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Amplitude recursions with an extra marked point

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Abstract

The recursive calculation of Selberg integrals by Aomoto and Terasoma using the Knizhnik–Zamolodchikov equation and the Drinfeld associator makes use of an auxiliary point and facilitates the recursive evaluation of string amplitudes at genus zero: open-string N-point amplitudes can be obtained from those at $N-1$ points.

We establish a similar formalism at genus one, which allows the recursive calculation of genus-one Selberg integrals using an extra marked point in a differential equation of Knizhnik–Zamolodchikov–Bernard type. Hereby genus-one Selberg integrals are related to genus-zero Selberg integrals. Accordingly, N-point open-string amplitudes at genus one can be obtained from $(N+2)$-point open-string amplitudes at tree level. The construction is related to and in accordance with various recent results in intersection theory and string theory.

Contents

1 Introduction

When calculating scattering amplitudes in quantum field theory and string theory, recycling and recursion are the most useful and powerful concepts to simplify and streamline calculations. Usually, the appearance of a new recursion algorithm [1–5] was preceded by establishing a new representation for a particular class of scattering amplitudes: this has for example been new variables or the idea of treating the color and kinematic part of an amplitude separately [6].

In this article, we would like to turn this reasoning around and discuss the question, whether there is a formalism inevitably leading to a recursion for scattering amplitudes?

The first question to be answered in this context is a geometrical one: we would like to find a parameter space for scattering amplitudes, which is powerful enough to represent all analytic properties for scattering amplitudes geometrically: soft and collinear limits including the color information, loop information, cutting. At the same time it would be desirable to not use Feynman graphs as model, as those lead to unphysical poles in intermediate steps of the amplitude calculation.

A further hint towards which geometry could be general enough to formalize amplitude recursions comes from the interplay between color an kinematics in QCD: in the purely kinematical part of amplitude calculations, we usually do not make explicit reference to color, albeit we carefully distinguish between planar and non-planar Feynman graphs. In other words, the color part implies boundary conditions for the calculation in terms of the ordering of external legs. In addition, the double-line formalism is a graphical way of keeping track how to multiply color matrices, which in turn allows to define what is called planar and non-planar. Accordingly, each Feynmn propagator dressed with double-lines effectively comes with a notion of left and right. When aiming at a formalized way to establish amplitude recursions, we should use a geometrical model, which allows to implement this information: the easiest way to do so, is to use bounded two-dimensional Riemann surfaces of fixed genus. While we are going to explore the formalism without reference to any physical theory, open-string amplitudes are naturally realized on these Riemann surfaces. As an example, we are going to construct an amplitude recursion for open strings at genus one to exemplify the expected features of the recursive formalism.

Having identified a suitable geometrical setup, one is led to think about the differential and integral structures on these Riemann surfaces. Solutions to scattering amplitudes can usually be represented in terms of iterated integrals on Riemann surfaces. Rather than recognizing certain Feynman integrals and string amplitudes as particular examples of a general class of iterated integrals on a particular surface, one could ask the question: which classes of iterated integrals do exist on a particular Riemann surface canonically? This question has been thoroughly investigated in mathematics and physics during the last decade: the most prominent examples are iterated integrals of abelian differentials on the particular Riemann surface in question. For a Riemann surface of genus zero those are various flavors of ordinary multiple polylogarithms [7,8], while a genus-one Riemann surface leads to elliptic polylogarithms [9, 10].

For Riemann surfaces of genus zero, a further particular class of canonical integrals sticks out: Selberg integrals [11] have been shown to generate all (harmonic) multiple polylogarithms, when properly expanded. Based on the Knizhnik–Zamolodchikov (KZ) equation and the Drinfeld associator [12–14], Aomoto and Terasoma have formalized the evaluation of Selberg integrals [15, 16] by considering limiting geometries parametrized by an auxiliary insertion point. Beautifully, their formalism addresses the questions of regularization of divergent integrals in a mathematically very simple and rigorous way. In other words: for canonical iterated integrals on a genus-zero surface a recursive algorithm delivering their solutions is known. Even more, since the algorithm is based on matrix representations of a certain free Lie algebra, the whole calculation is purely algebraically.

Correspondingly, all scattering amplitudes whose integral representations can be accommodated within the class of genus-zero Selberg integrals, can be calculated recursively. One recent application of this technique is the evaluation of all genus-zero open-string amplitudes in ref. [17].

In this article, we would like to explore the appropriate generalization of Aomoto's and Terasoma's formalism to genus one. We will consider bounded Riemann surfaces of genus one and identify the canonical generalization of Selberg integrals thereon. Simultaneously, we are going to consider the genus-one analogue of the Drinfeld associator and again establish a recursion facilitated by an extra insertion point. As an example, we are going to apply the formalism to open-string amplitudes at genus one.

Various recent results for scattering amplitudes are linked to the formalism established and explored in this article: the CHY formalism [18] is connected to Selberg integrals and twisted cohomology delivers basis choices for Selberg integrals at genus zero and genus one [19]. In particular, Mafra and Schlotterer established a formalism for the evaluation of open-string amplitudes at genus one [20,21], which is closely related to the construction in this article. We will comment on the connection to our genus-one formalism in subsection 3.5 and in section 5.

In section 2 we are going to review the recursive evaluation of Selberg integrals at genus zero. We will apply the technique to genus-zero open-string amplitudes in a way equivalent to the approach in ref. [17]. We are going to develop the genus-one formalism in section 3 and discuss the relation between genus-one objects and those at genus zero in section 4. In section 5 we conclude and point out several open questions.

2 Genus zero (tree-level)

In this section we are going to review the recursive construction of genus-zero Selberg integrals of Aomoto and Terasoma and relate it to the formalism for calculating open-string tree-level α'-corrections put forward in [17]. By doing so, we will reformulate the construction in different conventions, which are chosen to allow for a seamless generalization to genus one in section 3.

While reviewing, we are going to link and discuss various mathematical concepts and constructions having appeared recently, such as for example ref. [19]. Accompanying the review in this section, there is the article [22], in which the exact form of the matrices e_0 and e_1 appearing in [17] is derived starting from the braid matrices in refs. [15,16,19].

In this article, the Mandelstam variables

$$s_{i_1\dots i_r} = \alpha'(k_{i_1} + \dots + k_{i_r})^2 \tag{2.1}$$

for external momenta k_{i_p} are usually treated as formal parameters in the integrals to be considered. Only when applying our formalism to actual scattering amplitudes, we will impose momentum conservation for N massless external states:

$$\sum_{1\leq i<j\leq N-1} s_{ij} = 0\,, \qquad \sum_{\substack{i=1\\ i\neq j}}^{N} s_{ij} = 0 \quad \forall 1 \leq j \leq N\,. \tag{2.2}$$

2.1 Singularities, iterated integrals and multiple zeta values

The natural environment for the calculation of open-string amplitudes is the bounded disk: a bounded Riemann surface of genus zero. In general, a Green's function on a Riemann surface, which is going to serve as string propagator, is expected to diverge at zero separation of the insertion points. Simultaneously, the derivative of the propagator should have a simple pole. Both properties are obeyed by $\log x_{ij} = \log(x_i - x_j)$, where x_i denote the positions of marked points on the Riemann surface. In particular one finds

$$\frac{\partial}{\partial x_i}\log x_{ij} = \frac{1}{x_{ij}}\,, \tag{2.3}$$

which is very close to the Abelian differential of the second kind on the Riemann sphere:

$$\frac{dx_i}{x_i - a_j}\,. \tag{2.4}$$

Consequently, we will be dealing with iterated integrals over those differential forms,

$$G(a_1, a_2, \ldots, a_r; x) = \int_0^x dt \, \frac{1}{t - a_1} G(a_2, \ldots, a_r; t), \quad G(;x) = 1\,, \tag{2.5}$$

which are called Goncharov polylogarithms [7,8].

For the considerations below, we are going to confine the location of poles to $a_i \in \{0,1\}$: this will lead to the class of integrals sufficient to express the results of tree-level open-string integrals [23] as well as various results in numerous different quantum field theories.

Denoting the set of all words generated by the letters e_0 and e_1 by $\{e_0, e_1\}^\times$, multiple polylogarithms G_w are multi-valued functions on $\mathbb{C} \setminus \{0,1\}$ indexed by words of the form

$$w = e_0^{n_r-1} e_1 \ldots e_0^{n_1-1} e_1\,, \tag{2.6}$$

where $n_i \geq 1$:

$$G_w(x) = G(\underbrace{0,\ldots,0}_{n_r-1}, 1, \ldots, \underbrace{0,\ldots,0}_{n_1-1}, 1; x)\,. \tag{2.7}$$

The above definition differs by a sign from the sum representation of the multiple polylogarithm $\mathrm{Li}_{n_1,\ldots,n_r}(x)$ in one variable for $|x| < 1$

$$G_w(x) = (-1)^r \sum_{1 \leq k_1 < \cdots < k_r} \frac{x^{k_r}}{k_1^{n_1} \ldots k_r^{n_r}} = (-1)^r \, \mathrm{Li}_{n_1,\ldots,n_r}(x) \tag{2.8}$$

which indeed justifies the name multiple polylogarithm.

Words ending in e_0 have been excluded in definition (2.7), since the form dt/t in the original definition (2.5) of the iterated integral $G_w(x)$ would diverge at the lower integration boundary. However, definition (2.7) of multiple polylogarithms $G_w(x)$ may be extended to any word $w \in \{e_0, e_1\}^\times$ using the shuffle algebra for multiple polylogarithms

$$G_{w'}(x) G_{w''}(x) = G_{w' \sqcup\!\sqcup\, w''}(x)\,, \tag{2.9}$$

where $w', w'' \in \{e_0, e_1\}^\times$, and the definition

$$G_{e_0^n}(x) = \frac{\log^n(x)}{n!}\,. \tag{2.10}$$

This implies that for words ending in e_0, multiple polylogarithms exhibit a logarithmic divergence in the limit $x \to 0$, while they vanish in this limit for words ending in e_1

$$\lim_{x \to 0} G_{we_1}(x) = 0. \tag{2.11}$$

Moreover, a multiple polylogarithm indexed by a word $e_i w \in \{e_0, e_1\}^\times$ satisfies the differential equations

$$dG_{e_i w}(x) = \omega_i G_w(x)\,, \qquad \omega_0 = \frac{dx}{x}\,, \qquad \omega_1 = \frac{dx}{x-1}\,, \tag{2.12}$$

which follows from definitions (2.5) and (2.7).

Multiple zeta values (MZVs) are defined and labeled by words of the form

$$w = e_0^{n_r-1} e_1 \ldots e_0^{n_1-1} e_1, \quad n_r > 1\,, \tag{2.13}$$

which lead to convergent values of G_w at $x = 1$

$$\zeta_\omega = (-1)^r G_w(1) = \mathrm{Li}_{n_1,\ldots,n_r}(1)\,. \tag{2.14}$$

In addition to the divergence of dt/t at the lower integration boundary for words ending in e_0 discussed above, the integral $G_w(1)$ will also diverge at the upper integration boundary for words beginning with e_1 due to the pole in the differential $dt/(t-1)$ at $t = 1$. This is the reason for requiring $n_r > 1$ in the above definition.

In analogy to multiple polylogarithms, definition (2.14) can be extended to any word ending in e_0 using the above shuffle regularization (2.9) and (2.10) of $G_w(x)$ as well as a similar regularization for words beginning with e_1. This regularization[1] of MZVs turns out to be a genus-zero version of the tangential base point regularization [24, 25]: along the positive direction at $x = 0$ and in negative direction at $x = 1$, respectively [26].

This regularization of the MZVs effectively amounts to the definitions

$$\begin{aligned}\zeta_{e_0} &= G(0;1) = 0\,,\\ \zeta_{e_1} &= -G(1;1) = 0\,,\end{aligned} \tag{2.15}$$

the ordinary definition for the absolutely convergent sums for $n_r > 1$ and $n_i > 0$ for $i \in \{1, \ldots, r-1\}$

$$\zeta_{e_0^{n_r-1}e_1\ldots e_0^{n_1-1}e_1} = \sum_{1\leq k_1<\cdots<k_r} \frac{1}{k_1^{n_1}\ldots k_r^{n_r}} = (-1)^r G_{e_0^{n_r-1}e_1\ldots e_0^{n_1-1}e_1}\,, \tag{2.16}$$

and the use of the shuffle algebra to reduce the remaining cases to the former definitions

$$\zeta_{w'}\zeta_{w''} = \zeta_{w' ⧢ w''}\,. \tag{2.17}$$

2.2 Selberg Integrals

Even though the α'-expansion of open-string tree-level amplitudes can be finally phrased in terms of rational factors, polynomials of Mandelstam variables (2.1) and MZVs, in intermediate steps of the calculation, the notion of iterated integral in eq. (2.5) is not general enough. The class of integrals accommodating the relevant features is called *Selberg integrals* [11, 27, 16], and will be constructed in the following. Let us consider L points on the unit interval with the ordering

$$0 = x_1 < x_L < x_{L-1} < \cdots < x_3 < x_2 = 1 \tag{2.18}$$

and define[2] the empty Selberg integral or *Selberg seed*

$$\mathrm{S} = \mathrm{S}[](x_1, \ldots, x_L) = \prod_{0\leq x_i<x_j\leq 1} \exp\left(s_{ij}\log x_{ji}\right) = \prod_{0\leq x_i<x_j\leq 1} |x_{ij}|^{s_{ij}}\,, \tag{2.19}$$

[1] While the divergence for words ending in e_0 has been treated by the corresponding extension of the definition of multiple polylogarithms, using the shuffle algebra to extract the divergent contributions from $G_{e_1}(1)$ in $G_w(1)$, any multiple polylogarithm $G_w(x)$ can be written on the canonical branch for $x \in (0,1)$ such that it takes the form $G_w(x) = \sum_{k=0}^{|w|} c_k(x)\log(1-x)^k$, where $c_k(x)$ are holomorphic functions of x in a neighborhood of $x = 1$. Thus, for any word $w \in \{e_0, e_1\}^\times$, the multiple zeta value ζ_w can be defined by the regularized value of $G_w(x)$ at 1, which, in turn, is the coefficient $c_0(x)$: $\zeta_w = \mathrm{Reg}_{x=1}\left(G_w(x)\right) = c_0(1)$.

[2] We use the notation $\prod_{x_a\leq x_i<x_j\leq x_b} = \prod_{i,j\in\{1,2,\ldots,L\}:\, x_a\leq x_i<x_j\leq x_b}$.

with generic formal parameters s_{ij}. The empty integral S[] shall be integrated over various functions $1/x_{ij}$, which will be denoted by

$$\mathrm{S}[i_{k+1},\dots,i_L](x_1,\dots,x_k) = \int_0^{x_k} \frac{dx_{k+1}}{x_{k+1,i_{k+1}}}\, \mathrm{S}[i_{k+2},\dots,i_L](x_1,\dots,x_{k+1})\,, \tag{2.20}$$

where

$$1 \leq i_p < p \qquad \forall p \in \{k+1,\dots,L\}\,. \tag{2.21}$$

Note that the condition (2.21) is necessary in order to define honest iterated integrals: the integration kernel $1/x_{k+1,i_{k+1}}$ in eq. (2.20) can not depend on variables which have already been integrated out[3] In accordance with ref. [27], we call this property *admissibility* and an integral with an integrand proportional to $\prod_k 1/x_{k,i_k}$ satisfying eq. (2.21) *admissible*. By definition, the number of iterated integrations in a Selberg integral equals the number of entries in square brackets. As argued in subsection 2.5 and subsection 2.7, Selberg integrals of length $L-3$

$$\begin{aligned}\mathrm{S}[i_4,\dots,i_L](x_1,x_2,x_3) &= \int_0^{x_3} \frac{dx_4}{x_{4,i_4}}\, \mathrm{S}[i_5,\dots,i_L](x_1,\dots,x_4)\\ &= \int_{\mathcal{C}(x_3)} \prod_{i=4}^{L} dx_i\ \mathrm{S} \prod_{k=4}^{L} \frac{1}{x_{ki_k}}\,,\end{aligned} \tag{2.22}$$

where $\mathcal{C}(x_3)$ is the region of integration denoted by

$$\mathcal{C}(x_i) = \{0 = x_1 < x_L < x_{L-1} < \dots < x_i\} \tag{2.23}$$

for $0 = x_1 \leq x_i \leq x_2 = 1$, include all integrals appearing in the calculation of L-point open-string tree-level scattering amplitudes.

Besides of including all integrals for the calculation of tree-level string corrections, the main advantage of the integrals (2.22) is that their solutions can be obtained from a recursive procedure involving matrix operations [27, 16]. This construction is going to be described below.

2.3 Auxiliary marked point and a system of differential equations

The main idea of the recursive construction of solutions to Selberg integrals by Aomoto and Terasoma is the use of an auxiliary fixed insertion point x_3. The notion of *auxiliary* will become clear, when discussing certain limits of x_3 and their relation to string amplitudes below.

Prior to that, let us investigate the structure of the punctures appearing in the Selberg integrals. The Selberg integrals $\mathrm{S}[i_{k+1},\dots,i_L](x_1,\dots,x_k)$ are defined on the configuration space of the $(L+1)$-punctured Riemann sphere with $k+1$ fixed coordinates

$$\mathcal{F}_{L+1,k+1} = \{(x_{k+1},x_{k+2},\dots,x_L) \in (\mathbb{C}P^1)^{L-k} | \forall i \neq j : x_i \neq x_1,x_2,\dots,x_k,x_{L+1},x_j\} \tag{2.24}$$

in the following sense [27, 19]: the differential forms

$$\bigwedge_{p=k+1}^{L} \frac{dx_p}{x_{p,i_p}}\,, \tag{2.25}$$

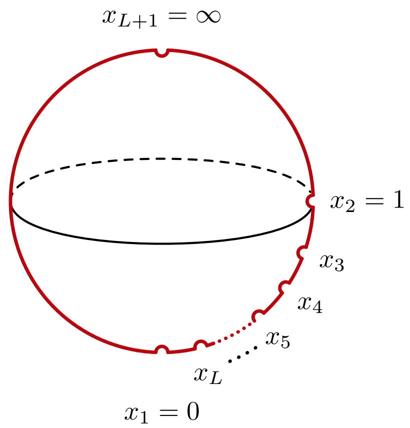
where $1 \leq i_p < p$, appearing in the definition (2.19) have integration variables $x_{k+1},x_{k+2},\dots,x_L$ and are defined for $x_i \neq x_j$ on the complex plane punctured by the k fixed coordinates

[3] In particular, the z-removal procedure from ref. [23] is not required.

$x_1, x_2, \ldots, x_k$. If this k-punctured complex plane is depicted on the Riemann sphere, an additional puncture at $x_{L+1} = \infty$ is introduced leading to the $k+1$ fixed coordinates in the configuration space $\mathcal{F}_{L+1,k+1}$. Then, the twisted cohomologies of the forms in $\mathrm{S}[i_{k+1}, \ldots, i_L](x_1, \ldots, x_k)$ span the twisted de Rham cohomology of $\mathcal{F}_{L+1,k+1}$. Three of the $k+1$ fixed coordinates are canonically chosen to be

$$(x_1, x_2, x_{L+1}) = (0, 1, \infty)\,, \tag{2.26}$$

such that for a configuration of the form (2.18), which is used in the integration domain of the Selberg integrals, the punctures in $\mathcal{F}_{L+1,k+1}$ can be depicted on a circle on the Riemann sphere as follows:



(2.27)

Accordingly, the genus-one Selberg integrals with $k = 3$ defined in eq. (2.22) is a class of integrals defined on the configuration space $\mathcal{F}_{L+1,4}$ with four fixed coordinates x_1, x_2, x_3, x_{L+1}. Since three of them are canonically fixed, the remaining fixed puncture parametrized by the coordinate x_3 will be the auxiliary point used in the amplitude recursion: if it is merged with the point $x_2 = 1$, one fixed puncture is removed such that the Selberg integrals on $\mathcal{F}_{L+1,4}$ degenerate to integrals $\mathrm{S}[i_4, \ldots, i_L](x_1 = 0, x_2 = 1, x_3 = x_2)$ defined on the configuration space $\mathcal{F}_{L,3}$. This is the moduli space of L-punctured Riemann spheres known from string calculations with the three coordinates being fixed by the $\mathrm{SL}(2,\mathbb{C})$ symmetry

$$\mathcal{M}_{0,L} = \{(x_4, \ldots, x_L) \in (\mathbb{C}P^1)^{L-3} | \forall i \neq j : x_i \neq x_1, x_2, x_j, x_{L+1}\} = \mathcal{F}_{L,3}\,, \tag{2.28}$$

on which L-point tree-level amplitudes are defined. Indeed, as shown below, they will be recovered in this limit of the Selberg integrals. The merging of $x_3 \to x_1 = 0$ is slightly more involved and will lead to the $(L-1)$-point integrals in a certain soft limit. Thus, the auxiliary puncture x_3 interpolates between the L- and $(L-1)$-point integrals. In the construction of Aomoto and Terasoma, these two boundary values are related using the differential equation satisfied by the integrals $\mathrm{S}[i_4, \ldots, i_L](x_1 = 0, x_2 = 1, x_3)$ with respect to x_3. In the rest of this subsection, we will review the investigation of these differential equations.

Thus we are considering the following Selberg integrals

$$\mathrm{S}[i_4, \ldots, i_L](x_1 = 0, x_2 = 1, x_3) = \int_0^{x_3} \frac{dx_4}{x_{4,i_4}}\, \mathrm{S}[i_5, \ldots, i_L](x_1 = 0, x_2 = 1, x_3, x_4)\,, \tag{2.29}$$

where we assume $x_3 \in (0,1)$. Attached to the point x_3 there is an auxiliary external momentum k_3. Correspondingly, there will be Mandelstam variables s_{3i}, $i \in \{1, 2, 4, 5, \ldots L\}$ and the Mandelstam variables $s_{i,L+1}$, $i \in \{1, 2, 3, \ldots L\}$ may be determined by momentum conservation. However, for the moment we are not imposing any conditions like the momentum conservation eq. (2.2) on any set or subset of the external states. Rather, the variables s_{ij} shall be considered as independent parameters whose interpretation as Mandelstam variables in a scattering amplitude context will become clear when considering the limits $x_3 \to 0$ and $x_3 \to 1$ below.

As a next step, let us explore differential equations with respect to the auxiliary point x_3 acting on the Selberg integrals (2.29):

$$\frac{d}{dx_3}\,\mathrm{S}[i_4,i_5,\ldots,i_L](0,1,x_3) = \frac{d}{dx_3}\int_{\mathcal{C}(x_3)}\prod_{i=4}^{L} dx_i\ \mathrm{S}\prod_{k=4}^{L}\frac{1}{x_{ki_k}}\,. \tag{2.30}$$

Noting that eq. (2.19) implies that the Selberg seed S converges to zero in the limit $x_i \to x_j$ for $i \neq j$

$$\mathrm{S}\,|_{x_i=x_j} = 0\,, \tag{2.31}$$

it follows that the derivative in eq. (2.30) only acts non-trivially on the integrand and not on the integration domain. The identity

$$\frac{\partial}{\partial x_i}\frac{1}{x_{ij}} = -\frac{\partial}{\partial x_j}\frac{1}{x_{ij}} \tag{2.32}$$

and integration by parts may be used to let partial derivatives act on the Selberg seed only:

$$\frac{d}{dx_3}\,\mathrm{S}[i_4,i_5,\ldots,i_L](0,1,x_3) = \int_{\mathcal{C}(x_3)}\prod_{i=4}^{L} dx_i\left(\sum_{j\in U_3}\frac{\partial}{\partial x_j}\mathrm{S}\right)\prod_{k=4}^{L}\frac{1}{x_{ki_k}}\,. \tag{2.33}$$

The set U_3 in the previous equation is defined as

$$U_3 = \Bigg\{ j\in\{3,4,\ldots,L\}\ \Bigg|\ j=3 \text{ or there exist labels } 3=j_1,j_2,\ldots,j_m=j \text{ such that } \prod_{i=1}^{m-1}\frac{1}{x_{j_{i+1},j_i}} \text{ is a factor of } \prod_{k=4}^{L}\frac{1}{x_{ki_k}}\Bigg\} \tag{2.34}$$

and is tailored to the labels $i_4, i_5, \ldots, i_L$ of the Selberg integral in eq. (2.33). Partial derivatives of the Selberg seed yield factors of s_{jl}/x_{jl}

$$\frac{\partial}{\partial x_j}\mathrm{S} = \sum_{l\neq j}\frac{s_{jl}}{x_{jl}}\,\mathrm{S}\,, \tag{2.35}$$

such that

$$\frac{d}{dx_3}\,\mathrm{S}[i_4,i_5,\ldots,i_L](0,1,x_3) = \int_{\mathcal{C}(x_3)}\prod_{i=4}^{L} dx_i\ \mathrm{S}\sum_{j\in U_3}\sum_{l\notin U_3}\frac{s_{jl}}{x_{jl}}\prod_{k=4}^{L}\frac{1}{x_{ki_k}}\,. \tag{2.36}$$

The structure of $\mathrm{S}[i_4,i_5,\ldots,i_L](0,1,x_3)$ as an iterated integral, in particular the condition $1 \leq i_k < k$, implies that upon consecutive applications of partial fractioning

$$\frac{1}{x_{k,l}}\frac{1}{x_{k,m}} = \left(\frac{1}{x_{k,l}} - \frac{1}{x_{k,m}}\right)\frac{1}{x_{l,m}}\,, \tag{2.37}$$

where $k > l > m$, we will again find (admissible) Selberg integrals $\mathrm{S}[i_4,i_5,\ldots,i_L](0,1,x_3)$ with $1 \leq i_k < k$ on the right-hand side of eq. (2.36), however, with different labels i_k.

Furthermore, all integrals on the right-hand side of eq. (2.30) will always contain a prefactor

of the form
$$\frac{s_{ij}}{x_{31}} = \frac{s_{ij}}{x_3} \quad \text{or} \quad \frac{s_{ij}}{x_{32}} = \frac{s_{ij}}{x_3 - 1}\,, \tag{2.38}$$
since the indices in x_{31} and x_{32} can no longer be reduced by partial fractioning. Accordingly, if we consider the vector of all the integrals
$$\mathbf{S}(x_3) = \Big(\mathrm{S}[i_4, i_5, \dots, i_L](0, 1, x_3)\Big)_{1\leq i_k<k}\,, \tag{2.39}$$
the result of an exhaustive application of the partial fractioning identity to the differential equation (2.36) can be phrased in terms of a vector equation
$$\frac{d}{dx_3}\,\mathbf{S}(x_3) = \Big(\frac{e_0}{x_3} + \frac{e_1}{x_3 - 1}\Big)\,\mathbf{S}(x_3)\,, \tag{2.40}$$
where the entries of the length $(L-1)!/2 \times (L-1)!/2$ matrices e_0 and e_1 either vanish or are homogeneous polynomials of degree one in the parameters s_{ij} for $i, j \in \{1, 2, \dots, L\}$. In an amplitude context later, this implies (cf. eq. (2.1)) that e_0 and e_1 are proportional to α'.

The fact that the derivative of $\mathrm{S}[i_4, i_5, \dots, i_L](0, 1, x_3)$ is expressible as linear combination of iterated integrals $\mathrm{S}[i_4, i_5, \dots, i_L](0, 1, x_3)$ originates in a property of the differential forms appearing in the integrand in eq. (2.30): they contain a basis of the twisted cohomology of $\mathcal{F}_{L+1,4}$, the so-called fibration basis [19]. Note that for each $4 \leq k \leq L$, one can get rid of one particular index $1 \leq i'_k < k$ by partial fractioning and integration by parts. Thus, one can identify a suitable basis of the iterated integrals $\mathrm{S}[i_4, i_5, \dots, i_L](0, 1, x_3)$ as
$$\mathcal{B}_{i'_4, i'_5, \dots, i'_L} = \{\mathrm{S}[i_4, i_5, \dots, i_L](0, 1, x_3) | 1 \leq i_k < k, i_k \neq i'_k\} \tag{2.41}$$
and reduce the vector in eq. (2.39) to the vector
$$\mathbf{S}(x_3)|_{\mathcal{B}_{i'_4, i'_5, \dots, i'_L}} = \Big(\mathrm{S}[i_4, i_5, \dots, i_L](0, 1, x_3)\Big)_{1\leq i_k<k, i_k\neq i'_k}\,. \tag{2.42}$$
In this case, the differential equation (2.40) for the reduced vector $\mathbf{S}(x_3)|_{\mathcal{B}_{i'_4, i'_5, \dots, i'_L}}$ is also of the form
$$\frac{d}{dx_3}\,\mathbf{S}(x_3)|_{\mathcal{B}_{i'_4, i'_5, \dots, i'_L}} = \Big(\frac{e_0}{x_3} + \frac{e_1}{x_3 - 1}\Big)\,\mathbf{S}(x_3)|_{\mathcal{B}_{i'_4, i'_5, \dots, i'_L}}\,, \tag{2.43}$$
where the entries of the matrices e_0 and e_1 are again either vanishing or homogeneous polynomials of degree one in the Mandelstam variables s_{ij} for $i, j \in \{1, 2, \dots, L\}$. Different than before, the dimension of the matrices is now $(L-2)! \times (L-2)!$. These matrices turn out to be braid matrices, that is, representations of the braid group of $L+1$ distinguishable strands with three strands held fixed. It is well known, how to obtain these matrices recursively [27, 16, 19].

Of course, the choice of the basis is a priori arbitrary. However, depending on the intended use, certain choices turn out to be much more beneficial than others in practice. For example, the recursive definition of the matrices in e_0 and e_1 in ref. [19] are constructed for the choice $\mathcal{B}_{1,1,\dots,1}$, i.e. $2 \leq i_k < k$. On the other hand, the limits considered in subsection 2.5 will conveniently be formulated in the basis $\mathcal{B}_{2,2,\dots,2}$.

Equation (2.40) is a first example of the type of differential equations we are going to deal with in the following: it is an equation of Knizhnik–Zamolodchikov (KZ) type [12]. The solution theory for this differential equation is well-known from refs. [13, 14]. In order to proceed, we will provide a short introduction to the KZ equation and its formal solutions in the next subsection.

