ANALYTIC THEORY OF HIGHER ORDER FREE CUMULANTS

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Abstract. We establish the functional relations between generating series of higher order free cumulants and moments in higher order free probability, solving an open problem posed fifteen years ago by Collins, Mingo, Śniady and Speicher. We propose an extension of free probability theory, which governs the all-order topological expansion in unitarily invariant matrix ensembles, with a corresponding notion of free cumulants and give as well their relation to moments via functional relations. Our approach is based on the study of a master transformation involving double monotone Hurwitz numbers via semi-infinite wedge techniques, building on the recent advances of the last-named author with Bychkov, Dunin-Barkowski and Kazarian.

Contents

1 Introduction

Voiculescu introduced free probability in the 1980's in the context of operator algebras. In this framework, the notion of independence of variables in classical probabilities is replaced by a notion of freeness, for which free cumulants are crucial objects. The first order moments and free cumulants of a random variable a are encoded in generating series:

$$M(X) \coloneqq 1 + \sum_{k \geq 1} \varphi_\ell[a] X^k\,, \qquad C(w) \coloneqq 1 + \sum_{k \geq 1} \kappa_\ell[a] w^k\,, \tag{1}$$

which satisfy the the following functional relation

$$C(X\,M(X)) = M(X)\,. \tag{2}$$

It is sometimes referred to as the R-transform machinery, originally given [Voi86] in a slightly different form: with the R-transform $R(w)$ and the Stieltjes transform $W(x)$ related to (1) by $C(w) = 1 + wR(w)$ and $W(x) = x^{-1}M(x^{-1})$. After the pioneering work of Voiculescu [Voi91], these notions brought a better understanding of the law of large numbers for the spectra of large random matrices, which in turn enriched the study of free probability.

Later, the study of random matrices of large size beyond the law of large numbers motivated, in [MS06; MŚS07; CMSS07], the development of a theory of higher order freeness. Given a higher order probability space, these works introduce higher order moments and free cumulants, the latter being obtained from the former via a convolution product involving non-crossing partitioned permutations. In the same manner as for the first order, one would like to relate the generating series of moments and free cumulants of order $n \geq 2$ of a variable a:

$$M_n(X_1, \ldots, X_n) \coloneqq \sum_{k_1,\ldots,k_n \geq 1} \varphi_n[a^{k_1}, \ldots, a^{k_n}] \prod_{i=1}^{n} X_i^{k_i}\,,$$
$$C_n(w_1, \ldots, w_n) \coloneqq \sum_{k_1,\ldots,k_n \geq 1} \kappa_{k_1,\ldots,k_n}[a, \ldots, a] \prod_{i=1}^{n} w_i^{k_i}\,.$$

In [CMSS07], second order freeness was studied in detail, and an explicit functional relation for $n = 2$ was found. It can be rewritten in the simple form

$$M_2(X_1, X_2) + \frac{X_1 X_2}{(X_1 - X_2)^2} = \frac{\mathrm{d}\ln w_1}{\mathrm{d}\ln X_1}\,\frac{\mathrm{d}\ln w_2}{\mathrm{d}\ln X_2}\left(C_2(w_1, w_2) + \frac{w_1 w_2}{(w_1 - w_2)^2}\right), \tag{3}$$

where $w_i = X_i M(X_i)$, or equivalently $X_i = w_i / C(w_i)$. However, due to the complicated combinatorics of partitioned permutations, the problem of obtaining such functional relations for $n \geq 3$ has remained open until now, limiting the practical applicability of higher order freeness. The present work remedies this by establishing in Theorem 3.9 the functional relations between generating series of moments and free cumulants to any order $n \geq 1$. They have the following structure.

Theorem 1.1. *Consider the change of variables $X_i = w_i / C(w_i)$. For $n \geq 3$, we have:*

$$M_n(X_1, \ldots, X_n) = \sum_{r_1,\ldots,r_n \geq 0} \sum_{T \in \mathcal{G}_{0,n}(\mathbf{r}+1)} \vec{\mathsf{O}}^{\vee}_{r_i}(w_i) \prod_{I \in \mathcal{I}(T)}{}^{\prime}\, C_{\# I}(w_I)\,, \tag{4}$$

where:

- *$\mathcal{G}_{0,n}(\mathbf{r}+1)$ is the set of bicoloured trees with white vertices labeled from 1 to n having valency $r_1 + 1, \ldots, r_n + 1$, and without univalent black vertices.*
- *The weight $\vec{\mathsf{O}}^{\vee}_{r_i}(w_i)$ of the i-th white vertex is a differential operator acting on the variable w_i. Its expression (Definition 3.8) involves only $C(w_i)$.*
- *$\mathcal{I}(T)$ is the set of black vertices, identified with the subset of white vertices they connect to.*
- *$\prod'$ means that any occurrence of $C_2(w_i, w_j)$ should be replaced with $C_2(w_i, w_j) + \frac{w_i w_j}{(w_i - w_j)^2}$.*
- *For a given monomial in the Xs, only finitely many terms of the right-hand side contribute.*

Formula (4) stems from combinatorics, but indirectly and with a convoluted history. In the case where the higher order probability space arises from the large size limit of an arbitrary (formal) unitarily-invariant measure of the space of Hermitian matrices, then moments are identified with generating series of ordinary maps [BIPZ78]. Later, for the same type of measure, the first and third-named authors [BGF20, Section 11.2] identified the free cumulants with generating series of planar fully simple maps. Using those identifications, the functional relations (2) and (3) admit combinatorial proofs [BGF20]. Generating series of maps of arbitrary genus g and number of boundaries n are known to satisfy a universal recurrence on $2g-2+n$, known under the name of topological recursion [EO09b; Eyn16]. Its initial data is encoded in the plane curve of equation $w = W(x) = x^{-1}M(x^{-1})$, called spectral curve. [BGF20] observed that (2) and (3) correspond to a transformation of the spectral curve called "symplectic exchange", which plays an important role in the theory of topological recursion [EO09b; EO08; EO13], and then conjectured that the topological recursion for the exchanged spectral curve $x = C(w)/w$ produces the generating series of fully simple maps of any topology. The functional relation (4) for $n = 3$ (in the special case of maps) can be derived from this conjecture, but the method could not be pushed to $n \geq 4$. The conjecture has received two proofs recently: using combinatorial tools in [BCGF21] by the three first-named authors; and for more general models of maps, *i.e.* more general unitarily-invariant ensembles of random matrices, using the Fock space formalism, in [BDBKS21a; BDBKS21b] by the fifth-named author with Bychkov, Dunin-Barkowski and Kazarian. The latter works are part of a general approach establishing, among other things, topological recursion for Hurwitz theory [BDBKS20].

The starting point of this article will be the *master relation* of the form

$$Z(\lambda) = z(\lambda) \sum_{\nu \vdash d} H^{<}(\lambda, \nu) Z^{\vee}(\nu) \tag{5}$$

between two functions Z and $Z^{\vee}$ on the set of integer partitions, where $H^{<}(\lambda, \mu)$ are the striclty monotone (double) Hurwitz numbers and $z(\lambda)$ is a symmetry factor (Sections 2.1 and 2.3). The weakly monotone Hurwitz numbers, encoded in $H^{\leq}(\lambda, \nu)$, which are in some sense (that will become explicit in Lemma 2.9) dual to the strict version, were introduced and studied in [GGPN14; GGPN13a; GGPN13b; Nov20]. As proved in [BGF20] via Weingarten calculus and recalled in Theorem 4.25, the master relation (5) materializes in the context of unitarily invariant ensembles of random hermitian matrices, $Z(\lambda)$ coming from moments of traces of powers and $Z^{\vee}(\nu)$ from expectation values of products of entries along cycles of a permutation. From the relation between map enumeration and formal matrix integrals, the master relation also relates the enumeration of (stuffed) maps (in $Z(\lambda)$) to the enumeration of fully simple (stuffed) maps (in $Z^{\vee}(\nu)$). This admits two bijective proofs and applies as well to hypermaps [BCDGF19].

We show that having the master relation (5) between two *arbitrary* generating series is equivalent to a moment-cumulant relation where moments are expressed as an extended convolution of the free cumulants with the zeta function on partitioned permutations (Theorem 2.13). By extended, we mean that we do not throw away the contribution of non-planar partitioned permutations, contrarily to the convolution used in [CMSS07, Section 5]. Representing these arbitrary generating series as expectation values in the Fock space, the universal relation amounts to inserting an operator creating the monotone Hurwitz numbers (Section 2.4). In Section 3, we then exploit the tools developed in [BDBKS21a] to obtain the core functional relations stated in Theorem 3.4. The extended convolution can be truncated to keep only the planar contributions as in [CMSS07], and by truncating accordingly we obtain the functional relations of Theorem 3.9 (or in the equivalent form of Theorem 3.12), summarised above as (4). These functional relations can be inverted to rather express free cumulants in terms of moments (Section 3.9).

The application of these results to higher order free probability is summarised in Sections 4.1-4.2. Besides, this leads us to propose in Section 4.5 a natural extension of the free probability theory to include non-planar cases, to all orders. It is based on surfaced permutations, which are essentially partitioned permutations with genus information (Section 4.3). In this context we define an extended convolution, a notion of surfaced moments and free cumulants related by extended convolution with the zeta function, and the functional relations are described by Theorem 3.4 (Section 4.5). We argue that this extension is the right one:

- We define (g_0, n_0)-freeness of variables by vanishing of mixed cumulants up to order (g_0, n_0), and this is equivalent to (g_0, n_0)-freeness of the algebras generated by those variables (Corollary 4.23).
- The (∞, ∞)-asymptotic freeness of two independent ensembles of random matrices, one of which is unitarily invariant, is guaranteed (Section 4.6).
- We stress that, compared to previous work in free probability where an emphasis was put on keeping only non-crossing or planar contributions, our conclusion is that the theory becomes simpler once one extends it to keep all genera. It is in this context that we derive functional relations, which we (only then) truncate to obtain the desired genus 0 relations (4).
- It is convenient to allow the genus to take half-integer values. Then, $(\frac{1}{2}, 1)$-freeness retrieves the notion of infinitesimal freeness of [FN09] coming from [BS12], and the known functional relations between the corresponding moments and free cumulants (Corollary 4.15).

Albeit not used in this article, the theory of the topological recursion whispered us that keeping all genera was the natural thing to do. It also led us to import for the present purposes the Fock space techniques recently developed to understand better the interplay between Hurwitz theory and topological recursion. Actually, we are led to formulate Conjecture 3.13, stating that the master relation (5) and the functional relations of Theorem 3.4 exactly describe the effect of symplectic exchange in topological recursion. We hope to return to the matter soon.

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2 Convolution on partitioned permutations and Hurwitz numbers

In Section 2.1, we set up some preliminary notations in order to review partitioned permutations in Section 2.2, in which we also introduce some new definitions needed for our purposes. In Section 2.3, following [ALS16; BCDGF19], we recall notations and basic facts concerning monotone Hurwitz numbers. This allows us to recast in Section 2.4 the extended convolution by the zeta function in the algebra of functions on partitioned permutations, in terms of transformations of topological partition functions using monotone Hurwitz numbers.

2.1 Integer and set partitions

Let d be a nonnegative integer. The set of permutations of $[d] \coloneqq \{1, \ldots, d\}$ is denoted $S(d)$. We say that $\lambda = (\lambda_i)_{i=1}^{\ell}$ is a partition of d (notation $\lambda \vdash d$) when it is a weakly decreasing sequence of positive integers such that $\sum_{i=1}^{\ell} \lambda_i = d$. We denote $\ell(\lambda) = \ell$ the length of the sequence. If $(k_1, \ldots, k_\ell)$ is a sequence of positive integers, we denote $\lambda(\mathbf{k})$ this sequence written in decreasing order. If $\lambda \vdash d$, we associate to it the permutation $\pi_\lambda \in S(d)$:

$$\pi_\lambda \coloneqq (1 \ \ldots \ \lambda_1)(\lambda_1 + 1 \ \ldots \ \lambda_1 + \lambda_2) \cdots (\lambda_1 + \cdots + \lambda_{\ell-1} + 1 \ \ldots \ \lambda_1 + \cdots + \lambda_\ell)\,.$$

Conversely, if σ is a permutation, we denote $\lambda(\sigma)$ the sequence of lengths of the cycles of σ, in weakly decreasing order. By convention, we define $S(0) = \{\emptyset\}$, and we declare $\emptyset$ to be the (only) partition of 0. Given $\lambda \vdash d$, let $C_\lambda \subseteq S(d)$ be the conjugacy class of π_λ, that is, the set of permutations $\sigma \in S[d]$ such that $\lambda(\sigma) = \lambda$, and

$$z(\lambda) = \frac{d!}{\# C_\lambda} = \prod_{i=1}^{\ell(\lambda)} \lambda_i \prod_{j \geq 1} m_j(\lambda)!\,,$$

where $m_j(\lambda)$ is the number of occurrences of j in the sequence λ.

