $g^{[k]}(z_0, z_1, \dots, z_k, t)$ with respect to $(z_0, z_1, \dots, z_k)$ and split it into

$$G^{[k]} = \begin{pmatrix} G_L^{[k]} & G_R^{[k]} \end{pmatrix},$$

$G_L^{[k]} \in \mathbb{R}^{8 \cdot k \times 8}$, $G_R^{[k]} \in \mathbb{R}^{8 \cdot k \times 8 \cdot k}$ (note that L and R stand for left-hand side and right-hand side, respectively).

Let us now consider the matrices

$$\mathcal{B}^{[k]} := \begin{pmatrix} P & 0 \\ G_L^{[k]} & G_R^{[k]} \end{pmatrix} \in \mathbb{R}^{8(k+1) \times 8(k+1)}. \tag{12}$$

According to [8], to determine the index we check whether the matrices $\mathcal{B}^{[k]}$ are 1-full with respect to the first 8 columns for $k = 1, 2, \dots$, i.e., whether

$$\ker \mathcal{B}^{[k]} \subseteq \left\{ \begin{pmatrix} s_0 \\ s_1 \end{pmatrix} \;:\; s_0 \in \mathbb{R}^8,\ s_0 = 0,\ s_1 \in \mathbb{R}^{8k} \right\}. \tag{13}$$

If $k = \mu$ is the smallest integer for which $\mathcal{B}^{[k]}$ is 1-full , then the index is μ.

For the Robotic Arm equations, $k = 1, 2, 3, 4$ do not lead to the required 1-fullness. We illustrate the 1-fullness of $\mathcal{B}^{[5]}$ by the patterns of a transformation into a block diagonal form, see Fig. 4. The orange dots represent 1, the blue dots -1, and the brown ones other nonzero elements. For the transformation we use rows with one nonzero entry only. The used rows are marked by small ellipses and arrows. In this procedure we have to exclude the singularities of the Jacobian matrix $\mathcal{N}$ from (5) first and later the singularities of the Jacobian matrix $\mathcal{M}$ from (4), too.

In order to characterize the different components of the solution we further analyze the matrices $G^{[k]}$ for $k = 1, 2, 3, 4, 5$.

- To decouple the undifferentiated component Q, for each k we consider a basis $W_R^{[k]}$ along $\operatorname{im} G_R^{[k]}$ and define T_k as the orthogonal projector onto

$$\ker \begin{pmatrix} P \\ W_R^{[k]} G_L^{[k]} \end{pmatrix} =: \operatorname{im} T_k.$$

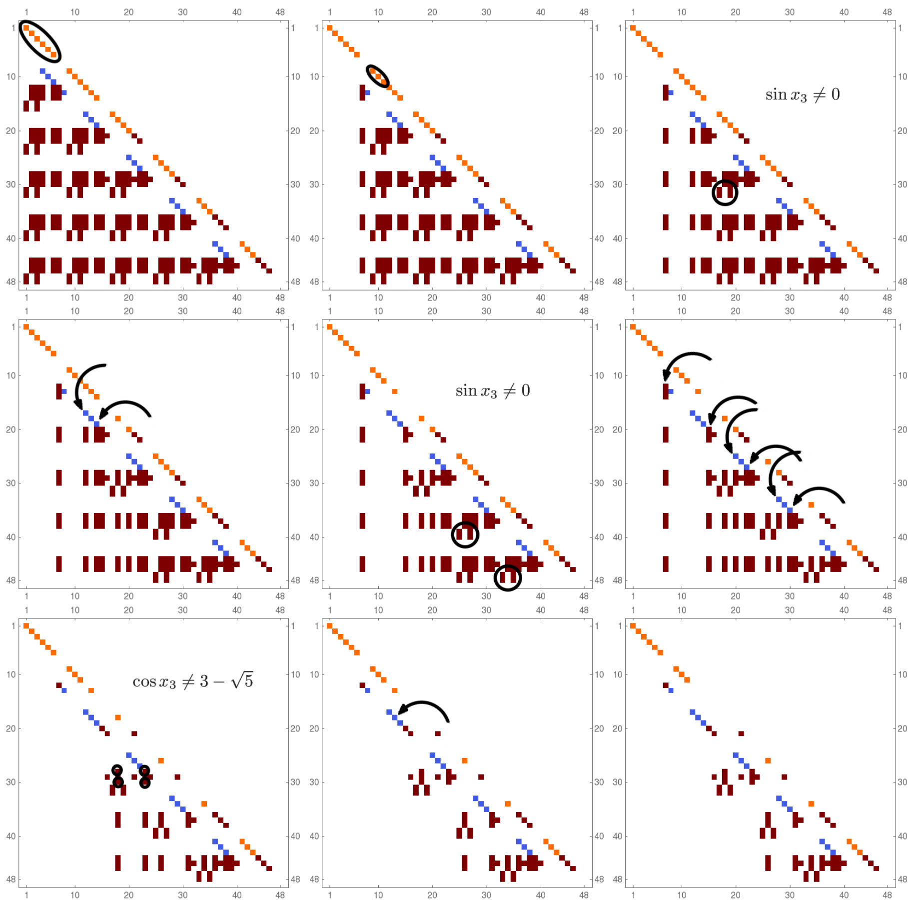

Figure 3: Illustration of the 1-fullness of (12) for the equations (3) and $k = 5$. Therefore, the index results to be 5.

Consequently, $T_k x$ corresponds to the part of the undifferentiated component Qx that cannot be represented as a function of (Px, t) after $k-1$ differentiations. Note that, by definition, $T_\mu = 0$, cf. [8].

If we define further $U_k := (Q - T_k)$, then we obtain the following decoupling for the Q-component:

$$Qx = QU_1 x + T_1 U_2 x + \cdots T_{\mu-2} U_{\mu-1} x + T_{\mu-1} x.$$

- To characterize the different parts of the differentiated component Px, in each step k we consider a basis $W^{[k]}_{LQ-R}$ along

  $$\operatorname{im} \begin{pmatrix} G^{[k]}_L Q & G^{[k]}_R \end{pmatrix}$$

  and define the orthogonal projector V_k onto

  $$\ker \begin{pmatrix} Q \\ W^{[k]}_{LQ-R} G^{[k]}_L \end{pmatrix} =: \operatorname{im} V_k.$$

  $V_k x$ represents then the part of the differentiated components Px not determined by the constraints that result after $k-1$ differentiations. By definition, the degree of freedom is $\operatorname{rank} V_{\mu-1}$.

  Defining $Z_k := (P - V_k)$ we also obtain a decoupling for the P-component:

  $$Px = PZ_1 x + V_1 Z_2 x + \cdots V_{\mu-2} Z_{\mu-1} x + V_{\mu-1} x.$$

We summarize the results obtained for the considered Robotic Arm DAE for $x_3 \neq k\pi$, $\cos x_3 \neq 3 - \sqrt{5}$ in the Tables 1 and 2:

- Table 1 corresponds to the reformulated equations with $(x_1, \ldots, x_6, x_7, x_8)$. Here, all projectors have diagonal form with only ones or zeros in the diagonal. Therefore, the different components correspond to particular rows of the vector x. The obtained splitting corresponds to the representations deduced in Section 3.

- In Table 2 we present the consequences of the original formulation with $(x_1, \ldots, x_6, u_1, u_2)$. For the first steps, we obtain identical results as for the reformulated equations. Hence, in Table 2, we present only the projectors obtained for $k = 3, 4$. There we can estimate that T_3 and T_4 do not have diagonal form, since we cannot assign this higher-index-property to a particular row, i.e., to either u_1 nor u_2. Indeed, the description of $u_1 + u_2$ by the corresponding projector results to be adequate.