Prior to that, we consider the simplest example $L = 4$ and show the above calculational steps explicitly for the basis

$$\mathcal{B}_2 = \{\mathrm{S}[1](0,1,x_3), \mathrm{S}[3](0,1,x_3)\}\,, \tag{2.44}$$

where

$$\mathrm{S}[i_4](0,1,x_3) = \int_0^{x_3} dx_4\ \mathrm{S}\,\frac{1}{x_{4,i_4}}\,, \qquad \mathrm{S} = x_{14}^{s_{14}} x_{13}^{s_{13}} x_{43}^{s_{43}} x_{12}^{s_{12}} x_{42}^{s_{42}} x_{32}^{s_{32}}\,. \tag{2.45}$$

These integrals are defined (by twisted forms) on (the twisted de Rham cohomology of) $\mathcal{F}_{5,4} = \{x_4 \in \mathbb{C}P^1 | x_4 \neq x_1, x_2, x_3, x_5\}$ with punctures

$$0 = x_1 < x_4 < x_3 < x_2 = 1 < x_5 = \infty\,, \tag{2.46}$$

where x_1, x_2, x_3, x_5 are fixed and x_4 is varying, i.e. x_4 is the integration variable in the integrals. First, note that $\mathcal{B}_2$ is indeed a basis, since the remaining Selberg integral $\mathrm{S}[2](0,1,x_3)$ with $i_4 = 2$ is a linear combination of the elements in $\mathcal{B}_2$ due to the integration by parts identity

$$s_{41}\,\mathrm{S}[1](0,1,x_3) + s_{42}\,\mathrm{S}[2](0,1,x_3) + s_{43}\,\mathrm{S}[3](0,1,x_3) = 0\,. \tag{2.47}$$

Now, let us calculate the derivatives of the entries of

$$\mathbf{S}(x_3)|_{\mathcal{B}_2} = \begin{pmatrix} \mathrm{S}[1](0,1,x_3) \\ \mathrm{S}[3](0,1,x_3) \end{pmatrix} \tag{2.48}$$

in order to recover the KZ equation (2.43) using our general analysis from above. Starting with $\mathrm{S}[1](0,1,x_3)$, we find that the set U_3 defined in eq. (2.34) is for $\mathrm{S}[1](0,1,x_3)$ given by

$$U_3(\mathrm{S}[1](0,1,x_3)) = \{3\}\,, \tag{2.49}$$

such that according to eq. (2.36)

$$\begin{aligned} \frac{d}{dx_3}\mathrm{S}[1](0,1,x_3) &= \int_0^{x_3} dx_4\ \mathrm{S}\left(\frac{s_{31}}{x_{31}} + \frac{s_{34}}{x_{34}} + \frac{s_{32}}{x_{32}}\right)\frac{1}{x_{41}} \\ &= \frac{s_{31}}{x_3}\mathrm{S}[1](0,1,x_3) + \frac{s_{32}}{x_3-1}\mathrm{S}[1](0,1,x_3) + \frac{s_{34}}{x_3}\left(\mathrm{S}[1](0,1,x_3) - \mathrm{S}[3](0,1,x_3)\right) \end{aligned} \tag{2.50}$$

where we have used the partial fractioning identity (2.37) for the third equality. Similarly, for $\mathrm{S}[3](0,1,x_3)$ we find

$$U_3(\mathrm{S}[3](0,1,x_3)) = \{3,4\}\,, \tag{2.51}$$

such that

$$\begin{aligned} \frac{d}{dx_3}\mathrm{S}[3](0,1,x_3) &= \int_0^{x_3} dx_4\ \mathrm{S}\left(\frac{s_{31}}{x_{31}} + \frac{s_{32}}{x_{32}} + \frac{s_{41}}{x_{41}} + \frac{s_{42}}{x_{42}}\right)\frac{1}{x_{43}} \\ &= \frac{s_{31}}{x_3}\mathrm{S}[3](0,1,x_3) + \frac{s_{32}}{x_3-1}\mathrm{S}[3](0,1,x_3) + \frac{s_{41}}{x_3}\left(\mathrm{S}[3](0,1,x_3) - \mathrm{S}[1](0,1,x_3)\right) \\ &\quad + \frac{1}{x_3-1}\left((s_{42}+s_{43})\,\mathrm{S}[3](0,1,x_3) + s_{41}\,\mathrm{S}[1](0,1,x_3)\right)\,, \end{aligned} \tag{2.52}$$

where we have again used partial fractioning (2.37) and integration by parts (2.47) for the fourth equality. From the above calculations, we find that the Selberg vector $\mathbf{S}(x_3)|_{\mathcal{B}_2}$ satisfies the differential equation

$$\frac{d}{dx_3}\mathbf{S}(x_3)|_{\mathcal{B}_2} = \left(\frac{1}{x_3}\begin{pmatrix} s_{31}+s_{34} & -s_{34} \\ -s_{41} & s_{31}+s_{41}\end{pmatrix} + \frac{1}{x_3-1}\begin{pmatrix} s_{32} & 0 \\ s_{41} & s_{432}\end{pmatrix}\right)\mathbf{S}(x_3)|_{\mathcal{B}_2}\,, \tag{2.53}$$

which is indeed of the form of the KZ equation (2.43) with the matrices

$$e_0 = \begin{pmatrix} s_{31}+s_{34} & -s_{34} \\ -s_{41} & s_{31}+s_{41}\end{pmatrix}\,, \qquad e_1 = \begin{pmatrix} s_{32} & 0 \\ s_{41} & s_{432}\end{pmatrix} \tag{2.54}$$

given by the braid matrices used in ref. [19].

2.4 Generating function for polylogarithms and the Drinfeld associator

Let us introduce the general solution strategy for a KZ equation such as (2.40) by considering a representation of some Lie algebra generators e_0 and e_1, as well as a function $\mathrm{F}(x)$ with $x \in (0,1)$ and values in the vector space the representations e_0 and e_1 act upon and which satisfies the KZ equation

$$\frac{d}{dx}\,\mathrm{F}(x) = \left(\frac{e_0}{x} + \frac{e_1}{x-1}\right)\mathrm{F}(x)\,. \tag{2.55}$$

Given this situation, one is often interested in calculating the limit of $\mathrm{F}(x)$ for $x \to 1$ while knowing the boundary value as $x \to 0$, which is what shall be understood her by solving the KZ equation (in spite of the fact that we still call $\mathrm{F}(x)$ a solution of eq. (2.55)). As will be reviewed in this section, there is an operator, the Drinfeld associator $\Phi(e_0, e_1)$ [13, 14], which parallel transports the (regularized) boundary value of $\mathrm{F}(x)$ at $x \to 0$ to its (regularized) value at $x \to 1$. It turns out that the Drinfeld associator is the generating series of the regularized MZVs, which was originally shown in ref. [28] and which is reviewed in this paragraph following the lines of ref. [26].

In order to construct the Drinfeld associator, we first investigate the following generating function of multiple polylogarithms

$$\mathrm{L}(x) = \sum_{w\in\{e_0,e_1\}^\times} w\, G_w(x)\,. \tag{2.56}$$

The differential equations (2.12) imply that the series $\mathrm{L}(x)$ satisfies the KZ equation

$$\frac{d}{dx}\,\mathrm{L}(x) = \left(\frac{e_0}{x} + \frac{e_1}{x-1}\right)\mathrm{L}(x)\,. \tag{2.57}$$

Furthermore, the boundary conditions (2.10) and (2.11) determine the asymptotic behavior as $x \to 0$

$$\mathrm{L}(x) \sim x^{e_0}\,. \tag{2.58}$$

By the symmetry $x \mapsto 1-x$ of the KZ equation, there is another solution L_1 of (2.57) with the asymptotic behavior

$$\mathrm{L}_1(x) \sim (1-x)^{e_1} \tag{2.59}$$

as $x \to 1$. Now, let $\mathrm{F}(x)$ be an arbitrary solution of the KZ equation (2.57). For this solution, regularized boundary values are defined via

$$C_0 = \lim_{x\to 0} x^{-e_0}\,\mathrm{F}(x)\,, \quad C_1 = \lim_{x\to 1}(1-x)^{-e_1}\,\mathrm{F}(x)\,. \tag{2.60}$$

Since for two functions $F_0(x)$ and $F_1(x)$ satisfying the KZ equation (2.55) the product $(F_1)^{-1} F_0$ is independent of x, and by the asymptotics (2.58), (2.59) of $L(x)$ and $L_1(x)$, respectively, the calculation

$$(L_1(x))^{-1} L(x) C_0 = \lim_{x\to 0} (L_1(x))^{-1} F(x) = \lim_{x\to 1} (L_1(x))^{-1} F(x) = C_1 \tag{2.61}$$

shows that the product

$$\Phi(e_0, e_1) = (L_1(x))^{-1} L(x) \tag{2.62}$$

maps the regularized boundary value C_0 to the regularized boundary value C_1

$$C_1 = \Phi(e_0, e_1)\, C_0 \,. \tag{2.63}$$

The operator $\Phi(e_0, e_1)$ is the Drinfeld associator which is defined in terms of the generating series of multiple polylogarithms $L(x)$ and the corresponding solution $L_1(x)$. In order to write it as a generating series of MZVs, its definition (2.62) can be evaluated in the limit $x \to 1$, since $\Phi(e_0, e_1)$ is independent of x: it is a product of a function satisfying the KZ equation and an inverse of such a function. This leads to the relation of the Drinfeld associator to the MZVs discovered in ref. [28],

$$\begin{aligned}
\Phi(e_0, e_1) &= \lim_{x\to 1} (1-x)^{-e_1} L(x) \\
&= \sum_{w \in \{e_0, e_1\}^\times} w \zeta_w \\
&= 1 - \zeta_2 [e_0, e_1] - \zeta_3 [e_0 + e_1, [e_0, e_1]] \\
&\quad + \zeta_4 ([e_1, [e_1, [e_1, e_0]]] + \tfrac{1}{4} [e_1, [e_0, [e_1, e_0]]] \\
&\quad - [e_0, [e_0, [e_0, e_1]]] + \tfrac{5}{4} [e_0, e_1]^2) + \ldots \quad ,
\end{aligned} \tag{2.64}$$

i.e. the Drinfeld associator is a generating series for the (regularized) MZVs defined in eqs. (2.15), (2.16) and (2.17). The limit $x \to 1$ is chosen to correspond to taking the tangential base point in negative direction at 1, such that the contributions from $(1-x)^{-e_1}$ lead to the discussed regularization of the divergent terms in $L(x)$ by canceling the positive integer powers of $\log(1-x)$ in the divergent multiple polylogarithms $G_w(x)$.

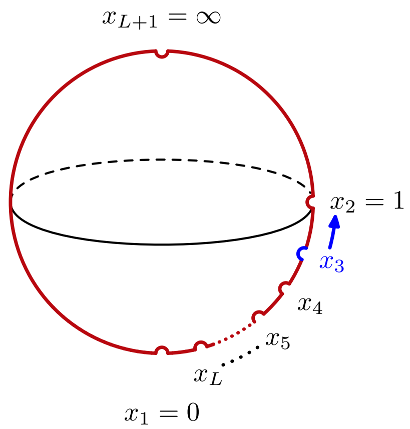
2.5 Regularized boundary values for the string tree-level KZ equation

Following the discussion in the previous subsection, let us investigate the vector-valued function $\mathbf{S}(x_3)$ defined in eq. (2.39) satisfying the KZ equation (2.40) and consider its limits when taking the auxiliary point x_3 to either zero or one[4]: in other words, we investigate the regularized limits (2.60) for $\mathbf{S}(x_3)$

$$\mathbf{C}_0 = \lim_{x_3\to 0} x^{-e_0}\, \mathbf{S}(x_3) \ , \quad \mathbf{C}_1 = \lim_{x_3 \to 1} (1-x_3)^{-e_1}\, \mathbf{S}(x_3) \,. \tag{2.65}$$

[4]The following paragraph is closely related to the original analysis of Selberg integrals in ref. [16], which serves as the prime reference for our investigation and led to the formulation of the amplitude recursion in ref. [17].

Boundary value $\mathbf{C}_1^{\mathbf{E}}$: Let us start by considering the limit $x_3 \to x_2 = 1$, which is depicted in the following figure:



(2.66)

The relevant integrals in the amplitude recursion in this limit turn out to be the Selberg integrals in $\mathcal{B}_{2,2,\ldots,2} \cap \mathcal{B}_{3,3,\ldots,3}$, i.e. integrals of the form

$$\mathrm{S}[i_4, i_5, \ldots, i_L](0,1,x_3) = \int_{\mathcal{C}(x_3)} \prod_{i=4}^{L} dx_i \; \mathrm{S} \prod_{k=4}^{L} \frac{1}{x_{k i_k}} \,, \qquad \mathrm{S} = \prod_{0=x_1 \leq x_j < x_l \leq x_2 = 1} x_{lj}^{s_{jl}} \,, \tag{2.67}$$

defined on the configuration space $\mathcal{F}_{L+1,4}$ with $1 \leq i_k < k$ and $i_k \neq 2,3$. For these integrals, the action of the prefactor $(1-x_3)^{-e_1}$ is particularly simple: on the one hand, the set U_3 in eq. (2.36) is simply $U_3 = \{3\}$. On the other hand, the only appearance of the insertion point x_2 in the integral $\mathrm{S}[i_4, i_5, \ldots, i_L](x_1 = 0, x_2 = 1, x_3)$ with $i_k \neq 2$ is in the Selberg seed. Therefore using partial fractioning to obtain the KZ form from eq. (2.36) does not introduce any factor of $1/x_{32}$ other than s_{23}/x_{32} obtained from differentiating the Selberg seed. Thus, for the basis $\mathcal{B}_{2,2,\ldots,2}$, the representation e_1 in the KZ equation (2.43) is of the form

$$e_1 = \begin{pmatrix} s_{23}\mathbb{I}_{(L-3)! \times (L-3)!} & 0_{(L-3) \times (L-3)!} \\ A_{(L-3)! \times (L-3)} & B_{(L-3) \times (L-3)} \end{pmatrix} , \tag{2.68}$$

where the upper left block proportional to the identity corresponds to the integrals in $\mathcal{B}_{2,2,\ldots,2} \cap \mathcal{B}_{3,3,\ldots,3}$, (cf. examples (2.54) or (2.100) below). For this subclass of integrals, the regularization factor $(1-x_3)^{-e_1}$ only contributes with the scalar $(1-x_3)^{-s_{23}} = x_{23}^{-s_{23}}$ and the corresponding entries of the regularized limit $\mathbf{C}_1$ can be calculated as

$$\begin{aligned} \lim_{x_3 \to x_2} x_{23}^{-s_{23}} \, \mathrm{S}[i_4, i_5, \ldots, i_L](0,1,x_3) &= \int_{\mathcal{C}(x_3 \to x_2)} \prod_{i=4}^{L} dx_i \prod_{0 \leq x_j < x_l < x_3} x_{lj}^{s_{jl}} \prod_{0 \leq x_n < x_3} x_{2n}^{s_{2n}+s_{3n}} \prod_{k=4}^{L} \frac{1}{x_{k i_k}} \\ &= \mathrm{S}[i_4, i_5, \ldots, i_L](0,1,x_3 = x_2)\big|^{\tilde{s}_{2n} = s_{2n}+s_{3n}}_{s_{23}=0} \,, \end{aligned} \tag{2.69}$$

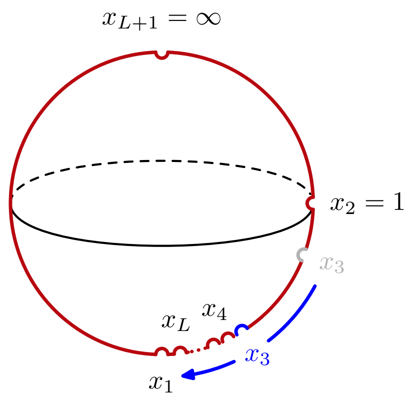
Thus the regularization $x_{23}^{-s_{23}}$ cancels the factor $x_{23}^{s_{23}}$ in the Selberg seed S, which would otherwise render the integral vanishing. Moreover, the punctures x_2 and x_3 have merged, such that the associated Mandelstam variables, and hence, the momenta of the external states, are added to yield effective Mandelstam variables $\tilde{s}_{2n} = s_{2n} + s_{3n}$ and $\tilde{s}_{mn} = s_{mn}$ for $m, n \neq 2, 3$.

The resulting differential form and integration domain in the integral (2.69) represent twisted cohomology and homology classes, respectively, of the moduli space $\mathcal{M}_{0,L} = \mathcal{F}_{L,3}$. Thus, in this limit, the forms in $\mathrm{S}(x_3)$ span the twisted cohomology class of $\mathcal{M}_{0,L}$ and can, in particular, be expressed as linear combinations of the Parke–Taylor forms of L-point string amplitudes, which are discussed in the next subsection. In terms of the disk picture 2.66, we are modifying the relative distances on the boundary by taking the limit $x_3 \to x_2 = 1$. Upon identification of the points x_2 and x_3 we find the transition

$$\mathcal{F}_{L+1,4} \to \mathcal{F}_{L,3} = \mathcal{M}_{0,L} \tag{2.70}$$

with the L insertion points $x_1, x_2, x_4, x_5, \dots, x_{L+1}$, which is the setup suitable for describing L-point amplitudes.

Boundary value $\mathbf{C}_0^{\mathbf{E}}$: For the limit $x_3 \to 0$, we are facing the following situation



(2.71)

Similarly as before, this limit can conveniently be described in the basis $\mathcal{B}_{2,2,\dots,2}$, since for this choice, the maximum eigenvalue of e_0 is given by

$$s_{\max} = s_{1,3,4,\dots,L} \,. \tag{2.72}$$

This can be seen by repeating the observation that led to eq. (2.68) for e_1: using partial fractioning to express the right-hand side of eq. (2.36) in terms of Selberg integrals $\mathrm{S}[i_4, \dots, i_L]$ with $i_k \neq 2$, i.e. to form the KZ eq. (2.43) for $\mathcal{B}_{2,2,\dots,2}$, assembles all the s_{ij} with $i, j \neq 2, L+1$ in the matrix e_0. Therefore, the regularization factor $z_0^{-e_0}$ in $\mathbf{C}_0$ can at most contribute with a factor $x_3^{-s_{\max}}$ to each integral.

The behavior of these entries for $x_3 \to x_1 = 0$ may be determined using the change of variables $x_i = x_3 w_i$ for $0 = x_1 \leq x_i < x_2 = 1$, such that in particular $w_1 = 0$ and $w_3 = 1$. This yields for $i_k \neq 2$

$$\begin{aligned}
&\lim_{x_3\to 0} x_3^{-s_{\max}} \mathrm{S}[i_4, \dots, i_L](0,1,x_3) \\
&= \lim_{x_3\to 0} x_3^{-s_{\max}} \int_{\mathcal{C}(x_3)} \prod_{i=4}^{L} dx_i \prod_{0\leq x_j<x_l<x_3} x_{lj}^{s_{jl}} \prod_{0\leq x_m<x_3} x_{3m}^{s_{3m}} \prod_{0\leq x_n<x_2} x_{2n}^{s_{2n}} \prod_{k=4}^{L} \frac{1}{x_{ki_k}} \\
&= \lim_{x_3\to 0} \int_{0=w_1<w_i<w_3=1} \prod_{i=4}^{L} dw_i \prod_{0\leq w_j<w_l<x_3} w_{lj}^{s_{jl}} \prod_{0\leq w_m<w_3} w_{3m}^{s_{3m}} \prod_{0\leq x_n<x_2} (1-x_3 w_n)^{s_{2n}} \prod_{k=4}^{L} \frac{1}{w_{ki_k}} \\
&= \int_{0=w_1<w_i<w_3=1} \prod_{i=4}^{L} dw_i \prod_{0\leq w_j<w_l<x_3} w_{lj}^{s_{jl}} \prod_{0\leq w_m<w_3} w_{3m}^{s_{3m}} \prod_{k=4}^{L} \frac{1}{w_{ki_k}} \\
&= \mathrm{S}[i_4, i_5, \dots, i_L](0,1,w_3=1)|_{s_{2n}=0} \,,
\end{aligned} \tag{2.73}$$

which is, as for the $x_3 \to 1$ limit, an integral $\mathrm{S}[i_4, i_5, \dots, i_L](0,1,w_3=1)|_{s_{2n}=0}$ defined on

$\mathcal{F}_{L,3} = \mathcal{M}_{0,L}$. Note that if we would not restrict to the basis $\mathcal{B}_{2,2,\dots,2}$ and there were r indices $k_j \in \{4, 5 \dots, L\}$ such that $i_{k_j} = 2$, then the change of variables would leave r factors of x_3 in the quotient of the measure and the denominator

$$\prod_{k=4}^{L} \frac{dx_k}{x_{ki_k}} = x_3^r \prod_{k=4, k \notin \{k_j\}}^{L} \frac{dw_k}{w_{ki_k}} \prod_{j=1}^{r} \frac{dw_{k_j}}{x_3 w_{k_j} - 1} \,, \tag{2.74}$$

which vanishes for $x_3 \to 0$. Therefore, the entries of $\mathbf{C}_0$ are linear combinations of integrals

$$\begin{aligned} &\lim_{x_3 \to 0} x_3^{-s_{\max}} \, \mathrm{S}[i_4, \dots, i_L](0, 1, x_3) \\ &= \begin{cases} \mathrm{S}[i_4, i_5, \dots, i_L](0, 1, w_3 = 1)|_{s_{2n}=0} & \text{if } \mathrm{S}[i_4, \dots, i_L](0, 1, x_3) \in \mathcal{B}_{2,2,\dots,2} \,, \\ 0 & \text{otherwise} \,. \end{cases} \end{aligned} \tag{2.75}$$

Mandelstam variables: According to eq. (2.69), the Mandelstam variables s_{3n} associated to the momentum of the auxiliary insertion point x_3 are redundant in $\mathbf{C}_1$: they simply appear as a splitting of the effective momentum $\tilde{s}_{2n} = s_{2n} + s_{3n}$ associated to the insertion point at $x_2 = 1$ and thus, may be chosen to be set to zero. This choice is more subtle in the boundary value $\mathbf{C}_0$ with the non-vanishing entries being calculated according to eq. (2.73): here, the Mandelstam variables s_{3n} are not at all redundant, i.e. an artificial splitting of the momentum contribution, but encode the full momentum of the insertion point $w_3 = 1$. Thus, it may be expected that setting this momentum to zero effectively removes one external state, leaving an integral defined on $\mathcal{M}_{0,L-1}$. This expectation will be shown to be true in subsection 2.7 for certain linear combinations of Selberg integrals.

Summary: To summarize, the vector of Selberg integrals $\mathrm{S}(x_3)$ encodes the $(N = L)$- and $(N - 1 = L - 1)$-point amplitudes in the regularized limits $\mathbf{C}_0$ and $\mathbf{C}_1$, which can be related to each other according to the previous subsection using the Drinfeld associator $\Phi(e_0, e_1)$, with e_0 and e_1 determined by the KZ eq. (2.40), as follows

$$\begin{tikzcd} & \mathbf{C}_1 \text{ on } \mathcal{F}_{N,3} = \mathcal{M}_{0,N} \arrow[r, "s_{3n} \to 0"] & N\text{-point amplitudes on } \mathcal{M}_{0,N} \\ \mathrm{S}(x_3) \text{ on } \mathcal{F}_{N+1,4} \arrow[ur, "x_3 \to x_2 = 1"'] \arrow[dr, "x_3 \to x_1 = 0"] & & \\ & \mathbf{C}_0 \text{ on } \mathcal{F}_{N,3} = \mathcal{M}_{0,N} \arrow[uu, "\Phi(e_0, e_1)"'] \arrow[r, "s_{3n} \to 0"] & (N-1)\text{-point amplitudes on } \mathcal{M}_{0,N-1} \arrow[uu, "\Phi(e_0, e_1)|_{s_{3n}=0}"'] \end{tikzcd} \tag{2.76}$$

where the exact degeneracy to the amplitudes as $s_{3n} \to 0$ and the corresponding map via $\Phi(e_0, e_1)|_{s_{3n}=0}$ will be explored in the next subsections.

2.6 Open string amplitudes at genus zero

In this section, we will finally relate the construction reviewed in the previous subsections to open-string tree-level amplitudes. These amplitudes arise as correlators between vertex-operators inserted at the boundary of the worldsheet, which is a bounded Riemann surface of genus zero. Usually, these correlators are referred to as disk correlators. In order to ease the calculation, one usually makes use of the conformal symmetry of the worldsheet in order to place the boundary of the disk at the real line.

Performing the calculation in this setup shows that $(L+1)$-point tree-level open-string subamplitudes split into a super-Yang–Mills A_{YM} part carrying all state dependence and a string correction $F^\sigma = F^\sigma(\alpha')$ [29,30]:

$$
\begin{aligned}
A_{\mathrm{open}}(1, L, L-1, ..., 2, L+1; \alpha') &= \sum_{\sigma \in S_{N-3}} F^\sigma(\alpha')\, A_{\mathrm{YM}}(1, \sigma(L, L-1, ..., 3), 2, L+1) \\
&= \mathbf{F}^T \cdot \mathbf{A}_{\mathrm{YM}}\,,
\end{aligned}
\tag{2.77}
$$

where the $(L+1)$-point string corrections are given by

$$
\hat{F}^\sigma(\alpha') = (-1)^L \int_{\mathcal{C}(x_2=1)} \prod_{i=3}^{L} dx_i\ \mathrm{S}\,\sigma \left(\prod_{k=3}^{L} \left(\sum_{j=k+1}^{L} \frac{s_{jk}}{x_{jk}} + \frac{s_{k1}}{x_{k1}} \right) \right), \tag{2.78}
$$

and where the permutation $\sigma \in S_{L-2}$ acts on the labels within the brackets on the right-hand side of σ. In eq. (2.77), the sum runs over all permutations σ of the labels $3, \ldots, L$, so there is a total of $(L-2)!$ different open-string tree-level subamplitudes. All subamplitudes not exhibiting labels of the form above can be obtained by using monodromy relations [31,32]. In the second line of eq. (2.77) we have rephrased the equation in terms of vectors of length $(L-2)!$ with obvious definitions. Note that our notation of the labeling of the insertion points x_i differs from standard string literature: as depicted in figure (2.27) we choose

$$
x_1 = 0 < x_L < x_{L-1} < \cdots < x_3 < x_2 = 1\,, \qquad x_{L+1} = \infty \tag{2.79}
$$

for the $L+1 = N$ states on the (positive) real line on the Riemann sphere $\mathbb{C}P^1$, rather than the usual $z_1 = 0 < z_2 < z_3 < \cdots < z_{N-1} = 1$ with $z_N = \infty$. Our choice is suited for the formulation of the iterated (Selberg) integrals defined in eq. (2.20) and in particular to state the admissibility condition (2.21).

While Yang–Mills tree-level amplitudes can be obtained (for example) from BCFW recursion relations [3, 4], the string corrections $F^\sigma(\alpha')$ are purely kinematical functions, which can be represented as iterated integrals over the remaining insertion points x_i and hence, as integrals defined on $\mathcal{M}_{0,L+1}$ (cf. eq. (2.28)). In ref. [23] it was recognized that a further simplification and formal improvement occurs, if the vector $\mathbf{F}$ is represented as an again $(L-2)!$-dimensional vector of so-called Z-integrals:

$$
A_{\mathrm{open}}(1, L, L-1, ..., 2, L+1; \alpha') = \mathbf{Z}^T \cdot \mathbf{MK} \cdot \mathbf{A}_{\mathrm{YM}}\,. \tag{2.80}
$$

The object $\mathbf{MK}$ is known as the momentum kernel and can be represented as a matrix of dimension $(L-2)! \times (L-2)!$. A recursive formula is known yielding all entries of this object [31,33]. The entries are products of Mandelstam variables (2.1), where the degree in the Mandelstam variables of each entry is $(L-2)$. Z-integrals have been introduced in ref. [23] as well: they are defined as

$$
Z(q_1, q_2, \ldots, q_{L+1}) = \int_{\mathcal{C}(x_2=1)} \prod_{i=3}^{L} dx_i\ \mathrm{KN}\, \frac{x_{1,L+1} x_{2,L+1} x_{12}}{x_{q_1 q_2} x_{q_2 q_3} \cdots x_{q_L q_{L+1}} x_{q_{L+1} q_1}}\,, \tag{2.81}
$$

where the factor $x_{1,L+1} x_{2,L+1} x_{12}$ in the numerator and the fixing of the coordinates $(x_1, x_2, x_{L+1}) = (0, 1, \infty)$ corresponds to dividing out the gauge volume $\mathcal{V}_{\mathrm{CKG}}$ of the conformal Killing group $\mathrm{SL}(2, \mathbb{C})$. The quotient together with the integration measure is called Parke–Taylor form, while

KN is called Koba–Nielsen factor and defined by

$$\mathrm{KN} = \prod_{0=x_1\le x_i<x_j\le x_2=1} x_{ji}^{s_{ij}} = \prod_{0=x_1\le x_i<x_j\le x_2=1} \exp(s_{ij}\log x_{ji})\,. \tag{2.82}$$

Note that we have defined the Selberg seed in eq. (2.19) in exactly the same way: it is constructed to equal the $(L+1)$-point Koba–Nielsen factor

$$\mathrm{S} = \mathrm{KN}\,. \tag{2.83}$$

Since $\log x_{ij}$ is (almost) the genus-zero string propagator, the Koba–Nielsen factor can easily be identified as a generating functional of graphs connecting the vertex operators, where each edge connecting vertex operators at positions x_i and x_j is weighted by the corresponding Mandelstam variable s_{ij}.

Acting with x_i-derivatives on the Koba–Nielsen factor will yield terms of the form (2.3), which were the starting point for the definition of the Selberg integrals in subsection 2.2. Iterated integrals in x_i over various derivatives of the Koba–Nielsen factor, in particular the Z-integrals defined in eq. (2.81), fall in the class of Selberg integrals [11]. It is only those integrals, which need to be calculated in order to determine the full open-string tree-level amplitude at any multiplicity.

2.7 Relation to the construction in 1304.7304

In this subsection, we review the construction in ref. [17] and relate it to the Selberg integrals (2.29), showing how they reproduce the string corrections F^σ in eq. (2.77) in the appropriate limits shown in the diagram (2.76).

The construction in ref. [17] is based on the definition

$$\hat{F}^\sigma_\nu(x_3) = (-1)^{L-1}\int_{\mathcal{C}(x_3)}\prod_{i=4}^{L} dx_i\;\mathrm{S}\,\sigma\left(\prod_{k=L+2-\nu}^{L}\left(\sum_{j=k+1}^{L}\frac{s_{jk}}{x_{jk}}+\frac{s_{1k}}{x_{1k}}\right)\prod_{m=4}^{L+1-\nu}\left(\sum_{l=4}^{m-1}\frac{s_{ml}}{x_{ml}}+\frac{s_{m2}}{x_{m2}}\right)\right)\,, \tag{2.84}$$

where $\sigma\in S_{L-3}$ acts on the indices of the Mandelstam variables and insertion points $i\in\{4,5,\dots,L\}$ and $\nu\in\{1,2,\dots,L-2\}$.[5] As argued in ref. [17], the vector

$$\hat{\mathbf{F}}(x_3) = (\hat{\mathbf{F}}_{L-2},\hat{\mathbf{F}}_{L-3},\dots,\hat{\mathbf{F}}_1)^T \tag{2.85}$$

of length $(L-2)!$ satisfies a KZ equation

$$\frac{d}{dx_3}\hat{\mathbf{F}}(x_3) = \left(\frac{\hat{e}_0}{x_3}+\frac{\hat{e}_1}{x_3-1}\right)\hat{\mathbf{F}}(x_3)\,, \tag{2.86}$$

where the subvectors $\hat{\mathbf{F}}_\nu$ of length $(L-3)!$ contain all the integrals (2.84) for a given ν, i.e. $\hat{\mathbf{F}}_\nu = (\hat{F}^\sigma_\nu(x_3))_{\sigma\in S_{L-3}}$.

Note that by definition (2.84), the integral $\hat{F}^\sigma_\nu(x_3)$ is not an explicit linear combination of Selberg integrals $\mathrm{S}(i_4,i_5,\dots,i_L)(x_1=0,x_2=1,x_3)$. However, using integration by parts and the Fay identity, any integral $\hat{F}^\sigma_\nu(x_3)$ can be rewritten as such a linear combination. This

[5] Note that in ref. [17], another notation for the insertion points has been used: $z_1=0<z_2<z_3<\cdots<z_{N-2}<z_0<z_{N-1}=1<z_N=\infty$, such that by comparing with our notation given in eq. (2.79) the auxiliary point x_3 is denoted by z_0 and the number of external states is shifted by one.

mechanism is discussed explicitly in ref. [22], where a basis transformation B such that

$$\hat{\mathbf{F}}(x_3) = B\ \mathbf{S}(x_3)|_{\mathcal{B}_{1,1,\dots,1}} \tag{2.87}$$

has been constructed. Accordingly, the matrices $\hat{e}_0$ and $\hat{e}_1$ in the KZ equation (2.86) of $\hat{\mathbf{F}}(x_3)$ are related to the matrices e_0 and e_1 in the KZ equation satisfied by $\mathrm{S}\,|_{\mathcal{B}_{1,1,\dots,1}}$ by

$$\hat{e}_0 = Be_0B^{-1}\,, \qquad \hat{e}_1 = Be_1B^{-1}\,. \tag{2.88}$$

In particular, the non-vanishing entries of the basis transformation B are homogeneous polynomials in s_{ij} of degree $L-3$. Therefore, the matrices $\hat{e}_0$ and $\hat{e}_1$ are proportional to the inverse string tension α' since the same holds for the matrices e_0 and e_1.