For us, a partition of $[d]$ is a set of non-empty and pairwise disjoint subsets of $[d]$ whose union is $[d]$. The set $P(d)$ of partitions of $[d]$ is endowed with the structure of a poset. Namely, let $\mathcal{A}$, $\mathcal{B} \in P(d)$, $\mathcal{A} = \{A_1, \ldots, A_a\}$, $\mathcal{B} = \{B_1, \ldots, B_b\}$. The partial order relation $\mathcal{A} \leq \mathcal{B}$ holds when every block A_i is contained in some block B_j. The trivial partition $\mathbf{1}_d \coloneqq \{[d]\} \in P(d)$ is the maximum of $P(d)$. If $\mathcal{A}, \mathcal{B} \in P(d)$, their merging is denoted $\mathcal{A} \vee \mathcal{B} \in P(d)$: it is the smallest $\mathcal{C} \in P(d)$ such that $\mathcal{A} \leq \mathcal{C}$ and $\mathcal{B} \leq \mathcal{C}$. In particular, we have $\mathcal{A} \vee \mathbf{1}_d = \mathbf{1}_d$. If $\sigma \in S(d)$, we define $\mathbf{0}_\sigma \in P(d)$ to be the partition of $[d]$ whose elements are the supports of cycles in σ. For instance $\mathbf{0}_{(126)(35)(4)} = \{\{1, 2, 6\}, \{3, 5\}, \{4\}\}$. Note that we have $\mathcal{A} \vee \mathbf{0}_{\mathrm{id}} = \mathcal{A}$. We also use the notation $\mathbf{0}_\lambda \coloneqq \mathbf{0}_{\pi_\lambda}$.

2.2 *Partitioned permutations*

Definition 2.1. A *partitioned permutation* of d elements is a pair $(\mathcal{A}, \alpha)$, where $\mathcal{A} \in P(d)$ and $\alpha \in S(d)$, such that $\mathbf{0}_\alpha \leq \mathcal{A}$. Let $PS(d)$ be the set of partitioned permutations of d elements and $PS = \bigcup_{d \geq 0} PS(d)$.

We define the *colength* of $\mathcal{A} \in P(d)$ and of $\alpha \in S(d)$ as

$$|\mathcal{A}| \coloneqq d - \#\mathcal{A} \quad \text{and} \quad |\alpha| \coloneqq d - \#\mathbf{0}_\alpha\,,$$

where $\#\mathcal{A}$ and $\#\mathbf{0}_\alpha$ are the number of blocks of $\mathcal{A}$ and the number of cycles of α, respectively. Then, $|\alpha|$ is the minimal number of transpositions in a factorisation of α. The colength of a partitioned permutation $(\mathcal{A}, \alpha) \in PS(d)$ is defined as:

$$|(\mathcal{A}, \alpha)| \coloneqq 2|\mathcal{A}| - |\alpha|\,.$$

It is a nonnegative integer, as the property $\mathbf{0}_\alpha \leq \mathcal{A}$ implies $\#\mathbf{0}_\alpha \geq \#\mathcal{A}$. Note the special case:

$$|(\mathbf{0}_\alpha, \alpha)| = |\alpha|\,, \tag{6}$$

and the additivity:

$$|(\mathcal{A}, \alpha)| = \sum_{A \in \mathcal{A}} |(\mathbf{1}_{\#A}, \alpha_{|A})|\,, \tag{7}$$

where we have made a choice of bijections $[\#A] \to A$ to consider $(\mathbf{1}_{\#A}, \alpha_{|A}) \in PS(\#A)$, but its colength appearing on the right-hand side of (7) is independent of these choices.

We recall the definitions of the product of partitioned permutations and the associated convolution, and introduce their extended version which is relevant for us. Hereafter R denotes a commutative ring.

Definition 2.2. Let $(\mathcal{A}, \alpha), (\mathcal{B}, \beta), (C, \gamma) \in PS(d)$ and $f_1, f_2\colon PS(d) \to R$ two functions.

- The multiplication[1] $\cdot$ and the extended multiplication $\odot$ of partitioned permutations are:

$$(\mathcal{A}, \alpha) \cdot (\mathcal{B}, \beta) \coloneqq \begin{cases} (\mathcal{A} \vee \mathcal{B}, \alpha \circ \beta) & \text{if} \quad |(\mathcal{A}, \alpha)| + |(\mathcal{B}, \beta)| = |(\mathcal{A} \vee \mathcal{B}, \alpha \circ \beta)|\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

$$(\mathcal{A}, \alpha) \odot (\mathcal{B}, \beta) \coloneqq (\mathcal{A} \vee \mathcal{B}, \alpha \circ \beta)\,.$$

[1]Strictly speaking, $\cdot$ is a multiplication on $PS(d) \cup \{0\}$, with 0 declared to be an absorbing element.

- The convolution product and the extended convolution of two functions are:

$$
\begin{aligned}
(f_1 * f_2)(\mathcal{C},\gamma) &:= \sum_{(\mathcal{A},\alpha)\cdot(\mathcal{B},\beta)=(\mathcal{C},\gamma)} f_1(\mathcal{A},\alpha)\, f_2(\mathcal{B},\beta)\,, \\
(f_1 \circledast f_2)(\mathcal{C},\gamma) &:= \sum_{(\mathcal{A},\alpha)\odot(\mathcal{B},\beta)=(\mathcal{C},\gamma)} f_1(\mathcal{A},\alpha)\, f_2(\mathcal{B},\beta)\,.
\end{aligned} \tag{8}
$$

It is easy to see that these products are associative, but for $d > 1$ they are not commutative. Elementary combinatorics — see *e.g.* [CMSS07, Lemma 4.7] — show that

$$|(\mathcal{A}\vee\mathcal{B},\alpha\circ\beta)| \leq |(\mathcal{A},\alpha)| + |(\mathcal{B},\beta)|\,. \tag{9}$$

Definition 2.3. The following functions on $PS(d)$ will play an important role.

- The *delta function* is:

$$\delta(\mathcal{A},\alpha) = \begin{cases} 1 & \text{if } \mathcal{A} = \mathbf{0}_{\mathrm{id}} \text{ and } \alpha = \mathrm{id}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

- The *zeta function* is:

$$\zeta(\mathcal{A},\alpha) := \begin{cases} 1 & \text{if } \mathcal{A} = \mathbf{0}_{\alpha}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

  The *extended zeta function* is $\zeta_\hbar(\mathcal{A},\alpha) := \hbar^{|\alpha|}\zeta(\mathcal{A},\alpha)$ and takes values in $R[\![\hbar]\!]$.

- The *Möbius function* $\mu\colon PS(d) \to R$ and the *extended Möbius function* $\mu_\hbar\colon PS(d) \to R[\![\hbar]\!]$ are uniquely determined by

$$
\begin{aligned}
\mu * \zeta = \zeta * \mu &= \delta\,, \\
\mu_\hbar \circledast \zeta_\hbar = \zeta_\hbar \circledast \mu_\hbar &= \delta\,.
\end{aligned}
$$

Lemma 2.4. *The Möbius functions μ and $\mu_\hbar$ exist.*

Proof. The existence of μ is known from [CMSS07]. For the existence of $\mu_\hbar$, we view functions $PS \to R[\![\hbar]\!]$ (and their $\circledast$ convolution) as elements of the group ring $R[\![\hbar]\!](PS(d))$ (with product induced by $\odot$). In particular, $\zeta_\hbar$ can be viewed as $\sum_{\alpha\in S(d)} \hbar^{|\alpha|}\,(\mathbf{0}_\alpha,\alpha)$. It is of the form $(\mathbf{0}_{\mathrm{id}},\mathrm{id})+O(\hbar)$ since the only permutation with zero colength is the identity, and $(\mathbf{0}_{\mathrm{id}},\mathrm{id})$ is the unit for $\odot$ thus invertible. Therefore, $\zeta_\hbar$ is invertible, and its inverse defines $\mu_\hbar$. ☐

Definition 2.5. A function $f\colon PS \to R$ is *multiplicative* if for any $d \in \mathbb{Z}_{\geq 0}$ and $\sigma \in S(d)$, $f(\mathbf{1}_d,\sigma)$ depends only on the conjugacy class of σ, and for any $(\mathcal{A},\alpha) \in PS$:

$$f(\mathcal{A},\alpha) = \prod_{A\in\mathcal{A}} f(\mathbf{1}_{\#A},\alpha_{|A})\,.$$

Here $\alpha_{|A}$ is the pre-composition of the restriction of α to A with some bijection $[\#A] \to A$, the result being independent of the choice of that bijection due to the first property.

The convolution of two functions $f_1, f_2\colon PS \to R$ is defined by requiring that $(f_1 * f_2)_{|PS(d)} = f_{1|PS(d)} * f_{2|PS(d)}$ for all $d \in \mathbb{Z}_{\geq 0}$, and likewise for $\circledast$. Then, multiplicative functions are stable under convolution (resp. extended convolution), and the zeta function, the extended zeta function and their corresponding Möbius functions are multiplicative. Besides, the convolution of two multiplicative functions is commutative:

Lemma 2.6. *If $f_1, f_2\colon PS \to R$ are two multiplicative functions, then*

$$f_1 * f_2 = f_2 * f_1\,, \qquad \text{and} \qquad f_1 \circledast f_2 = f_2 \circledast f_1\,.$$

Proof. Let $(\mathcal{A},\alpha),(\mathcal{B},\beta),(\mathcal{C},\gamma) \in PS$ such that $(\mathcal{A},\alpha)\odot(\mathcal{B},\beta) = (\mathcal{C},\gamma)$. Then $\mathcal{C} = \mathcal{A}\vee\mathcal{B}$ and $\alpha\circ\beta = \gamma$. This can also be written $\mathcal{C} = \mathcal{B}\vee\mathcal{A}$ and $\beta^{-1}\circ\alpha^{-1} = \gamma^{-1}$. The support of cycles of a permutation and its inverse are the same: $\mathbf{0}_\alpha = \mathbf{0}_{\alpha^{-1}}$, etc. Therefore $(\mathcal{A},\alpha^{-1}),(\mathcal{B},\beta^{-1}),(\mathcal{C},\gamma^{-1})$ are still partitioned permutations and $(\mathcal{B},\beta^{-1})\odot(\mathcal{A},\alpha^{-1}) = (\mathcal{C},\gamma^{-1})$. Since α and α^{-1} are conjugated, a multiplicative function takes the same values on $(\mathcal{A},\alpha)$ and on $(\mathcal{A},\alpha^{-1})$. Then, relabelling $(\mathcal{A},\alpha)$ into $(\mathcal{B},\beta^{-1})$ and $(\mathcal{B},\beta)$ into $(\mathcal{A},\alpha^{-1})$ in the definition (8) of the extended convolution shows that $f \circledast g = g \circledast f$. The same reasoning works for $*$ as $(\mathcal{C},\gamma)$ and $(\mathcal{C},\gamma^{-1})$ have the same colength. □

We give the following property for later use.