Since the diagnosis procedures of InitDAE are conceived for general DAEs, consistent initial values and the corresponding projectors can be computed for both formulations at regular points. We further observe that, for singular timepoints in case of the prescribed p_1, p_2,

1. $t_\star$ with $e^{t_\star} - t_\star = x_3(t_\star) = k\pi$
2. $t_\star$ with $e^{t_\star} - t_\star = x_3(t_\star) = \arccos(3 - \sqrt{5})$.

InitDAE cannot solve the minimization problem, i.e., no consistent values can be computed. Summarizing, the computed differentiation index and the detection of singularities are equivalent, independent of the chosen variables. This is a crucial difference to the structural index, where the introduction of the variable $x_7 = u_1 - u_2$ is essential for the correct index determination, cf. [18].

5 Direct Projector Based DAE Analysis and Tractability Index

Here we provide an admissible sequence of matrix functions and describe the regularity regions with their characteristic values, including the tractability index (cf. [14]). For this purpose, we rewrite the DAE in the proper form

$$A(Dx)'(t) + b(x(t), t) = 0, \tag{14}$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad D^- = A.$$

Following the projector based approach (e.g., [14]) we construct an admissible matrix function sequence to analyze the DAE. The matrix function sequence to be built pointwise for x, t comes from the given matrix functions

$$G_0 = AD,\ B_0 = b_x,\ P_0 = D^- D,\ Q_0 = I - P_0,\ \mathit{\Pi}_0 = P_0.$$

First we obtain the matrix function

$$G_1 = G_0 + B_0 Q_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & b_{47} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & b_{57} & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & b_{67} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and its nullspace

$$N_1 = \{z \in \mathbb{R}^8 : z_1 = 0, z_2 = 0, z_3 = 0, z_4 + b_{47} z_7 = 0, z_5 + b_{57} z_7 z_8 = 0, \\ z_6 + b_{67} z_7 = 0\}.$$

	$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$
A	$P = \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix}, \; Q = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix}$
$G^{[1]}$	$V_1 = \begin{pmatrix} 0 &&&&&&& \\ & 1 &&&&&& \\ && 0 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix}, \; T_1 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix}$
$G^{[2]}$	$V_2 = \begin{pmatrix} 0 &&&&&&& \\ & 1 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 1 &&& \\ &&&&& 0 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix}, \; T_2 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix}$
$G^{[3]}$	$V_3 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 1 &&& \\ &&&&& 0 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix}, \; T_3 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 0 & \\ &&&&&&& 1 \end{pmatrix}$
$G^{[4]}$	$V_4 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix}, \; T_4 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 0 & \\ &&&&&&& 1 \end{pmatrix}$
$G^{[5]}$	$V_5 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix}, \; T_5 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix}$

Table 1: Projectors associated to the derivative array analysis for the reformulated equations (3)

	$(x_1, x_2, x_3, x_4, x_5, x_6, u_1, u_2)$
$G^{[3]}$	$V_3 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 1 &&& \\ &&&&& 0 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix}, \ T_3 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 0.5 & 0.5 \\ &&&&&& 0.5 & 0.5 \end{pmatrix}$
$G^{[4]}$	$V_4 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix}, \ T_4 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 0.5 & 0.5 \\ &&&&&& 0.5 & 0.5 \end{pmatrix}$

Table 2: Projectors associated with the derivative array analysis: differences to Table 1 when using the original formulation (1).

Furthermore, the intersection $N_1 \cap \ker \mathit{\Pi}_0$ is trivial, thus there is a projector function Q_1 onto N_1 such that $\ker \mathit{\Pi}_0 \subseteq \ker Q_1$. It is evident that

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -rb_{57} & 0 & 0 \end{pmatrix}$$

is such a projector function. We also derive

$$\mathit{\Pi}_0 Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathit{\Pi}_1 = \mathit{\Pi}_0 - \mathit{\Pi}_0 Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$B_1 = B_0 \mathit{\Pi}_0 - G_1 D^-(D\mathit{\Pi}_1 D^-)' D = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & b_{42} & b_{43} & b_{44} & 0 & b_{46} + p' & 0 & 0 \\ 0 & b_{52} & b_{53} & b_{54} & 0 & b_{56} & 0 & 0 \\ 0 & b_{62} & b_{63} & b_{64} & 0 & b_{66} & 0 & 0 \\ b_{71} & 0 & b_{73} & 0 & 0 & 0 & 0 & 0 \\ b_{81} & 0 & b_{83} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the sign prime indicates the total derivative in jet variables. In particular, p' stands for the function $p'(x_3, x_3^1) = p'(x_3)x_3^1$ of $x_3 \in \mathbb{R}$ and the jet variable $x_3^1 \in \mathbb{R}$, see e.g., [14, Section 3.2].

Next we compute the matrix function

$$G_2 = G_1 + B_1 Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -p & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & b_{44}p + b_{46} + p' & b_{47} & 0 \\ 0 & 0 & 0 & 0 & 1 & b_{54}p + b_{56} & b_{57} & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 + b_{64}p + b_{66} & b_{67} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

its nullspace

$$\begin{aligned} N_2 = \{z \in \mathbb{R}^8 : & z_1 - pz_6 = 0,\ z_2 - z_5 = 0,\ z_3 - z_6 = 0, \\ & z_4 + (b_{44}p + b_{46} + p')z_6 + b_{47}z_7 = 0, \\ & z_5 + (b_{54}p + b_{56})z_6 + b_{57}z_7 - z_8 = 0, \\ & (1 + b_{64}p + b_{66})z_6 + b_{67}z_7 = 0\}, \end{aligned}$$

and the intersection

$$N_2 \cap \ker \mathit{\Pi}_1 = N_2 \cap \{z \in \mathbb{R}^8 : z_1 = 0,\ z_2 = 0,\ z_3 = 0, z_4 - pz_6 = 0\} = \{0\}.$$

With

$$Q_2 = \begin{pmatrix} 0 & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p + \mathcal{A} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r(1 + pb_{64} + b_{66}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & * & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} \mathcal{A} &= p^2 b_{64} + p(b_{66} - b_{44}) - b_{46} - p', \\ Q_{2,83} &= pb_{54} + b_{56} - rb_{57}(1 + pb_{64} + b_{66}), \end{aligned}$$

we find an admissible projector function onto N_2 such that $\ker \mathit{\Pi}_1 \subseteq \ker Q_2$. Then it results that

$$\mathit{\Pi}_1 Q_2 = \begin{pmatrix} 0 & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{A} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathit{\Pi}_2 = \mathit{\Pi}_1 - \mathit{\Pi}_1 Q_2 = \begin{pmatrix} 1 & 0 & -p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{A} & 1 & 0 & -p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

as well as

$$B_2 = B_1 \mathit{\Pi}_1 - G_2 D^-(D\mathit{\Pi}_2 D^-)' D\mathit{\Pi}_1 = \begin{pmatrix} 0 & 0 & +p' & -1 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{42} & b_{43}+\mathcal{A}' & b_{44} & 0 & -pb_{44} & 0 & 0 \\ 0 & b_{52} & b_{53} & b_{54} & 0 & -pb_{54} & 0 & 0 \\ 0 & b_{62} & b_{63} & b_{64} & 0 & -pb_{64} & 0 & 0 \\ b_{71} & 0 & b_{73} & 0 & 0 & 0 & 0 & 0 \\ b_{81} & 0 & b_{83} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_2 Q_2 = \begin{pmatrix} 0 & 0 & p' - \mathcal{A} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{42} & b_{43}+\mathcal{A}'+b_{44}\mathcal{A} & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{52} & b_{53}+b_{54}\mathcal{A} & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{62} & b_{63}+b_{64}\mathcal{A} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{73}+pb_{71} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{83}+pb_{81} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_3 = G_2 + B_2 Q_2 = \begin{pmatrix} 1 & 0 & p' - \mathcal{A} & 0 & 0 & -p & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & b_{42} & b_{43}+\mathcal{A}'+b_{44}\mathcal{A} & 1 & 0 & b_{44}p+b_{46}+p' & b_{47} & 0 \\ 0 & b_{52} & b_{53}+b_{54}\mathcal{A} & 0 & 1 & b_{54}p+b_{56} & b_{57} & -1 \\ 0 & b_{62} & b_{63}+b_{64}\mathcal{A} & 0 & 0 & 1+b_{64}p+b_{66} & b_{67} & 0 \\ 0 & 0 & b_{73}+pb_{71} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{83}+pb_{81} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the nullspace of G_3 is