Since $\hat{\mathbf{F}}(x_3)$ satisfies a KZ equation, as for the Selberg vector $\mathbf{S}(x_3)$, the regularized boundary values

$$\hat{\mathbf{C}}_0 = \lim_{x_3\to 0} x^{-\hat{e}_0}\,\hat{\mathbf{F}}(x_3)\ , \quad \hat{\mathbf{C}}_1 = \lim_{x_3\to 1}(1-x_3)^{-\hat{e}_1}\,\hat{\mathbf{F}}(x_3) \tag{2.89}$$

are well-defined and can be related by the Drinfeld associator $\Phi(\hat{e}_0,\hat{e}_1)$ to each other

$$\hat{\mathbf{C}}_1 = \Phi(\hat{e}_0,\hat{e}_1)\,\hat{\mathbf{C}}_0\,. \tag{2.90}$$

Note that according to eq. (2.88), this Drinfeld associator is related to the Drinfeld associator for the Selberg vector $\mathrm{S}\,|_{\mathcal{B}_{1,1,\dots,1}}$ by

$$\Phi(\hat{e}_0,\hat{e}_1) = B\Phi(e_0,e_1)B^{-1}\,. \tag{2.91}$$

From the results in subsection 2.5, it can be shown that the first $(L-3)!$ entries, corresponding to the subvector $\hat{\mathbf{F}}_{L-2}(x_3)$, of the regularized boundary value $\hat{\mathbf{C}}_1$ contain the open-string corrections $\mathbf{F}$ in eq. (2.77) for the L-point tree-level amplitudes[6]

$$\hat{\mathbf{C}}_1 = \begin{pmatrix} \mathbf{F}\,|_{L\text{-point}} \\ \vdots \end{pmatrix}\,. \tag{2.92}$$

This observation explains the meaning of the upper series of limits depicted (up to the basis transformation (2.87)) in the diagram (2.76). Following the same line of arguments which led to eq. (2.67), the entries of $\mathbf{F}\,|_{L\text{-point}}$, parametrized by $\sigma\in S_{L-3}$ acting on the indices $\{4,5,\dots,L\}$, are explicitly given by

$$\begin{aligned}
&\lim_{x_3\to 1}(1-x_3)^{-s_{23}}\hat{F}^{\sigma}_{L-2}(x_3)\\
&= (-1)^L\int_{\mathcal{C}(x_3\to x_2=1)}\prod_{i=4}^{L}dx_i\prod_{0\le x_j<x_l<x_3}x_{lj}^{s_{jl}}\prod_{0\le x_n<x_3}x_{2n}^{s_{2n}+s_{3n}}\,\sigma\left(\prod_{k=4}^{L}\left(\sum_{j=k+1}^{L}\frac{s_{jk}}{x_{jk}}+\frac{s_{k1}}{x_{k1}}\right)\right)\\
&= F^{\sigma}|_{L\text{-point}}\,,
\end{aligned}\tag{2.93}$$

where, as argued in subsection 2.5, the effective L-point Mandelstam variables of the insertion point at $x_2=1$ are given by $s_{2n}+s_{3n}$ for $n=1,4,5,\dots,L$, or solely by s_{2n} in the additional limit $s_{3n}\to 0$, and the L external strings states correspond to the L insertion points $(x_1 =$

[6]This was originally proven in ref. [16] in a different framework, restated in ref. [17] and proven using the notation of this article in ref. [22]

$0, x_L, x_{L-1}, \ldots, x_4, x_2 = 1, x_{L+1} = \infty)$ shown in figure 2.66.

Similarly, the only non-vanishing subvector $\hat{\mathbf{F}}_\nu(x_3)$ in the regularized boundary value $\hat{\mathbf{C}}_0$ is the one for $\nu = L-2$. Its non-vanishing entries degenerate in the limit $s_{3n} \to 0$ to the $(L-1)$-string corrections [16, 17, 22]

$$\lim_{s_{3n}\to 0} \hat{\mathbf{C}}_0 = \begin{pmatrix} \mathbf{F}\,|_{(L-1)\text{-point}} \\ 0_{(L-3)(L-3)!} \end{pmatrix} . \tag{2.94}$$

Explicitly, the non-vanishing entries are the ones for $\nu = L-2$ and parametrized by the permutations σ such that $\sigma 4 = 4$. They can be calculated using the same change of variables $x_i = x_3 w_i$ which lead to the result in eq. (2.75)

$$\begin{aligned}
&\lim_{s_{3n}\to 0}\lim_{x_3\to 0} x_3^{s_{\max}} \hat{F}^\sigma_{L-2}(x_3) \\
&= (-1)^{N-2} \int_{0=w_1<w_i<w_4=1} \prod_{i=5}^{L} dw_i \prod_{0\leq w_j<w_l<w_4=1} w_{lj}^{s_{jl}}\, \sigma \left(\prod_{k=5}^{L} \left(\sum_{j=k+1}^{L} \frac{s_{jk}}{w_{jk}} + \frac{s_{k1}}{w_{k1}} \right) \right) \\
&= F^\sigma|_{(L-1)\text{-point}} ,
\end{aligned} \tag{2.95}$$

where the $L-1$ external string states are described by the $L-1$ insertion points $(w_1 = 0, w_L, w_{L-1}, \ldots, w_4 = 1, w_{L+1} = \infty)$. Note that as discussed at the end of subsection 2.5, the limit $s_{3n} \to 0$ is necessary to recover the $(L-1)$-point amplitudes rather than the L-point situation (it is responsible for the merging $w_4 \to w_3 = 1$), this degeneracy is depicted in the bottom line of the diagram (2.76) (up to the basis transformation (2.87)).

Finally, the open-string recursion at tree level proposed in ref. [17] is given by the $s_{3n} \to 0$ limit of the associator equation (2.90), which takes the form

$$\begin{pmatrix} \mathbf{F}\,|_{L\text{-point}} \\ \vdots \end{pmatrix} = \Phi(\hat{e}_0, \hat{e}_1)|_{s_{3n}\to 0} \begin{pmatrix} \mathbf{F}\,|_{(L-1)\text{-point}} \\ 0_{(L-3)(L-3)!} \end{pmatrix} \tag{2.96}$$

and is up to the basis transformation (2.87) the right-most vertical map depicted in the diagram (2.76). According to ref. [17], eq. (2.96) can be used to calculate the α'-expansion of the L-point string corrections in $\mathbf{F}\,|_{L\text{-point}}$ solely by matrix multiplication from the $(L-1)$-point string corrections in $\mathbf{F}\,|_{(L-1)\text{-point}}$ as follows: first, the matrices $\hat{e}_0$ and $\hat{e}_1$ have to be determined by writing out the derivative of the vector $\hat{\mathbf{F}}(x_3)$ with respect to x_3 in KZ form (2.86). Then, all the words in $\{\hat{e}_0, \hat{e}_1\}^\times$ up to the maximal word length given by the desired maximal order in α', say $o_{\max}$, can be calculated to form the truncation of the associator $\Phi(\hat{e}_0, \hat{e}_1)$ at this length according to eq. (2.64). Since $\hat{e}_0$ and $\hat{e}_1$ are proportional to the Mandelstam variables, longer words would only contribute to higher powers in α'. This truncated Drinfeld associator can be used to calculate the required α'-expansion of $\mathbf{F}\,|_{L\text{-point}}$ up to order $o_{\max}$ by multiplication with the α'-expansion of $\mathbf{F}\,|_{(L-1)\text{-point}}$ up to order $o_{\max}$ using the associator equation (2.96).

As an example, we apply this recursion for $L = 5$ to calculate the five-point from the four-point string correction following the calculation in refs. [17, 22]. The vector $\hat{\mathbf{F}}(x_3)$ defined in eq. (2.85) is given by

$$\hat{\mathbf{F}}(x_3) = \begin{pmatrix} F_3^{\mathrm{id}} \\ F_3^{(5\,4)} \\ F_2^{\mathrm{id}} \\ F_2^{(5\,4)} \\ F_1^{\mathrm{id}} \\ F_1^{(5\,4)} \end{pmatrix} = \int_0^{x_3} dx_4 \int_0^{x_4} dx_5 \, \mathrm{S} \begin{pmatrix} \left(\frac{s_{54}}{x_{54}} + \frac{s_{14}}{x_{14}}\right) \frac{s_{15}}{x_{15}} \\ \left(\frac{s_{45}}{x_{45}} + \frac{s_{15}}{x_{15}}\right) \frac{s_{14}}{x_{14}} \\ \frac{s_{15}}{x_{15}} \frac{s_{42}}{x_{42}} \\ \frac{s_{14}}{x_{14}} \frac{s_{52}}{x_{52}} \\ \frac{s_{42}}{x_{42}} \left(\frac{s_{54}}{x_{54}} + \frac{s_{52}}{x_{52}}\right) \\ \frac{s_{52}}{x_{52}} \left(\frac{s_{45}}{x_{45}} + \frac{s_{42}}{x_{42}}\right) \end{pmatrix}, \tag{2.97}$$

which satisfies a KZ equation

$$\frac{d}{dx_3} \hat{\mathbf{F}}(x_3) = \left(\frac{\hat{e}_0}{x_3} + \frac{\hat{e}_1}{x_3 - 1}\right) \hat{\mathbf{F}}(x_3) \tag{2.98}$$

with the matrices

$$\hat{e}_0 = \begin{pmatrix} s_{5431} & 0 & -s_{41} - s_{54} & -s_{51} & -s_{51} & s_{51} \\ 0 & s_{5431} & -s_{41} & -s_{51} - s_{54} & s_{41} & -s_{41} \\ 0 & 0 & s_{531} & 0 & -s_{51} & 0 \\ 0 & 0 & 0 & s_{431} & 0 & -s_{41} \\ 0 & 0 & 0 & 0 & s_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{31} \end{pmatrix} \tag{2.99}$$

and

$$\hat{e}_1 = \begin{pmatrix} s_{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & s_{32} & 0 & 0 & 0 & 0 \\ -s_{42} & 0 & s_{432} & 0 & 0 & 0 \\ 0 & -s_{52} & 0 & s_{532} & 0 & 0 \\ -s_{42} & s_{42} & -s_{52} - s_{54} & -s_{42} & s_{5432} & 0 \\ s_{52} & -s_{52} & -s_{52} & -s_{42} - s_{54} & 0 & s_{5432} \end{pmatrix}. \tag{2.100}$$

The regularized boundary value $\mathbf{C}_0$ degenerates in the limit $s_{3n} \to 0$ according to eq. (2.95) to the four-point string correction F^{id} given in eq. (2.78) with the four insertion points being $0 = w_1 < w_5 < w_4 = 1 < w_6 = \infty$, which is the well-known Veneziano amplitude

$$\mathbf{C}_0 \, |_{s_{3n}=0} = \begin{pmatrix} \frac{\Gamma(1+s_{15})\Gamma(1+s_{54})}{\Gamma(1+s_{15}+s_{54})} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.101}$$

On the other hand, the regularized boundary value $\mathbf{C}_1$ degenerates as shown in eq. (2.93) to

$$\mathbf{C}_1 \, |_{s_{3n}=0} = \begin{pmatrix} F^{\mathrm{id}} \\ F^{(5\,4)} \\ \vdots \end{pmatrix} \tag{2.102}$$

containing the five-point string corrections F^σ defined in eq. (2.78) with the five insertion points denoted by $0 = x_1 < x_5 < x_4 < x_2 = 1 < x_6 = \infty$. The amplitude recursion of ref. [17] states that the α'-expansion of the five-point string corrections can be calculated by the four-point string correction using the associator eq. (2.96), which is in this case

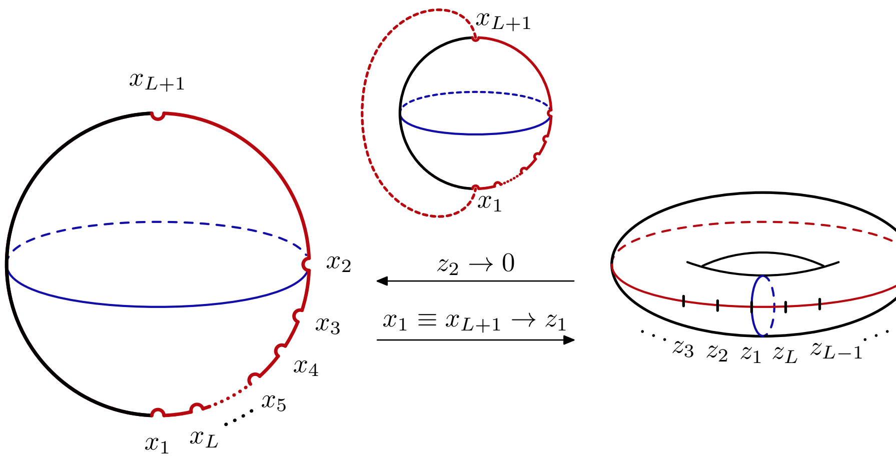
$$\begin{pmatrix} F^{\mathrm{id}} \\ F^{(5\,4)} \\ \vdots \end{pmatrix} = \Phi(\hat{e}_0|_{s_{3n}=0}, \hat{e}_1|_{s_{3n}=0}) \begin{pmatrix} \frac{\Gamma(1+s_{15})\Gamma(1+s_{54})}{\Gamma(1+s_{15}+s_{54})} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.103}$$

where the Drinfeld associator $\Phi(\hat{e}_0|_{s_{3n}=0}, \hat{e}_1|_{s_{3n}=0})$ can be calculated according to eq. (2.64) and truncated at the word length corresponding to the desired order in α'.

3 Genus one (one-loop)

In this section, we develop and explore the genus-one version of the concepts from section 2 and apply the resulting formalism to one-loop open string interactions. The genus-one recursion is similar to the recursion at genus zero from ref. [17] reviewed in subsection 2.7.

While the genus-zero recursion relates N-point amplitudes to $(N-1)$-point amplitudes and is thus a recursion in the number of external legs, the genus-one mechanism relates N-point one-loop string corrections to $(N+2)$-point tree-level corrections. Therefore it allows to relate objects occurring at different genera[7]:



(3.1)

In the genus-zero recursion, the Drinfeld associator effectively glues a three-point interaction to an $(N-1)$-point interaction by splitting one of the external string states into two separate states and, hence, increases the number of external states by one resulting in an N-point tree-level interaction. On the other hand, as shown below, the genus-one recursion amounts to two external states of the $(N+2)$-point tree-level interaction being glued together by the elliptic analogue of the Drinfeld associator, the elliptic Knizhnik-Zamolodchikov-Bernard (KZB) associator, to form a genus-one worldsheet of N external string states.

As will be discussed below, this geometrical interpretation in terms of scattering amplitudes (or string corrections) is dictated by the behavior of the canonical genus-zero and genus-one Selberg integrals.

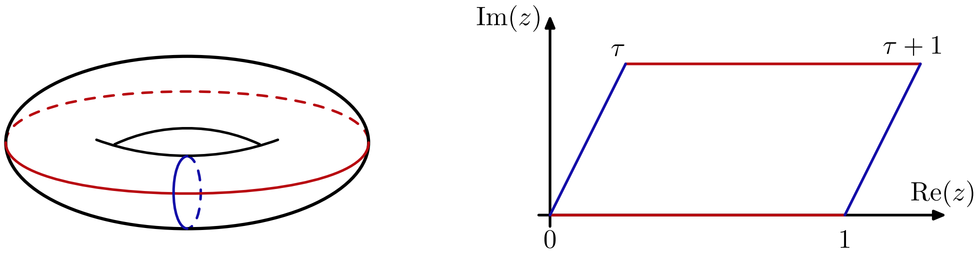
In subsection 3.1 up to subsection 3.5, we follow a similar structure as in section 2 and introduce the corresponding genus-one generalizations of the mathematical concepts: elliptic iterated integrals, a genus-one version of Selberg integrals, the genus-one KZB associator, the

[7] Notation and limits depicted in figure (3.1) will be introduced and explained in the course of this section.

KZB equation for an auxiliary marked point. In the subsequent subsection 3.6, the first orders in α' of the two-, three- and four-point one-loop string corrections are calculated using the genus-one associator mechanism and shown to agree with the known results.

3.1 Singularities, iterated integrals and elliptic multiple zeta values

In the following, we will envisage the genus-one Riemann surface as a torus with A-cycle (red) and B-cycle (blue), where the ratio of the respective lengths, the modular parameter, is denoted by τ.



(3.2)

By mimicking the formalism at genus zero described in section 2, let us start by considering canonical differentials on the torus: they are generated by the so-called Eisenstein–Kronecker series $F(z,\eta,\tau)$ [34, 10]

$$F(z,\eta,\tau) = \frac{\theta_1'(0,\tau)\theta_1(z+\eta,\tau)}{\theta_1(z,\tau)\theta_1(\eta,\tau)}\,, \tag{3.3}$$

where θ_1 is the odd Jacobi function and $'$ denotes a derivative with respect to the first argument. Expanding in the second complex argument η one finds

$$\eta F(z,\eta,\tau) = \sum_{n=0}^{\infty} g^{(n)}(z,\tau)\eta^n\,, \tag{3.4}$$

which – in distinction to the genus-zero scenario – defines an infinite number of differentials $g^{(n)}(z,\tau)dz$. The index n labeling the functions $g^{(n)}$ is called its weight. While $g^{(0)} = 1$ is trivial, the only function with poles, located at $z \in \mathbb{Z}\tau + \mathbb{Z}$, is $g^{(1)}$, which is nicely visible when writing down the q-expansion [35], where $q = \exp(2\pi i\tau)$:

$$g^{(1)}(z,\tau) = \pi\cot(\pi z) + 4\pi \sum_{m=1}^{\infty} \sin(2\pi m z) \sum_{n=1}^{\infty} q^{mn}\,. \tag{3.5}$$

All $g^{(n)}$ with $n \geq 2$ are holomorphic in the fundamental elliptic domain. However, due to the simple pole of $g^{(1)}$, these integration kernels can not be elliptic functions, i.e. meromorphic and one- as well as τ-periodic. They are only one-periodic

$$g^{(n)}(z+1,\tau) = g^{(n)}(z,\tau) \tag{3.6}$$

and, furthermore, have a well-defined symmetry property

$$g^{(n)}(-z,\tau) = (-1)^n g^{(n)}(z,\tau)\,. \tag{3.7}$$

Rather than defining elliptic functions, the functions $g^{(n)}$ can be considered to be genus-one generalizations of the integration kernels defining the multiple polylogarithms (2.7), which also lead to meromorphic, but multi-valued functions.

The integrals over the kernels $g^{(n)}$ lead to elliptic polylogarithms [9, 10]: due to their periodicity in eq. (3.6) they are single-valued functions on the cylinder, but can be thought of as multi-valued functions on the torus. This is equivalent to the behavior of the ordinary logarithm at genus zero: on each Riemann sheet the logarithm is single-valued, while it is a multi-valued function in the complex plane.

It will be precisely the integration kernels $g^{(1)}$, whose integrals need to be regularized and which will – in certain limits – act as the link between the string propagators at Riemann surfaces of genus zero and genus one. Corresponding to the differentials introduced in eq. (3.4), one can define a class of iterated integrals $\tilde{\Gamma}$ called elliptic multiple polylogarithms:

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1, n_2, \ldots, n_k \\ a_1, a_2, \ldots, a_k \end{smallmatrix} ; z, \tau \right) = \int_0^z dz' \, g^{(n_1)}(z' - a_1, \tau) \, \tilde{\Gamma}\left(\begin{smallmatrix} n_2, \ldots, n_k \\ a_2, \ldots, a_k \end{smallmatrix} ; z', \tau \right), \tag{3.8}$$

which due to their nature as iterated integrals obey shuffle relations

$$\tilde{\Gamma}(A_1, A_2, \ldots, A_j; z, \tau) \, \tilde{\Gamma}(B_1, B_2, \ldots, B_k; z, \tau) = \tilde{\Gamma}\left((A_1, A_2, \ldots, A_j) \sqcup\!\sqcup (B_1, B_2, \ldots, B_k); z, \tau \right) \tag{3.9}$$

in terms of combined letters $A_i = \begin{smallmatrix} n_i \\ a_i \end{smallmatrix}$.

The integral over $g^{(1)}$ will be of particular interest below: $\tilde{\Gamma}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} ; z, \tau \right)$ requires regularization because of an endpoint divergence at the lower integration boundary due to the pole at $z = 0$. The standard regularization procedure – which we are going to use here – is called *tangential basepoint regularization* and is discussed in detail for example in ref. [24, 36]. In short, we subtract the endpoint divergence by defining

$$\begin{aligned} \tilde{\Gamma}_{\text{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} ; z, \tau \right) &= \lim_{\epsilon \to 0} \int_\epsilon^z dz \, g^{(1)}(z, \tau) + \log(2\pi i \epsilon) \\ &= \log(1 - e^{2\pi i z}) - \pi i z + 4\pi \sum_{k,l>0} \frac{1}{2\pi k} \left(1 - \cos(2\pi k z)\right) q^{kl} \, . \end{aligned} \tag{3.10}$$

Considering $z \in (0, 1)$, the following properties can be read off from the above q-expansion

$$\tilde{\Gamma}_{\text{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} ; z \pm 1, \tau \right) = \tilde{\Gamma}_{\text{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} ; z, \tau \right) \mp \pi i \, , \qquad \tilde{\Gamma}_{\text{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} ; -z, \tau \right) = \tilde{\Gamma}_{\text{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} ; z, \tau \right) + \pi i \, , \tag{3.11}$$

where we place the branch cut of the logarithm such that $\log(-1) = \pi i$. This implies in particular invariance under $z \to 1 - z$ for $0 < z < 1$:

$$\tilde{\Gamma}_{\text{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} ; z, \tau \right) = \tilde{\Gamma}_{\text{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} ; 1 - z, \tau \right). \tag{3.12}$$

In addition, we find the following asymptotic behavior for $z \to 0$

$$\tilde{\Gamma}_{\text{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} ; z, \tau \right) \sim \log(2\pi i z) + s(z) \tag{3.13}$$

and $z \to 1$

$$\tilde{\Gamma}_{\text{reg}}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} ; z, \tau \right) \sim \log(2\pi i (1 - z)) + \mathcal{O}(1 - z) \, . \tag{3.14}$$

The above regularization procedure is an algebra homomorphism, e.g. compatible with the shuffle product. From now on, we will use the regularized iterated integrals exclusively and omit the subscript when noting $\tilde{\Gamma}$. Furthermore, we are going to keep the dependence on τ implicit for all integration kernels $g^{(n)}$ and all iterated elliptic integrals $\tilde{\Gamma}$.

In the same way as products of terms of the form $1/x_{ij}$ can be related by partial fractioning

(2.37), there is a genus-one analogue for the Kronecker series: the Fay identity. In terms of the functions $g^{(n)}(z)$ it can be phrased as

$$g^{(n_1)}(t-x)g^{(n_2)}(t) = -(-1)^{n_1} g^{(n_1+n_2)}(x) + \sum_{j=0}^{n_2} \binom{n_1-1+j}{j} g^{(n_2-j)}(x) g^{(n_1+j)}(t-x)$$
$$+ \sum_{j=0}^{n_1} \binom{n_2-1+j}{j} (-1)^{n_1+j} g^{(n_1-j)}(x) g^{(n_2+j)}(t) \tag{3.15}$$

and derived from a similar property obeyed by the generating function $F(z,\eta,\tau)$.

For compactness, we will use a notation similar to definition (2.7) in terms of words from an alphabet for the elliptic multiple polylogarithms $\tilde{\Gamma}$ defined in eq. (3.8) with $a_1 = a_2 = \cdots = a_k = 0$. Concretely, since there are infinitely many integration kernels $g^{(n)}$, the alphabet is infinite as well and denoted by $\{x^{(0)}, x^{(1)}, \ldots\}$. For a word $w = x^{(n_1)} \ldots x^{(n_k)} \in \{x^{(0)}, x^{(1)}, \ldots\}^\times$, we denote the corresponding elliptic multiple polylogarithm by

$$\tilde{\Gamma}_w(z) = \tilde{\Gamma}(x^{(n_1)} \cdots x^{(n_k)}; z) = \tilde{\Gamma}\left(\begin{smallmatrix} n_1, & \ldots, & n_k \\ 0, & \ldots, & 0 \end{smallmatrix} ; z \right) . \tag{3.16}$$

Denoting by $X = \{x^{(0)}, x^{(1)}, \ldots\}^\times$ the set of all words, we end up with the assignment

$$w \mapsto \begin{cases} \tilde{\Gamma}_w(z) = \tilde{\Gamma}(w; z) & \text{if } w \in X \setminus (X x^{(1)}) , \\ \frac{1}{n!} (\tilde{\Gamma}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; z))^n & \text{if } w = (x^{(1)})^n ,\ n \in \mathbb{N} . \end{cases} \tag{3.17}$$

For words $w \in X x_1$ with $w \neq (x^{(1)})^n$ for some $n \geq 1$, the map of the word w to an iterated integral is traced back to eq. (3.17) by the shuffle algebra. For $w \neq (x^{(1)})^n$, one finds

$$\lim_{z \to 0} \tilde{\Gamma}_w(z) = 0 , \tag{3.18}$$

while the regularization (3.10) implies logarithmic divergences for words $w = (x^{(1)})^n$ in this limit:

$$\tilde{\Gamma}_{(x^{(1)})^n}(z) \sim \frac{1}{n!} \log(2\pi i z)^n . \tag{3.19}$$

Due to the one-periodicity of $g^{(1)}$, this divergence also appears at the upper integration boundary for words $w = (x^{(1)})^n$ as $z \to 1$. The corresponding regularization procedure is particularly important for elliptic multiple zeta values to be discussed in the next paragraph.

Considering the limit $z \to 1$ leads to the genus-one analogues of MZVs defined in eq. (2.14). These so-called elliptic multiple zeta values (eMZVs) [37,38,35] are defined in terms of regularized iterated integrals $\tilde{\Gamma}_w$ with $w = x^{(n_1)} \ldots x^{(n_k)} \in X \setminus x^{(1)} X$, i.e. $n_1 \neq 1$, at $z = 1$:

$$\omega(n_k, \ldots, n_1; \tau) = \omega(w^t; \tau) = \lim_{z \to 1} \tilde{\Gamma}_w(z, \tau) = \lim_{z \to 1} \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ 0 & \cdots & 0 \end{smallmatrix} ; z, \tau \right) , \tag{3.20}$$

where w^t denotes the reversal of the word w. In order to extend this definition to all words $w \in X$, the singularity of $\tilde{\Gamma}_{x^{(1)} w}(z, \tau)$ at $z = 1$ has to be regularized. This can be done similarly as for the multiple polylogarithms in eq. (2.15), and is elaborated on in detail in appendix A. The main result is the following definition of the regularized eMZVs $\omega_{\text{reg}}(w^t; \tau)$: for any word

$w \in X \setminus x^{(1)}X$ they are defined by eq. (3.20), i.e.

$$w \mapsto \omega_{\text{reg}}(w^t;\tau) = \begin{cases} \omega(w^t;\tau) & \text{if } w \in X \setminus (x^{(1)}X)\,, \\ 0 & \text{if } w = (x^{(1)})^n,\ n \in \mathbb{N}\,. \end{cases} \tag{3.21}$$

Again, the remaining cases $w \in x^{(1)}X$ can be related to the above situations by use of the shuffle algebra. As for the elliptic multiple polylogarithms, from now on unless stated otherwise, all elliptic multiple zeta values are assumed to be regularized and simply denoted by $\omega(w^t)$ omitting the subscript and the τ-dependence in $\omega_{\text{reg}}(w^t;\tau)$.

In the same way as the shuffle algebra is preserved when regularizing iterated integrals $\tilde{\Gamma}$ in eq. (3.10), this is true for the corresponding MZVs: (regularized) eMZVs inherit the shuffle algebra, the properties implied by the Fay identity and some further properties from the elliptic multiple polylogarithms such as the reflection identity

$$\omega(n_k,\dots,n_1) = (-1)^{n_1+\dots+n_k}\omega(n_1,\dots,n_k) \tag{3.22}$$

due to the symmetry (3.7) of the integration kernels. Furthermore, even elliptic zeta values are related to the (genus-zero) zeta values according to

$$\omega(2m;\tau) = -2\zeta_{2m}\,. \tag{3.23}$$

Numerous other relations between eMZVs can be retrieved from [39].

3.2 Genus-one Selberg integrals

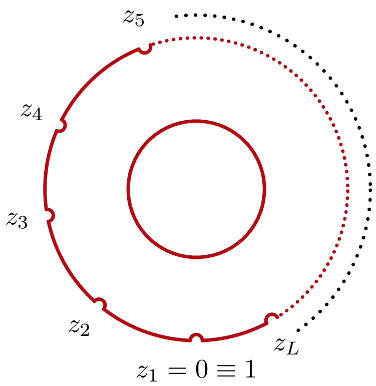
In order to repeat the construction described for genus zero in subsection 2.2, we will need to find a genus-one generalization of the Selberg seed function defined in eq. (2.19) which can be used to construct genus-one Selberg integrals. The genus-one Selberg seed should depend on the positions of insertion points inserted on the boundary of a genus-one Riemann surface: an annulus. Therefore – in analogy to the genus-zero scenario – we expect to find iterated integrals on the moduli space $\mathcal{M}_{1,L}$ of L-punctured tori with one fixed point.

For simplicity, we restrict our discussion to integrals, where all insertion points are on one boundary, which we choose to be the A-cycle: the real line between zero and one. This scenario corresponds to planar, open interactions at one loop. A generalization to the non-planar case, where points are allowed on both boundaries is not expected to pose any structural obstacles.

In contrast to the genus-zero labeling (2.18), the positions of the insertion points are going to be denoted by

$$0 = z_1 < z_L < z_{L-1} < \dots < z_2 < 1 = z_1 \bmod \mathbb{Z}\,, \tag{3.24}$$

where we have used the symmetries of the torus to fix $z_1 = 0$.



(3.25)

Remembering the basic properties of the genus-zero Selberg seed defined in eq. (2.19) when taking derivatives (cf. eq. (2.35)), the generalization of the Selberg seed to genus one is straightforward. Defining

$$\tilde{\Gamma}_{ij} = \tilde{\Gamma}\left(\begin{smallmatrix}1\\0\end{smallmatrix}; z_i - z_j, \tau\right) = \tilde{\Gamma}_{x^{(1)}}(z_{ij}, \tau), \tag{3.26}$$

one can simply replace $\log x_{ji} = G_{e_0}(x_{ji})$ in the genus-zero Selberg seed by the above expression to find

$$\mathrm{S} = \prod_{x_i < x_j} \exp\left(s_{ij} G_{e_0}(x_{ji})\right) \quad \rightarrow \quad \mathrm{S}^{\mathrm{E}} = \prod_{z_i < z_j} \exp\left(s_{ij}\, \tilde{\Gamma}_{x^{(1)}}(z_{ji}, \tau)\right). \tag{3.27}$$

Indeed, this expression is already very close to the one-loop Koba–Nielsen factor KN^{E} appearing in the one-loop string amplitudes below. In particular, G_{e_0} and $\tilde{\Gamma}_{x^{(1)}}$ are the regularized integrals as defined in eq. (2.10) and eq. (3.10), respectively. A key observation for our construction is the relation between these two functions which follows from eq. (3.19): the polylogarithm $G_{e_0}(2\pi i z)$ describes the asymptotic behavior of the elliptic polylogarithm $\tilde{\Gamma}_{x^{(1)}}(z, \tau)$ as $z \to 0$.

Accordingly, let us define a genus-one version of the Selberg integrals defined in eq. (2.20) starting from the *genus-one Selberg seed*

$$\mathrm{S}^{\mathrm{E}}(\tau) = \mathrm{S}^{\mathrm{E}}\big[\;\big](z_1, \ldots, z_N, \tau) = \prod_{0 = z_1 \leq z_i < z_j \leq z_2} \exp\left(s_{ij}\, \tilde{\Gamma}_{ji}\right). \tag{3.28}$$

Completely parallel to the genus-zero scenario, we can define *genus-one Selberg integrals* as follows: the empty integral $\mathrm{S}^{\mathrm{E}}\big[\;\big]$ is the genus-one Selberg seed, which shall be integrated over the integration kernels $g^{(n)}$. This class of integrals is denoted by

$$\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_{k+1}, & \ldots, & n_L \\ i_{k+1}, & \ldots, & i_L \end{smallmatrix}\right](z_1, \ldots, z_k) = \int_0^{z_k} dz_{k+1}\, g^{(n_{k+1})}_{k+1, i_{k+1}}\, \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_{k+2}, & \ldots, & n_L \\ i_{k+2}, & \ldots, & i_L \end{smallmatrix}\right](z_1, \ldots, z_{k+1}), \tag{3.29}$$

where we have defined

$$g^{(n)}_{ij} = g^{(n)}_{i,j} = g^{(n)}(z_i - z_j, \tau). \tag{3.30}$$

For all genus-one Selberg integrals as well as for the genus-one Selberg seed, we will again suppress the τ-dependence below. However, we will still indicate the dependence by using partial derivatives.

We call the sum $n_{k+1} + \cdots + n_L$ the *weight* of the Selberg integral. This notation, where instead of the actual shifts a_i from eq. (3.8) the index of a position variable z_i is used, will allow for rather compact equations when manipulating genus-one Selberg integrals. Moreover, as for the genus-zero Selberg integrals, the shift $z_{i_{k+1}}$ in the integration kernel $g^{(n_{k+1})}_{k+1, i_{k+1}}$ can only be a variable which has not yet been integrated out, which leads to the genus-one analogue of the admissibility condition in eq. (2.21):

$$1 \leq i_p < p \qquad \forall p \in \{k+1, \dots, L\} \tag{3.31}$$

As in the genus-zero setting, the corresponding integrals in the genus-one setting are also called admissible.

In order to be equipped for the next subsections, let us collect a couple of identities for genus-one Selberg integrals. Derivatives of the function $\tilde{\Gamma}_{ij}$ can be redirected to another index using the symmetry property (3.7) of $g^{(1)}$:

$$\frac{\partial}{\partial z_i} \tilde{\Gamma}_{ij} = g^{(1)}(z_i - z_j) = -\frac{\partial}{\partial z_j} \tilde{\Gamma}_{ij} \ . \tag{3.32}$$

In the above language, the Fay identity (3.15) takes the form

$$\begin{aligned} g^{(m)}_{kj} g^{(n)}_{ki} = (-1)^{m+1} g^{(m+n)}_{ji} &+ \sum_{r=0}^{n} \binom{m+r-1}{m-1} g^{(n-r)}_{ji} g^{(m+r)}_{kj} \\ &+ \sum_{r=0}^{m} (-1)^{m-r} \binom{n+r-1}{n-1} g^{(m-r)}_{ji} g^{(n+r)}_{ki} \,, \end{aligned} \tag{3.33}$$

The left-hand side of eq. (3.33) is admissible, when w.l.o.g. $i < j < k$: if this condition is met, the right-hand side will consist of admissible combinations only.