Lemma 2.7. *Let $\phi_{1,\hbar},\phi_{2,\hbar}\colon PS \to R[\![\hbar]\!]$ be two multiplicative functions such that*

$$\forall(\mathcal{A},\alpha)\in PS,\qquad \phi_{i,\hbar}(\mathcal{A},\alpha) = \hbar^{|(\mathcal{A},\alpha)|}\phi_i(\mathcal{A},\alpha) + o(\hbar^{(|\mathcal{A},\alpha)|}),\qquad i=1,2\,,$$

where $\phi_i\colon PS \to R$. The relation $\phi_{1,\hbar} = \zeta_\hbar \circledast \phi_{2,\hbar}$ implies $\phi_1 = \zeta * \phi_2$.

Proof. Let us evaluate $\phi_{1,\hbar} = \zeta_\hbar \circledast \phi_{2,\hbar}$ at $(\mathcal{A},\alpha)\in PS(d)$. In the left-hand side the leading term with respect to $\hbar$ is $\hbar^{|(\mathcal{C},\gamma)|}\phi_1(\mathcal{C},\gamma)$, by definition. Due to the definition of $\zeta_\hbar$, only the factorisations of the type $(\mathbf{0}_\alpha,\alpha)\odot(\mathcal{B},\beta) = (\mathcal{C},\gamma)$ contribute to the right-hand side, and the leading term with respect to $\hbar$ is $\hbar^{|\alpha|+|(\mathcal{B},\beta)|}\zeta(\mathbf{0}_\alpha,\alpha)\phi_2(\mathcal{B},\beta)$. Recalling (6) we have $|(\mathbf{0}_\alpha,\alpha)| = |\alpha|$, and together with the subadditivity of colength for the extended product (9), it shows that $|(\mathcal{C},\gamma)| \leq |\alpha| + |(\mathcal{B},\beta)|$, with equality if and only if $(\mathbf{0}_\alpha,\alpha)\cdot(\mathcal{B},\beta) = (\mathcal{C},\gamma)$. Therefore:

$$\phi_1(\mathcal{C},\gamma) = \sum_{(\mathbf{0}_\alpha,\alpha)\cdot(\mathcal{B},\beta)=(\mathcal{C},\gamma)} \phi_2(\mathcal{B},\beta) = (\zeta * \phi_2)(\mathcal{C},\gamma)\,.$$

□

2.3 *Monotone Hurwitz numbers*

Let r be a nonnegative integer. A sequence $\tau_1,\ldots,\tau_r$ of transpositions in $S(d)$ is called *strictly monotone* (resp. *weakly monotone*) if $\tau_i = (a_i\ b_i)$ with $a_i < b_i$ and the sequence $(b_i)_{i=1}^r$ is strictly (resp. weakly) increasing.

Definition 2.8. Let $\lambda,\nu \vdash d$. The *strictly monotone Hurwitz number* $H_r^<(\lambda,\nu)$ is $\frac{1}{d!}$ times the number of tuples $(\alpha,\tau_1,\ldots,\tau_r,\beta)$ of permutations in $S(d)$ such that:

- $\alpha \in C_\lambda$ and $\beta \in C_\nu$;
- $\tau_1,\ldots,\tau_r$ is a strictly monotone sequence of transpositions;
- $\alpha\circ\tau_1\circ\cdots\circ\tau_r\circ\beta = \mathrm{id}$.

The *weakly monotone Hurwitz number* $H_r^{\leq}(\lambda,\nu)$ is defined analogously[2]. Summing over all the possible numbers of transpositions, we introduce the generating series

$$(10)\qquad \begin{aligned} H^<(\lambda,\nu) &= \sum_{r=0}^{d-1} \hbar^r\, H_r^<(\lambda,\nu) \in \mathbb{Q}[\hbar]\,,\\ H^{\leq}(\lambda,\nu) &= \sum_{r\geq 0} (-\hbar)^r\, H_r^{\leq}(\lambda,\nu) \in \mathbb{Q}[\![\hbar]\!]\,. \end{aligned}$$

We recall a classical result, including a proof to be self-contained.

Lemma 2.9. *For any $d\in\mathbb{Z}_{\geq 0}$ and $\lambda,\nu\vdash d$, we have:*

$$\sum_{\rho\vdash d} z(\lambda)\,H^<(\lambda,\rho)\cdot z(\rho)H^{\leq}(\rho,\nu) = \delta_{\lambda,\nu}\,,$$
$$\sum_{\rho\vdash d} z(\lambda)\,H^{\leq}(\lambda,\rho)\cdot z(\rho)H^<(\rho,\nu) = \delta_{\lambda,\nu}\,.$$

[2]Weakly monotone Hurwitz numbers were first introduced and studied in a series of papers by Goulden, Guay-Paquet and Novak [GGPN14; GGPN13a; GGPN13b] as coefficients in the large N asymptotic expansion of the Harish-Chandra–Itzykson–Zuber (HCIZ) matrix integral over the unitary group $U(N)$.

Proof. This relation appears when we present monotone Hurwitz numbers using the Jucys–Murphy elements. These are the elements of the group algebra $\mathbb{Q}S(d)$ defined by $J_k = \sum_{i=1}^{k-1}(i\,k)$. They have the property [Juc74; Mur81] that symmetric polynomials in the $(J_k)_{k=2}^{d}$ belong to the center of $\mathbb{Q}S(d)$. In particular, we can use them in the r-th elementary symmetric and complete symmetric polynomials:

$$\mathsf{e}_r(x_2,\ldots,x_d) := \sum_{2\leq k_1<\ldots<k_r\leq d} x_{k_1}\cdots x_{k_r}\,, \qquad \mathsf{h}_r(x_2,\ldots,x_d) := \sum_{2\leq k_1\leq\cdots\leq k_r\leq d} x_{k_1}\cdots x_{k_r}\,.$$

By construction,

$$H_r^{<}(\lambda,\nu) = \frac{1}{d!}\cdot[\mathrm{id}]\ C_\lambda C_\nu\,\mathsf{e}_r(J_2,\ldots,J_d)\,, \qquad H_r^{\leq}(\lambda,\nu) = \frac{1}{d!}\cdot[\mathrm{id}]\ C_\lambda C_\nu\,\mathsf{h}_r(J_2,\ldots,J_d)\,,$$

where the operation $[\mathrm{id}]$ stands for the extraction of the coefficient of id in the group algebra, and C_λ is here used to denote the element of the group algebra $\sum_{\sigma\in C_\lambda}\sigma$. For the generating series (10) over r, this gives:

$$\begin{aligned} H^{<}(\lambda,\nu) &= \frac{1}{d!}\cdot[\mathrm{id}]\ C_\lambda C_\nu \prod_{k=2}^{d}(1+\hbar J_k)\,, \\ H^{\leq}(\lambda,\nu) &= \frac{1}{d!}\cdot[\mathrm{id}]\ C_\lambda C_\nu \frac{1}{\prod_{k=2}^{d}(1+\hbar J_k)}\,. \end{aligned} \tag{11}$$

In general, if B is in the center of $\mathbb{Q}S(d)$, we have

$$\frac{1}{d!}\cdot[\mathrm{id}]\ C_\lambda C_\nu B = [C_\lambda]\ \frac{C_\nu B}{z(\lambda)} = [C_\mu]\ \frac{C_\lambda B}{z(\mu)}\,,$$

where we recall $z(\lambda) = \frac{d!}{\#C_\lambda}$. We use this relation to compute for any $\lambda,\nu\vdash d$:

$$\begin{aligned} \delta_{\lambda,\nu} &= \frac{z(\lambda)}{d!}\cdot[\mathrm{id}]\ C_\lambda C_\nu = \frac{z(\lambda)}{d!}\cdot[\mathrm{id}]\ C_\lambda \prod_{k=2}^{d}(1+\hbar J_k)\cdot C_\nu\frac{1}{\prod_{k=2}^{d}(1+\hbar J_k)} \\ &= \frac{z(\lambda)}{d!}\cdot[\mathrm{id}]\bigg(\sum_{\rho,\rho'\vdash d} H^{<}(\lambda,\rho)\,z(\rho)C_\rho\cdot H^{\leq}(\rho',\nu)z(\rho')C_{\rho'}\bigg) \\ &= \sum_{\rho\vdash d} z(\lambda)H^{<}(\lambda,\rho)\cdot z(\rho)H^{\leq}(\rho,\nu)\,. \end{aligned}$$

The second relation is proved similarly. ☐

An alternative description of strictly monotone Hurwitz numbers will later come handy.

Definition 2.10 (See *e.g.* [ALS16]). Given $\lambda,\nu\vdash d$, the *free single Hurwitz number* $H_r^{\mid}(\lambda,\mu)$ is $\frac{1}{d!}$ times the number of triples (α,ψ,β) of permutations of $[d]$ such that

- $\alpha\in C_\lambda$ and $\beta\in C_\nu$;
- $\psi\in S(d)$ has colength r;
- $\alpha\circ\psi\circ\beta = \mathrm{id}$.

In other words, in the group algebra we have

$$H_r^{\mid}(\lambda,\nu) = \frac{1}{d!}\cdot[\mathrm{id}]\ C_\lambda C_\nu \sum_{\substack{\rho\vdash d \\ \ell(\rho)=d-r}} C_\rho\,. \tag{12}$$

The free single Hurwitz number is directly related to the weighted count $D_r(\lambda,\nu)$ of (possibly disconnected) dessins d'enfants having r more edges than there are vertices, namely $D_r(\lambda,\nu) = z(\lambda)H_r^{\mid}(\lambda,\nu)$, see [BCDGF19, Definition 2.4]. We also recall that dessins d'enfants is another name for bipartite maps (again, up to symmetry considerations which in this case amount to factors of 2 in the weighted count). The Harnad–Orlov correspondence [HO15] (see also [ALS16, Proposition

4.4]), or the more elementary observation that every permutation can be uniquely expressed as the product of transpositions along a strictly monotone sequence — see *e.g.* [BCDGF19, Lemma 2.5] — yield the following equality.

Lemma 2.11. *We have* $H_r^{<}(\lambda,\nu) = H_r^{|}(\lambda,\nu)$.

2.4 *Fock space preliminary*

We introduce the bosonic Fock space, *i.e.* the ring of formal series in countably many variables

$$\mathcal{F}_R \coloneqq \varprojlim_{d\in\mathbb{Z}_{\geq 0}} R[\![p_1,\ldots,p_d]\!]\,, \qquad \mathcal{F}_{R,\hbar} \coloneqq \mathcal{F}_R \otimes \mathbb{Q}(\!(\hbar)\!)\,.$$

We also introduce the vector $|\rangle \coloneqq 1 \in \mathcal{F}_R$ and the linear form $\langle|\colon \mathcal{F}_R \to R(\!(\hbar)\!)$.

As $R(\!(\hbar)\!)$-module, $\mathcal{F}_R$ admits the Schur basis s_λ, indexed by $\lambda \vdash d$ and $d \in \mathbb{Z}_{\geq 0}$. We say that $(i,j) \in \mathbb{Z}_{>0}^2$ belongs to λ when $i \leq \ell(\lambda)$ and $j \leq \lambda_i$, and consider the linear operator characterised by

$$\mathsf{D}\, s_\lambda = \prod_{(i,j)\in\lambda} (1 + \hbar(j-i))\, s_\lambda\,. \tag{13}$$

Note that D has a logarithm, and $\ln \mathsf{D}|\rangle = 0$ as well as $\langle|\ln\mathsf{D} = 0$. An explicit formula for D is provided in [ALS16, Section 5]. In fact, $\mathcal{F}_R$ can be identified (as a vector space) to the direct sum over $d \in \mathbb{Z}_{\geq 0}$ of the center of the group algebra of $S(d)$ over R, and the restriction of D to the d-th part is identified with the multiplication by $\prod_{k=2}^{d}(1+\hbar J_k)$, where J_k are the Jucys–Murphy elements already met in the proof of Lemma 2.9.