$$\begin{aligned} N_3 = \{z \in \mathbb{R}^8 : & z_3 = 0,\ z_2 - z_5 = 0,\ z_6 = 0,\ z_1 = 0,\ b_{42}z_2 + z_4 + b_{47}z_7 = 0, \\ & b_{52}z_2 + z_5 + b_{57}z_7 - z_8 = 0,\ b_{62}z_2 + b_{67}z_7 = 0\} \\ = \{z \in \mathbb{R}^8 : & z_1 = 0,\ z_3 = 0,\ z_6 = 0,\ z_5 = z_2,\ z_7 = -rb_{62}z_2, \\ & z_4 = (-b_{42} + pb_{62})z_2,\ z_8 = (b_{52} + 1 - b_{57}rb_{62})z_2\,\}. \end{aligned}$$

The intersection

$$\begin{aligned} N_3 \cap \ker \mathit{\Pi}_2 &= N_3 \cap \{z \in \mathbb{R}^8 : z_1 - pz_3 = 0,\ -\mathcal{A}z_3 + z_4 - pz_6 = 0\} \\ &= \{z \in \mathbb{R}^8 : z_3 = 0,\ z_6 = 0,\ z_1 = 0,\ z_4 = 0,\ z_5 = z_2, \\ &\quad z_8 = (1 + b_{52})z_2 + b_{57}z_7, b_{42}z_2 + b_{47}z_7 = 0,\ b_{62}z_2 + b_{67}z_7 = 0\} \end{aligned}$$

is trivial, precisely where the matrix $\mathcal{M}$ (see Lemma 1) is nonsingular, that means,

$$N_3(x) \cap \ker \mathit{\Pi}_2(x) = \{0\} \Leftrightarrow \cos x_3 \neq 3 - \sqrt{5}.$$

The planes in $\mathbb{R}^8$ described by $\cos x_3 = 3 - \sqrt{5}$ indicate critical points of the DAE. Denote the set of critical points arising at this level (cf. [14, Definition 2.75]) by

$$S_{crit}^{3-B} = \{x \in \mathbb{R}^8 : \cos x_3 = 3 - \sqrt{5}\}. \tag{15}$$

The function $\mathcal{S}$ (see Lemma 1) will play its role when constructing the next projector function Q_3 onto N_3 such that $\ker \Pi_2 \subseteq \ker Q_3$ for arguments outside the critical point set. We observe that there

$$N_3 = \operatorname{im} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -b_{42}+pb_{62} \\ 1 \\ 0 \\ -rb_{62} \\ \mathcal{B} \end{pmatrix} = \operatorname{im} \begin{pmatrix} 0 \\ -\mathcal{S} \\ 0 \\ 1 \\ -\mathcal{S} \\ 0 \\ \mathcal{S}rb_{62} \\ -\mathcal{S}\mathcal{B} \end{pmatrix}, \quad \mathcal{B} = 1 + b_{52} - rb_{57}b_{62},$$

leading to

$$Q_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{S}\frac{1}{p}\mathcal{A} & 0 & 0 & -\mathcal{S} & 0 & p\mathcal{S} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{p}\mathcal{A} & 0 & 0 & 1 & 0 & -p & 0 & 0 \\ \mathcal{S}\frac{1}{p}\mathcal{A} & 0 & 0 & -\mathcal{S} & 0 & p\mathcal{S} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -rb_{62}\mathcal{S}\frac{1}{p}\mathcal{A} & 0 & 0 & rb_{62}\mathcal{S} & 0 & -rb_{62}p\mathcal{S} & 0 & 0 \\ \mathcal{S}\frac{1}{p}\mathcal{A}\mathcal{B} & 0 & 0 & -\mathcal{S}\mathcal{B} & 0 & p\mathcal{S}\mathcal{B} & 0 & 0 \end{pmatrix},$$

$$\Pi_2 Q_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{p}\mathcal{A} & 0 & 0 & 1 & 0 & -p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \Pi_3 = \Pi_2 - \Pi_2 Q_3 = \begin{pmatrix} 1 & 0 & -p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{p}\mathcal{A} & 0 & -\mathcal{A} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned} B_3 &= B_2\Pi_2 - G_3 D^-(D\Pi_3 D^-)' D\Pi_2 \\ &= \begin{pmatrix} 0 & 0 & \mathcal{A} & -1 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -b_{44}\mathcal{A} & b_{44} & 0 & -pb_{44} & 0 & 0 \\ 0 & 0 & -b_{54}\mathcal{A} & b_{54} & 0 & -pb_{54} & 0 & 0 \\ 0 & 0 & -b_{64}\mathcal{A} & b_{64} & 0 & -pb_{64} & 0 & 0 \\ b_{71} & 0 & -pb_{71} & 0 & 0 & 0 & 0 & 0 \\ b_{81} & 0 & -pb_{81} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\frac{1}{p}\mathcal{A})' & 0 & -p(\frac{1}{p}\mathcal{A})' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

$$
= \begin{pmatrix}
0 & 0 & \mathcal{A} & -1 & 0 & p & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-(\frac{1}{p}\mathcal{A})' & 0 & -b_{44}\mathcal{A} + p(\frac{1}{p}\mathcal{A})' & b_{44} & 0 & -pb_{44} & 0 & 0 \\
0 & 0 & -b_{54}\mathcal{A} & b_{54} & 0 & -pb_{54} & 0 & 0 \\
0 & 0 & -b_{64}\mathcal{A} & b_{64} & 0 & -pb_{64} & 0 & 0 \\
b_{71} & 0 & -pb_{71} & 0 & 0 & 0 & 0 & 0 \\
b_{81} & 0 & -pb_{81} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Next we obtain

$$
B_3Q_3 = \begin{pmatrix}
\frac{1}{p}\mathcal{A} & 0 & 0 & -1 & 0 & p & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-b_{44}\frac{1}{p}\mathcal{A} & 0 & 0 & b_{44} & 0 & -pb_{44} & 0 & 0 \\
-b_{54}\frac{1}{p}\mathcal{A} & 0 & 0 & b_{54} & 0 & -pb_{54} & 0 & 0 \\
-b_{64}\frac{1}{p}\mathcal{A} & 0 & 0 & b_{64} & 0 & -pb_{64} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

and

$$
G_4 = G_3 + B_3Q_3 = \begin{pmatrix}
1 + \frac{1}{p}\mathcal{A} & 0 & p' - \mathcal{A} & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
-b_{44}\frac{1}{p}\mathcal{A} & b_{42} & b_{43} + \mathcal{A}' + b_{44}\mathcal{A} & 1 + b_{44} & 0 & b_{46} + p' & b_{47} & 0 \\
-b_{54}\frac{1}{p}\mathcal{A} & b_{52} & b_{53} + b_{54}\mathcal{A} & b_{54} & 1 & b_{56} & b_{57} & -1 \\
-b_{64}\frac{1}{p}\mathcal{A} & b_{62} & b_{63} + b_{64}\mathcal{A} & b_{64} & 0 & 1 + b_{66} & b_{67} & 0 \\
0 & 0 & b_{73} + pb_{71} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b_{83} + pb_{81} & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

as well as the nullspace

$$
\begin{aligned}
N_4 = \{z \in \mathbb{R}^8 : & z_3 = 0, z_6 = 0, (1 + \frac{\mathcal{A}}{p})z_1 - z_4 = 0, z_2 - z_5 = 0, \\
& - b_{44}\frac{\mathcal{A}}{p}z_1 + b_{42}z_2 + (1 + b_{44})z_4 + b_{47}z_7 = 0, \\
& - b_{54}\frac{\mathcal{A}}{p}z_1 + b_{52}z_2 + b_{54}z_4 + z_5 + b_{57}z_7 - z_8 = 0 \\
& - b_{64}\frac{\mathcal{A}}{p}z_1 + b_{62}z_2 + b_{64}z_4 + b_{67}z_7 = 0\}.
\end{aligned}
$$