The Fay identity is the reason why all integration kernels $g^{(n)}_{ij}$ are included in the definition of the genus-one Selberg integrals (3.29) rather than only $g^{(1)}_{ij}$: application of the Fay identity introduces weights $n \neq 1$, such that a closed system with respect to integration by parts and the Fay identity requires all integration kernels $g^{(n)}_{ij}$.

When discussing a recursive solution for genus-one Selberg integrals below, we will have to take various derivatives with respect to insertion points z_i, which is thoroughly discussed in appendix C.1. Here we would like to collect some key properties used in the calculations below. Taking the regularization prescription in eq. (3.10) into account, we find

$$\mathrm{S}^{\mathrm{E}} \,|_{z_i = z_j} = 0 \qquad \text{for } i \neq j \,, \tag{3.34}$$

which is the property analogous to eq. (2.31) in the genus-zero scenario. Taking a derivative of the one-loop Selberg seed with respect to a particular variable yields

$$\frac{\partial}{\partial z_i} \mathrm{S}^{\mathrm{E}} = \sum_{k \neq i} s_{ik} g^{(1)}_{ik} \, \mathrm{S}^{\mathrm{E}} \ . \tag{3.35}$$

For a fixed L, that is a fixed number of points z_i, $i \in \{1, \dots, L\}$, and a given number of integrations $L-k$, there is a large number of different Selberg integrals. It is natural to ask for a particular set of integrals constituting a basis in the space of genus-one Selberg integrals. In principle, there are two operations which can be performed on Selberg integrals: one can integrate by parts and one can apply Fay identities. The question of a basis for this type of integrals is a very old one and amounts to determining a basis of the corresponding twisted de Rham cohomology, similar to the fibration basis in genus zero mentioned in the discussion above definition (2.41) of the bases for (genus-zero) Selberg integrals.

Since a reduction to a basis is convenient, but not necessary in our construction, we do not try to rigorously provide a genus-one analogue of the fibration basis. However, we note certain observations for a class of genus-one Selberg integrals with fixed L and a fixed number of integrations $L-k$:

- for an index $n_p = 0$, the corresponding integration kernel $g^{(0)}_{p,i_p} = 1$ is a constant, thus, we can always choose $i_p = 1$ in this case.
- as for the genus-zero Selberg integrals, for an index $n_p = 1$, integration by parts yields a linear equation for the integrals due to the partial derivative of the Selberg seed (3.35). Hence, for each index $n_p = 1$, we expect to be able to reduce the class of integrals from $1 \leq i_p < p$ to $1 \leq i_p \neq i'_p < p$ for any $1 \leq i'_p < p$ by such an integration by parts identity and applications of the Fay identity (to recover admissible integrals). However, no further such simplifications are expected for the indices $n_p > 1$.

In subsection 3.4 below, we are going to consider a differential equation for a vector of genus-one Selberg integrals of integral length $L-2$, which are the relevant genus-one Selberg integrals containing the one-loop and tree-level string corrections. This is the class given by the integrals

$$\begin{aligned} \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L \end{smallmatrix}\right](z_1 = 0, z_2) &= \int_0^{z_2} dz_3\, g^{(n_3)}_{3,i_3}\, \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_4, & \dots, & n_L \\ i_4, & \dots, & i_L \end{smallmatrix}\right](z_1 = 0, z_2, z_3) \\ &= \int_{\mathcal{C}(z_2)} \prod_{i=3}^{L} dz_i\, \mathrm{S}^{\mathrm{E}} \prod_{k=3}^{L} g^{(n_k)}_{k,i_k}\,, \end{aligned} \tag{3.36}$$

where $1 \leq i_k < k$ and the integration region is still given by eq. (2.23) (however, insertion positions are labeled by z_i here):

$$\mathcal{C}(z_i) = \{0 = z_1 < z_L < z_{L-1} < \cdots < z_i\}\,, \tag{3.37}$$

such that the integral over this domain is given by

$$\int_{\mathcal{C}(z_2)} \prod_{i=3}^{L} dz_i = \int_0^{z_2} dz_3 \int_0^{z_3} dz_4 \cdots \int_0^{z_{L-1}} dz_L\,. \tag{3.38}$$

The integrals defined in eq. (3.36) are the genus-one generalization of the Selberg integrals relevant for the tree-level amplitude recursion, which are given in eq. (2.22). As for this genus-zero class, the differential equation satisfied by the vector of these genus-one Selberg integrals leads to an associator equation relating one-loop to tree-level string corrections.

Using the considerations about a fibration basis above, we will at least reduce the class of iterated integrals defined in eq. (3.36) to a spanning set

$$\begin{aligned} \mathcal{B}^{\mathrm{E}}_{i'_3,i'_4,\dots,i'_L} = \{\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L \end{smallmatrix}\right](0, z_2) | n_k \geq 0 \text{ and } 1 \leq i_k < k \text{ such that: } i_k \neq i'_k \text{ if } n_k = 1 \\ \text{and } i_k = 1 \text{ if } n_k = 0\} \end{aligned} \tag{3.39}$$

similar to the genus-zero basis (2.41). We also allow $i'_k = 0$ if we only intend to reduce the kernels with $n_k = 0$ and include all the kernels with $n_k = 1$, which certainly does not yield a basis, but a spanning set reduced by the redundant labeling of $g^{(0)}_{k,i_k} = 1$. In other words, the labels i'_k in $\mathcal{B}^{\mathrm{E}}_{i'_3,i'_4,\dots,i'_L}$ denote that the integrals defined by the set $\mathcal{B}^{\mathrm{E}}_{i'_3,i'_4,\dots,i'_L}$ are the genus-one Selberg integrals from eq. (3.36), where for $3 \leq k \leq L$ any kernel of the form $g^{(1)}_{k,i'_k}$ is rewritten in terms of the kernels $g^{(1)}_{k,i_k}$ with $1 \le i_k < k$ and $i_k \neq i'_k$ using integration by parts and the Fay identity. Similarly, any kernel $g^{(0)}_{k,i_k} = 1$ is simply denoted by $g^{(0)}_{k,1} = 1$.

3.3 Generating function for iterated integrals $\tilde{\Gamma}$ and the KZB associator

Before writing down a differential equation of KZB type for a vector of genus-one Selberg integrals in subsection 3.4 below, which is the genus-one generalization of the KZ equation (2.40), let us consider its formal solution[8] in terms of the so-called (elliptic) KZB associator, originally described in ref. [41]. [9] In fact, although usually represented in a language using a derivation algebra, we would like to point out that the equation as well as its formal solution is very naturally expressed in terms of the canonical iterated integrals $\tilde{\Gamma}$ on the genus-one Riemann surface.

By following exactly the same line of arguments as in subsection 2.4, let us start from a generating function [10]

$$\mathrm{L}^{\mathrm{E}}(z,\tau) = \sum_{w\in X} w\,\tilde{\Gamma}_w(z,\tau) \tag{3.40}$$

of the elliptic multiple polylogarithms $\tilde{\Gamma}_w(z,\tau)$, which can be shown to satisfy the differential equation

$$\frac{\partial}{\partial z}\,\mathrm{L}^{\mathrm{E}}(z,\tau) = \sum_{n\geq 0} g^{(n)}(z,\tau)x^{(n)}\,\mathrm{L}^{\mathrm{E}}(z,\tau)\,. \tag{3.41}$$

This differential equation is known as the Knizhnik–Zamolodchikov–Bernard equation (or KZB equation, for short) [42,43]. As for the genus-zero case, the asymptotic behavior around $z=0$ is determined by the asymptotics of the iterated integrals in eqs. (3.18) and (3.19) which amounts to

$$\mathrm{L}^{\mathrm{E}}(z,\tau) \sim \exp\left(x^{(1)}\,\tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix};z,\tau)\right) \sim (2\pi i z)^{x^{(1)}}\,. \tag{3.42}$$

Due to the one-periodicity (3.6) of the integration kernels $g^{(n)}$, the KZB equation is invariant under $z\mapsto z-1$ and, hence, there is another solution of the differential eq. (3.41), $\mathrm{L}^{\mathrm{E}}_1(z,\tau)$, with the following asymptotics near $z=1$

$$\mathrm{L}^{\mathrm{E}}_1(z,\tau) \sim \exp\left(x^{(1)}\,\tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix};z,\tau)\right) \sim (2\pi i(1-z))^{x^{(1)}}\,. \tag{3.43}$$

As for the genus-zero case, the associator

$$\Phi^{\mathrm{E}}(\tau) = \left(\mathrm{L}^{\mathrm{E}}_1(z,\tau)\right)^{-1}\mathrm{L}^{\mathrm{E}}(z,\tau) \tag{3.44}$$

is independent of z, which can be verified straightforwardly by taking the derivative of both sides of $L^{\mathrm{E}}_1\,\Phi^{\mathrm{E}} = L^{\mathrm{E}}$ and using the differential eq. (3.41). Thus, the elliptic associator $\Phi^{\mathrm{E}}(\tau)$

[8] As for the KZ equation, we are rather interested in relating a certain regularized boundary value to another regularized boundary value using an associator equation, than completely solving the equation. A rigorous discussion on solutions of the elliptic KZB equation can e.g. be found in ref. [40]

[9] KZB equations are the higher-genus generalization of the KZ equation [42,43]. In this article, we exclusively consider the elliptic KZB equation and the elliptic KZB associator. Therefore, we simply refer to these genus-one objects as KZB equation and KZB associator, respectively, while the genus-zero analogues are called KZ equation and Drinfeld associator.

[10] For this subsection, we explicitly denote the τ-dependence of the functions in order to keep track of the analytic behavior of certain limits. For example, in the asymptotic behavior shown in eqs. (3.42) and (3.43), the right-hand side is τ-independent.

can be expressed in the limit $z \to 1$, which yields the generating series of regularized eMZVs

$$\begin{aligned}\Phi^{\mathrm{E}}(\tau) &= \lim_{z\to 1} \exp\Big(-x^{(1)}\,\tilde{\Gamma}\big(\begin{smallmatrix}1\\0\end{smallmatrix};z,\tau\big)\Big)\,\mathrm{L}^{\mathrm{E}}(z,\tau)\\ &= \sum_{w\in X} w\,\omega(w^t;\tau)\,.\end{aligned} \tag{3.45}$$

The last equation follows from definition (3.40) and the cancellation of the divergent integrals due to the exponential prefactor in eq. (3.45). This is exactly the same mechanism which lead to the expression of the Drinfeld associator in terms of the regularized multiple zeta values in eq. (2.64) and effectively implements the appropriate regularization. Considering letters up to $x^{(2)}$ only, the first couple of terms of the KZB associator read

$$\begin{aligned}\Phi^{\mathrm{E}}(\tau) = 1 &+ x^{(0)}\omega(0;\tau) + x^{(1)}\omega(1;\tau) + x^{(2)}\omega(2;\tau)+\\ &+ x^{(0)}x^{(0)}\omega(0,0;\tau) + x^{(0)}x^{(1)}\omega(1,0;\tau) + x^{(0)}x^{(2)}\omega(2,0;\tau) + x^{(1)}x^{(0)}\omega(0,1;\tau)\\ &+ x^{(1)}x^{(1)}\omega(1,1;\tau) + x^{(1)}x^{(2)}\omega(2,1;\tau)\\ &+ x^{(2)}x^{(0)}\omega(0,2;\tau) + x^{(2)}x^{(1)}\omega(1,2;\tau) + x^{(2)}x^{(2)}\omega(2,2;\tau) + \cdots\\ = 1 &+ x^{(0)} - 2\zeta_2 x^{(2)}\\ &+ \frac{1}{2}x^{(0)}x^{(0)} - (x^{(0)}x^{(1)} - x^{(1)}x^{(0)})\omega(0,1;\tau) - \zeta_2(x^{(0)}x^{(2)} + x^{(2)}x^{(0)})\\ &+ \big(x^{(1)}x^{(2)} - x^{(2)}x^{(1)}\big)\big(\omega(0,3;\tau) - 2\zeta_2\omega(0,1;\tau)\big) + 5\zeta_4 x^{(2)}x^{(2)} + \cdots\end{aligned} \tag{3.46}$$

The elliptic associator $\Phi^{\mathrm{E}}(\tau)$ provides an associator equation similar to eq. (2.63) at genus zero: it connects the regularized boundary values of an arbitrary solution $\mathrm{F}^{\mathrm{E}}(z,\tau)$ of the KZB equation

$$\frac{\partial}{\partial z}\,\mathrm{F}^{\mathrm{E}}(z,\tau) = \sum_{n\geq 0} g^{(n)}(z,\tau)x^{(n)}\,\mathrm{F}^{\mathrm{E}}(z,\tau)\,, \tag{3.47}$$

which are regularized according to the asymptotic behavior shown in eqs. (3.42) and (3.43)

$$C_0^{\mathrm{E}}(\tau) = \lim_{z\to 0}(2\pi i z)^{-x^{(1)}}\,\mathrm{F}^{\mathrm{E}}(z,\tau)\,,\qquad C_1^{\mathrm{E}}(\tau) = \lim_{z\to 1}(2\pi i(z-1))^{-x^{(1)}}\,\mathrm{F}^{\mathrm{E}}(z,\tau)\,. \tag{3.48}$$

The calculation is similar to the genus-zero case (cf. eq. (2.61)) and the result is the genus-one associator equation

$$\begin{aligned}\Phi^{\mathrm{E}}(\tau)C_0^{\mathrm{E}}(\tau) &= \lim_{z\to 0}(\mathrm{L}_1^{\mathrm{E}}(z,\tau))^{-1}\,\mathrm{L}^{\mathrm{E}}(z,\tau)(2\pi i z)^{-x^{(1)}}\,\mathrm{F}^{\mathrm{E}}(z,\tau)\\ &= \lim_{z\to 1}(\mathrm{L}_1^{\mathrm{E}}(z,\tau))^{-1}\,\mathrm{F}^{\mathrm{E}}(z,\tau)\\ &= C_1^{\mathrm{E}}(\tau)\,.\end{aligned} \tag{3.49}$$

3.4 KZB equation for an auxiliary point

The one-loop version of the recursive construction of open-string amplitudes will again facilitate an extra marked point: the point z_2, which is the variable parametrizing the integration domain of the integrals in eq. (3.36).

In the limit $z_2 \to 1 = z_1 \mod \mathbb{Z}$, the integration domain closes and amounts to one complete boundary of the cylinder: it leads to $(L-1)$-point genus-one string corrections defined on $\mathcal{M}_{1,L-1}$. On the other hand, genus-one Selberg integrals degenerate to tree-level string corrections in the limit $z_2 \to 0 = z_1$, since the integration domain gets confined to a genus-zero domain. These two boundary values can be related by the genus-one associator equation (3.49) providing the genus-one analogue of the amplitude recursion of ref. [17].

Let us consider a vector of Selberg integrals with fixed upper labels, but lower labels stretching over all possible values:

$$\mathbf{S}^{\mathrm{E}(n_{k+1},\ldots,n_L)} = \begin{pmatrix} \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_{k+1}, & \ldots, & n_L \\ 1, & \ldots, & 1 \end{smallmatrix}\right](z_1,\ldots,z_k) \\ \vdots \\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_{k+1}, & \ldots, & n_L \\ k, & \ldots, & L-1 \end{smallmatrix}\right](z_1,\ldots,z_k) \end{pmatrix}. \tag{3.50}$$

Any of the one-loop string integrals, containing the $(L-1)$-point one-loop string corrections, to be calculated in subsection 3.6 below, will turn out to have $k=2$ and hence, we can restrict ourselves to the class of integrals defined in eq. (3.36). For the three-point example to be evaluated below, we have to consider integrals with $L=4$, such that we are going to work with vectors like

$$\mathbf{S}^{\mathrm{E}(2,1)} = \begin{pmatrix} \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 2, & 1 \\ 1, & 1 \end{smallmatrix}\right](z_1,z_2) \\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 2, & 1 \\ 1, & 2 \end{smallmatrix}\right](z_1,z_2) \\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 2, & 1 \\ 1, & 3 \end{smallmatrix}\right](z_1,z_2) \\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 2, & 1 \\ 2, & 1 \end{smallmatrix}\right](z_1,z_2) \\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 2, & 1 \\ 2, & 2 \end{smallmatrix}\right](z_1,z_2) \\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 2, & 1 \\ 2, & 3 \end{smallmatrix}\right](z_1,z_2) \end{pmatrix}. \tag{3.51}$$

The entries are going to be ordered canonically. As agreed on in the discussion of the spanning set $\mathcal{B}^{\mathrm{E}}_{i'_3,i'_4,\ldots,i'_L}$ defined in eq. (3.39), whenever there is an $n_k=0$, we write $i_k=1$ and we generally do not incorporate integration by parts identities to reduce the number of linearly independent integrals, i.e. we usually work with the set of integrals $\mathcal{B}^{\mathrm{E}}_{0,0,\ldots,0}$. Accordingly, if none of the labels $n_3,\ldots,n_L$ is zero, the vector $\mathbf{S}^{\mathrm{E}(n_3,\ldots,n_L)}$ has $(L-1)!$ components.

In establishing the KZB equation for a vector of Selberg integrals, we are going to take derivatives of $\mathbf{S}^{\mathrm{E}(n_3,\ldots,n_L)}(z_1=0,z_2)$ with respect to the auxiliary point z_2. While taking derivatives in the integral itself is elementary, combinatorics and in particular Fay identities kick in and lead to rather lengthy expressions. The guiding principle for achieving a canonical form can be deduced from the target KZB equation: we need to identify the analogue of the factors $g^{(n)}$ in eq. (3.47) in our scenario. In order to be able to pull the factors out of the Selberg integral, the indices i and j are confined to two and one: the factors will be $g^{(n)}_{21}=g^{(n)}(z_2-z_1)=g^{(n)}(z_2)$ with $n\geq 0$.

Accordingly, we take the z_2-derivative and afterwards apply Fay identities and partial fraction in order to find a factor $g^{(n)}_{21}$ in the integral, which then can be pulled out. A detailed discussion about this mechanism can be found in appendix C. In fact, a substantial part of the work in establishing the recursion at genus one consists of finding a suitable representation for the Selberg integrals, which leads to a nice and feasible form of the matrix coefficients in the KZB equation below.

In order to illustrate the procedure, let us consider the z_2-derivative of the Selberg vector $\mathbf{S}^{\mathrm{E}(0,1)}(z_1=0,z_2)$:

$$\frac{\partial}{\partial z_2}\mathbf{S}^{\mathrm{E}(0,1)} = g^{(0)}_{21}\begin{pmatrix} -s_{24} & -s_{24} & 0 & 0 & 0 & 0 & -s_{23} & 0 & 0 & 0 & 0 \\ s_{14} & s_{14}+s_{34} & s_{34} & 0 & 0 & 0 & 0 & -s_{23}-s_{34} & s_{34} & 0 & s_{34} \\ 0 & -s_{24} & -s_{24} & 0 & 0 & 0 & 0 & s_{24} & -s_{23}-s_{24} & 0 & -s_{24} \end{pmatrix}\begin{pmatrix} \mathbf{S}^{\mathrm{E}(0,2)} \\ \mathbf{S}^{\mathrm{E}(1,1)} \\ \mathbf{S}^{\mathrm{E}(2,0)} \end{pmatrix}$$

$$+\,g_{21}^{(1)}\begin{pmatrix} s_{12}+s_{24} & -s_{24} & 0 \\ -s_{14} & s_{12}+s_{14} & 0 \\ 0 & 0 & s_{12}\end{pmatrix}\mathbf{S}^{\mathrm{E}(0,1)} + g_{21}^{(2)}\begin{pmatrix} -s_{24} \\ s_{14} \\ 0\end{pmatrix}\mathbf{S}^{\mathrm{E}(0,0)}\,. \tag{3.52}$$

An immediate observation is in place: considering the weight of the derivative to be one, taking the weight n of each function $g_{21}^{(n)}$ into account and adding the weight of the genus-one Selberg integrals, the total weight is conserved in each term of the above equation.

Correspondingly, we collect all Selberg vectors of weight w into a larger vector $\mathbf{S}_w^{\mathrm{E}}(z_2)$:

$$\mathbf{S}_w^{\mathrm{E}}(z_2) = \Big(\mathbf{S}^{\mathrm{E}(n_3,n_4,\ldots,n_L)}(z_1=0,z_2)\Big)_{n_k\geq 0,\sum_{k=3}^{L} n_k=w}\,. \tag{3.53}$$

For the Selberg integrals in the above example, one could for example rewrite the

$$\mathbf{S}_2^{\mathrm{E}} = \begin{pmatrix}\mathbf{S}^{\mathrm{E}(0,2)} \\ \mathbf{S}^{\mathrm{E}(1,1)} \\ \mathbf{S}^{\mathrm{E}(2,0)}\end{pmatrix}, \tag{3.54}$$

where the three subvectors are given by

$$\mathbf{S}^{\mathrm{E}(0,2)} = \begin{pmatrix} \mathrm{S}^{\mathrm{E}}\begin{bmatrix}0, & 2\\ 1, & 1\end{bmatrix}(z_1,z_2) \\ \mathrm{S}^{\mathrm{E}}\begin{bmatrix}0, & 2\\ 1, & 2\end{bmatrix}(z_1,z_2) \\ \mathrm{S}^{\mathrm{E}}\begin{bmatrix}0, & 2\\ 1, & 3\end{bmatrix}(z_1,z_2)\end{pmatrix}, \quad \mathbf{S}^{\mathrm{E}(1,1)} = \begin{pmatrix} \mathrm{S}^{\mathrm{E}}\begin{bmatrix}1, & 1\\ 1, & 1\end{bmatrix}(z_1,z_2) \\ \mathrm{S}^{\mathrm{E}}\begin{bmatrix}1, & 1\\ 1, & 2\end{bmatrix}(z_1,z_2) \\ \mathrm{S}^{\mathrm{E}}\begin{bmatrix}1, & 1\\ 1, & 3\end{bmatrix}(z_1,z_2) \\ \mathrm{S}^{\mathrm{E}}\begin{bmatrix}1, & 1\\ 2, & 1\end{bmatrix}(z_1,z_2) \\ \mathrm{S}^{\mathrm{E}}\begin{bmatrix}1, & 1\\ 2, & 2\end{bmatrix}(z_1,z_2) \\ \mathrm{S}^{\mathrm{E}}\begin{bmatrix}1, & 1\\ 2, & 3\end{bmatrix}(z_1,z_2)\end{pmatrix}, \quad \mathbf{S}^{\mathrm{E}(2,0)} = \begin{pmatrix} \mathrm{S}^{\mathrm{E}}\begin{bmatrix}2, & 0\\ 1, & 1\end{bmatrix}(z_1,z_2) \\ \mathrm{S}^{\mathrm{E}}\begin{bmatrix}2, & 0\\ 2, & 1\end{bmatrix}(z_1,z_2)\end{pmatrix}. \tag{3.55}$$

So the vector $\mathrm{S}_2^{\mathrm{E}}$ captures the combinatorics from distributing weight two on the two slots (n_3,n_4) as well as the combinatorics of the labels i_k for each of those pairs. Neatly, the particular ordering does not play a role in the formalism to be described, however, we will follow the sorting convention in eq. (3.54).

As is visible from eq. (3.52), the z_2-derivative leads to Selberg integrals of different weight. Correspondingly, we are going to consider an infinitely large vector, where all vectors $\mathrm{S}_w^{\mathrm{E}}$ are joined in order of increasing w:

$$\mathbf{S}^{\mathrm{E}}(z_2) = \begin{pmatrix}\mathbf{S}_0^{\mathrm{E}} \\ \mathbf{S}_1^{\mathrm{E}} \\ \mathbf{S}_2^{\mathrm{E}} \\ \vdots\end{pmatrix}. \tag{3.56}$$

In appendix C, we prove that the form in eq. (3.52) can be achieved for any genus-one Selberg vector $\mathbf{S}^{\mathrm{E}}(z_2)$ for any number of insertion points L: it formally satisfies a KZB equation

$$\frac{\partial}{\partial z_2}\mathbf{S}^{\mathrm{E}}(z_2) = \sum_{n\geq 0} g_{21}^{(n)} x^{(n)}\,\mathbf{S}^{\mathrm{E}}(z_2)\,, \tag{3.57}$$

where the non-vanishing entries of the matrices $x^{(n)}$ are homogeneous polynomials of degree one in the parameters s_{ij} from the Selberg seed (3.28). The vector $\mathbf{S}^{\mathrm{E}}(z_2)$ is the genus-one analogue of the genus-zero Selberg vector $\mathbf{S}(x_3)$ defined in eq. (2.39), which satisfies the KZ eq. (2.40). Note that $\mathbf{S}^{\mathrm{E}}(z_2) = \mathbf{S}^{\mathrm{E}}(z_2,\tau)$ is actually τ-dependent, however, we will not denote the dependence explicitly.

Let us investigate the structure of eq. (3.57) a little further. As visible in example (3.52), taking a derivative of a Selberg integral will increase the weight by one. So taking a z_2-derivative on the Selberg vector $\mathbf{S}^{\mathrm{E}}_w$ yields

$$\frac{\partial}{\partial z_2}\mathbf{S}^{\mathrm{E}}_w(z_2) = \sum_{n=0}^{w+1} g_{21}^{(n)} x_w^{(n)}\, \mathbf{S}^{\mathrm{E}}_{w+1-n}(z_2)\,, \tag{3.58}$$

where the factor $x^{(n)}$ is linear in the parameters s_{ij} and does not contribute to the weight. From counting the weights, one can thus deduce that the matrices $x^{(n)}$ ought to be block-(off-)diagonal, where the size of the blocks corresponds the lengths of the Selberg vectors of weight w. Schematically, we find

$$\frac{\partial}{\partial z_2}\mathbf{S}^{\mathrm{E}} = g_{21}^{(0)}x^{(0)}\,\mathbf{S}^{\mathrm{E}} + g_{21}^{(1)} \underbrace{\left(\begin{array}{c|c|c|c|c} & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \ddots \end{array}\right)}_{x^{(1)}} \underbrace{\left(\begin{array}{c} \mathbf{S}^{\mathrm{E}}_0(z_2) \\ \hline \mathbf{S}^{\mathrm{E}}_1(z_2) \\ \hline \mathbf{S}^{\mathrm{E}}_2(z_2) \\ \hline \mathbf{S}^{\mathrm{E}}_3(z_2) \\ \hline \vdots \end{array}\right)}_{\mathbf{S}^{\mathrm{E}}} + g_{21}^{(2)}x^{(2)}\,\mathbf{S}^{\mathrm{E}} + \cdots\,, \tag{3.59}$$

where only the blue blocks are non-vanishing. Given the blocks in the above equation, the other matrices will have the following structure:

$$x^{(0)} = \left(\begin{array}{c|c|c|c|c} & x_0^{(0)} & & & \\ \hline & & x_1^{(0)} & & \\ \hline & & & x_2^{(0)} & \\ \hline & & & & x_3^{(0)} \\ \hline & & & & \ddots \end{array}\right), \qquad x^{(1)} = \left(\begin{array}{c|c|c|c|c} x_0^{(1)} & & & & \\ \hline & x_1^{(1)} & & & \\ \hline & & x_2^{(1)} & & \\ \hline & & & x_3^{(1)} & \\ \hline & & & & \ddots \end{array}\right),$$

$$x^{(2)} = \left(\begin{array}{c|c|c|c|c} & & & & \\ \hline x_1^{(2)} & & & & \\ \hline & x_2^{(2)} & & & \\ \hline & & x_3^{(2)} & & \\ \hline & & & x_3^{(2)} & \ddots \end{array}\right), \qquad \ldots\,, \tag{3.60}$$

where the blocks of the individual matrices are labeled by $x_w^{(n)}$.

In practice, one can not consider the infinitely long vector $\mathbf{S}^{\mathrm{E}}(z_2)$ and the corresponding infinitely many non-vanishing, infinite-dimensional matrices $x^{(n)}$. Instead, the vector $\mathbf{S}^{\mathrm{E}}(z_2)$

needs to be truncated at a certain maximal total weight $w_{\max}$

$$\mathbf{S}^{\mathrm{E}}_{\leq w_{\max}}(z_2) = \begin{pmatrix} \mathbf{S}^{\mathrm{E}}_0 \\ \mathbf{S}^{\mathrm{E}}_1 \\ \vdots \\ \mathbf{S}^{\mathrm{E}}_{w_{\max}} \end{pmatrix} . \tag{3.61}$$

Taking the z_2-derivative on the finite-length vector $\mathbf{S}^{\mathrm{E}}_{\leq w_{\max}}(z_2)$ leads to the differential equation

$$\frac{\partial}{\partial z_2}\mathbf{S}^{\mathrm{E}}_{\leq w_{\max}}(z_2) = \sum_{n=0}^{w_{\max}+1} g_{21}^{(n)} x^{(n)}_{\leq w_{\max}} \mathbf{S}^{\mathrm{E}}_{\leq w_{\max}}(z_2) + r_{w_{\max}} \mathbf{S}^{\mathrm{E}}_{w_{\max}+1}(z_2) \,, \tag{3.62}$$

where the remainder $r_{w_{\max}}$ prevents eq. (3.62) to be a complete KZB equation. However, as will be discussed below, this remainder may be disregarded when calculating one-loop string corrections up to a particular order in α'.

The matrices $x^{(n)}_{\leq w_{\max}}$ for $0 \leq n \leq w_{\max}+1$ correspond to the upper-left $(w_{\max}+1)\times(w_{\max}+1)$ block matrices of these matrices $x^{(n)}$. Explicitly:

$$x^{(0)}_{\leq w_{\max}} = \begin{pmatrix} & x^{(0)}_0 & & & \\ & & x^{(0)}_1 & & \\ & & & \ddots & \\ & & & & x^{(0)}_{w_{\max}-1} \\ & & & & \end{pmatrix} , \qquad x^{(1)}_{\leq w_{\max}} = \begin{pmatrix} x^{(1)}_0 & & & & \\ & x^{(1)}_1 & & & \\ & & x^{(1)}_2 & & \\ & & & \ddots & \\ & & & & x^{(1)}_{w_{\max}} \end{pmatrix} , \tag{3.63}$$

$$x^{(2)}_{\leq w_{\max}} = \begin{pmatrix} & & & & \\ x^{(2)}_1 & & & & \\ & x^{(2)}_2 & & & \\ & & \ddots & & \\ & & & x^{(2)}_{w_{\max}} & \end{pmatrix} , \ldots , x^{(w_{\max}+1)}_{\leq w_{\max}} = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ x^{(w_{\max}+1)}_{w_{\max}} & & & & \end{pmatrix} . \tag{3.64}$$

Moreover, we see that the remainder $r_{w_{\max}}$ is the $(w_{\max}+1)\times 1$ block submatrix of the first $w_{\max}+1$ blocks of the $(w_{\max}+2)$-column of the matrix $x^{(0)}_{\leq w_{\max}+1}$:

$$x^{(0)}_{\leq w_{\max}+1} = \begin{pmatrix} & x^{(0)}_0 & & & & \\ & & x^{(0)}_1 & & & \\ & & & \ddots & & \\ & & & & x^{(0)}_{w_{\max}-1} & \\ & & & & & x^{(0)}_{w_{\max}} \\ & & & & & \end{pmatrix} , \qquad r_{w_{\max}} = \begin{pmatrix} \\ \\ \vdots \\ \\ x^{(0)}_{w_{\max}} \end{pmatrix} . \tag{3.65}$$

3.5 Boundary values

Having a (modified) KZB equation for the genus-one Selberg integrals at hand, we would like to apply the genus-one associator equation (3.49). In order to do so, let us investigate the two regularized boundary values $\mathbf{C}_0^{\mathrm{E}}$ and $\mathbf{C}_1^{\mathrm{E}}$ of $\mathbf{S}^{\mathrm{E}}(z_2)$ for $z_2 \to 0, 1$. We will show, that these boundary values comprise the tree-level and the one-loop string corrections, respectively.