2.5 *Topological partition functions*

Let $\mathcal{F}_{R,\hbar}^0$ be the kernel of the ring homomorphism $\mathcal{F}_R \to R(\!(\hbar)\!)$ sending all p_is to 0. Clearly, if $F \in \mathcal{F}_{R,\hbar}^0$, then $e^F \in \mathcal{F}_{R,\hbar}$ is well-defined. If $\lambda \vdash d$, we define $p_\lambda = p_{\lambda_1}\cdots p_{\lambda_\ell}$. If σ is a permutation we also denote $p_\sigma \coloneqq p_{\lambda(\sigma)}$. Finally, given $F \in \mathcal{F}_{R,\hbar}$, we denote $[p_\sigma]\ F$ for the coefficient of p_σ in F.

Definition 2.12. A *topological partition function* is an element of $\mathcal{F}_{R,\hbar}$ of the form $Z = e^F$, where $F = \sum_{g\in\mathbb{Z}_{\geq 0}} \hbar^{2g-2} F_g$ and $F_g \in \mathcal{F}_R$.

Given a topological partition function $Z = e^F$, a multiplicative function $\Phi_{Z,\hbar}\colon PS \to R[\![\hbar]\!]$ can be constructed by setting, for $(\mathcal{A},\alpha) \in PS(d)$:

$$\begin{aligned}\Phi_{Z,\hbar}(\mathcal{A},\alpha) &\coloneqq \hbar^{d+\ell(\alpha)} \prod_{A\in\mathcal{A}} [p_{\alpha_{|A}}]\ F = \hbar^{|(\mathcal{A},\alpha)|} \sum_{g\colon \mathcal{A}\to\mathbb{Z}_{\geq 0}} \hbar^{2\sum_{A\in\mathcal{A}} g(A)} \prod_{A\in\mathcal{A}} [p_{\alpha_{|A}}]\ F_{g(A)} \\ &\coloneqq \hbar^{|(\mathcal{A},\alpha)|} \sum_{g\colon \mathcal{A}\to\mathbb{Z}_{\geq 0}} \hbar^{2g}\, \Phi_Z^{[g]}(\mathcal{A},\alpha)\,.\end{aligned} \tag{14}$$

Given $\lambda \vdash d$, we use the notation $Z(\lambda) = \hbar^{d+\ell(\lambda)}\cdot[p_\lambda]\ Z$ for the coefficients of Z. The multiplicative function completely determines the partition function, via the relation

$$Z(\lambda) = \sum_{\substack{\mathcal{A}\in P(d) \\ \mathbf{0}_\lambda \leq \mathcal{A}}} \Phi_{Z,\hbar}(\mathcal{A},\pi_\lambda)\,. \tag{15}$$

Note that $Z(\emptyset) = 1$.

Theorem 2.13. *Consider two topological partition functions Z_1, Z_2 and $d \in \mathbb{Z}_{>0}$. The following four properties are equivalent:*

- (i) $Z_1(\lambda) = z(\lambda)\sum_{\nu\vdash d} H^{<}(\lambda,\nu) Z_2(\nu)$ *holds for any* $\lambda \vdash d$*;*
- (ii) $\Phi_{Z_1,\hbar} = \zeta_\hbar \circledast \Phi_{Z_2,\hbar}$ *holds between functions on* $PS(d)$*;*
- (iii) $Z_2(\nu) = z(\nu)\sum_{\lambda\vdash d} H^{\leq}(\nu,\lambda) Z_1(\lambda)$ *holds for any* $\nu \vdash d$*;*
- (iv) $\Phi_{Z_2,\hbar} = \mu_\hbar \circledast \Phi_{Z_1,\hbar}$ *holds between functions on* $PS(d)$*.*

Besides, the property $Z_1 = \mathsf{D} Z_2$ is equivalent to any of these conditions simultaneously for all $d > 0$.

Corollary 2.14. *Let $d > 0$. If one of the four conditions above holds, then the relations $\Phi_{Z_1}^{[0]} = \zeta * \Phi_{Z_2}^{[0]}$ and $\Phi_{Z_2}^{[0]} = \mu * \Phi_{Z_1}^{[0]}$ between functions on $PS(d)$ hold.*

Remark 2.15. Due to Lemma 2.6, condition (ii) is also equivalent to $\Phi_{Z_1,\hbar} = \Phi_{Z_2,\hbar} \circledast \zeta_\hbar$, etc.

Proof of Theorem 2.13. We first prove that (ii) implies (i). Let $\lambda \vdash d$ be a partition of d. We start from the equality $\Phi_{Z_1,\hbar} = \zeta_\hbar \circledast \Phi_{Z_2,\hbar}$, which, by Equation (15), is equivalent to:

$$\begin{aligned}
Z_1(\lambda) &= \sum_{\substack{\mathcal{C}\in P(d)\\ \mathbf{0}_\lambda \leq \mathcal{C}}} (\zeta_\hbar \circledast \Phi_{Z_2,\hbar})(\mathcal{C},\pi_\lambda) \\
&= \sum_{\substack{\mathcal{C}\in P(d)\\ \mathbf{0}_\lambda \leq \mathcal{C}}} \sum_{(\mathbf{0}_\alpha,\alpha)\odot(\mathcal{B},\beta)=(\mathcal{C},\pi_\lambda)} \hbar^{|\alpha|}\, \Phi_{Z_2,\hbar}(\mathcal{B},\beta) \\
&= \sum_{\substack{\alpha,\beta\in S(d)\\ \alpha\circ\beta=\pi_\lambda}} \sum_{\substack{\mathcal{B}\in P(d)\\ \mathbf{0}_\beta\leq\mathcal{B}}} \hbar^{|\alpha|}\Phi_{Z_2,\hbar}(\mathcal{B},\beta) \\
&= \sum_{\nu\vdash d} \sum_{\alpha\in S(d)} \hbar^{|\alpha|} \sum_{\substack{\beta\in C_\nu\\ \alpha\circ\beta=\pi_\lambda}} \left(\sum_{\substack{\mathcal{B}\in P(d)\\ \mathbf{0}_\beta\leq\mathcal{B}}} \Phi_{Z_2,\hbar}(\mathcal{B},\beta) \right),
\end{aligned}$$

where we clustered the sum by the conjugacy class C_ν to which $\beta = \alpha^{-1}\circ\pi_\lambda$ belongs. By multiplicativity of $\Phi_{Z_2,\hbar}$, the sum inside the brackets only depends on the cycle structure of β. In particular, substituting β with π_ν does not change the sum, and by comparing with (15) we recognise the value of $Z_2(\nu)$. This yields:

$$Z_1(\lambda) = \sum_{\nu\vdash d}\sum_{\alpha\in S(d)} \hbar^{|\alpha|} \sum_{\substack{\beta\in C_\nu\\ \alpha\circ\beta=\pi_\lambda}} Z_2(\nu) = \sum_{\nu\vdash d} Z_2(\nu) \left(\sum_{\substack{\alpha\in S(d),\beta\in C_\nu\\ \alpha\circ\beta=\pi_\lambda}} \hbar^{|\alpha|} \right).$$

The last constraint can be written $\pi_\lambda^{-1}\circ\alpha\circ\beta = \mathrm{id}$. By comparison with Definition 2.10, we recognise the free single Hurwitz numbers:

$$Z_1(\lambda) = z(\lambda)\sum_{\nu\vdash d} Z_2(\nu)\left(\sum_{r=0}^{d-1} \hbar^r H_r^{!}(\lambda,\nu)\right).$$

Here, r is the colength of α, and the factor $z(\lambda) = \frac{d!}{\#C_\lambda}$ is explained as follows. The numerator compensates the $\frac{1}{d!}$ in the definition of Hurwitz numbers. The denominator comes from the fact that in the definition of free single Hurwitz numbers, we let the leftmost permutation to be any element of the conjugacy class C_λ, so we overcount by a factor of $\#C_\lambda$. Thanks to Lemma 2.11 and by comparison with the definition of the generating series of strictly monotone Hurwitz numbers in (10), we get:

$$(i) \;\; : \;\; Z_1(\lambda) = z(\lambda)\sum_{\nu\vdash d} H^{>}(\lambda,\nu) Z_2(\nu)\,.$$

As all steps are equivalences, this proves $(i) \Leftrightarrow (ii)$.

Extended convolution from the left by the Möbius function $\mu_\hbar$ proves $(ii) \Rightarrow (iv)$. The same operation with the zeta function $\zeta_\hbar$ proves the converse direction. The implication $(i) \Rightarrow (iii)$ is obtained by multiplying (i) by $z(\nu)H^{\leq}(\nu,\lambda)$, summing over $\lambda \vdash d$ and using the first line of Lemma 2.9, while the converse direction is obtained likewise using the second line of Lemma 2.9. This finishes the proof of all equivalences between $(i), (ii), (iii), (iv)$. The equivalence between (i) for all $d > 0$, and $Z_1 = \mathrm{D}Z_2$, is a well-known correspondence, see *e.g.* [BDBKS21b, Lemma 3.1]. □

Proof of Corollary 2.14. In light of Lemma 2.7 and the properties of the Möbius function for $*$, this is a direct consequence of the theorem we just proved. □

3 The functional relations

3.1 n-point functions

The bosonic Fock space $\mathcal{F}_R$ (and so $\mathcal{F}_{R,\hbar}$) is acted upon by the Heisenberg operators

$$\mathsf{J}_k = \begin{cases} k\partial_{p_k} & \text{if } k > 0\,, \\ 0 & \text{if } k = 0\,, \\ p_{-k} & \text{if } k < 0\,. \end{cases}$$

We collect them in generating series

$$\mathsf{J}(X) = \sum_{k>0} X^k \mathsf{J}_k\,, \qquad \widetilde{J}(X) = \sum_{k\in\mathbb{Z}} X^k \mathsf{J}_k\,.$$

A topological partition function $Z = e^F \in \mathcal{F}_{R,\hbar}$ as in Definition 2.12 can be decomposed as

(16) $$F = \sum_{\substack{n\geq 1 \\ g\geq 0}} \frac{\hbar^{2g-2}}{n!} \sum_{k_1,\ldots,k_n>0} F_{g;k_1,\ldots,k_n} \prod_{i=1}^n \frac{p_{k_i}}{k_i} = \sum_{\substack{d\geq 1 \\ g\geq 0}} \hbar^{2g-2} \sum_{\lambda\vdash d} F_{g;\lambda_1,\ldots,\lambda_\ell} \frac{p_\lambda}{z(\lambda)}\,,$$

with coefficients $F_{g;k_1,\ldots,k_n} \in R$ that are symmetric under permutation of the k_is. The operator

$$\mathsf{F} = \sum_{\substack{n\geq 1 \\ g\geq 0}} \frac{\hbar^{2g-2}}{n!} \sum_{k_1,\ldots,k_n>0} F_{g;k_1,\ldots,k_n} \prod_{i=1}^n \frac{\mathsf{J}_{-k_i}}{k_i}$$

is such that $Z = e^{\mathsf{F}}|\rangle$. For every $n \in \mathbb{Z}_{>0}$, we define the n-point functions G_n and their shifted version $\widetilde{G}_n$:

(17) $$\begin{aligned} G_n(X_1,\ldots,X_n) &= \hbar^{-1}\delta_{n,1} + \hbar^n \left\langle \Big| \prod_{i=1}^n \mathsf{J}(X_i) \cdot e^{\mathsf{F}} \Big| \right\rangle^{\circ}, \\ \widetilde{G}_n(X_1,\ldots,X_n) &= \hbar^{-1}\delta_{n,1} + \hbar^n \left\langle \Big| \prod_{i=1}^n \widetilde{\mathsf{J}}(X_i) \cdot e^{\mathsf{F}} \Big| \right\rangle^{\circ}. \end{aligned}$$

Here, $\langle|\cdots|\rangle^\circ$ refers to the connected expectation value, defined for any tuple of linear operators $(\mathsf{A}_i)_{i=1}^n$ by

$$\langle|\mathsf{A}_1\cdots\mathsf{A}_n \cdot e^{\mathsf{F}}|\rangle^\circ := \partial_{t_1=0}\cdots\partial_{t_n=0} \ln\Big(\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\, e^{\mathsf{F}}|\rangle\Big)\,.$$

Equivalently, we have the inclusion-exclusion formulas:

$$\begin{aligned} \langle|\mathsf{A}_1\cdots\mathsf{A}_n\, e^{\mathsf{F}}|\rangle &= \sum_{\mathcal{I}\in\mathcal{P}(n)} \prod_{I\in\mathcal{I}} \left\langle \Big| \prod_{i\in I} \mathsf{A}_i \cdot e^{\mathsf{F}} \Big| \right\rangle^{\circ}, \\ \langle|\mathsf{A}_1\cdots\mathsf{A}_n \cdot e^{\mathsf{F}}|\rangle^\circ &= \sum_{\mathcal{I}\in\mathcal{P}(n)} (-1)^{\#\mathcal{I}-1}\,(\#\mathcal{I}-1)! \prod_{I\in\mathcal{I}} \left\langle \Big| \prod_{i\in I} \mathsf{A}_i\, e^{\mathsf{F}} \Big| \right\rangle. \end{aligned}$$

In concrete terms, we have

(18) $$\begin{aligned} G_n(X_1,\ldots,X_n) &= \hbar^{-1}\delta_{n,1} + \sum_{g\geq 0} \sum_{k_1,\ldots,k_n>0} \hbar^{2g-2+n}\, F_{g;k_1,\ldots,k_n} X_1^{k_1}\cdots X_n^{k_n}\,, \\ \widetilde{G}_n(X_1,\ldots,X_n) &= G_n(X_1,\ldots,X_n) + \delta_{n,2}\frac{X_1X_2}{(X_1-X_2)^2}\,. \end{aligned}$$

The second equation can be obtained by elementary manipulations with the Heisenberg commutation relations, see [BDBKS21a, Proposition 4.1]. Collecting the coefficients of powers of $\hbar$, we obtain a decomposition

(19) $$G_n = \sum_{g\geq 0} \hbar^{2g-2+n}\, G_{g,n} = \hbar^{n-2}\, G_{0,n} + o(\hbar^{n-2})\,,$$

and likewise for $\widetilde{G}$.