The intersection

$$
\begin{aligned}
N_4 \cap \ker \mathit{\Pi}_3 = \{z \in \mathbb{R}^8 : z_1 = 0, z_3 = 0, z_6 = 0, z_4 = 0, z_2 = z_5, \\
b_{42}z_2 + b_{47}z_7 = 0, b_{62}z_2 + b_{67}z_7 = 0, (b_{52} + 1)z_2 + b_{57}z_7 - z_8 = 0\}
\end{aligned}
$$

becomes trivial exactly where the matrix $\mathcal{M}$ (see Lemma 1) is nonsingular. Thus, $\operatorname{rank} G_4 = 7$ and $N_4 \cap \ker \mathit{\Pi}_3 = \{0\}$ on $\{x \in \mathbb{R}^8 : x \notin S_{crit}^{3-B}\}$, and we find a projector matrix Q_4 onto N_4 such that $\ker \mathit{\Pi}_3 \subseteq \ker Q_4$.

For $z \in N_4$, it holds, in particular that $z_4 = (1 + \frac{\mathcal{A}}{p})z_1$ and

$$b_{42}z_2 + b_{47}z_7 = b_{44}\frac{\mathcal{A}}{p}z_1 - (1 + b_{44})(1 + \frac{\mathcal{A}}{p})z_1,$$
$$b_{62}z_2 + b_{67}z_7 = b_{64}\frac{\mathcal{A}}{p}z_1 - b_{64}(1 + \frac{\mathcal{A}}{p})z_1.$$

Since here the coefficient matrix $\mathcal{M}$ is nonsingular, we obtain the expressions[1]

$$z_2 = g\, z_1, \quad z_7 = h\, z_1,$$

with functions g and h being continuous outside of S^{3-B}_{crit}. Denoting further

$$f = b_{54} + (1 + b_{52})g + b_{57}h,$$

we arrive at

$$N_4 = \mathrm{im}\begin{pmatrix} 1 \\ g \\ 0 \\ 1 + \frac{\mathcal{A}}{p} \\ g \\ 0 \\ h \\ f \end{pmatrix}.$$

Regarding that $\ker \mathit{\Pi}_3 = \{z \in \mathbb{R}^8 : z_1 - pz_3 = 0\}$ we choose

$$Q_4 = \begin{pmatrix} 1 & 0 & -p & 0 & 0 & 0 & 0 & 0 \\ g & 0 & -pg & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 + \frac{\mathcal{A}}{p} & 0 & -p - \mathcal{A} & 0 & 0 & 0 & 0 & 0 \\ g & 0 & -pg & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h & 0 & -ph & 0 & 0 & 0 & 0 & 0 \\ f & 0 & -pf & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This yields

$$\mathit{\Pi}_3 Q_4 = \begin{pmatrix} 1 & 0 & -p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{p}\mathcal{A} & 0 & -\mathcal{A} & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathit{\Pi}_4 = \mathit{\Pi}_3 - \mathit{\Pi}_3 Q_4 = 0,$$

[1]In detail $g = -\mathcal{S}(1 + \frac{1}{p}\mathcal{A} + b_{44}) + p\mathcal{S}b_{64}$ and $h = rb_{62}\mathcal{S}(1 + \frac{1}{p}\mathcal{A} + b_{44}) - (r + rb_{62}p\mathcal{S})b_{64}$.

and

$$B_4 = B_3\Pi_3 = \begin{pmatrix} -\frac{1}{p}\mathcal{A} & 0 & \mathcal{A} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{44}\frac{1}{p}\mathcal{A} - (\frac{1}{p}\mathcal{A})' & 0 & -\mathcal{A}b_{44} + p(\frac{1}{p}\mathcal{A})' & 0 & 0 & 0 & 0 & 0 \\ b_{54}\frac{1}{p}\mathcal{A} & 0 & -\mathcal{A}b_{54} & 0 & 0 & 0 & 0 & 0 \\ b_{64}\frac{1}{p}\mathcal{A} & 0 & -\mathcal{A}b_{64} & 0 & 0 & 0 & 0 & 0 \\ b_{71} & 0 & -pb_{73} & 0 & 0 & 0 & 0 & 0 \\ b_{81} & 0 & -pb_{83} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_4Q_4 = B_3\Pi_3Q_4 = B_3\Pi_3,$$

$$G_5 = G_4 + B_4Q_4 = \begin{pmatrix} 1 & 0 & p' & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ -(\frac{1}{p}\mathcal{A})' & b_{42} & b_{43} + \mathcal{A}` + p(\frac{1}{p}\mathcal{A})' & 1 + b_{44} & 0 & b_{46} + p' & b_{47} & 0 \\ 0 & b_{52} & b_{53} & b_{54} & 1 & b_{56} & b_{57} & -1 \\ 0 & b_{62} & b_{63} & b_{64} & 0 & 1 + b_{66} & b_{67} & 0 \\ b_{71} & 0 & b_{73} & 0 & 0 & 0 & 0 & 0 \\ b_{81} & 0 & b_{83} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It remains to check if G_5 is nonsingular. $z \in N_5 = \ker G_5$ implies

$$\begin{aligned} z_5 &= z_2, \\ z_6 &= z_3, \\ z_4 &= z_1 + p'z_3, \\ \begin{pmatrix} b_{42} & b_{47} \\ b_{62} & b_{67} \end{pmatrix}\begin{pmatrix} z_2 \\ z_7 \end{pmatrix} &= -\begin{pmatrix} 1 + b_{44} - (\frac{1}{p}\mathcal{A})' & b_{43} + \mathcal{A}' + p(\frac{1}{p}\mathcal{A})' + (1+b_{44})p' + b_{46} + p' \\ b_{64} & 1 + b_{66} + b_{64}p' \end{pmatrix}\begin{pmatrix} z_1 \\ z_3 \end{pmatrix}, \\ z_8 &= \begin{pmatrix} b_{52}+1 & b_{57} \end{pmatrix}\begin{pmatrix} z_2 \\ z_7 \end{pmatrix} + \begin{pmatrix} b_{54} & b_{53} + b_{54}p' + b_{56} \end{pmatrix}\begin{pmatrix} z_1 \\ z_3 \end{pmatrix}, \end{aligned}$$

and

$$\begin{pmatrix} b_{71} & b_{73} \\ b_{81} & b_{83} \end{pmatrix}\begin{pmatrix} z_1 \\ z_3 \end{pmatrix} = 0.$$

This shows that G_5 becomes nonsingular precisely if the matrix functions $\mathcal{M}$ and $\mathcal{N}$ are nonsingular (see Lemma 1). The matrix function $\mathcal{N}$ depends only on x_1 and x_3. However, since $\det \mathcal{N}(x_1, x_3) = \sin x_3$, the DAE features also critical points of type 5-A, (cf. [14, Definition 2.75])

$$S_{crit}^{5-A} = \{x \in \mathbb{R}^8 : \sin x_3 = 0\}.$$

We summarize the results as a proposition:

Proposition 1. *The definition domain $\mathbb{R}^8 \times \mathbb{R}$ of the data of the given DAE (14) decomposes into an infinite number of regularity regions $\mathfrak{G}$, each of which is an open connected set determined by*

$$\mathfrak{G} = \{(x,t) \in \mathbb{R}^8 \times \mathbb{R} : \cos x_3 \neq 3 - \sqrt{5},\ \sin x_3 \neq 0\}.$$

On all these regularity regions the DAE has the tractability index 5 and the characteristic values $r_0 = 6$, $r_1 = 6$, $r_2 = 6$, $r_3 = 7$, $r_4 = 7$, $r_5 = 8$.
The regularity regions are separated by hyperplanes corresponding to the sets of critical points S^{3-B}_{crit} and S^{5-A}_{crit}, respectively.