Following the definition of the regularized boundary values in eq. (3.48), we will have to evaluate

$$\mathbf{C}_0^{\mathrm{E}} = \lim_{z_2\to 0}(2\pi i z_2)^{-x^{(1)}}\,\mathbf{S}^{\mathrm{E}}(z_2) = \lim_{z_2\to 0}\begin{pmatrix}(2\pi i z_2)^{-x_0^{(1)}}\,\mathbf{S}_0^{\mathrm{E}}(z_2)\\(2\pi i z_2)^{-x_1^{(1)}}\,\mathbf{S}_1^{\mathrm{E}}(z_2)\\ \vdots\end{pmatrix} = \begin{pmatrix}\mathbf{C}_{0,0}^{\mathrm{E}}\\ \mathbf{C}_{0,1}^{\mathrm{E}}\\ \vdots\end{pmatrix},$$

$$\mathbf{C}_1^{\mathrm{E}} = \lim_{z_2\to 1}(2\pi i(1-z_2))^{-x^{(1)}}\,\mathbf{S}^{\mathrm{E}}(z_2) = \lim_{z_2\to 1}\begin{pmatrix}(2\pi i(1-z_2))^{-x_0^{(1)}}\,\mathbf{S}_0^{\mathrm{E}}(z_2)\\(2\pi i(1-z_2))^{-x_1^{(1)}}\,\mathbf{S}_1^{\mathrm{E}}(z_2)\\ \vdots\end{pmatrix} = \begin{pmatrix}\mathbf{C}_{1,0}^{\mathrm{E}}\\ \mathbf{C}_{1,1}^{\mathrm{E}}\\ \vdots\end{pmatrix}, \tag{3.66}$$

where $\mathbf{C}_{0,w}^{\mathrm{E}}$ and $\mathbf{C}_{1,w}^{\mathrm{E}}$ denote the regularized limits of the subvectors $\mathbf{S}_w^{\mathrm{E}}(z_2)$ of weight w and the second equality in the above equations follows from the block-diagonal form of $x^{(1)}$. Switching again to finite matrix size, we define

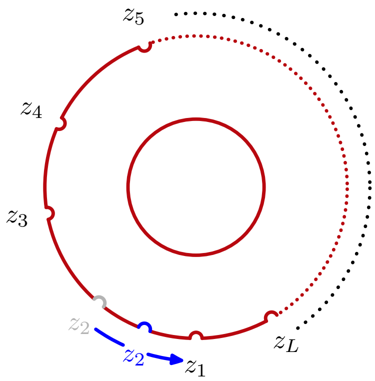
$$\begin{aligned}\mathbf{C}_{0,\leq w_{\max}}^{\mathrm{E}} &= \lim_{z_2\to 0}(2\pi i z_2)^{-x^{(1)}_{\leq w_{\max}}}\,\mathbf{S}_{\leq w_{\max}}^{\mathrm{E}}(z_2)\,,\\ \mathbf{C}_{1,\leq w_{\max}}^{\mathrm{E}} &= \lim_{z_2\to 1}(2\pi i(1-z_2))^{-x^{(1)}_{\leq w_{\max}}}\,\mathbf{S}_{\leq w_{\max}}^{\mathrm{E}}(z_2)\,.\end{aligned} \tag{3.67}$$

Boundary value $\mathbf{C}_1^{\mathrm{E}}$: Considering the limit $z_2 \to 1$, we first determine the behavior of $\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}n_3, & \ldots, & n_L\\ i_3, & \ldots, & i_L\end{smallmatrix}\right](z_2)$ and include the regularization factor $2\pi i(1-z_2)^{-x^{(1)}}$ afterwards. According to eq. (3.14), the genus-one Selberg seed degenerates as follows

$$\begin{aligned}&\lim_{z_2\to 1}\mathrm{S}^{\mathrm{E}}\\ &= \lim_{z_2\to 1}\prod_{0=z_1<z_i<z_j<z_2}\exp\left(s_{ij}\,\tilde\Gamma_{ji}\right)\prod_{j>2}\exp\left(s_{1j}\,\tilde\Gamma_{j1}\right)\prod_{i\neq 2}\exp\left(s_{i2}\,\tilde\Gamma_{2i}\right)\\ &= \lim_{z_2\to 1}(2\pi i(1-z_2))^{s_{12}}\prod_{0=z_1<z_i<z_j<z_2}\exp\left(s_{ij}\,\tilde\Gamma_{ji}\right)\prod_{j>2}\exp\left((s_{1j}+s_{2j})\,\tilde\Gamma_{j1}\right) + \mathcal{O}\Big((1-z_2)^{s_{12}+1}\Big)\\ &= \lim_{z_2\to 1}(2\pi i(1-z_2))^{s_{12}}\,\mathrm{S}^{\mathrm{E}}\Big|_{(L-1)\text{-point}}^{\tilde s_{1j}=s_{1j}+s_{2j}} + \mathcal{O}\Big((1-z_2)^{s_{12}+1}\Big)\,,\end{aligned} \tag{3.68}$$

where we have used the symmetry property described in eq. (3.12). The factor denoted by $\mathrm{S}^{\mathrm{E}}\Big|_{(L-1)\text{-point}}^{\tilde s_{1j}=s_{1j}+s_{2j}}$ is the genus-one Selberg seed for $L-1$ insertion points on the cylinder boundary

$0 = z_1 < z_L < z_{L-1} < \cdots < z_3 < 1 = z_1 \bmod \mathbb{Z}$.



(3.69)

Since the insertion points z_2 and z_1 merge in the limit, we assign effective Mandelstam variables $\tilde{s}_{1j}$ to z_1

$$\tilde{s}_{1j} = s_{1j} + s_{2j} \, . \tag{3.70}$$

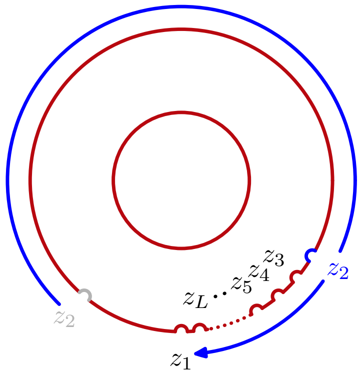
In terms of a momentum interpretation, we find the same behavior as in the genus-zero case in eq. (2.69): the momentum of the external state which corresponds to one of the fixed insertion points receives two contributions, one coming from the state at $z_1 = 0$ and the other from a state at the same position of the cylinder boundary $z_2 = z_1 \mod \mathbb{Z}$ due to the merged auxiliary insertion point z_2.

Accordingly, the genus-one Selberg integral defined in eq. (3.36) on the configuration space of the torus with two positions fixed, degenerates at lowest order in $(1-z_2)$ up to a (vanishing) prefactor to an integral on $\mathcal{M}_{1,L-1}$:

$$\begin{aligned}
&\lim_{z_2\to 1} \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3, & \ldots, & n_L \\ i_3, & \ldots, & i_L \end{smallmatrix}\right](z_1=0, z_2) \\
&= \lim_{z_2\to 1} (2\pi i(1-z_2))^{s_{12}} \int_{\mathcal{C}(z_2\to 1)} \prod_{i=3}^{L} dz_i \, \mathrm{S}^{\mathrm{E}}\Big|^{\tilde{s}_{1j}=s_{1j}+s_{2j}}_{(L-1)\text{-point}} \prod_{k=3}^{L} g^{(n_k)}_{k,i_k}\big|_{z_2=z_1=0} + \mathcal{O}\Big((1-z_2)^{s_{12}+1}\Big) \\
&= \lim_{z_2\to 1} (2\pi i(1-z_2))^{s_{12}} \, \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3, & \ldots, & n_L \\ i_3, & \ldots, & i_L \end{smallmatrix}\right](0, z_2 = 1 = 0 \bmod \mathbb{Z})\Big|^{\tilde{s}_{1j}=s_{1j}+s_{2j}}_{z_2=z_1=0} + \mathcal{O}\Big((1-z_2)^{s_{12}+1}\Big) \, .
\end{aligned} \tag{3.71}$$

By the same arguments which led to eq. (2.68), we find that the relevant eigenvalues of the matrices $x_w^{(1)}$ which correspond to the subspace containing the one-loop string corrections are s_{12}, such that the regularization factor $2\pi i(1-z_2)^{-x_w^{(1)}}$ cancels the otherwise vanishing prefactor $(2\pi i(1-z_2))^{s_{12}}$ in eq. (3.71) and the entries of $\mathbf{C}_1^{\mathrm{E}}$ are given by the degenerate genus-one Selberg integrals $\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3, & \ldots, & n_L \\ i_3, & \ldots, & i_L \end{smallmatrix}\right](0, z_2 = 1 = 0 \bmod \mathbb{Z})|^{\tilde{s}_{1j}=s_{1j}+s_{2j}}_{z_2=z_1=0}$.

Boundary value $\mathbf{C}_0^{\mathrm{E}}$: The boundary value $\mathbf{C}_0^{\mathrm{E}}$ is obtained by confining the region of integration to an infinitesimal interval as $z_2 \to 0 = z_1$. As for the genus-zero calculation in eq. (2.73), the main tool to investigate this degeneration and the corresponding behavior of genus-one Selberg integrals is a change of variables $z_i = z_2 x_i$, where x_i are points in the unit interval on the real line whereas z_i lay on the boundary of a cylinder.



(3.72)

According to the discussion after eq. (3.27) and as a consequence of eq. (3.13), the seed S^{E} degenerates at lowest order in z_2 for $z_2 \to 0$ up to a proportionality factor to the genus-zero Selberg seed S for the L points $0 = x_1 < x_L < x_{L-1} < \cdots < x_2 = 1$ on the unit interval, which is (cf. eq. (2.83)) precisely the $(L+1)$-point genus-zero Koba–Nielsen factor defined in eq. (2.82):

$$\begin{aligned}
\lim_{z_2\to 0} \mathrm{S}^{\mathrm{E}} &= \lim_{z_2\to 0} \prod_{x_i<x_j} \exp\Big(s_{ij}\, \tilde{\Gamma}_{x^{(1)}}(z_2 x_{ji}, \tau)\Big) \\
&= \lim_{z_2\to 0} (2\pi i z_2)^{s_{12\ldots L}} \prod_{0=x_1<x_i<x_j<x_2=1} x_{ji}^{s_{ij}} + \mathcal{O}\Big((z_2)^{s_{12\ldots L}+1}\Big) \\
&= \lim_{z_2\to 0} (2\pi i z_2)^{s_{12\ldots L}}\, \mathrm{S}\,|_{L\text{-point}} + \mathcal{O}\Big((z_2)^{s_{12\ldots L}+1}\Big) \\
&= \lim_{z_2\to 1} (2\pi i z_2)^{s_{12\ldots L}}\, \mathrm{KN}\,|_{(L+1)\text{-point}} + \mathcal{O}\Big((z_2)^{s_{12\ldots L}+1}\Big)\,.
\end{aligned} \tag{3.73}$$

The discussion of the eigenvalues of $x_w^{(1)}$ is analogous to the genus-zero case. It turns out that the maximal and therefore dominant eigenvalue of $x_w^{(1)}$ is $s_{12\ldots L}$, such that the regularization $(2\pi i z_2)^{-x_w^{(1)}}$ cancels the prefactor $(2\pi i z_2)^{s_{12\ldots L}}$ in eq. (3.73). Thus, the entries of $\mathbf{C}_0^{\mathrm{E}}$ are given by

$$\begin{aligned}
&\lim_{z_2\to 0} (2\pi i z_2)^{-s_{12\ldots L}}\, \mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix} n_3, & \ldots, & n_L \\ i_3, & \ldots, & i_L \end{smallmatrix}\Big](z_1 = 0, z_2) \\
&= \lim_{z_2\to 0} z_2^{L-2} \int_{\mathcal{C}(x_2=1)} \prod_{i=3}^{L} dx_i\, \mathrm{S}\,|_{L\text{-point}} \prod_{k=3}^{L} g^{(n_k)}(z_2 x_{k,i_k}, \tau) \\
&= \begin{cases} \int_{\mathcal{C}(x_2=1)} \prod_{i=3}^{L} dx_i\, \mathrm{S}\,|_{L\text{-point}} \prod_{k=3}^{L} \frac{1}{x_{k i_k}} & \text{if } n_1 = n_2 = \cdots = n_k = 1\,, \\ 0 & \text{otherwise}\,. \end{cases}
\end{aligned} \tag{3.74}$$

The only non-vanishing entries are the ones for which all integration kernels have weight one, i.e. $n_k = 1$, since only their pole can compensate for the z_2^{L-2} factor from the measure.

A similar behavior was observed for the genus-zero boundary value which led to eq. (2.75). Moreover, these simple poles ensure that the only non-vanishing integrals are exactly the degenerate genus-zero Selberg integrals $\mathrm{S}[i_3, i_4, \ldots, i_L](0, 1, x_2 = 1)$ found in the genus-zero regularized boundary values $\mathbf{C}_0$ and $\mathbf{C}_1$ in eqs. (2.75) and (2.69), respectively. However, here we recover integrals defined for L points on the unit interval, which constitute a basis of the twisted de Rham cohomology of $\mathcal{M}_{0,L+1}$ with $(L+1)$ insertion points $0 = x_1 < x_L < x_{L+1} < \cdots < x_2 = 1 < x_{L+1} = \infty$. As discussed in subsection 2.7, these integrals are related to the $(L+1)$-point genus-zero string corrections by a basis transformation.

Mandelstam variables: In contrast to both the genus-zero discussion and the limiting situation $\mathbf{C}_1^{\mathrm{E}}$, in the boundary value $\mathbf{C}_0^{\mathrm{E}}$ the Mandelstam variables s_{2j} in eq. (3.73) associated to the auxiliary insertion point z_2 are not redundant: the auxiliary genus-one momentum $k_2^{\text{1-loop}}$ associated to z_2 encodes the genus-zero momentum k_2^{tree} associated to the tree-level insertion point x_2

$$k_2^{\text{1-loop}} = k_2^{\text{tree}} . \tag{3.75}$$

In order to keep track of how this momentum contributes to the one-loop momenta, two distinct processes have to be considered: first, the topological change by the identification of x_1 with x_{L+1} giving the genus-one insertion point z_1 depicted in figure (3.1) and second, the merging of $z_2 \to 1 = z_1 \bmod \mathbb{Z}$ shown in figure (3.69). In the first case, the momenta k_1^{tree} and k_{L+1}^{tree} associated to x_1 and x_{L+1}, respectively, yield the joint contribution to the one-loop momentum associated to z_1

$$k_1^{\text{1-loop}} = k_1^{\text{tree}} + k_{L+1}^{\text{tree}} . \tag{3.76}$$

The second limit is the merging of z_2 to z_1, which adds the momentum $k_2^{\text{1-loop}}$ associated to z_2 to the momentum $k_1^{\text{1-loop}}$ and we expect to find the effective momentum

$$\tilde{k}_1^{\text{1-loop}} = k_1^{\text{1-loop}} + k_2^{\text{1-loop}} = k_1^{\text{tree}} + k_{L+1}^{\text{tree}} + k_2^{\text{tree}} \tag{3.77}$$

for the insertion point $z_1 = z_2 \bmod \mathbb{Z}$ of the $(L-1)$-point one-loop interaction in the regularized boundary value $\mathbf{C}_1^{\mathrm{E}}$, where we denote the one-loop momenta $k_i^{\text{1-loop}}$ in the limit $z_2 \to z_1 = 1 \bmod \mathbb{Z}$ by a tilde as depicted in figure eq. (3.82). However, from our calculations of $\mathbf{C}_1^{\mathrm{E}}$ in eq. (3.68) we see that the Mandelstam variables associated to $\tilde{k}_1^{\text{1-loop}}$ are

$$\tilde{s}_{1j} = s_{1j} + s_{2j} . \tag{3.78}$$

Therefore, the actual one-loop momentum associated to $z_1 = z_2 \bmod \mathbb{Z}$ turns out to be

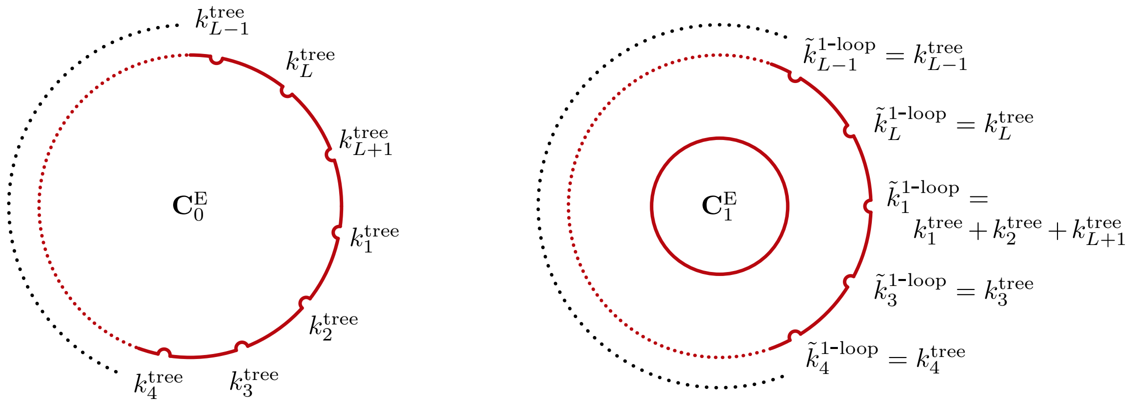
$$\tilde{k}_1^{\text{1-loop}} = k_1^{\text{tree}} + k_2^{\text{tree}} . \tag{3.79}$$

This is in agreement with simultaneous momentum conservation in the tree-level and one-loop interaction if and only if

$$k_{L+1}^{\text{tree}} = 0 , \tag{3.80}$$

which can be interpreted as follows: going the first procedure discussed above, which is depicted in figure (3.1), in the other direction from genus one to genus zero, the momentum $k_1^{\text{1-loop}}$ associated to z_1 is expected to split in a certain way and to contribute to the two tree-level momenta k_1^{tree} and k_{L+1}^{tree} accordingly. From eq. (3.80) follows that these two contributions are very unequal: while the momentum associated to x_1 obtains the full contribution $k_1^{\text{tree}} = k_1^{\text{1-loop}}$, the momentum associated to x_{L+1} goes away empty-handed $k_{L+1}^{\text{tree}} = 0$. Note that the momenta associated the remaining tree-level insertion points x_i for $i = 3, 4, \dots, L$ are exactly the one-loop momenta associated to the punctures z_i for any $0 < z_2 \leq 1 = z_1 \bmod \mathbb{Z}$:

$$\tilde{k}_i^{\text{1-loop}} = k_i^{\text{1-loop}} = k_i^{\text{tree}} \quad \text{for } i = 3, 4, \dots, L . \tag{3.81}$$



(3.82)

Summary: The regularized boundary value $\mathbf{C}_0^{\text{E}}$ is found to only have finitely many non-vanishing entries which are degenerate genus-zero Selberg integrals and hence linear combinations of $(N+2) = (L+1)$-point tree-level string corrections. In turn, as will be discussed in detail in the next subsection, the entries of $\mathbf{C}_1^{\text{E}}$ given by eq. (3.71) contain the $N = (L-1)$-point one-loop string corrections.

Therefore, the genus-one Selberg vector $\mathbf{S}^{\text{E}}(z_2)$ indeed interpolates between the genus-zero and genus-one string corrections and the corresponding associator equation

$$\mathbf{C}_1^{\text{E}} = \Phi^{\text{E}} \, \mathbf{C}_0^{\text{E}} \tag{3.83}$$

provides a recursion linking genus zero and genus one and generalizing the genus-zero recursion from ref. [17].

The consideration about the contributions of the insertion points defining the genus-one Selberg integrals to the Mandelstam variables in the string corrections appearing in the boundary values $\mathbf{C}_0^{\text{E}}$ and $\mathbf{C}_1^{\text{E}}$ leads to a geometric interpretation of the associator eq. (3.83): the N-point one-loop worldsheet is obtained from the $(N+2)$-point tree-level worldsheet by an effective gluing of the two legs corresponding to the insertion points $x_1 = 0$ and $x_{L+1} = \infty$ on the Riemann sphere. By momentum conservation the Mandelstam variables associated to the insertion point z_1 in the one-loop string corrections of $\mathbf{C}_1^{\text{E}}$ are given by the sum $\tilde{s}_{1j} = s_{1j} + s_{2j}$.

3.6 Open-string amplitudes at genus one

The associator equation (3.83) can be employed to calculate the α'-expansion of the N-point one-loop string corrections up to any desired order in α' from $(N+2)$-point tree-level integrals.

While setting up the calculation and relating various entries of the regularized boundary values to known integral representations for string corrections at genus zero and genus one, we will simultaneously single out the relevant parts of the matrix equation (3.83) and thereby substantially improve applicability of our method.

The main goal is the calculation of the N-point one-loop string-correction up to order $o_{\text{max}}^{\text{1-loop}}$ in α'. As observed in the previous subsection, integrals on $\mathcal{M}_{1,N}$ defining the N-point one-loop string corrections arise in the $z_2 \to 1$ limit of genus-one Selberg integrals with $L = N+1$ marked points. Simultaneously, $(N+2)$-point tree-level string corrections are encoded in the $z_2 \to 0$ limit of the same genus-one Selberg integrals.

As pointed out at the end of subsection 3.4 above, for practical calculations we wil have to truncate the infinite genus-one Selberg vector to $\mathbf{S}^{\text{E}}_{\leq w_{\text{max}}}(z_2)$. Given the target values N and

$o^{\text{1-loop}}_{\max}$ for the calculation, let us determine $w_{\max}$ as well as various other parameters for the calculation.

Each of the objects on the right-hand side in eq. (3.83) has an expansion in the parameter α': since $x^{(n)} \propto \alpha'$ (cf. eq. (2.1), the expansion in word length of the elliptic KZB associator is exactly its α'-expansion. The α'-expansion of the tree-level integrals in $\mathbf{C}_0^E$ can be obtained from the recursions in refs. [44, 17]. Therefore, the maximal target α'-order $o^{\text{1-loop}}_{\max}$ of the one-loop string corrections on the right-hand side is reached, when the KZB-associator is expanded up to α'-order

$$l_{\max} = o^{\text{1-loop}}_{\max} - o^{\text{tree}}_{\min} \tag{3.84}$$

where $o^{\text{tree}}_{\min}$ denotes the leading (e.g. minimal) order in the α'-expansion of tree-level integrals in $\mathbf{C}_0^{\mathrm{E}}$. This order turns out to be given by [30]

$$o^{\text{tree}}_{\min} = 2 - L = 3 - N\,. \tag{3.85}$$

In order to determine $w_{\max}$, we need to think about the positions of the relevant information within the vectors $\mathbf{C}_0^{\mathrm{E}}$ and $\mathbf{C}_1^{\mathrm{E}}$: on the one hand, according to eq. (3.74) the non-vanishing subvector of $\mathbf{C}_0^{\mathrm{E}}$ which includes the tree-level string corrections is contained in the weight

$$w_0 = L - 2 \tag{3.86}$$

subvector $\mathbf{C}_{0,w_0}^{\mathrm{E}}$ of $\mathbf{C}_0^{\mathrm{E}}$. On the other hand, the one-loop string corrections are contained in the weight

$$w_1 = L - 5 - d \tag{3.87}$$

subvector $\mathbf{C}_{1,w_1}^{\mathrm{E}}$. The quantity d denotes the number of additional factors of $g^{(n)}$ appearing in higher-point one-loop string integrals: $d = 0$ for $L \leq 8$ and $d \geq 0$ [35]. For all calculations in this article, $d = 0$ holds. The relevant part of the elliptic KZB associator is the submatrix $\Phi^{\mathrm{E}}_{w_1,w_0}$, which satisfies the equation

$$\mathbf{C}_{1,w_1}^{\mathrm{E}} = \Phi^{\mathrm{E}}_{w_1,w_0}\,\mathbf{C}_{0,w_0}^{\mathrm{E}}\,. \tag{3.88}$$

Since for all amplitude situations we find $w_1 < w_0$, the submatrix $\Phi^{\mathrm{E}}_{w_1,w_0}$ is located above the diagonal of Φ^{E}.

Here comes the block-(off-)diagonal form of the matrices $x^{(n)}$ depicted in (3.60) into play, which ensures that for a certain word length l, only finitely many words $w = x^{(n_1)} \dots x^{(n_l)}$ contribute non-trivially to $\Phi^{\mathrm{E}}_{w_1,w_0}$. A detailed discussion, where sufficient and necessary conditions for a word to contribute non-trivially to $\Phi^{\mathrm{E}}_{w_1,w_0}$ are formulated, is given in appendix C.2.

The α'-expansion of $\Phi^{\mathrm{E}}_{w_1,w_0}$ up to some maximal order $l_{\max}$ in α' or maximal word length, respectively, can be calculated by finite-dimensional submatrices of $x^{(n)}$, which are the matrices $x^{(n)}_{\leq w_{\max}}$ for some maximal weight $w_{\max} \geq w_0$:

$$\Phi^{\mathrm{E}}(x^{(n)})_{w_1,w_0} = \Phi^{\mathrm{E}}(x^{(n)}_{\leq w_{\max}})_{w_1,w_0} + \mathcal{O}\Big((\alpha')^{l_{\max}+1}\Big)\,. \tag{3.89}$$

The integer $w_{\max} = w_{\max}(l_{\max})$ is determined in appendix C.2 by carefully sorting out, which blocks $x^{(n)}_w$ contribute to $\Phi^{\mathrm{E}}(x^{(n)})_{w_1,w_0}$. The result is that the α'-expansion of the $(L-1)$-point one-loop string corrections up to order $o^{\text{1-loop}}_{\max}$ can be calculated from the associator eq. (3.88) using words up to the maximal word length $l_{\max}$ defined in eq. (3.84) and the maximal weight

$$w_{\max} = \max(l_{\max} + w_1 - w_0, w_0)\,. \tag{3.90}$$

In other words, the associator submatrix $\Phi^{\mathrm{E}}(x^{(n)})_{w_1,w_0}$ can be deduced from a truncated associator, which is determined by evaluating the matrix products of truncated representations of letters, taking only words up to length $l_{\max}$ and weight $w_{\max}$ into account. The truncated matrix representations $x^{(n)}_{\leq w_{\max}}$ of the letters can be obtained from the modified KZB eq. (3.62). Since word length $l_{\max}$ and maximal weight $w_{\max}$ are finite quantities, all sums consist of a finite number of terms and all matrices are of finite size. The process yields the finite-dimensional, truncated associator equation

$$\mathbf{C}^{\mathrm{E}}_{1,\leq w_{\max}} + \mathcal{O}\Big((\alpha')^{o^{\text{1-loop}}_{\max}+1}\Big) = \Phi^{\mathrm{E}}_{l_{\max}}(x^{(n)}_{\leq w_{\max}})\; \mathbf{C}^{\mathrm{E}}_{0,\leq w_{\max}}\,, \tag{3.91}$$

where $\Phi^{\mathrm{E}}_{l_{\max}}$ is the truncation of Φ^{E} at the maximal word length $l_{\max}$. The finite subvectors

$$\begin{aligned}\mathbf{C}^{\mathrm{E}}_{0,\leq w_{\max}} &= \lim_{z_2\to 0}(2\pi i z_2)^{-x^{(1)}_{\leq w_{\max}}}\,\mathbf{S}^{\mathrm{E}}_{\leq w_{\max}}(z_2)\,,\\ \mathbf{C}^{\mathrm{E}}_{1,\leq w_{\max}} &= \lim_{z_2\to 1}(2\pi i(1-z_2))^{-x^{(1)}_{\leq w_{\max}}}\,\mathbf{S}^{\mathrm{E}}_{\leq w_{\max}}(z_2)\end{aligned} \tag{3.92}$$

of $\mathbf{C}^{\mathrm{E}}_0$ and $\mathbf{C}^{\mathrm{E}}_1$, respectively, contain the $(L+1)$-point tree-level string corrections at weight $w_0 = L-2 \leq w_{\max}$ and the $(L-1)$-point one-loop corrections at $w_1 = L-5-d$. Thus, denoting by $\Phi^{\mathrm{E}}_{l_{\max}}(x^{(n)}_{\leq w_{\max}})_{w_1,w_0}$ the weight-(w_1,w_0) submatrix of the truncated KZB associator $\Phi^{\mathrm{E}}_{l_{\max}}(x^{(n)}_{\leq w_{\max}})$, the relevant truncated vector equation which relates the string corrections to each other is

$$\mathbf{C}^{\mathrm{E}}_{1,w_1} + \mathcal{O}\Big((\alpha')^{o^{\text{1-loop}}_{\max}+1}\Big) = \Phi^{\mathrm{E}}_{l_{\max}}(x^{(n)}_{\leq w_{\max}})_{w_1,w_0}\; \mathbf{C}^{\mathrm{E}}_{0,w_0}\,. \tag{3.93}$$

where $\Phi^{\mathrm{E}}_{l_{\max}}(x^{(n)}_{\leq w_{\max}})_{w_1,w_0}$ is the weight-(w_1,w_0) submatrix of the truncated elliptic KZB associator $\Phi^{\mathrm{E}}_{l_{\max}}(x^{(n)}_{\leq w_{\max}})$.

3.6.1 Two points

As a first example, let us calculate the two-point one-loop string correction. This correction is non-trivial only, if we treat the Mandelstam variables s_{ij} as independent parameters of the integrals, which do not satisfy any constraints like the ones imposed by momentum conservation. The two-point string-correction is given by the integral [21]

$$S^{\text{1-loop}}_{\text{2-point}}(\tilde{s}_{13}) = \int_0^1 dz_3 \exp\left(\tilde{s}_{13}\,\tilde{\Gamma}_{31}\right) = \sum_{n\geq 0}\tilde{s}^n_{13}\,\omega(\underbrace{1,\ldots,1}_{n},0)\,, \tag{3.94}$$

where $\tilde{s}_{13}$ is the Mandelstam variable associated to the loop momentum. Since the integral requires two vertex insertion points, the appropriate genus-one Selberg integral with an extra insertion point z_2 is of length $L=3$ and the insertion points are ordered as

$$0 = z_1 < z_3 < z_2 < 1 = z_1 \bmod \mathbb{Z} \tag{3.95}$$

on the boundary of the cylinder. Indeed, in the limit $z_2 \to 1$, the punctures z_2 and z_1 merge, leaving the two punctures relevant for the one-loop string corrections. Thus, we consider the iterated integrals

$$\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3\\ i_3\end{smallmatrix}\right](0,z_2) = \int_0^{z_2} dz_3\, \mathrm{S}^{\mathrm{E}}\, g^{(n_3)}_{3i_3}\,,\quad \mathrm{S}^{\mathrm{E}} = \exp\left(s_{13}\,\tilde{\Gamma}_{31} + s_{12}\,\tilde{\Gamma}_{21} + s_{23}\,\tilde{\Gamma}_{23}\right)\,,\quad 1\leq i_3 < 3\,. \tag{3.96}$$

In this section, we point out and explain the different steps of the calculation of the α'-expansion of the two-point one-loop string correction from the four-point tree-level integral and write down the explicit results of each step necessary to obtain the expansion up to order $o^{\text{1-loop}}_{\max}=2$ in α'. Additional details of the calculation are collected in appendix B.1.

According to eq. (3.87), the two-point one-loop correction can be found in the weight $w_1=0$ entry $\mathbf{C}^{\mathrm{E}}_{1,w_1}$, while the tree-level correction resides at weight $w_0=1$ (cf. eq. (3.86)). The α'-expansion of the four-point tree-level correction turns out to start at order $o^{\text{tree}}_{\min}=-1$, (cf. eq. (3.104)). Therefore, consulting eq. (3.90), it is sufficient to consider the truncated Selberg vector at maximal weight $w_{\max}=2$ to calculate the one-loop string corrections up to second order in α', i.e. we only need to consider the vector

$$\mathbf{S}^{\mathrm{E}}_{\leq 2}(z_2)=\begin{pmatrix}\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,z_2)\\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,z_2)\\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right](0,z_2)\\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right](0,z_2)\end{pmatrix} \tag{3.97}$$

where we use the reduced set of integrals $\mathcal{B}^{\mathrm{E}}_2$ obtained from the relations

$$\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,z_2)=\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right](0,z_2)\,,\qquad s_{13}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,z_2)=-s_{23}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right](0,z_2) \tag{3.98}$$

to exclude the integrals $\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right](0,z_2)$ and $\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right](0,z_2)$ from our analysis.