Remark 3.1. Note that G_n for all n and $\widetilde{G}_n$ for $n \neq 2$ are honest formal power series in $X_1, \ldots, X_n$, while $\widetilde{G}_2$ should be considered as a formal series when $X_i \to 0$ in the sector $|X_1| < |X_2| < \cdots < |X_n|$, *i.e.* an element of $R(\!(\hbar)\!)[X_1; \ldots; X_n] \coloneqq R(\!(\hbar)\!)[\![X_1]\!](\!(X_2)\!) \cdots (\!(X_n)\!)$.

To a multiplicative function $\phi_\hbar \colon PS \to R[\![\hbar]\!]$ such that

$$\forall (\mathcal{A}, \alpha) \in PS, \qquad \phi_\hbar(\mathcal{A}, \alpha) \in \hbar^{|(\mathcal{A},\alpha)|} R[\![\hbar^2]\!]\,,$$

we can associate a topological partition function Z by comparison with (14)-(16), and thus a collection of n-point functions (18). Their coefficients are given for any $\lambda \vdash d$ of length n by the formula

$$F_{g;\lambda_1,\ldots,\lambda_n} = z(\lambda) \cdot [\hbar^{2g-2+n}]\ \phi_\hbar(\mathbf{1}_{\lambda_1+\cdots+\lambda_n}, \pi_\lambda)\,.$$

In particular, given a multiplicative function $\phi \colon PS \to R$, we can put ourselves in the previous situation by multiplying it by $\hbar^{\text{colength}}$, and thus associate to it n-point functions (18), in which the $g > 0$ sector is zero and:

$$F_{0;\lambda_1,\ldots,\lambda_n} = z(\lambda)\, \phi(\mathbf{1}_{\lambda_1+\cdots+\lambda_n}, \pi_\lambda)\,.$$

3.2 *Main formulas*

Topological partition functions, multiplicative functions and collections of n-point functions are different ways to encode the same information. In Theorem 2.13 we have given equivalent forms, in terms of multiplicative functions, of the relation $Z = \mathsf{D}Z^\vee$ between two topological partition functions Z and $Z^\vee$. Our aim is now to translate this relation into functional relations between their respective n-point functions G_n and $G_n^\vee$. The result is expressed as weighted sums over graphs and the formal power series:

$$\varsigma(w) = \frac{\sinh(w/2)}{w/2} = 1 + \frac{w^2}{24} + O(w^4)\,. \tag{20}$$

Definition 3.2 (Graphs)**.** If $n > 0$, we let $\mathcal{G}_n$ be the set of connected bicoloured graphs such that

- the white vertices are labelled from 1 to n;
- edges only connect vertices of different colour;
- black vertices have valency ≥ 2;

The three conditions imply that the set of graphs in $\mathcal{G}_n$ is infinite, but it is finite if we fix their first Betti number. Black vertices are characterised by multisets[3] I in $[n]$ — also called *hyperedges* — recording the white vertices they connect to. The last condition implies $\#I \geq 2$. If $\Gamma \in \mathcal{G}_n$, we denote $\mathcal{I}(\Gamma)$ its set of hyperedges and $\mathrm{Aut}(\Gamma)$ the automorphism group, consisting of permutations of the edges respecting the structure of Γ and the labelling of white vertices.

Definition 3.3 (Weights)**.** Let $(w_i)_{i=1}^n$ be an n-tuple of variables, and if I is a multiset in $[n]$, denote $w_I = (w_i)_{i\in I}$ the corresponding collection of variables with multiplicity.

- To a hyperedge I of $[n]$ that is not of the form $I = \{j, j\}$, we assign the weight

  $$\mathsf{c}^\vee(u_I, w_I) = \Big(\prod_{i\in I} \hbar u_i\, \varsigma(\hbar u_i w_i \partial_{w_i})\Big) \widetilde{G}^\vee_{\#I}(w_I)\,.$$

  These are series depending on variables $(u_i, w_i)_{i\in I}$, and following Remark 3.1 they are only considered in the sector $|w_i| < |w_j|$ for $i < j$.
- For a hyperedge of the form $I = \{j, j\}$, the above expression would be ill-defined due to the double pole in the shifted 2-point function (18). We rather assign the weight:

  $$\mathsf{c}^\vee(u_I, w_I) = \big(\hbar u_j\, \varsigma(\hbar u_j w_j \partial_{w_j})\big)^2\, G_2^\vee(w_j, w_j)\,.$$

[3] A multiset I in $[n]$ is a function $f_I \colon [n] \to \mathbb{Z}_{\geq 0}$. We say that i is an element of I when $f_I(i) > 0$, but elements may have multiplicity $f_I(i)$ greater than 1. The cardinality is defined to take into account the multiplicity: $\#I = \sum_{i\in I} f(i)$.

- To the i-th white vertex, we attach the operator weight

(21)
$$\begin{aligned}\vec{\mathsf{O}}^\vee(w_i) = \sum_{m\geq 0} & \big(P^\vee(w_i)w_i\partial_{w_i}\big)^m P^\vee(w_i) \\ & \cdot [v_i^m] \sum_{r\geq 0}\Big(\partial_y + \frac{v_i}{y}\Big)^r \exp\bigg(v_i\,\frac{\varsigma(\hbar v_i\partial_y)}{\varsigma(\hbar\partial_y)}\ln y - v_i\ln y\bigg)\Big|_{y=G^\vee_{0,1}(w_i)} \\ & \cdot [u_i^r]\,\frac{\exp\big(\hbar u_i\varsigma(\hbar u_i w_i\partial_{w_i})(G^\vee_1(w_i)-\hbar^{-1}) - u_i(G^\vee_{0,1}(w_i)-1)\big)}{\hbar u_i\,\varsigma(\hbar u_i)}\,,\end{aligned}$$

where $P^\vee(w_i)$ is a power series specified later. As $y = G^\vee_{0,1}(w_i) = 1 + O(w_i)$, $\ln y$ is a well-defined power series in w_i. The $-\hbar^{-1}$ and -1 in the last line cancel the conventional constant added in the definition of the 1-point function (18). This operator acts from the left on series depending on the variables u_i, w_i and gives as output a series in w_i.

Theorem 3.4. *Let $Z, Z^\vee$ be two topological partition functions and $G_n, G^\vee_n$ their respective n-point functions. Suppose that $Z = \mathsf{D}Z^\vee$. Then, under the substitution*

$$X_i = \frac{w_i}{G^\vee_{0,1}(w_i)}, \qquad P^\vee(w_i) = \frac{\mathrm{d}\ln w_i}{\mathrm{d}\ln X_i}\,,$$

we have

(22)
$$\begin{aligned}G_{0,1}(X_1) &= G^\vee_{0,1}(w_1)\,, \\ G_{0,2}(X_1,X_2) &= P^\vee(w_1)P^\vee(w_2)\bigg(G^\vee_{0,2}(w_1,w_2) + \frac{w_1w_2}{(w_1-w_2)^2}\bigg) - \frac{X_1X_2}{(X_1-X_2)^2}\,,\end{aligned}$$

for $2g-2+n>0$:

(23)
$$G_{g,n}(X_1,\ldots,X_n) = \delta_{n,1}\Delta^\vee_g(X_1) + [\hbar^{2g-2+n}] \sum_{\Gamma\in\mathcal{G}_n}\frac{1}{\#\mathrm{Aut}(\Gamma)}\prod_{i=1}^n \vec{\mathsf{O}}^\vee(w_i)\prod_{I\in\mathcal{I}(\Gamma)}\mathsf{c}^\vee(u_I,w_I)\,.$$

The correction term appearing for $n=1$ is:

(24)
$$\begin{aligned}\Delta^\vee_g(X) = [\hbar^{2g}]\sum_{m\geq 0}\big(P^\vee(w)w\partial_w\big)^m\,[v^{m+1}]\exp\bigg(v\,\frac{\varsigma(\hbar v\partial_y)}{\varsigma(\hbar\partial_y)}\ln y - v\ln y\bigg)\Big|_{y=G^\vee_{0,1}(w)} \\ \cdot P^\vee(w)w\partial_w G^\vee_{0,1}(w)\,.\end{aligned}$$

Remark 3.5. Equation (23) remains valid for $(g,n) = (0,2)$ provided the left-hand side is replaced with $\widetilde{G}_{0,2}(X_1,X_2)$. This recovers the second equation in (22).

Remark 3.6. Equation (23) remains also valid for $(g,n) = (0,1)$ provided we extend the summation over m to $m\geq -1$ in the definition of $\Delta^\vee_0(X_1)$ and identify $\big(P^\vee(w_1)w_1\partial_{w_1}\big)^{-1}P^\vee(w_1)w_1\partial_{w_1}G^\vee_{0,1}(w_1)$ with $G^\vee_{0,1}(w_1)$. This is the only contribution (the sum over graphs does not contribute as it does yield an $\hbar^{-1}$). This recovers the first equation in (22).

This theorem is a special case of [BDBKS21b, Theorem 4.14 and Remark 4.15], or more precisely, this theorem is a special case of the statement made in the first line of the proof of [BDBKS21b, Theorem 4.14]. For completeness, and since this theorem is of crucial importance for us, we shall give its full proof in Section 3.6.

There are several simplifications in the genus 0 sector. We are going to present the result in terms of multiplicative functions.

Definition 3.7. If $\mathbf{r} = (r_1,\ldots,r_n)\in\mathbb{Z}^n_{\geq 0}$, let $\mathcal{G}_{0,n}(\mathbf{r}+1)$ be the subset of $\mathcal{G}_n$ consisting of trees in which the i-th vertex has valency r_i+1.

Note that such trees do not have non-trivial automorphisms. Observe as well that for fixed n, the set $\mathcal{G}_{0,n}(\mathbf{r}+1)$ is non-empty only for finitely many n-tuples $\mathbf{r}$.

Definition 3.8. Let us introduce the r-th piece of the genus 0 version of the operator weight of Definition 3.3:

$$\vec{\mathsf{O}}_r^\vee(w) = \sum_{m \geq 0} (P^\vee(w) w\partial_w)^m P^\vee(w) \cdot [v^m] \left(\partial_y + \frac{v}{y}\right)^r \cdot 1\Big|_{y = G_{0,1}^\vee(w)} .$$

This is an operator acting from the left on series in the variable w.