Note that the given DAE (14) has no dynamics owing to the fact that $\Pi_4 = 0$. The solution $x_*(\cdot)$ decomposes according to $I = Q_0 + \Pi_0 Q_1 + \Pi_1 Q_2 + \Pi_2 Q_3 + \Pi_3 Q_4$,

$$Q_0 x_* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ x_{*\,7} \\ x_{*\,8} \end{pmatrix}, \quad \Pi_0 Q_1 x_* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ p x_{*\,6} \\ x_{*\,5} \\ x_{*\,6} \\ 0 \\ 0 \end{pmatrix}, \quad \Pi_1 Q_2 x_* = \begin{pmatrix} p x_{*3} \\ x_{*\,2} \\ x_{*\,3} \\ \mathcal{A} x_{*\,3} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\Pi_2 Q_3 x_* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_{*\,4} - \frac{1}{p}\mathcal{A} x_{*\,1} - p x_{*\,6} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Pi_3 Q_4 x_* = \begin{pmatrix} x_{*\,1} - p x_{*3} \\ 0 \\ 0 \\ \frac{1}{p}\mathcal{A} x_{*\,1} - \mathcal{A} x_{*\,3} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

It is worth noting that the projector functions Π_0, Π_1, Π_2 are continuous, but Π_3, Π_4 have continuous extensions through the critical points. Therefore, also the matrix functions G_i, B_i are continuous or have continuous extensions. This fact is closely related to the approach in [17] and seems to be helpful for further critical point studies.

We want to emphasize that according to [14, Theorem 3.39] the characteristic values $r_0, \ldots, r_5$ are invariant under regular transformations. Therefore, analogous results are obtained if the original equation (1) is used instead of (3).

6 Types of Singularities and Numerical Experiments

The different ways of investigation of the Robotic Arm problem in the Sections 3–5 discover the same two types of singularities. This underlines that the singularities belong to the problem and are not owed to the used technical procedure.

- In [16, 17] a classification of singularities is introduced. Using the corresponding nomenclature, we concluded in Section 5 that two singularity sets appear:

  1. $S^{3-B}_{crit} = \{x \in \mathbb{R}^8 : \cos x_3 = 3 - \sqrt{5}\}$

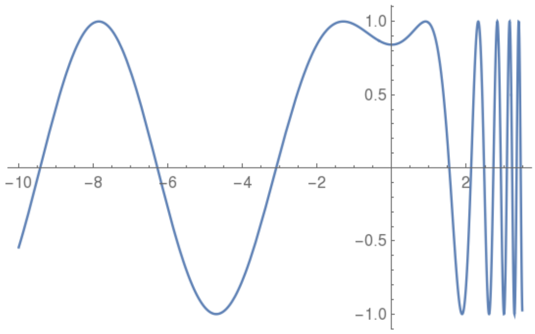

Figure 4: Graph of $\sin x_3(t)$

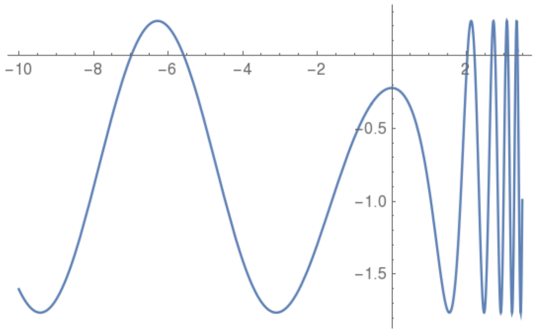

Figure 5: Graph of $\cos x_3(t) - (3 - \sqrt{5})$

2. $S^{5-A}_{crit} = \{x \in \mathbb{R}^8 : \sin x_3 = 0\}$.

Singularities arise, if the DAE solution crosses a singularity set.

- In terms of the nomenclature used in Section 4, we observed that
  1. the projector V_1 can only be obtained if x_3 is not in the set of critical points $\{x \in \mathbb{R}^8 : \sin x = 0\}$.
  2. the projectors V_3 and T_3 can only be obtained if, furthermore, x_3 is not in the set of critical points $\{x \in \mathbb{R}^8 : \cos x_3 = 3 - \sqrt{5}\}$.

For the prescribed p_1, p_2, one has $x_3(t) = \mathrm{e}^t - t$ and singularities arise whenever $\cos(e^t - t) = 3 - \sqrt{5}$ and $\sin(e^t - t) = 0$.

- The graph and the zeros of $\sin x_3(t) = \sin(\mathrm{e}^t - t)$ can be found in Figure 4. Here we confirm that the singularity in the interval $[0, 2]$ is at 1.5446260000352112.
- The zeros of $\cos x_3(t) - (3 - \sqrt{5}) = \cos(\mathrm{e}^t - t) - (3 - \sqrt{5})$ are represented in Figure 5.

The singularities of the Robotic Arm DAE (1) are the zeros of both functions. In the interval $[-6, 2.21967]$ we have both types of singularities, as shown in Figure 6(top).

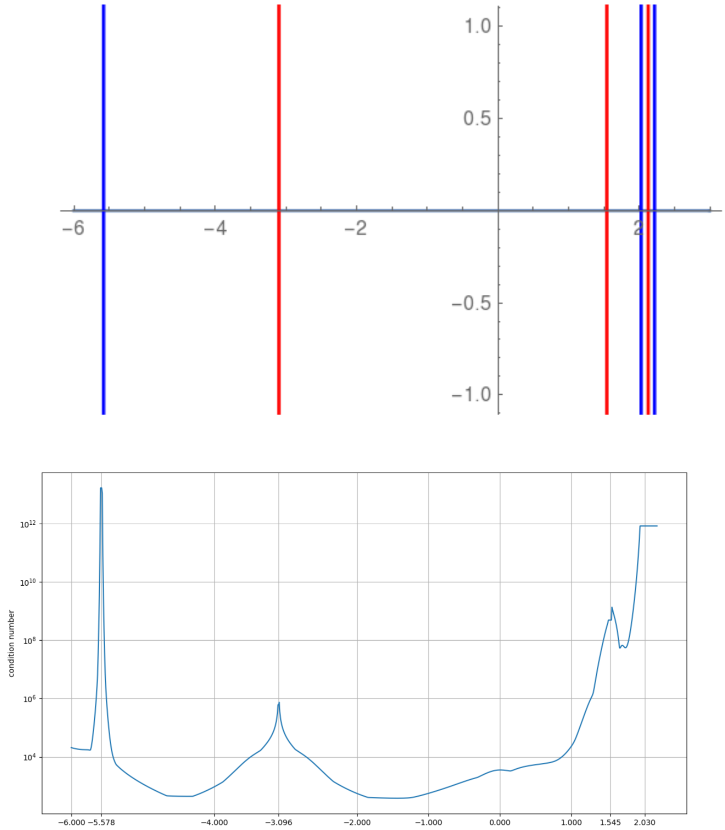

Figure 6: The singularities in $[-6, 2.21967]$ (top) and the condition number of $\mathcal{B}^{[5]}$ (bottom)

According to [8], the condition number of $\mathcal{B}^{[5]}$ is an indicator for singularities of DAEs, whereas a more detailed analysis can be obtained observing the ranks of the associated projectors.

In our numerical experiments with the DAE (1) the condition number perfectly showed the position of the singularities, see Figure 6. In this case, the theoretical analysis presented in the Sections 3–5 confirmed, indeed, the existence of the singularities noticed monitoring this condition number.