Before we can explicitly check that the regularized boundary values indeed reproduce the tree-level and one-loop string corrections and apply the associator eq. (3.93), we have to determine the matrices $x^{(0)}_{\leq 2}$, $x^{(1)}_{\leq 2}$ and $x^{(2)}_{\leq 2}$ appearing in the modified KZB equation satisfied by $\mathbf{S}^{\mathrm{E}}_{\leq 2}(z_2)$. Following the general algorithm in appendix C.1 and performing the corresponding calculations shown in appendix B.1, the partial differential equation can indeed be written in the form (3.62):

$$\frac{\partial}{\partial z_2}\,\mathbf{S}^{\mathrm{E}}_{\leq 2}(z_2)=\left(g^{(0)}_{21}x^{(0)}_{\leq 2}+g^{(1)}_{21}x^{(0)}_{\leq 2}+g^{(2)}_{21}x^{(0)}_{\leq 2}\right)\mathbf{S}^{\mathrm{E}}_{\leq 2}(z_2)+r_2\,\mathbf{S}^{\mathrm{E}}_{3}(z_2)\,, \tag{3.99}$$

where $\mathbf{S}^{\mathrm{E}}_3(z_2)=\left(\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right](0,z_2),\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right](0,z_2)\right)^T$ and the matrices are given by

$$x^{(0)}_{\leq 2}=\begin{pmatrix}0&s_{13}&0&0\\0&0&-s_{23}&-s_{23}\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad x^{(1)}_{\leq 2}=\begin{pmatrix}s_{12}&0&0&0\\0&s_{123}&0&0\\0&0&s_{12}+s_{23}&-s_{23}\\0&0&-s_{13}&s_{12}+s_{13}\end{pmatrix} \tag{3.100}$$

and

$$x^{(2)}_{\leq 2}=\begin{pmatrix}0&0&0&0\\-s_{23}&0&0&0\\0&s_{13}&0&0\\0&s_{13}&0&0\end{pmatrix},\qquad r_2=\begin{pmatrix}0&0\\0&0\\-2s_{23}&-s_{23}\\-s_{13}&2s_{13}\end{pmatrix}. \tag{3.101}$$

Now, we can evaluate the relevant entries of the regularized boundary values $\mathbf{C}^{\mathrm{E}}_{0,w_0=1}$ and $\mathbf{C}^{\mathrm{E}}_{1,w_1=0}$ explicitly: the latter involves the weight $w_1=0$ eigenvalue $x^{(1)}_0=s_{12}$ of $x^{(1)}_{\leq 2}$ in the regularization factor $(2\pi i(1-z_2))^{-x^{(1)}_{\leq 2}}$, which leads to the boundary value

$$\mathbf{C}^{\mathrm{E}}_{1,0} = \lim_{z_2\to 1}(2\pi i(1-z_2))^{-s_{12}}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,z_2) = S^{\text{1-loop}}_{\text{2-point}}(\tilde{s}_{13})\,, \tag{3.102}$$

given by the one-loop string correction $S^{\text{1-loop}}_{\text{2-point}}(\tilde{s}_{13})$ with effective Mandelstam variable

$$\tilde{s}_{13} = s_{13} + s_{23}\,, \tag{3.103}$$

which is in agreement with our general considerations in eq. (3.68). On the other hand, the relevant eigenvalue of $x^{(1)}_{\leq 2}$ for the boundary value $\mathbf{C}^{\mathrm{E}}_{0,1}$ is $x^{(1)}_{1} = s_{123}$, such that

$$\mathbf{C}^{\mathrm{E}}_{0,1} = \lim_{z_2\to 0}(2\pi i z_2)^{-s_{123}}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,z_2) = \frac{1}{s_{13}}\frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})} \tag{3.104}$$

yields indeed the well-known Veneziano amplitude for the four-point amplitude of open strings at tree-level. Since each Mandelstam variable comes with a factor of α', we find the leading order to be $o^{\text{tree}}_{\text{min}} = -1$.

Since according to eq. (3.84), the maximal order in α' or, equivalently, the maximal word length in the KZB associator is $l_{\text{max}} = 3$, the truncated associator eq. (3.91) reads

$$\begin{pmatrix} S^{\text{1-loop}}_{\text{2-point}}(\tilde{s}_{13}) \\ * \\ * \\ * \end{pmatrix} + \mathcal{O}\Big((\alpha')^3\Big) = \Phi^{\mathrm{E}}_3(x^{(n)}_{\leq 2}) \begin{pmatrix} 0 \\ \frac{1}{s_{13}}\frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})} \\ 0 \\ 0 \end{pmatrix}. \tag{3.105}$$

From the matrices given in eqs. (3.100) and (3.101) and the truncation Φ^{E}_3 of the associator Φ^{E} given by the generating series of eMZVs in eq. (3.45), we find that the only words contributing to the relevant $(w_1, w_0) = (0,1)$-submatrix $\Phi^{\mathrm{E}}_3(x^{(n)}_{\leq 2})_{0,1}$ are at

- word length 1: $x^{(0)}_{\leq 2}$
- word length 2: the commutator

$$[x^{(1)}_{\leq 2}, x^{(0)}_{\leq 2}] = \begin{pmatrix} 0 & -s_{13}(s_{13}+s_{23}) & 0 & 0 \\ 0 & 0 & -2s_{13}s_{23} & -2s_{23}^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.106}$$

- word length 3: the nested commutator

$$[x^{(1)}_{\leq 2}, [x^{(1)}_{\leq 2}, x^{(0)}_{\leq 2}]] = \begin{pmatrix} 0 & s_{13}(s_{13}+s_{23})^2 & 0 & 0 \\ 0 & 0 & -2s_{13}s_{23}(s_{13}+s_{23}) & 2s_{23}^2(s_{13}+s_{23}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.107}$$

and the products

$$x^{(0)}_{\le 2}x^{(0)}_{\le 2}x^{(2)}_{\le 2} = \begin{pmatrix} 0 & -2s_{13}^2 s_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad x^{(0)}_{\le 2}x^{(2)}_{\le 2}x^{(0)}_{\le 2} = \begin{pmatrix} 0 & -s_{13}^2 s_{23} & 0 & 0 \\ 0 & 0 & 2s_{13}s_{23}^2 & 2s_{13}s_{23}^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.108}$$

The above list of contributions can be easily obtained from our general analysis in appendix C.2.

Evaluating all matrix products, the relevant (w_1, w_0)-submatrix of the truncated KZB associator is explicitly given by the entry

$$\begin{aligned} \Phi_3^{\mathrm{E}}(x^{(n)}_{\le 2})_{0,1} = s_{13}(&\omega(0) + (s_{13}+s_{23})\omega(1,0) + (s_{13}+s_{23})^2\omega(1,1,0) \\ &- s_{13}s_{23}(\omega(0,2,0) + 2\omega(2,0,0)))\,. \end{aligned} \tag{3.109}$$

The α'-expansion of the Veneziano amplitude can be obtained from the identity

$$\begin{aligned} \frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})} &= \exp\left(\sum_{n\ge 2}(-1)^n \frac{\zeta_n}{n}(s_{13}^n + s_{23}^n - (s_{13}+s_{23})^n)\right) \\ &= 1 - \zeta_2 s_{13}s_{23} + \mathcal{O}\Big((\alpha')^3\Big)\,. \end{aligned} \tag{3.110}$$

Using these two α'-expansions, the right-hand side of the relevant part of the truncated associator eq. (3.105) is given by

$$\begin{aligned} &S^{\text{1-loop}}_{\text{2-point}}(\tilde{s}_{13}) + \mathcal{O}\Big((\alpha')^3\Big) \\ &\quad = \Phi_3^{\mathrm{E}}(x^{(n)}_{\le 2})_{0,1}\frac{1}{s_{13}}\frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})} \\ &\quad = 1 + (s_{13}+s_{23})\omega(1,0) + (s_{13}+s_{23})^2\omega(1,1,0) + \mathcal{O}\Big((\alpha')^3\Big)\,, \end{aligned} \tag{3.111}$$

where we have used the identity $\omega(0,2,0) = -\zeta_2 - 2\omega(2,0,0)$ for the regularized eMZVs [39]. This reproduces indeed the two-point one-loop string correction $S^{\text{1-loop}}_{\text{2-point}}(\tilde{s}_{13})$ given in eq. (3.94) with the effective Mandelstam variable $\tilde{s}_{13} = s_{13} + s_{23}$ up to second order in α'. Simultaneously, this result approves the validity of the (relevant part) of the truncated associator eq. (3.111).

We have performed the calculation up to the order $o^{\text{1-loop}}_{\max} = 4$ in α'. In order to compare our result with the literature, in particular with ref. [21], we translate our result into iterated integrals of Eisenstein series[11] γ_0 and use the one-loop open Green's function

$$\mathcal{G}_{ij} = \tilde{\Gamma}_{ij} + \omega(0,1) \tag{3.112}$$

in the definition (3.27) of the Selberg seed $\mathbf{S}^{\mathrm{E}}$ and in the one-loop string corrections $S^{\text{1-loop}}_{N\text{-point}}(\tilde{s}_{ij})$ rather than just $\tilde{\Gamma}_{ij}$. The additional term $\omega(0,1)$ vanishes in the sum $\sum_{i<j} s_{ij}\left(\tilde{\Gamma}_{ji} + \omega(0,1)\right)$ if momentum conservation is imposed and is, thus, physically irrelevant. Using these two adjustments, we find that the relevant part of the right-hand side of the associator eq. (3.105) up to

[11] The conversion from the ω-form of eMZVs to their representation in terms of iterated integrals of Eisenstein series γ_0 is thoroughly explained in ref. [45].

order $(\alpha')^4$ is given by

$$S^{\text{1-loop}}_{\text{2-point}}(\tilde{s}_{13})\Big|_{\mathcal{G}_{ij}} = 1 + \tilde{s}_{13}^2 \left(\frac{1}{4}\zeta_2 - 3\gamma_0(4,0)\right) + \tilde{s}_{13}^3 \left(10\gamma_0(6,0,0) - 24\zeta_2\gamma_0(4,0,0) - \frac{1}{4}\zeta_3\right)$$
$$+ \tilde{s}_{13}^4\Big(9\gamma_0(4,0,4,0) - 18\gamma(4,4,0,0) - 126\gamma_0(8,0,0,0) - \frac{3}{4}\zeta_2\gamma_0(4,0)$$
$$- 144\zeta_4\gamma_0(4,0,0,0) + 240\zeta_2\gamma_0(6,0,0,0) + \frac{19}{64}\zeta_4\Big) + \mathcal{O}\Big((\alpha')^5\Big)\,. \quad (3.113)$$

Note that eqs. (3.111) and (3.113) show nicely on a simple example, how using the associator eq. (3.83) relating the $(L+1)$-point tree-level to $(L-1)$-point one-loop string corrections may geometrically be interpreted in terms of a gluing mechanism of worldsheets as discussed at the end of subsection 3.5: starting with the four-point Veneziano amplitude, gluing together the external legs of the string worldsheet which correspond to the two external states labelled by the positions $x_1 = 0$ and $x_4 = \infty$ on the Riemann sphere yields a two-point genus-one worldsheet with punctures $z_1 = z_2 \bmod \mathbb{Z}$ and z_3. The effective momentum propagating between $z_1 = z_2 \bmod \mathbb{Z}$ and z_3 yields the Mandelstam variable $\tilde{s}_{13} = s_{13} + s_{23}$ of the two-point one-loop interaction.

3.6.2 Three points

The calculation for three points proceeds in analogy to the two-point example without structural difficulties and complications. Naturally, the dimensionality of the relevant matrices and vectors is larger, such that we do not write them down explicitly but rather provide the results of the computation.

The recursive algorithm requires one extra point on top of the three insertion points present in three-point one-loop string correction integrals. Correspondingly, we are going to consider the class of genus-one Selberg integrals with $L = 4$. The relevant integral is of the form

$$S^{\text{1-loop}}_{\text{3-point}}(\tilde{s}_{ij}) = \int_0^1 dz_3 \int_0^{z_3} dz_4 \exp\Big(\tilde{s}_{13}\,\tilde{\Gamma}_{31} + \tilde{s}_{14}\,\tilde{\Gamma}_{41} + \tilde{s}_{34}\,\tilde{\Gamma}_{34}\Big)\,, \quad (3.114)$$

The above integral resides in the weight $w_1 = 0$ subvector of $\mathbf{C}^{\mathrm{E}}_1$. We are going to perform the calculation up to order $o^{\text{1-loop}}_{\text{max}} = 3$ in α'. Since the corresponding five-point tree-level integrals start at order $o^{\text{tree}}_{\text{min}} = -2$ and appear at weight $w_0 = 2$ in $\mathbf{C}^{\mathrm{E}}_0$, the required maximal weight for the truncation of the genus-one Selberg vector is $w_{\text{max}} = 3$ according to eq. (3.90). The relevant finite-dimensional matrices $x^{(n)}_{\leq 3}$ for $n = 0,1,2,3$ are obtained from the algorithm in appendix C.1, which leads to the modified KZB equation

$$\frac{\partial}{\partial z_2}\mathbf{S}^{\mathrm{E}}_{\leq 3}(z_2) = \sum_{n=0}^{4} g^{(n)}_{21}\, x^{(n)}_{\leq 3}\mathbf{S}^{\mathrm{E}}_{\leq 3}(z_2) + r_3\mathbf{S}^{\mathrm{E}}_{4}(z_2)\,. \quad (3.115)$$

Regularized boundary values can be calculated from the $x^{(1)}_{w_0=2}$ and $x^{(1)}_{w_1=0}$ submatrices of $x^{(1)}_{\leq 3}$, which results in the expected subvectors

$$\mathbf{C}^{\mathrm{E}}_{0,2} = \lim_{z_2\to 0}(2\pi i z_2)^{-x_2^{(1)}}\,\mathbf{S}^{\mathrm{E}}_2(z_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathrm{S}[1,1](0,1,x_2=1) \\ \vdots \\ \mathrm{S}[2,3](0,1,x_2=1) \\ 0 \\ 0 \end{pmatrix} \tag{3.116}$$

containing the five-point, genus-zero Selberg integrals for $z_2 \to 0$ at weight $w_0 = 2$ and the three-point one-loop string correction for $z_2 \to 1$ at weight $w_1 = 0$:

$$\mathbf{C}^{\mathrm{E}}_{1,0} = \lim_{z_2\to 1}(2\pi i(1-z_2))^{-x_0^{(1)}}\,\mathbf{S}^{\mathrm{E}}_0(z_2) = \left(S^{\text{1-loop}}_{\text{3-point}}(\tilde{s}_{ij})\right) \tag{3.117}$$

with the effective Mandelstam variables

$$\tilde{s}_{1j} = s_{1j} + s_{2j}\,, \qquad \tilde{s}_{ij} = s_{ij} \tag{3.118}$$

for $i,j \in \{3,4\}$. The truncation of the KZB associator at $l_{\max} = 5$ (cf. eq. (3.84)), is required in order to use the finite associator eq. (3.93)

$$\mathbf{C}^{\mathrm{E}}_{1,0} + \mathcal{O}\Big((\alpha')^4\Big) = \Phi^{\mathrm{E}}_5(x^{(n)}_{\leq 3})_{0,2}\,\mathbf{C}^{\mathrm{E}}_{0,2}\,. \tag{3.119}$$

The words contributing to the weight-$(0,2)$ submatrix $\Phi^{\mathrm{E}}_5(x^{(n)}_{\leq 3})_{0,2}$ of this truncation are determined with the mechanism described in appendix C.2. The resulting α'-expansion of the right-hand side of eq. (3.119) up to order $o^{\text{1-loop}}_{\max} = 3$ reads in terms of iterated integrals of Eisenstein series and the redefinition $\tilde{\Gamma}_{ij} \mapsto \tilde{\Gamma}_{ij} + \omega(0,1) = \mathcal{G}_{ij}$ in the Selberg seed as follows:

$$\begin{aligned} S^{\text{1-loop}}_{\text{3-point}}(\tilde{s}_{ij})\Big|_{\mathcal{G}_{ij}} &= \frac{1}{2} + \frac{1}{8}\left(\tilde{s}^2_{13} + \tilde{s}^2_{14} + \tilde{s}^2_{34}\right)\left(\zeta_2 - 12\gamma_0(4,0)\right) \\ &\quad + \frac{1}{8}\Big(- \tilde{s}_{13}\tilde{s}_{34}\tilde{s}_{14}\big(- 240\gamma_0(6,0,0) + 144\zeta_2\gamma_0(4,0,0) + \zeta_3\big) \\ &\qquad - \left(\tilde{s}^3_{13} + \tilde{s}^3_{14} + \tilde{s}^3_{34}\right)\left(-40\gamma_0(6,0,0) + 96\zeta_2\gamma_0(4,0,0) + \zeta_3\right)\Big) + \mathcal{O}\Big((\alpha')^4\Big)\,, \end{aligned} \tag{3.120}$$

which agrees with the known α'-expansion of the three-point string correction.

3.6.3 Four points

If momentum conservation is imposed at the one-loop level, the first non-trivial example is the four-point one-loop string correction. It is given by the integral [35]

$$S^{\text{1-loop}}_{\text{4-point}}(\tilde{s}_{ij}) = \int_0^1 dz_3 \int_0^{z_3} dz_4 \int_0^{z_4} dz_5 \prod_{0\leq z_i<z_j\leq z_3} \exp\left(\tilde{s}_{ij}\,\tilde{\Gamma}_{ji}\right)\,, \tag{3.121}$$

where $i,j \in \{1,3,4,5\}$. The calculation of the α'-expansion is exactly the same as for the previous integrals: the one-loop integral is found in the weight $w_1 = 0$ subvector of $\mathbf{C}^{\mathrm{E}}_1$ and the six-point tree-level integrals at the weight $w_0 = 3$ with $o^{\text{tree}}_{\text{min}} = -3$. Hence, in order to obtain the expansion up to order $o^{\text{1-loop}}_{\text{max}} = 2$, the KZB associator can be truncated at the maximal word length $l_{\text{max}} = 5$ and eq. (3.90) requires the maximal weight $w_{\text{max}} = w_0 = 3$. The matrices $x^{(n)}_{\leq 3}$ for $n = 0, 1, 2, 3$ are obtained by forming the modified KZB eq. (3.62)

$$\frac{\partial}{\partial z_2}\mathbf{S}^{\text{E}}_{\leq 3}(z_2) = \sum_{n=0}^{4} g^{(n)}_{21} x^{(n)}_{\leq 3}\mathbf{S}^{\text{E}}_{\leq 3}(z_2) + r_3\mathbf{S}^{\text{E}}_{4}(z_2)\,. \tag{3.122}$$

As before, the subvectors of the regularized boundary values which contain the six-point, three-level Selberg integrals for $z_2 \to 0$ at weight $w_0 = 3$ and the four-point one-loop string correction for $z_2 \to 1$ at weight $w_1 = 0$ can be calculated using the appropriate submatrices of $x^{(1)}_{\leq 3}$ and read

$$\mathbf{C}^{\text{E}}_{0,3} = \lim_{z_2\to 0}(2\pi i z_2)^{-x^{(1)}_3}\,\mathbf{S}^{\text{E}}_3(z_2) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{S}[1,1,1](0,1,x_2=1) \\ \vdots \\ \mathrm{S}[2,3,4](0,1,x_2=1) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.123}$$

and

$$\mathbf{C}^{\text{E}}_{1,0} = \lim_{z_2\to 1}(2\pi i(1-z_2))^{-x^{(1)}_0}\,\mathbf{S}^{\text{E}}_0(z_2) = \left(S^{\text{1-loop}}_{\text{4-point}}(\tilde{s}_{ij})\right)\,, \tag{3.124}$$

respectively, with the effective Mandelstam variables

$$\tilde{s}_{1j} = s_{1j} + s_{2j}\,, \qquad \tilde{s}_{ij} = s_{ij} \tag{3.125}$$

for $i, j \in \{3, 4, 5\}$. The truncated elliptic KZB associator at the maximal length $l_{\text{max}} = 5$, with the contributing words calculated as usually, leads to the finite associator eq. (3.93)

$$\mathbf{C}^{\text{E}}_{1,0} + \mathcal{O}\Big((\alpha')^3\Big) = \Phi^{\text{E}}_5(x^{(n)}_{\leq 3})_{0,3}\;\mathbf{C}^{\text{E}}_{0,3}\,. \tag{3.126}$$

Expressed in terms of iterated integrals of Eisenstein series and using the redefinition $\tilde{\Gamma}_{ij} \mapsto \tilde{\Gamma}_{ij} + \omega(0,1) = \mathcal{G}_{ij}$ in the Selberg seed, the α'-expansion of the right-hand side of eq. (3.126) turns out to be

$$\begin{aligned} S^{\text{1-loop}}_{\text{4-point}}(\tilde{s}_{ij})\Big|_{\mathcal{G}_{ij}} &= \frac{1}{6} - \frac{\zeta(3)}{4\pi^2}\left(\tilde{s}_{1,2} - 2\tilde{s}_{1,3} + \tilde{s}_{1,4} + \tilde{s}_{2,3} - 2\tilde{s}_{2,4} + \tilde{s}_{3,4}\right) \\ &\quad - 6\left(\tilde{s}_{1,2} - 2\tilde{s}_{1,3} + \tilde{s}_{1,4} + \tilde{s}_{2,3} - 2\tilde{s}_{2,4} + \tilde{s}_{3,4}\right)\gamma_0(4,0,0) + \mathcal{O}\Big((\alpha')^2\Big)\,, \end{aligned} \tag{3.127}$$

which has been checked up to order $(\alpha')^2$ to agree with the expected α'-expansion of the four-point string correction.

4 Relating genus zero and genus one

In this section, we briefly discuss how the regularized boundary value C_0^{E} of a function satisfying a KZB equation is related to a corresponding genus-zero limit C_0 of a solution of a KZ equation. This provides an explanation why in our construction of the recursion relating loop-level string corrections to tree-level string corrections described in the previous section, genus-zero string corrections are discovered from the genus-one Selberg integrals.

Before we focus on genus-one quantities, we determine the origin of the regularization used for the regularized genus-zero boundary value

$$C_0 = \lim_{x\to 0} x^{-e_0} F(x) \tag{4.1}$$

of a solution $F(x)$ of the KZ equation

$$\frac{d}{dx} F(x) = \left(\frac{e_0}{x} + \frac{e_1}{x-1}\right) F(x)\,. \tag{4.2}$$

In order to estimate the behavior of $F(x)$ close to zero, the change of variables $x = \epsilon w$ and the limit $\epsilon \to 0$ are used, such that the KZ equation can be written as

$$\frac{1}{\epsilon}\frac{d}{dw} F(\epsilon w) = \left(\frac{e_0}{\epsilon w} - e_1 + \mathcal{O}(\epsilon)\right) F(\epsilon w) \tag{4.3}$$

up to linear order. Using this differential equation and the fact that $[e_0, e^{\epsilon w e_1}] = \mathcal{O}(\epsilon)$, the function $F(\epsilon w)$ can be approximated by

$$F(\epsilon w) = e^{-\epsilon w e_1} (\epsilon w)^{e_0} f_0 + \mathcal{O}(\epsilon) \tag{4.4}$$

for some constant f_0 in a neighborhood of zero. The regularization in C_0 ensures that this constant is exactly the regularized boundary value

$$C_0 = f_0\,. \tag{4.5}$$

The genus-one calculation can be carried out analogously, which naturally leads to a close relation to the constant f_0. For a function $F^{\mathrm{E}}(z,\tau)$ satisfying the KZB equation

$$\frac{\partial}{\partial z} F^{\mathrm{E}}(z,\tau) = \sum_{n\geq 0} g^{(n)}(z,\tau) x^{(n)} F^{\mathrm{E}}(z,\tau)\,, \tag{4.6}$$

the change of variables $z = \epsilon x$ and letting $\epsilon \to 0$ lead to a similar situation as above: from the q-expansion of the integration kernels $g^{(n)}(z,\tau)$, we find that [35]

$$g^{(n)}(\epsilon x,\tau) = \begin{cases} 1 & \text{if } n = 0\,, \\ \frac{1}{\epsilon x} + \mathcal{O}(\epsilon) & \text{if } n = 1\,, \\ -2\zeta_{2m} - 2\frac{(2\pi i)^{2m}}{(2m-1)!}\sum_{k,l>0} l^{2m-1} q^{kl} + \mathcal{O}(\epsilon^2) & \text{if } n = 2m > 0\,, \\ \mathcal{O}(\epsilon) & \text{if } n = 2m+1 > 1\,. \end{cases} \tag{4.7}$$

Therefore, we can assemble the even generators $x^{(2m)}$ and the corresponding order-zero prefactors into

$$x^{(e)}(\tau) = x^{(0)} - 2\sum_{m>0}\left(\zeta_{2m} + \frac{(2\pi i)^{2m}}{(2m-1)!}\sum_{k,l>0} l^{2m-1}q^{kl}\right)x^{(2m)} \tag{4.8}$$

in order to write the KZB eq. (4.6) as

$$\frac{1}{\epsilon}\frac{d}{dx}F^{\mathrm{E}}(\epsilon x,\tau) = \left(\frac{x^{(1)}}{\epsilon x} + x^{(e)}(\tau) + \mathcal{O}(\epsilon)\right)F^{\mathrm{E}}(\epsilon x,\tau) \tag{4.9}$$

in a neighborhood of zero. This is a differential equation of the form (4.3) of the KZ equation in the same regime. In other words, in the limit $\epsilon \to 0$, the operator

$$\nabla^{\mathrm{KZB}}(x^{(n)}) = \sum_{n\geq 0} g^{(n)}(\epsilon x)x^{(n)} \tag{4.10}$$

on the right-hand side in the KZB equation (4.6) degenerates to the operator

$$\nabla^{\mathrm{KZ}}(e_0,e_1) = \frac{e_0}{\epsilon x} + \frac{e_1}{\epsilon x - 1} \tag{4.11}$$

in the KZ equation (4.2) with $e_0 = x^{(1)}$ and $e_1 = x^{(e)}$:

$$\nabla^{\mathrm{KZB}}(x^{(n)}) = \nabla^{\mathrm{KZ}}(x^{(1)},x^{(e)}) + \mathcal{O}(\epsilon)\,. \tag{4.12}$$

Thus, as before for $F(x)$, the function $F^{\mathrm{E}}(\epsilon x,\tau)$ can be approximated by

$$F^{\mathrm{E}}(\epsilon x,\tau) = e^{\epsilon x x^{(e)}(\tau)}(\epsilon x)^{x^{(1)}}f_0^{\mathrm{E}} + \mathcal{O}(\epsilon)\,, \tag{4.13}$$

where f_0^{E} is some constant. Note that a similar degeneration to the genus-zero framework occurs for the generating series $L^{\mathrm{E}}(z)$ of elliptic multiple polylogarithms defined in eq. (3.40): according to eq. (3.42), for $e_0 = x^{(1)}$ the series has at lowest order the same behavior as the generating series $L(z)$ of the multiple polylogarithms

$$L^{\mathrm{E}}(\epsilon x) = (2\pi i\epsilon x)^{x^{(1)}}(1+\mathcal{O}(\epsilon)) = (2\pi i)^{x^{(1)}}L(\epsilon x)|_{e_0=x^{(1)}}(1+\mathcal{O}(\epsilon))\,. \tag{4.14}$$

We can conclude that the regularized boundary value

$$C_0^{\mathrm{E}} = \lim_{z\to 0} z^{-x^{(1)}}F^{\mathrm{E}}(z,\tau) = f_0^{\mathrm{E}} \tag{4.15}$$

is indeed independent of τ and, upon comparing eq. (4.4) with eq. (4.13), it is proportional (up to a constant matrix) to the corresponding genus-zero boundary value $C_0 = f_0$ for a function $F(x)$ satisfying a KZ equation with $e_0 = x^{(1)}$

$$C_0^{\mathrm{E}} = \lim_{z\to 0} z^{-x^{(1)}}F^{\mathrm{E}}(z,\tau) = f_0^{\mathrm{E}} \propto f_0 = C_0\,. \tag{4.16}$$

Note that if $e_0 \neq x^{(1)}$, but they have the same maximal eigenvalue, then the above argument modifies slightly but still applies analogously such that the elements of C_0^{E} turn out to be some linear combinations of the elements of C_0, which is exactly the situation observed in the recursion described in the previous section.

5 Summary and Outlook

In this article, we have generalized the recursive formalism for the evaluation of genus-zero Selberg integrals by Aomoto and Terasoma to genus one. After establishing and discussing the genus-one formalism, we have put it to work to evaluate one-loop open-string scattering amplitudes.

The original construction at genus zero is based on relating two boundary values of a Knizhnik-Zamolodchikov equation by the Drinfeld associator. The boundary values arise as two different limits of Selberg integrals and can be shown to contain integrals constituting the N-point and $(N-1)$-point open-string tree-level amplitudes respectively. Accordingly, the method allows to determine all tree-level string corrections at arbitrary order in α' recursively using a suitable representation of the Drinfeld associator.

Our genus-one formalism is based on canonical generalizations of the above construction: at the heart there is now the elliptic Knizhnik-Zamolodchikov-Bernard equation, whose boundary values are related by the genus-one analogue of the Drinfeld associator, the elliptic KZB associator. The boundary values arise as limits of genus-one Selberg integrals and can be shown to contain the one-loop N-point and the tree-level $(N+2)$-point open-string integrals. Thus all one-loop open-string corrections can be calculated using the elliptic associator equation (3.83) to any desired order in α'. Our results so obtained match the known expressions at multiplicity two, three and four.

The original recursion at genus zero as well as our recursion at genus one have clear geometrical interpretations in terms of degenerations of bounded Riemann surfaces: the extra marked point serves as variable in the KZ and KZB equations and thereby simultaneously parametrize the degeneration of the Riemann surfaces in the limits, which define the boundary values. The class of iterated integrals leading to the Selberg integrals as well as the respective integration domains are very naturally defined in terms of the de Rham cohomology of the Riemann surface in question: at genus zero, the twisted forms appearing in the Selberg integrals form a basis of the twisted de Rham cohomology of the configuration space of punctured Riemann spheres with fixed points. Similarly, the twisted forms in the genus-one Selberg integrals form a closed system with respect to integration by parts, the Fay identity and taking derivatives.

The following points deserve further investigation:

- Very likely, recursions with an extra marked point can not only be constructed for corrections to open-string amplitudes as done in this article. Rather, it seems the formalism is extendable to a wide range of string- and quantum field theories. An application or translation to the calculation of scattering amplitudes in $\mathcal{N}=4$ super-Yang–Mills theory in the multi-Regge limit might be a first testing ground: several recursive structures as well as numerous formal similarities are already visible in refs. [46, 47]. Another environment for amplitude recurrences, similar to our current construction, is discussed and applied in refs. [48, 49]. It would be very interesting to understand the relation between the two approaches.

- Considering the step from genus zero to genus one, all generalizations have been completely canonical. We do not see any structural obstructions for establishing a similar recursion for higher genera. Given the algebraic complexity of the genus-one construction already, combinatorics will not only cause large matrix sizes, but also originate from considering three geometric parameters in the period matrix at genus two.

- Our construction makes use of several genus-zero tools developed in the context of [19], the most prominent example being the matching of dimensions of the respective matrices, which correspond to a basis of Selberg vectors w.r.t. partial fraction and integration by parts: the respective dimensions are exactly as predicted by twisted de Rham theory.

- A substantial part in establishing our genus-one recursion was devoted to finding a useful and feasible way to single out a basis for Selberg vectors. For higher orders in α' as well as for higher multiplicity, a formulation of genus-one Selberg integrals in terms of weighted graphs and Fay identities using weighted adjacency matrices analogous to the genus-zero description in [22] might be the correct computational framework.

- Most importantly, a formalism for calculating one-loop open-string amplitudes from a differential equation has been put forward in refs. [20, 21]. The constructions are formally rather similar: both rely on an elliptic KZB equation. While we are using an extra insertion point as differentiation variable, Mafra and Schlotterer employ the modular parameter τ for this purpose. Our formulation employs iterated integrals for the insertion points and the ω-representations of eMZVs, while in refs. [20, 21] iterated τ-integrals, Eisenstein series and the γ_0-representation of eMZVs is employed. There is little doubt that the formalisms can be shown to be equivalent.

- Our genus-one recursion is tailored to the calculation of *planar* open-string corrections, where vertex insertions are allowed on only one of the boundaries of the annulus. An extension to non-planar open-string amplitudes is expected to be straightforward: in particular one ought to use doubly-periodic integration kernels instead of the functions $g^{(n)}$. In particular does a construction for non-planar one-loop string corrections already exist in refs. [20, 21].

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Appendix

A Regularization of elliptic multiple zeta values

In this section, we give a brief description how eMZVs may be regularized analogously to the regularization of the (genus-zero) MZVs.