Theorem 3.9. *Let $\phi, \phi^\vee : PS \to R$ be multiplicative functions and $G_{0,n}, G_{0,n}^\vee$ their respective n-point functions. Suppose that $\phi = \zeta * \phi^\vee$. Then, under the substitution*

$$X_i = \frac{w_i}{G_{0,1}^\vee(w_i)}, \qquad P^\vee(w_i) = \frac{\mathrm{d}\ln w_i}{\mathrm{d}\ln X_i},$$

we have (22) for $n = 1, 2$, and for any $n \geq 3$:

$$G_{0,n}(X_1, \ldots, X_n) = \sum_{r_1,\ldots,r_n \geq 0} \prod_{i=1}^n \vec{\mathsf{O}}_{r_i}^\vee(w_i) \sum_{T \in \mathcal{G}_{0,n}(\mathbf{r}+1)} \prod_{I \in \mathcal{I}(T)}' G_{0,\#I}^\vee(w_I) .$$

where $\prod'$ means that one should replace each occurrence of $G_{0,2}^\vee(w_i, w_j)$ for $i \neq j$ with $\widetilde{G}_{0,2}^\vee(w_i, w_j)$.

This will be proved in Section 3.7.

Remark 3.10. The formula for $(g, n) = (0, 1)$ corresponds to the Voiculescu R-transform and was derived from the combinatorics of non-crossing partitions in [Spe94]. For $(g, n) = (0, 2)$ it was derived from the combinatorics of non-crossing partitions in [CMSS07], and obtained differently from the combinatorics of fully simple maps in [BGF20; BCDGF19]. Their generalisation for $g = 0$ and $n \geq 3$ was an open problem from [CMSS07], to which Theorem 3.9 answers. Its application to free probability will be discussed in Section 4.

We can present the relation in terms of the coefficients of the n-point functions. Although we state it only in genus 0, the interested reader can easily derive the formula in higher genus from Theorem 3.4 or the more convenient preliminary form Lemma 3.15 appearing later in the text.

Definition 3.11. Let $\mathcal{T}_n$ be the set of trees T obtained by connecting to a $T' \in \mathcal{G}_{0,n}$ finitely many univalent black vertices. The difference with $\mathcal{G}_{0,n}$ is therefore that we allow hyperedges I with $\#I = 1$. This makes the set $\mathcal{T}_n$ infinite. We denote $\mathcal{T}_n(\mathbf{r}+1) \subset \mathcal{T}_n$ the subset of trees in which the i-th white vertex has valency $r_i + 1$. Note that if $\ell_i(T)$ is the number of univalent black vertices incident to the i-th white vertex, we have $\#\mathrm{Aut}(T) = \prod_{i=1}^n \ell_i(T)!$.

Theorem 3.12. *Let $\phi, \phi^\vee : PS \to R$ be multiplicative functions and $G_{0,n}, G_{0,n}^\vee$ their respective n-point functions. Suppose that $\phi = \zeta * \phi^\vee$. For any $k_1, \ldots, k_n > 0$, we have for $n \geq 3$:*

$$F_{0;k_1,\ldots,k_n} = \left[\prod_{i=1}^n w_i^{k_i}\right] \sum_{\substack{0 \leq r_i \leq k_i \\ i \in [n]}} \prod_{i=1}^n \frac{k_i!}{(k_i - r_i)!} \sum_{T \in \mathcal{T}_n(\mathbf{r}+1)} \frac{\prod''_{I \in \mathcal{I}(T)} G_{0,\#I}^\vee(w_I)}{\#\mathrm{Aut}(T)} ,$$

where $\prod''$ means that one should replace each occurrence of $G_{0,1}^\vee(w_j)$ with $G_{0,1}^\vee(w_j) - 1$, and each occurrence[4] of $G_{0,2}^\vee(w_i, w_j)$ with $\widetilde{G}_{0,2}^\vee(w_i, w_j)$.

This will be proved in Section 3.8.

The dual statements of these three results, *i.e.* the relations giving $G_n^\vee$ in terms of G_n, have a similar structure and will be given in Section 3.9. Before turning to the proof, we explain how to use the formulas in practice.

[4]Trees do not have hyperedges of the type $\{i, i\}$, so if 2-point functions occur it is only in their shifted version.

3.3 *Reformulation*

It is sometimes more convenient (see [BGF20]) to use a different convention for the definition of the n-point functions, namely for $(g,n)=(0,1)$ we define

$$X_{0,1}(w) = w^{-1}G^{\vee}_{0,1}(w)$$

and for $(g,n)\neq(0,1)$ we introduce the differential forms

$$\begin{aligned} \omega_{g,n}(x_1,\ldots,x_n) &= \prod_{i=1}^{n}\frac{\mathrm{d}x_i}{x_i}\,G_{g,n}(x_1^{-1},\ldots,x_n^{-1}) = \sum_{k_1,\ldots,k_n>0} F_{g;k_1,\ldots,k_n}\prod_{i=1}^{n}\frac{\mathrm{d}x_i}{x_i^{k_i+1}}\,, \\ \omega^{\vee}_{g,n}(w_1,\ldots,w_n) &= \prod_{i=1}^{n}\frac{\mathrm{d}w_i}{w_i}\,G^{\vee}_{g,n}(w_1,\ldots,w_n) = \sum_{k_1,\ldots,k_n>0} F^{\vee}_{g;k_1,\ldots,k_n}\prod_{i=1}^{n} w_i^{k_i-1}\mathrm{d}w_i\,, \end{aligned} \tag{25}$$

and their shifted version for $(g,n)=(0,2)$:

$$\widetilde{\omega}_{0,2}(x_1,x_2) = \omega_{0,2}(x_1,x_2) + \frac{\mathrm{d}x_1\mathrm{d}x_2}{(x_1-x_2)^2}\,, \qquad \widetilde{\omega}^{\vee}_{0,2}(w_1,w_2) = \omega^{\vee}_{0,2}(w_1,w_2) + \frac{\mathrm{d}w_1\mathrm{d}w_2}{(w_1-w_2)^2}\,.$$

Let us present the equivalent form it gives to Theorem 3.9. The substitution rule is

$$w_i = W_{0,1}(x_i) \qquad \Longleftrightarrow \qquad x_i = X_{0,1}(w_i)\,. \tag{26}$$

In other words, the relation between the 1-point functions in genus 0 becomes the functional inverse. For $(g,n)=(0,2)$, (22) becomes:

$$\widetilde{\omega}_{0,2}(x_1,x_2) = \widetilde{\omega}^{\vee}_{0,2}(w_1,w_2)\,.$$

The alternative convention makes these relations particularly simple to remember. For $n\geq 3$, using the variable $x = X^{-1}$, the operator of Definition 3.8 becomes:

$$\vec{\mathsf{O}}^{\vee}_r(w) = \sum_{m\geq 0}(-x\partial_x)^m\frac{-x\mathrm{d}w}{w\mathrm{d}x}\cdot[v^m]\Big(\partial_y+\frac{v}{y}\Big)^r\cdot 1\Big|_{y=xw}\,,$$

and we get

$$\omega_{0,n}(x_1,\ldots,x_n) = \sum_{r_1,\ldots,r_n\geq 0}\prod_{i=1}^{n}\frac{\mathrm{d}x_i}{x_i}\,\vec{\mathsf{O}}^{\vee}_{r_i}(w_i)\Big(\frac{w_i}{\mathrm{d}w_i}\Big)^{r_i+1}\sum_{T\in\mathcal{G}_{0,n}(\mathbf{r}+1)}\prod_{I\in\mathcal{I}(T)}{}^{'}\,\omega^{\vee}_{0,\#I}(w_I)\,,$$

where $\prod'$ means that any occurrence of $\omega^{\vee}_{0,2}(w_i,w_j)$ with $i\neq j$ should be replaced with $\widetilde{\omega}^{\vee}_{0,2}(w_i,w_j)$.

3.4 *Examples*

For $(g,n)=(0,3)$, there are exactly 4 trees in $\mathcal{G}_{0,3}$ (Figure 1), and we get from Theorem 3.9:

$$\begin{aligned} &G_{0,3}(x_1,x_2,x_3) \\ &= \Big(\prod_{a=1}^{3}P^{\vee}(w_a)\Big)\Bigg[G^{\vee}_{0,3}(w_1,w_2,w_3) + \sum_{i=1}^{3}\frac{w_i}{G^{\vee}_{0,1}(w_i)x'(w_i)}\partial_{w_i}\Bigg(\frac{\prod_{j\neq i}\big(G^{\vee}_{0,2}(w_i,w_j)+\frac{w_iw_j}{(w_i-w_j)^2}\big)}{G^{\vee}_{0,1}(w_i)x'(w_i)}\Bigg)\Bigg]. \end{aligned}$$

In terms of the differential forms, the form is also slightly simpler to remember:

$$\omega_{0,3}(x_1,x_2,x_3) = -\omega^{\vee}_{0,3}(w_1,w_2,w_3) - \sum_{i=1}^{3}\mathrm{d}_{w_i}\Bigg(\frac{\prod_{j\neq i}\widetilde{\omega}^{\vee}_{0,2}(w_i,w_j)}{\mathrm{d}x_i\,\mathrm{d}w_i}\Bigg)\,. \tag{27}$$

For $(g,n)=(1,1)$, we need to extract the coefficient of $\hbar^1$ in the formula of Theorem 3.4. Recall that the leading order was $\hbar^{-1}$, so this contribution may come from one of the three following possibilities:

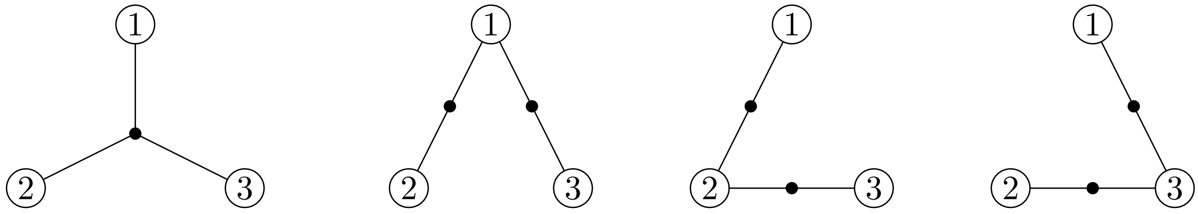

FIGURE 1. The trees in $\mathcal{G}_{0,3}$

- The vertex without hyperedges. Its weight is the coefficient of $\hbar^1$ in (21). If we pick the leading order $\varsigma(z) = 1 + O(z^2)$ everywhere, we can get a contribution by picking one $G^{\vee}_{1,1}(w)$ in the last line. This comes without powers of u, so selects $r = 0$; as ς was replaced by 1, the exponential in the second line of (21) is 1, thus forcing $m = 0$. Another contribution comes from picking the second term $\frac{z^2}{24}$ from one occurrence of $\varsigma(z)$ and replacing the other occurrences of ς by 1. If we pick this second term in ς from the denominator of the third line of (21), we will get a linear term in u, thus selecting $r = 1$, while in the second line the exponential will be 1, thus selecting $m = 1$. If we rather pick it from the exponential in the third line, we get u^2, thus selecting $r = 2$, while in the second line the exponential will be 1, and as $(\partial_y + v/y)^2 \cdot 1 = (v^2 - v)/y$, we get contributions from $m = 1$ and $m = 2$.
- The graph consisting of one white vertex connected in two ways to a black vertex (which admits 2 automorphisms), with ς always replaced by 1. In this case, we get $\vec{\mathsf{O}}^{\vee}_r(w) u^2 G^{\vee}_{0,2}(w,w)$, which selects $r = 1$ and subsequently $m = 1$.
- The correction term $\Delta^{\vee}_1(X_1)$ given by (24). We must select one term $\frac{z^2}{24}$ in $\varsigma(z)$ either from the numerator or the denominator in the exponential, hence get contributions from $m = 0$ and $m = 2$.