The singularities of the Robotic Arm problem depend on the solution component x_3 only. x_3 is fixed by p_1 and p_2 (cf. (2)), which determine the prescribed path of the robot. If we choose functions $\bar{p}_1, \bar{p}_2$ such that the resulting solution component x_3 never crosses a singularity plane, we obtain a singularity-free solution. We illustrate that by

$$\begin{aligned} \bar{p}_1(t) &= \cos(1-t) + \cos(3 + \tfrac{\sin t}{2} - t), \\ \bar{p}_2(t) &= \sin(1-t) + \sin(3 + \tfrac{\sin t}{2} - t), \end{aligned} \tag{16}$$

which leads to $x_3(t) = 2 + \frac{\sin t}{2}$. We integrate the Robotic Arm problem with this modification over the interval $[-5, 5]$, see Figure 7. The computed condition number, see Figure 8, varies inside of noncritical values, i.e., no singularities

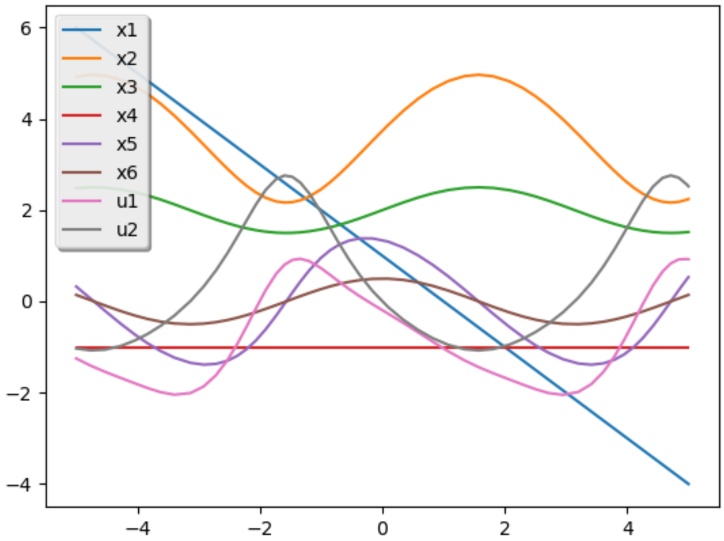

Figure 7: Singularity free solution of modified Robotic Arm problem (cf. (16))

appear.

7 Singularities of a more General Formulation

Only recently, the robotic arm was described in more general terms as a test for tracking problems, [5]. Therefore, for completeness, we describe the two types of singularities that were detected above in these general terms.

The general form of the equations reads

$$
\begin{aligned}
x_1' &= x_4,\\
x_2' &= x_5,\\
x_3' &= x_6,\\
x_4' &= f_4(x_2, x_3, x_4, x_6) + g_{41}(x_3)u_1 - g_{41}(x_3)u_2,\\
x_5' &= f_5(x_2, x_3, x_4, x_6) - g_{41}(x_3)u_1 + g_{52}(x_3)u_2,\\
x_6' &= f_6(x_2, x_3, x_4, x_6) + g_{61}(x_3)u_1 - g_{61}(x_3)u_2,\\
p_1(t) &= l_1 \cos x_1 + l_2 \cos(x_1 + x_3),\\
p_2(t) &= l_1 \sin x_1 + l_2 \sin(x_1 + x_3),
\end{aligned}
\tag{17}
$$

where $(p_1(t), p_2(t))$ is again the endpoint of the outer arm in Cartesian coordinates, cf. (2), l_1, l_2 are possibly different lengths of the two links and f_i, g_{ij} are suitable functions resulting from the dynamic model of the Robotic Arm, they

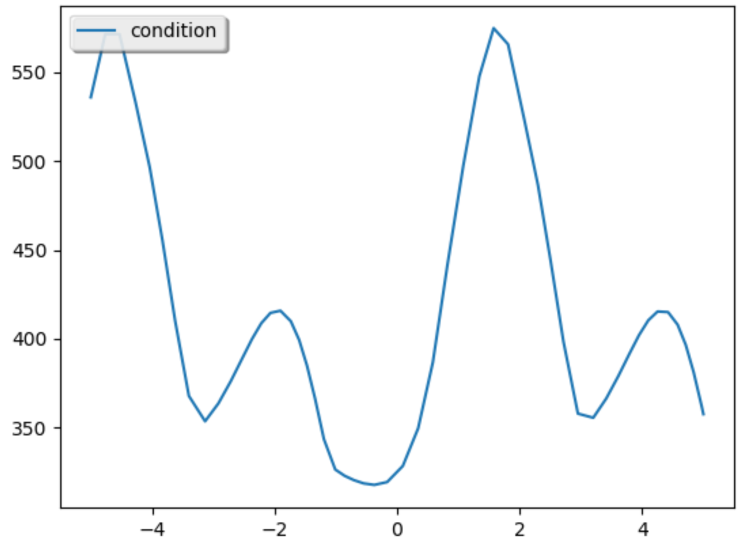

Figure 8: Condition number of the modified Robotic Arm problem(cf. (16))

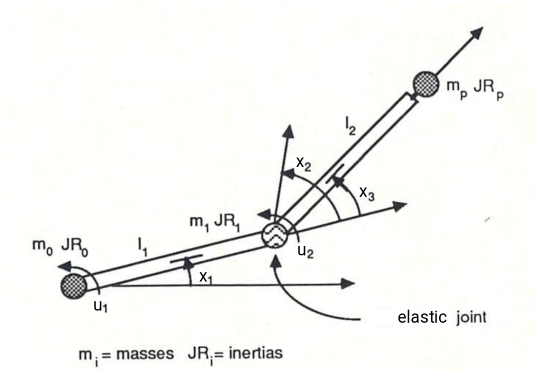

Figure 9: Two-link planar robot arm with second joint elastic. (Modification of a graphic from [6])

read

$$
\begin{aligned}
g_{41}(x_3) &= \frac{A_2}{A_3(A_4 - A_3\cos^2 x_3)} \\
g_{52}(x_3) &= g_{41}(x_3) + \frac{1}{JR_1} \\
g_{61}(x_3) &= -g_{41}(x_3) - \frac{\cos x_3}{A_4 - A_3\cos^2 x_3} \\
f_4(x_2, x_3, x_4, x_6) &= \frac{A_2 \sin x_3 (x_4 + x_6)^2 + A_3 x_4^2 \sin x_3 \cos x_3}{A_4 - A_3\cos^2 x_3} \\
&\quad + \frac{K\left(x_3 - \frac{x_2}{NT}\right)\left(\frac{A_2}{A_3}\left(\frac{NT-1}{NT}\right) + \cos x_3\right)}{A_4 - A_3\cos^2 x_3} \\
f_5(x_2, x_3, x_4, x_6) &= -f_4(x_2, x_3, x_4, x_6) + \frac{K}{NT}\left(x_3 - \frac{x_2}{NT}\right)\left(\frac{1}{JR_1} - 2g_{41}(x_3)\right), \\
f_6(x_2, x_3, x_4, x_6) &= -f_4(x_2, x_3, x_4, x_6) - \frac{K\left(x_3 - \frac{x_2}{NT}\right)\left(\frac{A_5}{A_3} - \left(\frac{3NT+1}{NT}\right)\cos x_3\right)}{A_4 - A_3\cos^2 x_3} \\
&\quad - \frac{A_5 x_4^2 \sin x_3 + A_3 \sin x_3 \cos x_4 (x_4 + x_6)^2}{A_4 - A_3\cos^2 x_3},
\end{aligned}
$$

where

- K is the coefficient of elasticity of the second joint,
- NT is the transmission ratio at the second joint,
- m_p is the mass of the object being held,
- m_0 and m_1 are the masses of the arms viewed as concentrated at the joints,
- JR_1 and JR_p are corresponding rotor inertias, and
- the constants are defined by

$$
\begin{aligned}
A_2 &= JR_p + m_p l_2^2, \\
A_3 &= m_p l_1 l_2, \\
A_4 &= (m_1 + m_p) l_1 l_2, \\
A_5 &= (m_1 + m_p) l_2^2.
\end{aligned}
$$

Analogously to our above results, in [5], $x_3 \neq k\pi$ is identified to be a necessary condition for reasonable rank properties.