The reversal of the ordering in the definition (3.20) and the regularization of the iterated integrals $\tilde{\Gamma}$ implies that only the eMZVs

$$\omega(n_k, \ldots, n_1; \tau) = \omega(w^t; \tau) = \lim_{z \to 1} \tilde{\Gamma}_w(z, \tau) = \lim_{z \to 1} \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ 0 & \cdots & 0 \end{smallmatrix}; z, \tau\right), \tag{A.1}$$

labeled by the word $w = x^{(n_1)} \dots x^{(n_k)} \in X$ with $n_1 = 1$ inherit the end point divergence at the upper integration boundary due to the $1/(z-1)$ asymptotics of $g^{(1)}(z,\tau)$ in the limit $z \to 1$. For example the definition (3.10) and the asymptotic behavior (3.14) imply that if we would allow for $n_1 = 1$ in the definition of the eMZVs, then

$$\omega(1;\tau) = \lim_{z\to 1} \tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix}; z,\tau) = \lim_{z\to 1} \log(2\pi i(1-z))\,, \qquad \omega(\underbrace{1,\dots,1}_{n};\tau) = \frac{1}{n!}\omega(1;\tau)^n \tag{A.2}$$

are divergent and the q-expansion of $g^{(1)}$ implies

$$\begin{aligned}\omega(0,1;\tau) &= \lim_{z\to 1} \tilde{\Gamma}(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}; z,\tau) &\text{(A.3)}\\ &= \lim_{z\to 1} \int_0^z dz'\, g^{(1)}(z',\tau) z' \\ &= \lim_{z\to 1} \log(2\pi i(1-z)) - \frac{i\pi}{2} - 2\sum_{k,l>0} \frac{q^{kl}}{k}\,, &\text{(A.4)}\end{aligned}$$

such that

$$\begin{aligned}\omega(1,0;\tau) &= \lim_{z\to 1} \tilde{\Gamma}(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}; z,\tau) \\ &= \lim_{z\to 1} \Big(\tilde{\Gamma}(\begin{smallmatrix}0\\0\end{smallmatrix}; z,\tau)\,\tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix}; z,\tau) - \tilde{\Gamma}(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}; z,\tau)\Big) \\ &= \omega(1;\tau) - \omega(0,1;\tau) \\ &= \frac{i\pi}{2} + 2\sum_{k,l>0} \frac{q^{kl}}{k} &\text{(A.5)}\end{aligned}$$

is free of any logarithmic divergence. Using the shuffle algebra, any (divergent) elliptic multiple zeta value can be expanded in powers of $\omega(1;\tau)$, such that the regularized eMZVs ω_{reg} can be defined as being the convergent coefficient (of 1) in this expansion. For example from above, we find at depth one

$$\omega_{\mathrm{reg}}(1;\tau) = 0\,, \tag{A.6}$$

at depth two

$$\omega(0,1;\tau) = -\omega(1,0;\tau) + \omega(0)\omega(1;\tau)\,, \quad \text{such that} \quad \omega_{\mathrm{reg}}(0,1;\tau) = -\omega(1,0;\tau) = -\omega_{\mathrm{reg}}(1,0;\tau) \tag{A.7}$$

and further examples of divergent eMZVs are at depth three and weight one

$$\begin{aligned}\omega(0,0,1;\tau) &= -\omega(0,1,0;\tau) - \omega(1,0,0;\tau) + \omega(0,0;\tau)\omega(1;\tau) \\ &= -\omega(1,0,0;\tau) + \omega(0,0;\tau)\omega(1;\tau) &\text{(A.8)}\end{aligned}$$

and at weight 2

$$\omega(1,0,1;\tau) = -2\omega(1,1,0;\tau) + \omega(1,0;\tau)\omega(1;\tau)\,, \tag{A.9}$$

as well as

$$\omega(0,1,1;\tau) = -\omega(1,1,0;\tau) - \omega(1,0,1;\tau) + \omega(0;\tau)\omega(1,1;\tau)$$

$$= \omega(1,1,0;\tau) - \omega(1,0;\tau)\omega(1;\tau) + \omega(0;\tau)\omega(1,1;\tau)\,, \tag{A.10}$$

such that

$$\begin{aligned}
\omega_{\mathrm{reg}}(0,0,1;\tau) &= -\omega_{\mathrm{reg}}(0,0,1;\tau)\,,\\
\omega_{\mathrm{reg}}(1,0,1;\tau) &= -2\omega_{\mathrm{reg}}(1,1,0;\tau)\,,\\
\omega_{\mathrm{reg}}(0,1,1;\tau) &= \omega_{\mathrm{reg}}(1,1,0;\tau)\,.
\end{aligned} \tag{A.11}$$

As for the regularized elliptic multiple polylogarithms, we generally omit the subscript in ω_{reg} and always refer to the regularized versions when we write an elliptic multiple zeta value ω.

B Explicit Calculations

This appendix provides the explicit calculations of some of the results stated in the main part of this article.

B.1 Two-point String Corrections

In this subsection, we give the detailed calculations for the two-point example in subsection 3.6.1. The two-point amplitude is described by the class of genus-one Selberg integrals with $L = 3$, thus, we consider the iterated integrals

$$\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3 \\ i_3 \end{smallmatrix}\right](0,z_2) = \int_0^{z_2} dz_3\, \mathrm{S}^{\mathrm{E}}\, g_{3i_3}^{(n_3)}\,, \qquad \mathrm{S}^{\mathrm{E}} = \exp\left(s_{13}\,\tilde{\Gamma}_{31} + s_{12}\,\tilde{\Gamma}_{21} + s_{23}\,\tilde{\Gamma}_{23}\right)\,, \qquad 1 \leq i_3 < 3\,. \tag{B.1}$$

The two-point one-loop amplitude with Mandelstam variable $s = s_{13} + s_{23}$ is reproduced for $n = 0$, $i_3 = 1$ as the first entry of the boundary value

$$\mathbf{C}_1^{\mathrm{E}} = \lim_{z_2 \to 1} (2\pi i(1-z_2)) \begin{pmatrix} \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](0,z_2) \\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](0,z_2) \\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right](0,z_2) \\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right](0,z_2) \\ \vdots \end{pmatrix}\,. \tag{B.2}$$

In order to evaluate the first entry of $\mathbf{C}_1^{\mathrm{E}}$ we can use the block-diagonal form of $x^{(1)}$ with the first block being $x_0^{(1)} = s_{12}$ as shown below. Thus, the relevant entry of the regularization factor for $z_2 \to 1$ is $(2\pi i(1-z_2))^{-x_1^{(1)}} \sim e^{-s_{12}\,\tilde{\Gamma}_{21}}$ and the integral is given by

$$\begin{aligned}
&\lim_{z_2 \to 1} (2\pi i(1-z_2))^{-s_{12}}\, \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](0,z_2)\\
&= \lim_{z_2 \to 1} e^{-s_{12}\,\tilde{\Gamma}_{21}} \int_0^{z_2} dz_3 \exp\left(s_{13}\,\tilde{\Gamma}_{31} + s_{12}\,\tilde{\Gamma}_{21} + s_{23}\,\tilde{\Gamma}_{23}\right)\\
&= \int_0^1 dz_3 \exp\left((s_{13}+s_{23})\,\tilde{\Gamma}_{31}\right)\\
&= \sum_{n \geq 0} \frac{(s_{13}+s_{23})^n}{n!} \int_0^1 dz_3\, \tilde{\Gamma}_{31}^n
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\geq 0} \frac{(s_{13}+s_{23})^n}{n!} \int_0^1 dz_3\, n!\, \tilde{\Gamma}(\underbrace{\begin{smallmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{smallmatrix}}_{n}; z_3, \tau) \\
&= \sum_{n\geq 0} (s_{13}+s_{23})^n\, \omega(\underbrace{1,\ldots,1}_{n},0)\,.
\end{aligned} \tag{B.3}
$$

The regularization of the above boundary value corresponds to the first eigenvalue s_{12} of $x^{(1)}$, which can be determined by bringing the derivative of $\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3 \\ i_3 \end{smallmatrix}\right](0,z_2)$ in KZB form

$$
\begin{aligned}
\frac{\partial}{\partial z_2} \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](0,z_2) &= \int_0^{z_2} dz_3 s_{21} g_{21}^{(1)} S + \int_0^{z_2} dz_3 s_{23} g_{23}^{(1)} S \\
&= s_{21} g_{21}^{(1)}\, \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](0,z_2) + \int_0^{z_2} dz_3 s_{31} g_{31}^{(1)} S \\
&= s_{21} g_{21}^{(1)}\, \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](0,z_2) + s_{31} g_{31}^{(0)}\, \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](0,z_2)\,,
\end{aligned} \tag{B.4}
$$

such that the first columns of the matrices $x^{(0)}$ and $x^{(1)}$ are given by

$$
x^{(0)} = \begin{pmatrix} 0 & s_{31} & 0 & 0 & \ldots \\ \vdots & & & & \end{pmatrix}, \qquad x^{(1)} = \begin{pmatrix} s_{21} & 0 & 0 & 0 & \ldots \\ \vdots & & & & \end{pmatrix}. \tag{B.5}
$$

Note that we have used the integration by parts identity

$$
s_{23}\, \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right](0,z_2) + s_{13}\, \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](0,z_2) = 0\,. \tag{B.6}
$$

The boundary value for $z_2 \to 0$ is more subtle. In this limit, the one-loop propagator degenerates to the tree level propagator and, in particular, loses its τ-dependence at the lowest order in z_2

$$
\lim_{z_2\to 0} \tilde{\Gamma}_{\mathrm{reg}}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; z_2, \tau) = \log(2\pi i z_2) + \mathcal{O}\Big(z_2^2\Big)\,, \qquad g^{(1)}(z_2,\tau) = \frac{1}{z_2} + \mathcal{O}(z_2) \tag{B.7}
$$

such that, using the change of variables $z_i = z_2 w_i$, the unregularized limit for $n_3 = 1$, $i_3 = 1$ is given by

$$
\begin{aligned}
&\lim_{z_2\to 0} \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](0,z_2) \\
&= \lim_{z_2\to 0} \int_0^{z_2} dz_3\, \exp\Big(s_{13}\, \tilde{\Gamma}_{31} + s_{12}\, \tilde{\Gamma}_{21} + s_{23}\, \tilde{\Gamma}_{23}\Big)\, g_{31}^{(1)} \\
&= \lim_{z_2\to 0} \int_0^1 dw_3\, z_2 (2\pi i z_2 w_3)^{s_{13}} (2\pi i z_2)^{s_{12}} (2\pi i z_2(1-w_3))^{s_{23}} \frac{1}{z_2 w_3} (1+\mathcal{O}(z_2)) \\
&= \lim_{z_2\to 0} (2\pi i z_2)^{s_{123}} \int_0^1 dw_3\, w_3^{s_{13}} (1-w_3)^{s_{23}} \frac{1}{w_3} (1+\mathcal{O}(z_2)) \\
&= \lim_{z_2\to 0} (2\pi i z_2)^{s_{123}} \left(\frac{1}{s_{13}} \frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})}\right) (1+\mathcal{O}(z_2))\,.
\end{aligned} \tag{B.8}
$$

Therefore, at the lowest order in z_2, the integral $\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](0,z_2)$ degenerates to the four-point tree-level amplitude with Mandelstam variables s_{13} and s_{23}. Now, let us check that the regularization by the factor $(2\pi i z_2)^{-x^{(1)}}$ projects out that lowest-order coefficient of z_2. In order to obtain the appropriate eigenvalue of $x^{(1)}$, the differential equation satisfied by $\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](0,z_2)$ has to be brought in KZB form and the coefficient of $\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,z_2)$ itself has to be determined

$$\begin{aligned}\frac{\partial}{\partial z_2}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,z_2) &= \int_0^{z_2} dz_3\,\exp\left(s_{13}\,\tilde\Gamma_{31}+s_{12}\,\tilde\Gamma_{21}+s_{23}\,\tilde\Gamma_{23}\right) g_{31}^{(1)}\left(s_{12}g_{21}^{(1)}+s_{23}g_{23}^{(1)}\right)\\ &= s_{12}g_{21}^{(1)}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,z_2) - s_{23}\int_0^{z_2} dz_3\,\mathrm{S}^{\mathrm{E}}\,g_{31}^{(1)}g_{32}^{(1)}.\end{aligned} \tag{B.9}$$

In order to bring the second integral into the appropriate form, the Fay identity

$$g_{31}^{(1)}g_{32}^{(1)} = g_{21}^{(2)}+g_{31}^{(2)}+g_{32}^{(2)}+g_{21}^{(1)}g_{32}^{(1)}-g_{21}^{(1)}g_{31}^{(1)} \tag{B.10}$$

has to be used, followed by an application of eq. (B.6)

$$\begin{aligned}\frac{\partial}{\partial z_2}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,z_2) &= -s_{23}g_{21}^{(2)}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,z_2) - s_{23}g_{21}^{(0)}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right](0,z_2) - s_{23}g_{21}^{(0)}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right](0,z_2)\\ &\quad + s_{12}g_{21}^{(1)}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,z_2) - s_{23}g_{21}^{(1)}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right](0,z_2) + s_{23}g_{21}^{(1)}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,z_2)\\ &= -s_{23}g_{21}^{(2)}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,z_2) - s_{23}g_{21}^{(0)}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right](0,z_2) - s_{23}g_{21}^{(0)}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right](0,z_2)\\ &\quad + (s_{12}+s_{13}+s_{23})\,g_{21}^{(1)}\,\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,z_2)\,.\end{aligned} \tag{B.11}$$

Therefore, we find that the appropriate eigenvalue of $x^{(1)}$ is indeed $s_{123}=s_{12}+s_{13}+s_{23}$, such that according to eq. (B.8) the second, i.e. the weight-one, entry of $\mathbf{C}_0^{\mathrm{E}}$ is given by the four-point tree-level amplitude

$$\mathbf{C}_0^{\mathrm{E}} = \lim_{z_2\to 0} e^{-x^{(1)}\,\tilde\Gamma_{21}} \begin{pmatrix} \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,z_2)\\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,z_2)\\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right](0,z_2)\\ \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right](0,z_2)\\ \vdots \end{pmatrix} = \begin{pmatrix} * \\ \frac{1}{s_{13}}\frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})} \\ * \\ * \\ \vdots \end{pmatrix}. \tag{B.12}$$

As discussed in subsection 3.5, since the eigenvalue of $x^{(1)}$ can not be bigger than s_{123} and we can only compensate the Jacobian z_2 in eq. (B.8) from the change of variables $z_3=z_2w_3$ by the singular asymptotic behavior of $g^{(1)}(z_3,\tau)\to\frac{1}{z_2w_3}$ for $z_2\to 0$, if there would be another integration kernel $g^{(n_3)}(z_{3i_3},\tau)$ with $n_3\neq 1$ which is regular close to the origin, there would not be such a compensation. Thus, all other entries of the boundary value $\mathbf{C}_0^{\mathrm{E}}$ which do not correspond to a singular integration kernel $g^{(1)}(z_{3i_3},\tau)$ vanish and we obtain

$$\mathbf{C}_0^{\mathrm{E}} = \begin{pmatrix} 0 \\ \frac{1}{s_{13}}\frac{\Gamma(1+s_{13})\Gamma(1+s_{23})}{\Gamma(1+s_{13}+s_{23})} \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{B.13}$$

In order to check the consistency of the first entry of the vector equation

$$\mathbf{C}_1^{\mathrm{E}} = \Phi^{\mathrm{E}}\,\mathbf{C}_0^{\mathrm{E}} \tag{B.14}$$

up to order $(\alpha')^2$, we also need to calculate the derivative of $\mathbf{S}_2^{\mathrm{E}}(z_2)$, which includes the following two derivatives: the first one is

$$\begin{aligned}\frac{\partial}{\partial z_2}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\1\end{smallmatrix}\Big](0,z_2) &= \int_0^{z_2} dz_3\ \mathrm{S}^{\mathrm{E}}\, g_{31}^{(2)}\left(s_{21}g_{21}^{(1)}+s_{23}g_{23}^{(1)}\right)\\ &= s_{12}g_{21}^{(1)}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\1\end{smallmatrix}\Big](0,z_2)-s_{23}\int_0^{z_2} dz_3\ \mathrm{S}^{\mathrm{E}}\, g_{31}^{(2)}g_{32}^{(1)}\,, \end{aligned}\tag{B.15}$$

where we can apply again the Fay identity

$$\begin{aligned}g_{32}^{(1)}g_{31}^{(2)} &= -(-1)^2 g_{12}^{(3)}+\sum_{r=0}^{2}\binom{r}{0}g_{21}^{(2-r)}g_{k2}^{(1+r)}+\sum_{r=0}^{1}\binom{r+1}{1}g_{12}^{(1-r)}g_{k1}^{(2+r)}\\ &= g_{21}^{(3)}+g_{21}^{(2)}g_{32}^{(1)}+g_{21}^{(1)}g_{32}^{(2)}+g_{21}^{(0)}g_{32}^{(3)}-g_{21}^{(1)}g_{31}^{(2)}+2g_{12}^{(0)}g_{31}^{(3)}\,.\end{aligned}\tag{B.16}$$

Therefore, we find

$$\begin{aligned}\frac{\partial}{\partial z_2}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\1\end{smallmatrix}\Big] &= s_{12}g_{21}^{(1)}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\1\end{smallmatrix}\Big]-s_{23}\big(g_{21}^{(3)}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}0\\1\end{smallmatrix}\Big]+g_{21}^{(2)}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}1\\2\end{smallmatrix}\Big]+g_{21}^{(1)}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\2\end{smallmatrix}\Big]\\ &\quad+g_{21}^{(0)}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}3\\2\end{smallmatrix}\Big]-g_{21}^{(1)}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\1\end{smallmatrix}\Big]+2g_{12}^{(0)}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}3\\1\end{smallmatrix}\Big]\big)\\ &= g_{21}^{(0)}\left(-2s_{23}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}3\\1\end{smallmatrix}\Big]-s_{23}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}3\\2\end{smallmatrix}\Big]\right)+g_{21}^{(1)}\left((s_{12}+s_{23})\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\1\end{smallmatrix}\Big]-s_{23}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\2\end{smallmatrix}\Big]\right)\\ &\quad+g_{21}^{(2)}\left(-s_{32}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}1\\2\end{smallmatrix}\Big]\right)+g_{21}^{(3)}\left(-s_{32}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}0\\1\end{smallmatrix}\Big]\right)\\ &= g_{21}^{(0)}\left(-2s_{23}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}3\\1\end{smallmatrix}\Big]-s_{23}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}3\\2\end{smallmatrix}\Big]\right)+g_{21}^{(1)}\left((s_{12}+s_{23})\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\1\end{smallmatrix}\Big]-s_{23}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\2\end{smallmatrix}\Big]\right)\\ &\quad+g_{21}^{(2)}\left(s_{13}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}1\\1\end{smallmatrix}\Big]\right)+g_{21}^{(3)}\left(-s_{32}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}0\\1\end{smallmatrix}\Big]\right)\end{aligned}\tag{B.17}$$

and similarly

$$\begin{aligned}\frac{\partial}{\partial z_2}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\2\end{smallmatrix}\Big](0,z_2) &= \int_0^{z_2} dz_3\ \mathrm{S}^{\mathrm{E}}\, g_{32}^{(2)}\left(s_{21}g_{21}^{(1)}+s_{23}g_{23}^{(1)}\right)+\int_0^{z_2} dz_3\ \mathrm{S}^{\mathrm{E}}\,\frac{\partial}{\partial z_2}g_{32}^{(2)}\\ &= s_{21}g_{21}^{(1)}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\2\end{smallmatrix}\Big]-s_{23}\int_0^{z_2} dz_3\ \mathrm{S}^{\mathrm{E}}\, g_{32}^{(2)}g_{32}^{(1)}-\int_0^{z_2} dz_3\ \mathrm{S}^{\mathrm{E}}\,\frac{\partial}{\partial z_3}g_{32}^{(2)}\\ &= s_{21}g_{21}^{(1)}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\2\end{smallmatrix}\Big]-s_{23}\int_0^{z_2} dz_3\ \mathrm{S}^{\mathrm{E}}\, g_{32}^{(2)}g_{32}^{(1)}\\ &\quad+\int_0^{z_2} dz_3\ \mathrm{S}^{\mathrm{E}}(s_{31}g_{31}^{(1)}+s_{32}g_{32}^{(1)})g_{32}^{(2)}\\ &= s_{21}g_{21}^{(1)}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\2\end{smallmatrix}\Big]+s_{13}\int_0^{z_2} dz_3\ \mathrm{S}^{\mathrm{E}}\, g_{32}^{(2)}g_{31}^{(1)}\,,\end{aligned}\tag{B.18}$$

where we can again use

$$\begin{aligned}g_{31}^{(1)}g_{32}^{(2)} &= g_{21}^{(3)}+g_{12}^{(2)}g_{31}^{(1)}+g_{12}^{(1)}g_{31}^{(2)}+g_{12}^{(0)}g_{31}^{(3)}-g_{12}^{(1)}g_{32}^{(2)}+2g_{21}^{(0)}g_{32}^{(3)}\\ &= -g_{21}^{(3)}+g_{21}^{(2)}g_{31}^{(1)}-g_{21}^{(1)}g_{31}^{(2)}+g_{21}^{(0)}g_{31}^{(3)}+g_{21}^{(1)}g_{32}^{(2)}+2g_{21}^{(0)}g_{32}^{(3)}\,,\end{aligned}\tag{B.19}$$

such that

$$\begin{aligned}\frac{\partial}{\partial z_2}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\2\end{smallmatrix}\Big](0,z_2) &= g_{21}^{(0)}\left(s_{13}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}3\\1\end{smallmatrix}\Big]+2s_{13}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}3\\2\end{smallmatrix}\Big]\right)+g_{21}^{(1)}\left(-s_{13}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\1\end{smallmatrix}\Big]+(s_{12}+s_{13})\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}2\\2\end{smallmatrix}\Big]\right)\\ &\quad+g_{21}^{(2)}\left(s_{13}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}1\\1\end{smallmatrix}\Big]\right)+g_{21}^{(3)}\left(-s_{13}\,\mathrm{S}^{\mathrm{E}}\Big[\begin{smallmatrix}0\\1\end{smallmatrix}\Big]\right)\,.\end{aligned}\tag{B.20}$$

The relevant 4×4-submatrices $x^{(n)}_{\leq 2}$ of $x^{(n)}$ for $n \in \{0, 1, 2\}$ appearing in the differential eq. (3.99) of $\mathbf{S}^{\mathrm{E}}_{\leq 2}(z_2)$, i.e.

$$\frac{\partial}{\partial z_2} \mathbf{S}^{\mathrm{E}}_{\leq 2}(z_2) = \left(g^{(0)}_{21} x^{(0)}_{\leq 2} + g^{(1)}_{21} x^{(0)}_{\leq 2} + g^{(2)}_{21} x^{(0)}_{\leq 2} \right) \mathbf{S}^{\mathrm{E}}_{\leq 2}(z_2) + r_2 \, \mathbf{S}^{\mathrm{E}}_{3}(z_2) \,, \tag{B.21}$$

can now be read off from the differential equations (B.4), (B.11), (B.17) and (B.20), which gives the matrices in eqs. (3.100) and (3.101).

C KZB equation of the genus-one Selberg integrals

In the first part of this appendix, the KZB equation satisfied by the genus-one Selberg integrals is derived and discussed. In the second part, it is shown how a modified KZB equation can be used in practice to calculate the α'-expansion of the genus-one string corrections from the genus-zero string corrections.

C.1 Partial differential equations

In this subsection, a combinatorial algorithm to express the derivative $\frac{\partial}{\partial z_2} \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3, & \dots, & n_N \\ i_3, & \dots, & i_N \end{smallmatrix}\right](0, z_2)$ of the genus-one Selberg integrals in KZB-type form is provided, where

$$\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L \end{smallmatrix}\right](0, z_2) = \int_{\mathcal{C}(z_2)} \prod_{i=3}^{L} dz_i \; \mathrm{S}^{\mathrm{E}} \prod_{k=3}^{L} g^{(n_k)}_{k,i_k} \tag{C.1}$$

with $1 \leq i_k < k$, and, in particular, we show how the KZB equation is recovered. This will allow us to calculate the matrices $x^{(n)}_{\leq w_{\max}}$ in the partial differential eq. (3.62) up to any desired weight $w_{\max}$. The algorithm involves two steps: the first one is based on integration by parts such that any partial derivative in the integrand of $\frac{\partial}{\partial z_2} \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L \end{smallmatrix}\right](0, z_2)$ only acts on the Selberg seed $\mathrm{S}^{\mathrm{E}} = \prod_{0 \leq z_i < z_j \leq z_2} \exp\left(s_{ij} \, \tilde{\Gamma}_{ji}\right)$. The second step is an iterative application of the Fay identity to recover admissible products $\prod_{k=3}^{L} g^{(n_k)}_{k,i_k}$ in the integrand, such that the integral can be written as a linear combination of genus-one Selberg integrals $\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L \end{smallmatrix}\right](0, z_2)$, where the coefficients are a product of a polynomial of degree one in the Mandelstam variables with rational coefficients and one factor of $g^{(n)}_{21}$ for some $n \in \mathbb{N}$. These polynomials in front of $g^{(n)}_{21} \, \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L \end{smallmatrix}\right](0, z_2)$ will be the entries of the matrices $x^{(n)}_{(\leq w_{\max})}$.

The first step can conveniently be described using the following definitions in analogy to the graphical notation of ref. [22] for the genus-zero recursion. We call a product of the form

$$\prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} \,, \qquad \text{where} \quad k_{i+1} > k_i \,, \tag{C.2}$$

a *g-chain* from k_1 to k_r with *weights* $(n_{k_1}, n_{k_2}, \dots, n_{k_r})$. Furthermore, a *g-chain with a branch* at k_j is a product of the form

$$\Big(\prod_{i=1}^{j-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i}\Big) g^{(n_{l_1})}_{l_1,k_j} \prod_{i=1}^{s-1} g^{(n_{l_{i+1}})}_{l_{i+1},l_i} \, g^{(n_{m_1})}_{m_1,k_j} \prod_{i=1}^{t-1} g^{(n_{m_{i+1}})}_{m_{i+1},m_i} \,, \tag{C.3}$$

with the *g-subchains* from k_1 to k_j, from k_j to l_s and from k_j to m_t. If there exists a g-chain in the product $\prod_{k=3}^{L} g^{(n_k)}_{k,i_k}$ from k_1 to k_s, k_s is said to be *g-chain connected* to k_1. In order to formulate the first step in the algorithm, we define for $1 \leq k \leq L$ the set of all the integers which are g-chain connected to k

$$U_k^{\vec{n},\vec{i}} = \{k \leq k' \leq L | k' \text{ is } g\text{-chain connected to } k \text{ in } \prod_{k=3}^{L} g_{k,i_k}^{(n_k)}\}\,, \tag{C.4}$$

which, as indicated by the superscripts $\vec{n} = (n_3, \ldots, n_L)$ and $\vec{i} = (i_3, \ldots, i_L)$, depends on the product $\prod_{k=3}^{L} g_{k,i_k}^{(n_k)}$ and is the genus-one analogue of the set defined in eq. (2.34). Similarly, we define the set of all the integers to which k is g-connected

$$D_k^{\vec{n},\vec{i}} = \{3 \leq k' \leq k | k \text{ is } g\text{-chain connected to } k' \text{ in } \prod_{k=3}^{L} g_{k,i_k}^{(n_k)}\}\,. \tag{C.5}$$

Thus, the set $U_k^{\vec{n},\vec{i}}$ goes up the g-chain with possible branches beginning at k and the set $D_k^{\vec{n},\vec{i}}$ goes down the g-chain beginning at k. Using these definitions, the derivative of $\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3, & \ldots, & n_L \\ i_3, & \ldots, & i_L \end{smallmatrix}\right](0, z_2)$ with respect to z_2 can be expressed as

$$\begin{aligned} \frac{\partial}{\partial z_2} \mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3, & \ldots, & n_L \\ i_3, & \ldots, & i_L \end{smallmatrix}\right](0, z_2) &= \int_{\mathcal{C}(z_2)} \prod_{i=3}^{L} dz_i \left(\sum_{l \in U_2^{\vec{n},\vec{i}}} \frac{\partial}{\partial z_l} \mathrm{S}^{\mathrm{E}} \right) \prod_{l=3}^{L} g_{k,i_k}^{(n_k)} \\ &= \int_{\mathcal{C}(z_2)} \prod_{i=3}^{L} dz_i \; \mathrm{S}^{\mathrm{E}} \left(\sum_{l \in U_2^{\vec{n},\vec{i}}} \sum_{j \in U_1^{\vec{n},\vec{i}}} s_{lj} g_{lj}^{(1)} \right) \prod_{k=3}^{L} g_{k,i_k}^{(n_k)}\,. \end{aligned} \tag{C.6}$$

This can be seen as follows: first, we note that since $1 \leq i_k < k$, the product $\prod_{k=3}^{L} g_{k,i_k}^{(n_k)}$ is a product of g-chains starting at 1 and g-chains starting at 2

$$\prod_{k=3}^{L} g_{k,i_k}^{(n_k)} = \prod_{k \in U_1^{\vec{n},\vec{i}}, k \geq 3} g_{k,i_k}^{(n_k)} \prod_{k \in U_2^{\vec{n},\vec{i}}, k \geq 3} g_{k,i_k}^{(n_k)}\,. \tag{C.7}$$

The partial derivative of the integrand of $\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3, & \ldots, & n_L \\ i_3, & \ldots, & i_L \end{smallmatrix}\right](0, z_2)$ with respect to z_2 only acts on S^{E} and the g-chains starting at 2

$$\frac{\partial}{\partial z_2} \left(\mathrm{S}^{\mathrm{E}} \prod_{k=3}^{L} g_{k,i_k}^{(n_k)} \right) = \left(\frac{\partial}{\partial z_2} \mathrm{S}^{\mathrm{E}} \right) \prod_{k=3}^{L} g_{k,i_k}^{(n_k)} + \mathrm{S}^{\mathrm{E}} \prod_{k \in U_1^{\vec{n},\vec{i}}, k \geq 3} g_{k,i_k}^{(n_k)} \left(\frac{\partial}{\partial z_2} \prod_{k \in U_2^{\vec{n},\vec{i}}, k \geq 3} g_{k,i_k}^{(n_k)} \right)\,. \tag{C.8}$$

Moreover, the product $\prod_{k \in U_2^{\vec{n},\vec{i}}, k \geq 3} g_{k,i_k}^{(n_k)}$ can be split into a product of all the (disjoint) g-chains (possibly with branches) starting at 2 and ending at some $k \in U_2^{\vec{n},\vec{i}}$ (or several such terminal values in case of branches). If we consider one such g-chain without a branch $g_{k,k_r}^{n_k} \prod_{i=1}^{r-1} g_{k_{i+1},k_i}^{(n_{k_{i+1}})} g_{k_1,2}^{n_{k_1}}$ for $k > k_{i+1} > k_i > 2$, the partial derivative with respect to z_2 acts as follows

$$\begin{aligned} &\mathrm{S}^{\mathrm{E}} \left(\frac{\partial}{\partial z_2} g_{k,k_r}^{n_k} \prod_{i=1}^{r-1} g_{k_{i+1},k_i}^{(n_{k_{i+1}})} g_{k_1,2}^{n_{k_1}} \right) \\ &= \mathrm{S}^{\mathrm{E}}\, g_{k,k_r}^{n_k} \prod_{i=1}^{r-1} g_{k_{i+1},k_i}^{(n_{k_{i+1}})} \frac{\partial}{\partial z_2} g_{k_1,2}^{n_{k_1}} \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{S}^{\mathrm{E}}\, g^{n_k}_{k,k_r} \prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} \Big(-\frac{\partial}{\partial z_{k_1}} g^{n_{k_1}}_{k_1,2}\Big) \\
&= \left(\frac{\partial}{\partial z_{k_1}} \mathrm{S}^{\mathrm{E}}\right) g^{n_k}_{k,k_r} \prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} g^{n_{k_1}}_{k_1,2} + \mathrm{S}^{\mathrm{E}}\, g^{n_k}_{k,k_r} \prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} \Big(\frac{\partial}{\partial z_{k_1}} g^{n_{k_2}}_{k_2,k_1}\Big) g^{n_k}_{k_1,2} \\
&= \left(\frac{\partial}{\partial z_{k_1}} \mathrm{S}^{\mathrm{E}}\right) g^{n_k}_{k,k_r} \prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} g^{n_{k_1}}_{k_1,2} + \mathrm{S}^{\mathrm{E}}\, g^{n_k}_{k,k_r} \prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} \Big(-\frac{\partial}{\partial z_{k_2}} g^{n_{k_2}}_{k_2,k_1}\Big) g^{n_k}_{k_1,2}
\end{aligned} \tag{C.9}
$$

where we have used integration by parts for the second last equation and omitted the boundary terms, since they vanish in the iterated integral $\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix} n_3, & \dots, & n_L \\ i_3, & \dots, & i_L \end{smallmatrix}\right](0, z_2)$. The above calculation can iteratively be repeated until any partial derivative only acts on the factor S^{E}, such that due to the product rule of the derivative we obtain

$$
\mathrm{S}^{\mathrm{E}} \left(\frac{\partial}{\partial z_2} g^{n_k}_{k,k_r} \prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} g^{n_{k_1}}_{k_1,2} \right) = \left(\left(\sum_{i=1}^{r} \frac{\partial}{\partial z_{k_i}} + \frac{\partial}{\partial z_k} \right) \mathrm{S}^{\mathrm{E}} \right) \prod_{i=1}^{r-1} g^{(n_{k_{i+1}})}_{k_{i+1},k_i} g^{n_{k_1}}_{k_1,2} \,. \tag{C.10}
$$

The product rule ensures that the same holds for the g-chains with branches as well. Therefore, we can continue with the calculation (C.8) and use the above procedure such that all the partial derivatives only act on the Selberg seed. The calculation is the following

$$
\begin{aligned}
&\frac{\partial}{\partial z_2} \left(\mathrm{S}^{\mathrm{E}} \prod_{k=3}^{L} g^{(n_k)}_{k,i_k} \right) \\
&= \left(\frac{\partial}{\partial z_2} \mathrm{S}^{\mathrm{E}}\right) \prod_{k=3}^{L} g^{(n_k)}_{k,i_k} + \mathrm{S}^{\mathrm{E}} \prod_{k\in U_1^{\vec{n},\vec{i}},k\geq 3} g^{(n_k)}_{k,i_k} \left(\frac{\partial}{\partial z_2} \prod_{k\in U_2^{\vec{n},\vec{i}},k\geq 3} g^{(n_k)}_{k,i_k} \right) \\
&= \left(\frac{\partial}{\partial z_2} \mathrm{S}^{\mathrm{E}}\right) \prod_{k=3}^{L} g^{(n_k)}_{k,i_k} + \left(\sum_{l\in U_2^{\vec{n},\vec{i}},l\geq 3} \frac{\partial}{\partial z_l} \mathrm{S}^{\mathrm{E}} \right) \prod_{k\in U_1^{\vec{n},\vec{i}},k\geq 3} g^{(n_k)}_{k,i_k} \prod_{k\in U_2^{\vec{n},\vec{i}},k\geq 3} g^{(n_k)}_{k,i_k} \\
&= \left(\sum_{l\in U_2^{\vec{n},\vec{i}}} \frac{\partial}{\partial z_l} \mathrm{S}^{\mathrm{E}} \right) \prod_{k=3}^{L} g^{(n_k)}_{k,i_k} \\
&= \mathrm{S}^{\mathrm{E}} \left(\sum_{l\in U_2^{\vec{n},\vec{i}}} \sum_{j=1,j\neq l}^{L} s_{lj} g^{(1)}_{lj} \right) \prod_{k=3}^{L} g^{(n_k)}_{k,i_k} \\
&= \mathrm{S}^{\mathrm{E}} \left(\sum_{l\in U_2^{\vec{n},\vec{i}}} \left(\sum_{j\in U_2^{\vec{n},\vec{i}}\setminus\{l\}} s_{lj} g^{(1)}_{lj} + \sum_{j\in U_1^{\vec{n},\vec{i}}} s_{lj} g^{(1)}_{lj} \right) \right) \prod_{k=3}^{L} g^{(n_k)}_{k,i_k} \\
&= \mathrm{S}^{\mathrm{E}} \left(\sum_{l\in U_2^{\vec{n},\vec{i}}} \sum_{j\in U_1^{\vec{n},\vec{i}}} s_{lj} g^{(1)}_{lj} \right) \prod_{k=3}^{L} g^{(n_k)}_{k,i_k} \,,
\end{aligned} \tag{C.11}
$$

where we have used the antisymmetry $g^{(1)}_{lj} = -g^{(1)}_{jl}$ for the last equality. This completes the proof of eq. (C.6).