This results in:

$$\begin{aligned} G_{1,1}(X) = {} & P^{\vee}(w) G^{\vee}_{1,1}(w) - \frac{1}{24} P^{\vee}(w) w\partial_w \left(\frac{P^{\vee}(w)}{G^{\vee}_{0,1}(w)} \right) \\ & + \frac{1}{24} P^{\vee}(w) w\partial_w (P^{\vee}(w) w\partial_w - 1) \left(\frac{P^{\vee}(w)}{(G^{\vee}_{0,1}(w))^2} (w\partial_w)^2 G^{\vee}_{0,1}(w) \right) \\ & + \frac{1}{2} P^{\vee}(w) w\partial_w \left(\frac{P^{\vee}(w)}{G^{\vee}_{0,1}(w)} G^{\vee}_{0,2}(w,w) \right) \\ & - \frac{1}{24} \big((P^{\vee}(w) w\partial_w)^2 - 1 \big) \frac{P^{\vee}(w) w\partial_w G^{\vee}_{0,1}(w)}{(G^{\vee}_{0,1}(w))^2} \,. \end{aligned}$$

After a tedious algebra, we observe many simplifications:

$$\omega_{1,1}(x) + \omega^{\vee}_{1,1}(w) = \mathrm{d}\left[\frac{1}{2} \frac{\omega^{\vee}_{0,2}(w,w)}{\mathrm{d}x\,\mathrm{d}w} - \frac{1}{24} \frac{\mathrm{d}(\partial_w^2 x / \partial_w x)}{\mathrm{d}x} \right] . \tag{28}$$

3.5 *Relation to topological recursion*

When $\omega_{g,n}$ is the generating series of ordinary maps of genus g with n boundaries, $\omega^{\vee}_{g,n}$ enumerates fully simple maps of the same topology [BGF20; BCDGF19]. It is known since Eynard [Eyn04; Eyn11; Eyn16] that $\omega_{g,n}$s are computed by the topological recursion [EO07; EO09b] applied to the spectral curve

$$\mathcal{S} = \left(\mathbb{P}^1, x, w, \tfrac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2} \right) ,$$

while it was conjectured in [BGF20] and proved in [BDBKS21b; BCGF21] that $\omega^{\vee}_{g,n}$s are computed by the topological recursion applied to

$$\mathcal{S}^{\vee} = \left(\mathbb{P}^1, w, x, \tfrac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2} \right) .$$

This gives a class of examples to test our functional relations numerically. The $(0,3)$ formula (27) was indeed shown to hold in this situation in [BGF20, Section 6], and here we see that in fact it holds in the greater generality provided by the master relation (5). To test the $(1,1)$ formula (28), let us consider the case of quadrangulations (maps with faces of degree 4), which corresponds to the spectral curve in parametric form

$$x(z) = c\Big(z + \frac{1}{z}\Big)\,, \qquad w(z) = \frac{z^2 - \tau}{cz^3}\,, \qquad \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1 - z_2)^2} = \widetilde{\omega}_{0,2}(z_1,z_2) = \widetilde{\omega}^{\vee}_{0,2}(z_1,z_2)\,,$$

where t is the weight per quadrangle, and

$$c = \sqrt{\frac{1 - \sqrt{1-12t}}{6t}}\,, \qquad \tau = ct^4\,.$$

From this we deduce

$$\omega^{\vee}_{0,2}(z_1,z_2) = \widetilde{\omega}^{\vee}_{0,2}(z_1,z_2) - \frac{w'(z_1)w'(z_2)\mathrm{d}z_1\mathrm{d}z_2}{(w(z_1) - w(z_2))^2} = \frac{\tau\big(z_1^2z_2^2 + \tau(z_1^2 + 4z_1z_2 + z_2^2)\big)\mathrm{d}z_1\mathrm{d}z_2}{\big(\tau(z_1^2 + z_1z_2 + z_2^2) - z_1^2z_2^2\big)^2}\,,$$

$$\frac{\omega^{\vee}_{0,2}(z,z)}{2\,\mathrm{d}x(z)\,\mathrm{d}w(z)} = -\frac{\tau z^4(z^2 + 6\tau)}{2(z^2 - 3\tau)^3(z^2 - 1)}\,,$$

and

$$\begin{aligned}\frac{1}{24}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial_w^2 x}{\partial_w x}\right) &= \frac{1}{24\,x'(z)}\partial_z\left(\frac{1}{w'(z)}\partial_z \ln\left(\frac{x'(z)}{w'(z)}\right)\right)\\ &= -\frac{z^4\big(z^8 - 3z^6(3\tau+1) + 18\tau z^4(3\tau+1) + 3\tau z^2(-33\tau+1) + 27\tau^2\big)}{12(z^2-1)^3(z^2-3\tau)^3}\,.\end{aligned}$$

Besides, [BGF20, Section 5.2] computed from topological recursion[5]:

$$\begin{aligned}\omega_{1,1}(z) &= \frac{z\big(\tau z^4 + z^2(1-5\tau) + \tau\big)}{(z^2-1)^4(3\tau-1)^2}\,\mathrm{d}z\,,\\ \omega^{\vee}_{1,1}(z) &= \frac{3\tau^2 z\big((3\tau-2)z^4 + 3\tau(9\tau-1)z^2 - 27\tau^3\big)}{(z^2-3\tau)^4(3\tau-1)^2}\,\mathrm{d}z\,,\end{aligned} \tag{29}$$

and we checked that (28) is indeed satisfied.

We suggest that the master relation (5), or rather the functional relations given in Theorem 3.4 when written in terms of differentials, in fact describe the effect of the symplectic exchange $x \mapsto w$ in topological recursion.

Conjecture 3.13. *Let $\mathcal{C}$ be a compact Riemann surface, x, w are meromorphic functions on C such that $\mathrm{d}x$ and $\mathrm{d}w$ do not have common zeroes, and $\mathcal{B}$ is a fundamental bidifferential of the second kind. We call $\omega_{g,n}$ the differentials obtained from the topological recursion with the spectral curve $(\mathcal{C}, x, w, \mathcal{B})$, and $\omega^{\vee}_{g,n}$ the ones associated to the spectral curve $(\mathcal{C}, w, x, \mathcal{B})$, and define $\widetilde{\omega}_{0,2} = \widetilde{\omega}^{\vee}_{0,2} = \mathcal{B}$. Then, these differentials will satisfy for all $2g - 2 + n \geq 0$ the functional relations of Theorem 3.4 (after they are converted in relations between meromorphic differentials on $\mathcal{C}$).*

This conjecture holds by definition for $(g,n) = (0,2)$. It holds for $(0,3)$ (formula justified in [BGF20]) as well as for $(1,1)$ by comparison of (28) with [EO07, Lemma C.1] where one can use w as a coordinate. Besides, as we have just discussed, it holds for all (g,n) in the case of spectral curves corresponding to map enumeration. This conjecture should be a step towards understanding the symplectic invariance property of topological recursion envisioned in [EO08; EO13], which one should extract from the analytic properties of $\omega_{g,1} + \omega^{\vee}_{g,1}$ for all $g \geq 1$.

Furthermore, the conjecture remains true in the more general context of the matrix model with external field. The generalised resolvents of the matrix model with external field were proved to

[5]There is a misprint in [BGF20], namely the left-hand side of formula (5.2) should be divided by $x'(z)$ and the left-hand side of formula (5.3) should be divided by $w'(z)$.

satisfy topological recursion in [EO07; EO09a], while the topological recursion for the same spectral curve, but after exchanging the role of x and w, produces diagonal correlation functions of the matrix model with external field [BCEGF21, Section 4.1], which combinatorially corresponds to the enumeration of the so-called ciliated maps[6]. As proved in [BGF20] and recalled in Theorem 4.25, the master relation appears in general unitarily invariant ensembles of random hermitian matrices, so it also applies to the generalised resolvents (of the type p_λ in Theorem 4.25) and the diagonal correlation functions (of the type $\mathcal{P}_\nu$ in Theorem 4.25) of the matrix model with external field.

3.6 All genus: proof of Theorem 3.4

We rely on the description of the projective representation of $\widehat{\mathfrak{gl}}_\infty(R)$ in terms of differential operators on $\mathcal{F}_R$ [Kac90]. Our starting point is the formula found *e.g.* in [BDBKS21a, Proposition 3.1], expressing the conjugation of the Heisenberg field with the operator D or D^{-1}:

$$\begin{aligned}
&\mathsf{D}^{\mp 1}\,\widetilde{\mathsf{J}}(X)\,\mathsf{D}^{\pm 1} \\
&= \sum_{k\in\mathbb{Z}} X^k[w^k] \sum_{r\geq 0} \partial_y^r \exp\bigg(\pm k\,\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\bigg)\Big|_{y=0} \\
&\qquad \cdot [u^r]\,\frac{1}{\hbar u\varsigma(\hbar u)}\exp\bigg(\sum_{m>0}\hbar u\,\varsigma(\hbar u w\partial_w)\mathsf{J}_{-m}w^{-m}\bigg)\exp\bigg(\sum_{m>0}\hbar u\,\varsigma(\hbar u w\partial_w)\mathsf{J}_m w^m\bigg),
\end{aligned} \tag{30}$$

where we recall $\varsigma(w) = 2w^{-1}\sinh(w/2) = 1 + \frac{w^2}{24} + O(w^4)$. Let use this formula with the sign $+$, and substitute the latter formula in Equation (17), and then use the commutation relations $[\mathsf{J}_l, \mathsf{J}_m] = l\delta_{l+m,0}$, as well as the properties $\langle|\mathsf{J}_{-m} = 0$ and $\mathsf{J}_m|\rangle = 0$, for $m > 0$. As explained in [BDBKS21b, Section 2] and [BDBKS21a, Section 3], we obtain a sum over the graphs of Definition 3.2, where the weights consist of the following elementary blocks.

Definition 3.14. We fix variables $(u_i, v_i, w_i)_{i=1}^n$.

- To a hyperedge $I \subseteq [n]$, we attach the weight (already in Definition 3.3):
$$\mathsf{c}^\vee(u_I, w_I) = \prod_{i\in I} \hbar u_i \varsigma(\hbar u_i w_i \partial_{w_i}) \widetilde{G}^\vee_{\#I}(w_I)\,.$$
- To the i-th white vertex, we attach the operator weight:

$$\begin{aligned}
\vec{\mathsf{U}}^\vee(X_i) &= \sum_{k\in\mathbb{Z}} X_i^k\cdot[w_i^k]\sum_{r\geq 0}\partial_y^r\exp\bigg(k\,\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\bigg)\Big|_{y=0} \\
&\qquad \cdot [u_i^r]\,\frac{\exp\big(\hbar u_i\,\varsigma(\hbar u_i w_i\partial_{w_i})(G_1^\vee(w_i) - \hbar^{-1})\big)}{\hbar u_i\,\varsigma(\hbar u_i)}\,.
\end{aligned} \tag{31}$$

This operator acts from the left on series depending on variables u_i and w_i and gives as output a series in X_i. The subtraction of $\hbar^{-1}$ in the last line kills the conventional constant added in the definition of the 1-point function in (18).

Lemma 3.15 (Key combinatorial identity)**.** *The relation $Z = \mathsf{D}Z^\vee$ is equivalent to the relations:*

$$\forall n > 0, \qquad \widetilde{G}_n(X_1,\ldots,X_n) = \sum_{\Gamma\in\mathcal{G}_n}\frac{1}{\#\mathrm{Aut}(\Gamma)}\prod_{i=1}^n \vec{\mathsf{U}}^\vee(X_i)\prod_{I\in\mathcal{I}(\Gamma)}\mathsf{c}^\vee(u_I, w_I)\,, \tag{32}$$

where the first product is taken from left to right with i increasing.

To pass from (32) to the statement of Theorem 3.4, in particular to (23), we proceed in three steps, which were suggested by Kazarian in an unpublished manuscript (see [Kaz19; Kaz21; Kaz20]). In our exposition we follow [BDBKS21a, Section 4.4].

[6]Ciliated maps were introduced in [BCEGF21] in order to study r-spin intersection numbers on the moduli space of curves in the context of topological recursion and in relation to map enumeration.