The second type of singularities that are described in the previous sections depends, in general, on several parameters.

In terms of the notation used before, for the general equations we obtain

$$
\begin{aligned}
b_{42} &= -\frac{\partial f_4}{\partial x_2} = \frac{K\left(\cos x_3 + \frac{A_2\,(NT-1)}{A_3\,NT}\right)}{NT\ (A_4 - A_3\ \cos^2 x_3)},\\
b_{47} &= -g_{41} = -\frac{A_2}{A_3\ (A_4 - A_3\ \cos^2 x_3)},\\
b_{62} &= -\frac{\partial f_6}{\partial x_2} = \frac{K\ (A_2 + A_3\ \cos x_3 - A_2\,NT - A_5\,NT + 2\,A_3\,NT\ \cos x_3)}{A_3\,NT^2\ (A_4 - A_3\ \cos^2 x_3)},\\
b_{67} &= -g_{61} = \frac{A_2 + A_3\ \cos x_3}{A_3\ (A_4 - A_3\ \cos^2 x_3)},\\
b_{71} &= -l_1 \sin x_1 - l_2 \sin(x_1 + x_3),\\
b_{73} &= -l_2 \sin(x_1 + x_3),\\
b_{81} &= l_1 \cos x_1 + l_2 \cos(x_1 + x_3),\\
b_{83} &= l_2 \cos(x_1 + x_3),
\end{aligned}
$$

such that analogous structural properties result directly.

Lemma 2. (a) *The function $g_{61}(x_3) = -b_{67}$ is smooth and has no zeros. It depends on x_3 only.*

(b) *The functions*

$$p := \frac{g_{41}(x_3)}{g_{61}(x_3)} = \frac{b_{47}}{b_{67}} \quad \text{and} \quad r := -\frac{1}{g_{61}(x_3)} = \frac{1}{b_{67}}$$

are smooth and depend on x_3 only. They have no zeros.

(c) *The matrix function*

$$
\begin{aligned}
\mathcal{M}(x_3) &:= -\begin{pmatrix} \frac{\partial f_4}{\partial x_2}(x_3) & g_{41}(x_3) \\ \frac{\partial f_6}{\partial x_2}(x_3) & g_{61}(x_3) \end{pmatrix} = -\begin{pmatrix} b_{42} & b_{47} \\ b_{62} & b_{67} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{K\left(\cos x_3 + \frac{A_2\,(NT-1)}{A_3\,NT}\right)}{NT\ (A_4 - A_3\ \cos^2 x_3)} & \frac{A_2}{A_3\ (A_4 - A_3\ \cos^2 x_3)} \\ -\frac{K\ (A_2 + A_3\ \cos x_3 - A_2\,NT - A_5\,NT + 2\,A_3\,NT\ \cos x_3)}{A_3\,NT^2\ (A_4 - A_3\ \cos^2 x_3)} & -\frac{A_2 + A_3\ \cos x_3}{A_3\ (A_4 - A_3\ \cos^2 x_3)} \end{pmatrix},
\end{aligned}
$$

has smooth entries depending on x_3 only. $\mathcal{M}(x_3)$ is nonsingular precisely if

$$b_{42}(x_3) - p(x_3) b_{62}(x_3) = \frac{\partial f_4}{\partial x_2}(x_3) - \frac{g_{41}(x_3)}{g_{61}(x_3)} \frac{\partial f_6}{\partial x_2}(x_3) \neq 0.$$

(d) *For*

$$
\begin{aligned}
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} -\frac{2\,A_2 + \sqrt{A_2\,(4\,A_2 + A_5)}}{A_3} \\ -\frac{2\,A_2 - \sqrt{A_2\,(4\,A_2 + A_5)}}{A_3} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{2\,JR_p + 2\,l_2{}^2\,m_p + \sqrt{(m_p\,l_2{}^2 + JR_p)\,(4\,JR_p + l_1{}^2\,m_1 + l_1{}^2\,m_p + 4\,l_2{}^2\,m_p)}}{l_1\,l_2\,m_p} \\ -\frac{2\,JR_p + 2\,l_2{}^2\,m_p - \sqrt{(m_p\,l_2{}^2 + JR_p)\,(4\,JR_p + l_1{}^2\,m_1 + l_1{}^2\,m_p + 4\,l_2{}^2\,m_p)}}{l_1\,l_2\,m_p} \end{pmatrix}.
\end{aligned}
$$

it holds that $\cos x_3 = z_1$ *or* $\cos x_3 = z_2$ *imply* $b_{42}(x_3) - p(x_3)b_{62}(x_3) = 0$ *and vice versa.*

(e) *The function*

$$\mathcal{S}(x_3) = \frac{1}{b_{42}(x_3) - p(x_3)b_{62}(x_3)}, \quad x_3 \in \operatorname{dom} \mathcal{S} = \{\tau \in \mathbb{R} : \cos\tau \neq z_{1,2}\},$$

is smooth on its definition domain, and so is

$$\mathcal{M}^{-1} = \begin{pmatrix} \mathcal{S} & -p\mathcal{S} \\ -rb_{62}\mathcal{S} & r + rpb_{62}\mathcal{S} \end{pmatrix}.$$

(f) *The matrix function*

$$\mathcal{N} = \begin{pmatrix} b_{71} & b_{73} \\ b_{81} & b_{83} \end{pmatrix} = \begin{pmatrix} -l_1 \sin x_1 - l_2 \sin(x_1 + x_3) & -l_2 \sin(x_1 + x_3) \\ l_1 \cos x_1 + l_2 \cos(x_1 + x_3) & l_2 \cos(x_1 + x_3) \end{pmatrix} \tag{18}$$

depends only on x_1 *and* x_3. $\mathcal{N}(x_1, x_3)$ *is nonsingular precisely if* $\det \mathcal{N}(x_1, x_3) = l_1 l_2 \sin x_3 \neq 0$.

Proof. Assertion (d): Since

$$A_4 - A_3 \cos^2 x_3 = m_1 l_1 l_2 + m_p l_1 l_2 (1 - \cos^2 x_3),$$

we can assume that all denominators are nonzero and, analogously to Lemma 1, focus on the singularities of the matrix $\mathcal{H}(x_3) := (A_4 - A_3 \cos^2 x_3)\mathcal{M}$, i.e.,

$$\mathcal{H}(x_3) = \begin{pmatrix} -\frac{K\left(\cos x_3 + \frac{A_2\,(NT-1)}{A_3 NT}\right)}{NT} & \frac{A_2}{A_3} \\ -\frac{K\,(A_2 + A_3\,\cos x_3 - A_2\,NT - A_5\,NT + 2\,A_3\,NT\,\cos x_3)}{A_3\,NT^2} & -\frac{A_2 + A_3\,\cos x_3}{A_3} \end{pmatrix},$$

with the determinant

$$\det(\mathcal{H}(x_3)) = ({A_3}^2\,\cos^2 x_3 + 4\,A_2\,A_3\,\cos x_3 - A_2\,A_5)\underbrace{K/(A_3^2 NT)}_{\neq 0}.$$