As an example, let us consider $L = 6$ and the following product $p(z)$ with a branch at $k = 3$

$$
p(z) = \mathrm{S}^{\mathrm{E}}\, g^{(n_6)}_{62} g^{(n_5)}_{53} g^{(n_4)}_{43} g^{(n_3)}_{32} \,. \tag{C.12}
$$

Upon discarding boundary terms, the partial derivative of $p(z)$ with respect to z_2 is

$$\begin{aligned}
\frac{\partial}{\partial z_2}p(z) &= \frac{\partial}{\partial z_2}\left(\mathrm{S}^{\mathrm{E}}\, g_{62}^{(n_6)}g_{53}^{(n_5)}g_{43}^{(n_4)}g_{32}^{(n_3)}\right) \\
&= \left(\frac{\partial}{\partial z_2}\mathrm{S}^{\mathrm{E}}\right) g_{62}^{(n_6)}g_{53}^{(n_5)}g_{43}^{(n_4)}g_{32}^{(n_3)} + \mathrm{S}^{\mathrm{E}}\left(\frac{\partial}{\partial z_2}g_{62}^{(n_6)}\right) g_{53}^{(n_5)}g_{43}^{(n_4)}g_{32}^{(n_3)} \\
&\quad + \mathrm{S}^{\mathrm{E}}\, g_{62}^{(n_6)}g_{53}^{(n_5)}g_{43}^{(n_4)}\left(\frac{\partial}{\partial z_2}g_{32}^{(n_3)}\right) \\
&= \left(\frac{\partial}{\partial z_2}\mathrm{S}^{\mathrm{E}}\right) g_{62}^{(n_6)}g_{53}^{(n_5)}g_{43}^{(n_4)}g_{32}^{(n_3)} + \mathrm{S}^{\mathrm{E}}\left(-\frac{\partial}{\partial z_6}g_{62}^{(n_6)}\right) g_{53}^{(n_5)}g_{43}^{(n_4)}g_{32}^{(n_3)} \\
&\quad + \mathrm{S}^{\mathrm{E}}\, g_{62}^{(n_6)}g_{53}^{(n_5)}g_{43}^{(n_4)}\left(-\frac{\partial}{\partial z_3}g_{32}^{(n_3)}\right) \\
&= \left(\left(\frac{\partial}{\partial z_2}+\frac{\partial}{\partial z_6}+\frac{\partial}{\partial z_3}\right)\mathrm{S}^{\mathrm{E}}\right) g_{62}^{(n_6)}g_{53}^{(n_5)}g_{43}^{(n_4)}g_{32}^{(n_3)} \\
&\quad + \mathrm{S}^{\mathrm{E}}\, g_{62}^{(n_6)}\left(\frac{\partial}{\partial z_3}g_{53}^{(n_5)}\right) g_{43}^{(n_4)}g_{32}^{(n_3)} + \mathrm{S}^{\mathrm{E}}\, g_{62}^{(n_6)}g_{53}^{(n_5)}\left(\frac{\partial}{\partial z_3}g_{43}^{(n_4)}\right) g_{32}^{(n_3)} \\
&= \left(\left(\frac{\partial}{\partial z_2}+\frac{\partial}{\partial z_6}+\frac{\partial}{\partial z_3}\right)\mathrm{S}^{\mathrm{E}}\right) g_{62}^{(n_6)}g_{53}^{(n_5)}g_{43}^{(n_4)}g_{32}^{(n_3)} \\
&\quad + \mathrm{S}^{\mathrm{E}}\, g_{62}^{(n_6)}\left(-\frac{\partial}{\partial z_5}g_{53}^{(n_5)}\right) g_{43}^{(n_4)}g_{32}^{(n_3)} + \mathrm{S}^{\mathrm{E}}\, g_{62}^{(n_6)}g_{53}^{(n_5)}\left(-\frac{\partial}{\partial z_4}g_{43}^{(n_4)}\right) g_{32}^{(n_3)} \\
&= \left(\left(\frac{\partial}{\partial z_2}+\frac{\partial}{\partial z_6}+\frac{\partial}{\partial z_3}+\frac{\partial}{\partial z_5}+\frac{\partial}{\partial z_4}\right)\mathrm{S}^{\mathrm{E}}\right) g_{62}^{(n_6)}g_{53}^{(n_5)}g_{43}^{(n_4)}g_{32}^{(n_3)} \\
&= \mathrm{S}^{\mathrm{E}}\left(\sum_{k=2}^{L} s_{k1}g_{k1}^{(1)}\right) g_{62}^{(n_6)}g_{53}^{(n_5)}g_{43}^{(n_4)}g_{32}^{(n_3)}\,,
\end{aligned} \tag{C.13}$$

which is exactly the result expected from eq. (C.6) since $U_2^{\vec{n},(2,3,3,2)} = \{2,3,4,5,6\}$.

However, the integrals in eq. (C.6) do not yet have the desired form, i.e. a factor of $g_{21}^{(n)}$ times a product of the form $g_{k,i_k}^{(n_k)}$ with $1 \le i_k < k$ for all $k \in \{3,\dots,L\}$. This form can be obtained in a second step using the Fay identity (3.33). Due to the decomposition in eq. (C.7), any term in eq. (C.6) can be split into a product of a g-chain from 1 to j labeled by $D_j^{\vec{n},\vec{i}} = \{j_1 < j_2 < \cdots < j_s < j\}$ and a g-chain from 2 to l labeled by $D_l^{\vec{n},\vec{i}} = \{l_1 < l_2 < \cdots < l_r < l\}$ and the remaining factors:

$$s_{lj}g_{lj}^{(1)}\prod_{k=3}^{L} g_{k,i_k}^{(n_k)} = s_{kj}g_{kj}^{(1)}\, g_{j,j_s}^{(n_j)}\prod_{i=1}^{s-1} g_{j_{i+1}j_i}^{(n_{j_{i+1}})}\, g_{j_1,1}^{(n_{j_1})}\, g_{l,l_r}^{(n_l)}\prod_{i=1}^{r-1} g_{l_{i+1}l_i}^{(n_{l_{i+1}})}\, g_{l_1,2}^{(n_{l_1})} \prod_{k=3,k\notin D_l^{\vec{n},\vec{i}}\cup D_j^{\vec{n},\vec{i}}}^{L} g_{k,i_k}^{(n_k)}\,. \tag{C.14}$$

The factor $g_{lj}^{(1)}$ connects the two g-chains starting at 1 and 2, such that applying the Fay identity iteratively, the product

$$g_{lj}^{(1)}g_{j,j_s}^{(n_j)}\prod_{i=1}^{s-1} g_{j_{i+1}j_i}^{(n_{j_{i+1}})}\, g_{j_1,1}^{(n_{j_1})}\, g_{l,l_r}^{(n_l)}\prod_{i=1}^{r-1} g_{l_{i+1}l_i}^{(n_{l_{i+1}})}\, g_{l_1,2}^{(n_{l_1})} \tag{C.15}$$

can be written as a factor $g_{21}^{(n)}$ times a linear combination of admissible factors. The correct procedure is the following:

- First, assume (without loss of generality, rename the labels otherwise) that $l < j$, such that the subscript j in $g_{lj}^{n_l}$ can be lowered to j_s using the Fay identity as follows:

$$g_{lj}^{n_l} g_{j,j_s}^{n_j} = (-1)^{n_l} g_{jl}^{n_l} g_{j,j_s}^{n_j} = (-1)^{n_l} g_{l,j_1} \begin{pmatrix} g_{j,j_s} \\ g_{jl} \end{pmatrix}_{n_l,n_j} , \tag{C.16}$$

where the product on the right-hand side is defined to be the sum obtained by the Fay identity (3.33). It is a linear combination of $g_{l,j_s}^{(n_l+n_j-i)} g_{j,j_s}^{(i)}$ and $g_{l,j_s}^{(n_l+n_j-i)} g_{jl}^{(i)}$ for $0 \leq i \leq n_k + n_j$ with rational coefficients. Importantly, it is a linear combination of admissible factors and the index j in $g_{lj}^{(n_l)}$ has been lowered to j_s.

- Now, if $l < j_s$, we repeat this step with the products $g_{l,j_s}^{(n_l+n_j-i)} g_{j_s,j_{s-1}}^{(n_{j_s})}$. Similarly for lower indices j_t, unless we arrive at $g_{j_1,1}^{(n_{j_1})}$, where another application of the Fay identity leaves us with a linear combination of $g_{l,1}^{(n)}$ and admissible factors times the product $g_{l,l_r}^{(n_l)} \prod_{i=1}^{r-1} g_{l_{i+1},l_i}^{(n_{l_{i+1}})} g_{l_1,2}^{(n_{l_1})}$. Now, the same procedure can be applied to $g_{l,1}^{(n)} g_{l,l_r}^{(n_l)} \prod_{i=1}^{r-1} g_{l_{i+1},l_i}^{(n_{l_{i+1}})} g_{l_1,2}^{(n_{l_1})}$ such that we are left with a linear combination of admissible factors times a factor $g_{21}^{(n)}$ and some rational coefficients. However, if we arrive at some j_t such that $l > j_t$, we have to apply the Fay identity earlier to the product $g_{l,l_r}^{(n_l)} \prod_{i=1}^{r-1} g_{l_{i+1},l_i}^{(n_{l_{i+1}})} g_{l_1,2}^{(n_{l_1})}$ in order to recover admissible factors.

- Thus, if we arrive at some j_t with $l > j_t$, we apply the above procedure to the product $g_{l,l_r}^{(n_l)} \prod_{i=1}^{r-1} g_{l_{i+1},l_i}^{(n_{l_{i+1}})} g_{l_1,2}^{(n_{l_1})}$ beginning with the factor

$$g_{l,j_t}^{(n)} g_{l,l_r}^{(n_l)} = g_{j_t,l_r} \begin{pmatrix} g_{l,l_r} \\ g_{l,j_t} \end{pmatrix}_{n,n_l} . \tag{C.17}$$

  As above, this process can be applied to lower l_i unless we arrive either at $g_{l_1,2}^{(n_{l_1})}$ or at $l_i < j_t$. In the latter case, we again proceed with the application of the Fay identity with respect to the j_t index as in the previous step. In the former case, we arrive at a linear combination of $g_{j_t,2}^{(m)}$ and we are left with applying the procedure to the j_t index unless we hit j_1.

- The above process ends once we could rewrite the product in eq. (C.15) as a linear combination of $g_{21}^{(n)}$ times solely admissible factors and some rational coefficients.

Writing the weights of the genus-one Selberg vector as $\vec{w} = (w_3, \ldots, w_L) \in \mathbb{N}^{L-2}$, such that the total weight is given by $w = |\vec{w}| = w_3 + \ldots w_L$, and the admissible labelings $\vec{i} = (i_3, \ldots, i_L) \in \mathbb{N}^{L-2}$ with $1 \leq i_k < k$, this algorithm converts the derivative of the genus-one Selberg integral $\mathrm{S^E}\left[\begin{smallmatrix} w_3, & \ldots, & w_L \\ i_3, & \ldots, & i_L \end{smallmatrix}\right](0, z_2) = \mathrm{S^E}\left[\begin{smallmatrix} \vec{w} \\ \vec{i} \end{smallmatrix}\right](0, z_2)$ given in eq. (C.6) to a form similar to the KZB equation

$$\frac{\partial}{\partial z_2} \mathrm{S^E}\left[\begin{smallmatrix} \vec{w} \\ \vec{i} \end{smallmatrix}\right](0, z_2) = \sum_{n=0}^{w+1} g_{21}^{(n)} \sum_{\substack{\vec{m} \in \mathbb{N}^{L-2}: \\ m = w+1-n}} \sum_{\vec{j} \text{ adm}} x_{\vec{m},\vec{j}}^{\vec{w},\vec{i}} \mathrm{S^E}\left[\begin{smallmatrix} \vec{m} \\ \vec{j} \end{smallmatrix}\right](0, z_2) , \tag{C.18}$$

where $m = |\vec{m}|$ and the sum over $\vec{j} \in \mathbb{N}^{L-2}$ runs over the admissible labelings, i.e. the vectors $\vec{j}$ such that $1 \leq (\vec{j})_i = j_i < 2 + i$. Each coefficient $x_{\vec{m},\vec{j}}^{\vec{w},\vec{i}} \in \mathbb{Q}[s_{ij}]$ either vanishes or is a polynomial of degree one in the Mandelstam variables over the rational numbers, determined by the above algorithm. Note that all the terms $g_{21}^{(n)} \mathrm{S^E}\left[\begin{smallmatrix} \vec{m} \\ \vec{j} \end{smallmatrix}\right](0, z_2)$ are of total weight $w + 1 = n + m$, since

$m = w + 1 - n$. This is a consequence of the above algorithm: the partial derivatives in the last line of eq. (C.6) only act on the Selberg seed S^{E}, which effectively multiplies S^{E} with some $g_{lj}^{(1)}$. Hence, the integrand $\mathrm{S}^{\mathrm{E}} \prod_{k=3}^{L} g_{k,i_k}^{(n_k)}$ is multiplied with $g_{lj}^{(1)}$ which increases the total weight by one. The application of the Fay identity in the second step of the algorithm preserves this weight, which leads to the differential eq. (C.18).

The differential eq. (C.18) can be turned into a matrix equation by collecting the iterated integrals of a given weight $w = |\vec{w}|$ and all the possible admissible labelings $\vec{i}$ in a vector

$$\mathbf{S}_w^{\mathrm{E}}(z_2) = \left(\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}\vec{w}\\ \vec{i}\end{smallmatrix}\right](0, z_2)\right)_{|\vec{w}|=w, \vec{i}\text{ adm}}, \tag{C.19}$$

such that eq. (C.18) reads

$$\frac{\partial}{\partial z_2}\mathbf{S}_w^{\mathrm{E}}(z_2) = \sum_{n=0}^{w+1} g_{21}^{(n)} x_w^{(n)} \mathbf{S}_{w+1-n}^{\mathrm{E}}(z_2)\,, \tag{C.20}$$

where the entries of the matrices $x_w^{(n)}$ are given by the coefficients $x_{\vec{m},\vec{j}}^{\vec{w},\vec{i}}$ according to

$$\left(x_w^{(n)}\right)_{\vec{m},\vec{j}\text{ adm}}^{\vec{w},\vec{i}\text{ adm}} = x_{\vec{m},\vec{j}}^{\vec{w},\vec{i}} \tag{C.21}$$

and $|\vec{m}| = w + 1 - n$, $|\vec{w}| = w$. This partial differential equation is exactly eq. (3.58) with the matrices being determined by the above algorithm. However, this is not yet a KZB equation, but if the vector with subvectors $\mathbf{S}_w^{\mathrm{E}}(z_2)$ up to a maximal weight $w_{\max}$

$$\mathbf{S}_{\leq w_{\max}}^{\mathrm{E}}(z_2) = \left(\mathbf{S}_w^{\mathrm{E}}(z_2)\right)_{0\leq w\leq w_{\max}} \tag{C.22}$$

is differentiated, the partial differential eq. (3.62), i.e.

$$\frac{\partial}{\partial z_2}\mathbf{S}_{\leq w_{\max}}^{\mathrm{E}}(z_2) = \sum_{n=0}^{w_{\max}+1} g_{21}^{(n)} x_{\leq w_{\max}}^{(n)} \mathbf{S}_{\leq w_{\max}}^{\mathrm{E}}(z_2) + r_{w_{\max}} \mathbf{S}_{w_{\max}+1}^{\mathrm{E}}(z_2)\,, \tag{C.23}$$

is recovered, which is almost a KZB equation up to the remainder $r_{w_{\max}}$. As discussed in subsection 3.4, the matrices $x_{\leq w_{\max}}^{(n)}$ are block-(off-)diagonal with respect to the weight-(w_0, w_1) blocks and are given by

$$x_{\leq w_{\max}}^{(0)} = \left(\begin{array}{c|c|c|c|c} & x_0^{(0)} & & & \\ \hline & & x_1^{(0)} & & \\ \hline & & & \ddots & \\ \hline & & & & x_{w_{\max}-1}^{(0)} \\ \hline & & & & \end{array}\right), \qquad x_{\leq w_{\max}}^{(1)} = \left(\begin{array}{c|c|c|c|c} x_0^{(1)} & & & & \\ \hline & x_1^{(1)} & & & \\ \hline & & x_2^{(1)} & & \\ \hline & & & \ddots & \\ \hline & & & & x_{w_{\max}}^{(1)} \end{array}\right), \tag{C.24}$$

$$
x^{(2)}_{\leq w_{\max}} = \begin{pmatrix} & & & & \\ x^{(2)}_1 & & & & \\ & x^{(2)}_2 & & & \\ & & \ddots & & \\ & & & x^{(2)}_{w_{\max}} & \end{pmatrix}, \ldots, x^{(w_{\max}+1)}_{\leq w_{\max}} = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ x^{(w_{\max}+1)}_{w_{\max}} & & & & \end{pmatrix}. \tag{C.25}
$$

where the blank blocks are zero submatrices. The remainder $r_{w_{\max}}$ corresponds to the weight-$(w_{\max}+2)$ column of the matrix $x^{(0)}_{\leq w_{\max}+1}$ as follows

$$
x^{(0)}_{\leq w_{\max}+1} = \begin{pmatrix} & x^{(0)}_0 & & & & \\ & & x^{(0)}_1 & & & \\ & & & \ddots & & \\ & & & & x^{(0)}_{w_{\max}-1} & \\ & & & & & x^{(0)}_{w_{\max}} \\ & & & & & \end{pmatrix}, \qquad r_{w_{\max}} = \begin{pmatrix} \\ \\ \vdots \\ \\ x^{(0)}_{w_{\max}} \end{pmatrix}. \tag{C.26}
$$

The matrix $x^{(n)}_{w_{\max}}$ has $(w_{\max}+1)^2$ blocks of weights (w_0, w_1) and is block-(off-)diagonal, shifted by $n-1$ to the bottom. This is a consequence of the increase in the weight of the subvectors $\mathbf{S}^{\mathrm{E}}_w(z_2)$ by one, when differentiated with respect to z_2. Moreover, this also leads to the fact that the differential eq. (C.23) is not exactly in KZB form. It differs from the KZB form by the term which comes from the differentiation of the highest weight subvector $\mathbf{S}^{\mathrm{E}}_{w_{\max}}(z_2)$ in $\mathbf{S}^{\mathrm{E}}_{\leq w_{\max}}(z_2)$: the factor proportional to $1 = g^{(0)}_{21}$ is a linear combination of iterated integrals of weight $w_{\max}+1$, but $\mathbf{S}^{\mathrm{E}}_{w_{\max}+1}(z_2)$ is not included in $\mathbf{S}^{\mathrm{E}}_{\leq w_{\max}}(z_2)$. Thus, we have to account for this contribution by the matrix $r_{w_{\max}}$. Therefore, in principle, only the infinite vector $\mathbf{S}^{\mathrm{E}}(z_2) = \lim_{w_{\max}\to\infty} \mathbf{S}^{\mathrm{E}}_{\leq w_{\max}}(z_2)$ satisfies the proper KZB eq. (3.57),

$$
\frac{\partial}{\partial z_2} \mathbf{S}^{\mathrm{E}}(z_2) = \sum_{n\geq 0} g^{(n)}_{21} x^{(n)}\, \mathbf{S}^{\mathrm{E}}(z_2) \tag{C.27}
$$

with $x^{(n)} = \lim_{w_{\max}\to\infty} x^{(n)}_{\leq w_{\max}}$. This is how the KZB equation is recovered and satisfied by the genus-one Selberg integrals.

C.2 α'-expansion of genus-one string corrections

The KZB eq. (C.27) satisfied by the Selberg vector $\mathbf{S}^{\mathrm{E}}(z_2)$ is an infinite-dimensional vector equation. However, as mentioned in subsection 3.6, in order to calculate the α'-expansion of the one-loop string corrections up to any desired order $o^{\text{1-loop}}_{\max}$, we may in practice truncate the vector $\mathbf{S}^{\mathrm{E}}(z_2)$ at a certain weight $w_{\max}$ and simply work with the finite-dimensional, modified KZB eq. (C.23). In this section this truncation is discussed and, in particular, an expression for the required maximal weight $w_{\max}$ depending on the desired order $o^{\text{1-loop}}_{\max}$ is given.

Recall that due to the block-diagonality of $x^{(1)}$, the regularized boundary values (3.66) of the genus-one Selberg integral,

$$\mathbf{C}_0^{\mathrm{E}} = \lim_{z_2\to 0}(2\pi i z_2)^{-x^{(1)}}\,\mathbf{S}^{\mathrm{E}}(z_2) = \begin{pmatrix}\mathbf{C}_{0,0}^{\mathrm{E}}\\ \mathbf{C}_{0,1}^{\mathrm{E}}\\ \vdots\end{pmatrix},$$
$$\mathbf{C}_1^{\mathrm{E}} = \lim_{z_2\to 1}(2\pi i(1-z_2))^{-x^{(1)}}\,\mathbf{S}^{\mathrm{E}}(z_2) = \begin{pmatrix}\mathbf{C}_{1,0}^{\mathrm{E}}\\ \mathbf{C}_{1,1}^{\mathrm{E}}\\ \vdots\end{pmatrix}, \tag{C.28}$$

are ordered with respect to the total weight

$$\mathbf{C}_{0,w}^{\mathrm{E}} = \lim_{z_2\to 0}(2\pi i z_2)^{-x_w^{(1)}}\,\mathbf{S}_w^{\mathrm{E}}(z_2)\,, \qquad \mathbf{C}_{1,w}^{\mathrm{E}} = \lim_{z_2\to 1}(2\pi i(1-z_2))^{-x_w^{(1)}}\,\mathbf{S}_w^{\mathrm{E}}(z_2)\,. \tag{C.29}$$

Furthermore, from our discussion in subsection 3.5, we have learned that on the one hand, the only non-vanishing subvector of $\mathbf{C}_0^{\mathrm{E}}$ is $\mathbf{C}_{0,w_0}^{\mathrm{E}}$ with the weight $w_0 = L-2$, which contains the Selberg integrals $\mathrm{S}^{\mathrm{E}}\left[\begin{smallmatrix}1, & \dots, & 1\\ i_3, & \dots, & i_L\end{smallmatrix}\right](0,z_2)$ that degenerate to the genus-zero integrals in the regularized limit. On the other hand, the $(L-1)$-point, genus-one string corrections reside in the subvector $\mathbf{C}_{1,w_1}^{\mathrm{E}}$ of $\mathbf{C}_1^{\mathrm{E}}$ which corresponds to the weight $w_1 = \max(L-5,0)$ for $L\leq 8$ and $w_1 = L-5-d$, where $0\leq d\leq L-5$, for eight-point and higher string corrections at $L>8$ [35]. Hence, the relevant part of the associator equation is

$$\mathbf{C}_{1,w_1}^{\mathrm{E}} = \Phi_{w_1,w_0}^{\mathrm{E}}\,\mathbf{C}_{0,w_0}^{\mathrm{E}}\,, \tag{C.30}$$

and we only require the weight-(w_1,w_0) submatrix $\Phi_{w_1,w_0}^{\mathrm{E}}$ of the KZB associator. But since $w_1 < w_0$ and due to the block-(off-)diagonal form of the matrices $x^{(n)}$, this relevant (w_1,w_0)-block can be calculated using the finite matrices $x_{\leq w_{\max}}^{(n)}$ for some sufficiently large $w_{\max}\geq w_0$: as shown below, the non-trivial contribution to $\Phi_{w_1,w_0}^{\mathrm{E}}$ at each word length l, which is the order l in the α'-expansion of the associator since $x^{(n)}\propto\alpha'$, is a finite sum $\sum_w w\,\omega(w^t)$ of products $w = x_{\leq w_{\max}}^{(n_1)}x_{\leq w_{\max}}^{(n_2)}\dots x_{\leq w_{\max}}^{(n_l)}$, where

$$w_{\max} = \max(l+w_1-w_0, w_0) \tag{C.31}$$

and $(n_1,n_2,\dots,n_l)$ is a length-l, ordered partition of $w_{\max}$, i.e. $n_1+n_2+\dots+n_l = w_{\max}$, which satisfies for each $r\in\{1,2,\dots,l\}$ the additional conditions

$$0\leq i-\sum_{s=1}^{r-1}(n_s-1)\leq w_{\max}\,, \qquad 0\leq j+n_l-1\leq w_{\max}\,. \tag{C.32}$$

Therefore, in order to calculate the α'-expansion of $\Phi_{w_1,w_0}^{\mathrm{E}}$ up to order $l_{\max}$, we need to determine the matrices $x_{\leq w_{\max}}^{(n)}$ for $0\leq n\leq w_{\max}$ in the partial differential eq. (3.62) with $w_{\max} = l_{\max}+w_1-w_0$. Moreover, if $o_{\min}^{\text{tree}}$ is the minimal order at which the α'-expansion of the tree-level integrals in $\mathbf{C}_0^{\mathrm{E}}$ begins, eq. (C.30) implies that the maximal word length is

$$l_{\max} = o_{\max}^{\text{1-loop}} - o_{\min}^{\text{tree}}\,, \tag{C.33}$$

which yields together with eq. (C.31) an expression for the maximal weight depending on the desired order $w_{\max} = w_{\max}(o_{\max}^{\text{1-loop}})$.

These statements can be shown as follows: first, we note that the weight-(i,j) submatrix of

$x^{(n)}_{\leq w_{\max}}$ is

$$(x^{(n)}_{\leq w_{\max}})_{i,j} = x^{(n)}_i \delta_{i,j+n-1}\,, \tag{C.34}$$

where $x^{(n)}_i$ is given by eq. (C.21). Therefore, the product of two such matrices has the weight-(i,j) submatrix

$$\begin{aligned}(x^{(n_1)}_{\leq w_{\max}} x^{(n_2)}_{\leq w_{\max}})_{i,j} &= \sum_{k=0}^{w_{\max}} x^{(n_1)}_i x^{(n_2)}_k \delta_{i,k+n_1-1}\delta_{k,j+n_2-1} \\ &= x^{(n_1)}_i x^{(n_2)}_{i-(n_1-1)} \delta_{i-(n_1-1),j+n_2-1}\,. \end{aligned} \tag{C.35}$$

Note that this vanishes in particular for weights n_1 and n_2 which do not satisfy

$$0 \leq i-(n_1-1), j+n_2-1 \leq w_{\max}\,. \tag{C.36}$$

Iterating this calculation, it turns out that the weight-(i,j) submatrix of the matrix product $w = x^{(n_1)}_{\leq w_{\max}} x^{(n_2)}_{\leq w_{\max}} \dots x^{(n_l)}_{\leq w_{\max}}$ is

$$w_{ij} = \prod_{r=1}^{l} x^{(n_r)}_{i-\sum_{s=1}^{r-1}(n_s-1)}\, \delta_{i-\sum_{r=1}^{l-1}(n_r-1),j+n_l-1}\,, \tag{C.37}$$

where for each $r \in \{1,2,\dots,l\}$ the weights n_i have to satisfy

$$0 \leq i - \sum_{s=1}^{r-1}(n_s-1) \leq w_{\max}\,, \qquad 0 \leq j+n_l-1 \leq w_{\max} \tag{C.38}$$

in order to have a possibly non-vanishing submatrix w_{ij}. Therefore, taking $(i,j) = (w_1,w_0)$, we can conclude that the product $w = x^{(n_1)}_{\leq w_{\max}} x^{(n_2)}_{\leq w_{\max}} \dots x^{(n_l)}_{\leq w_{\max}}$ contributes non-trivially at length l to the (w_1,w_0)-submatrix $\Phi^{\mathrm{E}}_{(w_1,w_0)}$ of the KZB associator only if

$$w_0 - w_1 + \sum_{r=1}^{l}(n_r-1) = 0\,. \tag{C.39}$$

This gives for $n_1 = w_{\max}$ and $n_r = 0$ for $r > 1$ the maximal weight which has to be considered

$$w_{\max} = l + w_1 - w_0\,. \tag{C.40}$$

Hence, in order to calculate the contribution at order l in the α'-expansion of $\Phi^{\mathrm{E}}_{(w_1,w_0)}$, we have to include all the words $w = x^{(n_1)}_{\leq w_{\max}} x^{(n_2)}_{\leq w_{\max}} \dots x^{(n_l)}_{\leq w_{\max}}$ with $w_{\max} = l + w_1 - w_0$ and n_r given by the ordered, length-l partitions $(n_1,n_2,\dots,n_l)$ of $w_{\max}$. However, not all such partitions actually contribute: a partition $(n_1,n_2,\dots,n_l)$ of $w_{\max}$ can only contribute if it satisfy the conditions (C.38). [12] This completes the proof of the statements in eqs. (C.31) and (C.32).

[12]For example in the two-point calculation in subsection 3.6.1 with $L=3$, $w_0=1$, $w_1=0$, $w_{\max}=2$ and at word length $l=3$, the word $x^{(2)}_{\leq 2} x^{(0)}_{\leq 2} x^{(0)}_{\leq 2}$ gives no non-trivial contribution to the submatrix $\Phi^{\mathrm{E}}_3(x^{(n)}_{\leq 2})_{0,1}$ since it fails to satisfy the necessary condition (C.32) for $r=2$. By comparison, the words $x^{(0)}_{\leq 2} x^{(2)}_{\leq 2} x^{(0)}_{\leq 2}$ and $x^{(0)}_{\leq 2} x^{(0)}_{\leq 2} x^{(2)}_{\leq 2}$ satisfy the conditions and are indeed found to contribute non-trivially to $\Phi^{\mathrm{E}}_3(x^{(n)}_{\leq 2})_{0,1}$.

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