For the substitution $z = \cos x_3$ we obtain ${A_3}^2\,z^2 + 4\,A_2\,A_3\,z - A_2\,A_5$ with the roots

$$\begin{aligned} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} -\frac{2\,A_2 + \sqrt{A_2\,(4\,A_2 + A_5)}}{A_3} \\ -\frac{2\,A_2 - \sqrt{A_2\,(4\,A_2 + A_5)}}{A_3} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2\,JR_p + 2\,{l_2}^2\,m_p + \sqrt{(m_p\,{l_2}^2 + JR_p)\,(4\,JR_p + {l_1}^2\,m_1 + {l_1}^2\,m_p + 4\,{l_2}^2\,m_p)}}{l_1\,l_2\,m_p} \\ -\frac{2\,JR_p + 2\,{l_2}^2\,m_p - \sqrt{(m_p\,{l_2}^2 + JR_p)\,(4\,JR_p + {l_1}^2\,m_1 + {l_1}^2\,m_p + 4\,{l_2}^2\,m_p)}}{l_1\,l_2\,m_p} \end{pmatrix}. \end{aligned}$$

This proves the assertion. All other assertions follow straightforward. □

Parameters			Critical values
$m_1 = 1,\ m_p = 1$	$l_1 = 1,\ l_2 = 1$	$JR_p = 1$	$z_\star = 2\sqrt{5} - 4 = 0.4721$
$m_1 = 1,\ m_p = 10$	$l_1 = 1,\ l_2 = 1$	$JR_p = 1$	$z_\star = \frac{11\sqrt{5}}{10} - \frac{11}{5} = 0.2597$
$m_1 = 10,\ m_p = 10$	$l_1 = 1,\ l_2 = 1$	$JR_p = 1$	$z_\star = \frac{4\sqrt{11}}{5} - \frac{11}{5} = 0.4533$
$m_1 = 1,\ m_p = 1$	$l_1 = 1,\ l_2 = 2$	$JR_p = 1$	$z_\star = \frac{\sqrt{110}}{2} - 5 = 0.2440$
$m_1 = 1,\ m_p = 1$	$l_1 = 1,\ l_2 = 0.5$	$JR_p = 1$	$z_\star = \sqrt{35} - 5 = 0.9161$
$m_1 = 1,\ m_p = 1$	$l_1 = 1,\ l_2 = 0.3$	$JR_p = 1$	$-$
$m_1 = 1,\ m_p = 1$	$l_1 = 1,\ l_2 = 1$	$JR_p = 0.9$	$z_\star = \frac{2\sqrt{114}}{5} - \frac{19}{5} = 0.4708$
$m_1 = 1,\ m_p = 1$	$l_1 = 1,\ l_2 = 1$	$JR_p = 1.1$	$z_\star = \frac{\sqrt{546}}{5} - \frac{21}{5} = 0.4733$

Table 3: For the specified parameters, $\cos x_3 = z_\star$ leads to a singularity.

At regular points, these structural properties imply that all the results from the previous sections can be applied also to the general equations. In particular, the shape of all the described projectors is analogous.

If z_1 or z_2 belongs to the interval $[-1, 1]$, then the corresponding singularities appear. In Table 3 we present some critical points in dependence of some values for the parameters.

This means that, in general, there actually appear singularities in configurations, depending on the particular values for JR_p, m_p , m_1, l_1, l_2.

Remark 1. *Writing this article, we noticed that there seems to be a misprint in one sign of* (1)*, since we could not fit parameters to obtain that specific equation. We highly appreciate that this was confirmed by [2]. However, since the equations* (1) *were discussed in several publications over the last decades, we focused on them on the first part of this article to facilitate the comparison.*

In practice, the results from the present section should be considered for the corresponding parameters.

For $m_1 = 1,\ m_p = 1,\ l_1 = 1,\ l_2 = 1,\ JR_p = 1$*, we obtain the singularity for* $\cos x_3 = 0.4721$*. If* $p_1(t)$*,* $p_2(t)$ *are chosen according to Section 2.1, and therefore* $x_3 = e^t - t$*, then we have a singularity at* $t = 0.372999$*.*

8 Conclusions

In this article, we applied two different methodologies to characterize singular points of higher-index DAEs considering the Robotic Arm equations, the well-known benchmark from literature.

The two methodologies, which are related to the projector based differentiation index and the tractability index, are based on rank consideration of matrices that are constructed in accordance to both index concepts. For the differentiation index, the 1-fullness of the expanded derivative array is considered. For the tractability index, the corresponding matrix sequence has to deliver a nonsingular matrix.

Although the matrices considered in both approaches are constructed in very different ways, both give us, in the end, hints to the same two types of singular points. The existence of these singularities depends on the particular values of the variable x_3, which describes the angular coordinate of the outer arm.

The detected singularity for $\cos x_3 = z_\star$ means that, for the Robotic Arm, there are singular configurations that, to our knowledge, have not been described so far in the DAE literature. These particular values are influenced by several model parameters.

References

[1] A. Barrlund. Constrained least squares methods for non-linear differential algebraic systems. Technical report, University of Umeåa Sweden, Institute of Information Processing, June 1991.

[2] S. L. Campbell. Private communication.

[3] S. L. Campbell. A general method for nonlinear descriptor systems: An example from robotic path control. Technical Report CRSC 090588-01, North Carolina State University Raleigh, 1988.

[4] S. L. Campbell and E. Griepentrog. Solvability of general differential algebraic equations. *SIAM J. Sci.Comput.*, 16(2):257–270, 1995.

[5] S. L. Campbell and P. Kunkel. *Applications of Differential-Algebraic Equations: Examples and Benchmarks*, chapter General Nonlinear Differential Algebraic Equations and Tracking Problems: A Robotics Example. Differential-Algebraic Equations Forum. Springer, Cham, 2018.

[6] A. De Luca. Control properties of robot arms with joint elasticity. Analysis and control of nonlinear systems, Sel. Pap. 8th Int. Symp. Math. Networks Syst., Phoenix/Ariz. 1987, 61-70 (1988)., 1988.

[7] D. Estévez Schwarz and R. Lamour. InitDAE's documentation. `https://www.mathematik.hu-berlin.de/~lamour/software/python/InitDAE/html/`.

[8] D. Estévez Schwarz and R. Lamour. A new projector based decoupling of linear DAEs for monitoring singularities. *Numer. Algorithms*, 73(2):535–565, 2016.

[9] D. Estévez Schwarz and R. Lamour. Consistent initialization for higher-index DAEs using a projector based minimum-norm specification. Technical Report 1, Institut für Mathematik, Humboldt-Universität zu Berlin, 2016.

[10] D. Estévez Schwarz and R. Lamour. A new approach for computing consistent initial values and Taylor coefficients for DAEs using projector-based constrained optimization. *Numer. Algorithms*, 78(2):355–377, 2018.

[11] D. Estévez Schwarz and R. Lamour. `https://www.mathematik.hu-berlin.de/~lamour/software/python/RoboticArmSolution.py`, 2019.

[12] D. Estévez Schwarz and R. Lamour. InitDAE: Computation of consistent values, index determination and diagnosis of singularities of DAEs using automatic differentiation in Python. Technical Report 1, Institut für Mathematik, Humboldt-Universität zu Berlin, 2019.

[13] D. Estévez Schwarz and R. Lamour. Projected explicit and implicit Taylor series methods for DAEs. *In preparation.*, 2019.

[14] R. Lamour, R. März, and C. Tischendorf. *Differential-Algebraic Equations: A Projector Based Analysis.* Differential-Algebraic Equations Forum. Springer-Verlag Berlin Heidelberg New York Dordrecht London, 2013. Series Editors: A. Ilchman, T. Reis.

[15] A. De Luca and A. Isidori. Feedback linearization of invertible systems. In *2nd Duisburger Kolloquium Automation und Robotik*, 1987.

[16] R. März and R. Riaza. Linear differential–algebraic equations with properly stated leading term: A-critical points. *Mathematical and Computer Modelling of Dynamical Systems*, 13(3):291–314, 2007.

[17] R. März and R. Riaza. Linear differential-algebraic equations with properly stated leading term: B-critical points. *Dynamical Systems*, 23(4):505–522, 2008.

[18] J. Pryce. Solving high-index DAEs by Taylor series. *Numer. Algorithms*, 19:195–211, 1998.