First, if $\Phi(y)$, $\Psi(u)$ and $Y(w)$ are three formal power series with $Y(w) = O(w)$, we observe that

$$\sum_{r\geq 0}(\partial_y^r\Phi)(0)\cdot[u^r]\; e^{uY(w)}\Psi(u) = \sum_{r\geq 0}(\partial_y^r\Phi)(Y(w))\cdot[u^r]\;\Psi(u)\,, \tag{33}$$

which can be checked by direct expansion of the left-hand side in the variable w. This observation applied to $Y(w_i) = G^{\vee}_{0,1}(w_i) - 1$ allows us to remove the otherwise infinitely many[7] contributions $\exp\big(u_i(G^{\vee}_{0,1}(w_i) - 1)\big)$ in the second line of (31). Note that this second line of (31) starts with $1/u$ for $n = 1$. We will discuss this case separately later.

Second, we have to remove the explicit dependence on ks coming from the weight of the black vertices (31). To this end, with $Y(w)$ as before, we use for $r \geq 0$ and $k \in \mathbb{Z}$:

$$\begin{aligned} &\partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\log(1+y)\right)\Big|_{y=Y(w)} \\ &= (1+Y(w))^k\cdot\left(\exp(-k\log(1+y))\partial_y^r\exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\log(1+y)\right)\right)\Big|_{y=Y(w)}. \end{aligned} \tag{34}$$

The second factor here has a polynomial dependence on k in each degree in $\hbar$, and we can use the following trick to assemble it: if $Q(k)$ is a polynomial in k, we have

$$\sum_k Q(k)\,z^k = Q(z\partial_z)\sum_k X^k = \sum_{j\geq 0}(z\partial_z)^j\cdot[v^j]\;\sum_k z^kQ(v)\,.$$

For every given monomial in $\hbar, X_1,\ldots,X_n$, we need to use such a formula when the summation ranges over a set of integers k bounded from below, in which case it makes sense in the framework explained in Remark 3.1.

Third, to remove the first factor in (34), we use the Lagrange inversion formula: given a Laurent series $L(w)$, we have for any $k \in \mathbb{Z}$,

$$\sum_{k\in\mathbb{Z}} X^k\cdot[\tilde{w}^k]\;(1+Y(\tilde{w}))^kL(\tilde{w}) = \frac{\mathrm{d}\ln X}{\mathrm{d}\ln w}\,L(w)\,,\qquad \text{where}\;\; X = \frac{w}{1+Y(w)}\,. \tag{35}$$

Using these three observations with $Y(w_i) = G^{\vee}_{0,1}(w_i) - 1$ for each $i \in [n]$, we obtain for $n \geq 2$ and $g \geq 0$:

$$\widetilde{G}_{g,n}(X_1,\ldots,X_n) = [\hbar^{2g-2+n}]\sum_{\Gamma\in\mathcal{G}_n}\frac{1}{\#\mathrm{Aut}(\Gamma)}\prod_{i=1}^n\vec{\mathsf{O}}^{\vee}(w_i)\prod_{I\in\mathcal{I}(\Gamma)}\mathsf{c}^{\vee}(u_I,w_I)\,, \tag{36}$$

using the substitutions

$$X_i = \frac{w_i}{G^{\vee}_{0,1}(w_i)}\,,\qquad P^{\vee}(w_i) = \frac{\mathrm{d}\ln w_i}{\mathrm{d}\ln X_i}\,. \tag{37}$$

The operators $\vec{\mathsf{O}}^{\vee}(X_i)$ are those introduced in (21), and they involve the above choice of $P^{\vee}(w_i)$. For $2g - 2 + n > 0$, the left-hand side is simply $G_{g,n}(X_1,\ldots,X_n)$ and we recover (23). Up to the comparison of the right-hand side in the case $(g,n) = (0,2)$ to the second formula of (22) and the study of the special case $(g,n) = (0,1)$, which will both be carried out in the next section, this concludes the proof of Theorem 3.4 for $n \geq 2$.

In the case $n = 1$ we have to make one more extra step. There is a graph that contains just one vertex and no hyperedges. The expression that we assign to this graph by Definition 3.14 is equal to the operator (31) applied to the constant 1. Then, in order to apply Equation (33), we have to remove

[7] More precisely, for every fixed degree of X_i this amounted to finitely many contributions, but their number increases with the degree of the monomial in Xs, resulting into infinitely many contributions in the functional relation.

manually the singularity in u as it is done in [BDBKS21a, Section 6.2], that is, we have:

$$
\begin{aligned}
& G_1(X) - \hbar^{-1} \\
& = \sum_{k\geq 1} X^k \cdot [w^k] \sum_{r\geq 0} \partial_y^r \exp\left(k\,\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\,\ln(1+y)\right)\Big|_{y=0} \\
& \qquad \cdot [u^r] \left(\frac{\exp\big(\hbar u\,\varsigma(\hbar u w\partial_w)(G_1^\vee(w)-\hbar^{-1})\big)}{\hbar u\,\varsigma(\hbar u)} - \frac{\exp\big(u(G_{0,1}^\vee(w)-1)\big)}{\hbar u}\right) \\
& \quad + \sum_{k\geq 1} X^k \cdot [w^k] \sum_{r\geq 0} \partial_y^r \exp\left(k\,\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\,\ln(1+y)\right)\Big|_{y=0} \cdot [u^r]\,\frac{\exp\big(u\,(G_{0,1}^\vee(w)-1)\big)}{\hbar u}\,.
\end{aligned}
$$

The first summand here can be evaluated by the three steps performed above in the general case, and it gives the contribution of the graph with one vertex and no edges in the statement of Theorem 3.4. The second summand is equal to $G_{0,1}(w) - \hbar^{-1} + \sum_{g\geq 1}\hbar^{2g-1}\Delta_g^\vee(X)$.

3.7 *Genus* 0*: proof of Theorem 3.9*

Let $\phi,\phi^\vee\colon PS\to R$ be two multiplicative functions such that $\phi=\zeta*\phi^\vee$. As described at the end of Section 3.1, we associate to $\phi^\vee$ a topological partition function $Z^\vee$ whose n-point functions $G_{g,n}^\vee$ vanish for genus $g>0$. Then, we *define* a topological partition function $Z=\mathsf{D}Z^\vee$. The corresponding multiplicative function $\Phi_{Z,\hbar}\colon PS\to R[\![\hbar]\!]$ is a priori different from ϕ, but thanks to Corollary 2.14 its genus 0 part is

$$\Phi_Z^{[0]} = \zeta * \Phi_{Z^\vee}^{[0]} = \zeta*\phi^\vee = \phi\,.$$

Therefore, the n-point functions $G_{n,0}$ of ϕ and $G_{n,0}^\vee$ are related by the restriction to genus 0 of the expressions found in Section 3.6.

Let us come back to (36) (which is valid for $n\geq 2$) and specialise it to genus 0. Recall the $\hbar$-expansion of the n-point functions (19). We need to isolate the leading order in $\hbar$ in the building blocks from Definition 3.3:

$$
\begin{aligned}
\mathsf{c}^\vee(u_i,w_i) &= \hbar^{\#I-2}\Big(\prod_{i\in I}u_i\Big)\widetilde{G}^\vee_{\#I,0}(w_I) + O(\hbar^{\#I-1}) \\
\vec{\mathsf{O}}^\vee(w) &= \sum_{m\geq 0}(P^\vee(w)w\partial_w)^m\cdot[v^m]\sum_{r\geq 0}\left(\partial_y+\frac{v}{y}\right)^r 1\Big|_{y=G_{0,1}^\vee(w)}\cdot[u^r]\,u^{-1} + O(\hbar) \\
&= \sum_{r\geq 0}\vec{\mathsf{O}}_r^\vee(w)\cdot[u^{r+1}] + O(\hbar)\,,
\end{aligned}
$$

where, in the last line, we refer to Definition 3.8. Substituting this in (36), we find that for a particular graph $\Gamma\in\mathcal{G}_n$ the minimal degree of $\hbar$ in (36) is equal to

$$-n + \sum_{I\in\mathcal{I}(\Gamma)}(2\#I-2)\,.$$

This value is minimal for trees, for which it is equal to $n-2$, hence they give the only contribution to the genus 0 part of $\widetilde{G}_n$. Note also that trees have no non-trivial automorphisms. The variable u_i only appear from the hyperedge contributions, and its power is the valency of the i-th white vertex. Therefore, the extraction of powers of u_i prescribed by the operators at the vertices restricts the sum to the set $\mathcal{G}_{0,n}(\mathbf{r}+1)$ of trees where the i-th white vertex has valency r_i+1. All in all, (36) in genus 0 and for $n\geq 2$ becomes

$$\widetilde{G}_{0,n}(X_1,\ldots,X_n) = \sum_{r_1,\ldots,r_n\geq 0}\prod_{i=1}^n\vec{\mathsf{O}}_{r_i}^\vee(w_i)\sum_{T\in\mathcal{G}_{0,n}(\mathbf{r}+1)}\prod_{I\in\mathcal{I}(T)}\widetilde{G}_{0,\#I}^\vee(w_I) \tag{38}$$

with the substitutions (37), as announced in Theorem 3.9. For $n = 2$, there is only one tree in $\mathcal{G}_{0,n}$, namely the one where the two white vertices are connected to a single black vertex. Then, only $\vec{\mathsf{O}}_0^\vee(w_i) = P^\vee(w_i)$ for $i = 1, 2$ contribute to (38):

$$\widetilde{G}_{0,2}(X_1, X_2) = P^\vee(w_1)P^\vee(w_2)G^\vee_{0,2}(w_1, w_2)\,.$$

Coming back to the non-shifted 2-point functions via (18) yields the $(0, 2)$ case of Theorems 3.4 and 3.9.

It remains to treat the $(g, n) = (0, 1)$ case, and for this we return to Lemma 3.15. Compared to the previous argument, only the weight of the white vertices is different (it is given by (31) instead of (21)), and its leading order in $\hbar$ is:

$$\begin{aligned}\vec{\mathsf{U}}(X) &= \sum_{k\in\mathbb{Z}} X^k \cdot [w^k] \sum_{r\geq 0} \partial_y^r (1+y)^k\big|_{y=0} \cdot [u^r]\, \frac{\exp\big(u(G^\vee_{0,1}(w)-1)\big)}{u} + O(\hbar) \\ &= \sum_{k\geq 0} X^k \cdot [w^k] \sum_{r=0}^{k} \frac{k!}{(k-r)!} \sum_{l\geq 0} \frac{(G^\vee_{0,1}(w)-1)^{l+1}}{(l+1)!} \cdot [u^{r-l}] + O(\hbar)\,.\end{aligned} \tag{39}$$

It is still true that only the trees will contribute to (32) for $g = 0$, and for $n = 1$ there is a single tree, namely the one without hyperedges. Therefore, we obtain:

$$\begin{aligned}G_{0,1}(X_1) - 1 = \widetilde{G}_{0,1}(X_1) - 1 &= \sum_{k\in\mathbb{Z}} X_1^k \cdot [w^k] \sum_{r\geq 0} \frac{k!}{(k-r)!} \frac{(G^\vee_{0,1}(w)-1)^{r+1}}{(r+1)!} \\ &= \sum_{k\geq 0} X_1^k \cdot [w^k] \sum_{r=0}^{k} \frac{k!}{(k-r)!r!} \frac{(G^\vee_{0,1}(w)-1)^{r+1}}{r+1} \\ &= \sum_{k\geq 0} X_1^k \cdot [w^k]\, \frac{(G^\vee_{0,1}(w))^{k+1}-1}{k+1} \\ &= G^\vee_{0,1}(w_1) - 1\,, \qquad \text{where} \quad X_1 = \frac{w_1}{G^\vee_{0,1}(w_1)}\,,\end{aligned}$$

with the help of the Lagrange inversion formula in the last line. This also completes the proof of the $(0, 1)$ case, and of Theorem 3.4 and 3.9.

Remark 3.16. We encourage the readers who want to understand better the three tricks of Section 3.6 to derive by themselves the $(0, 2)$ case from Lemma 3.15 as we did here with $(0, 1)$.

3.8 *Genus 0 coefficient-wise: proof of Theorem 3.12*

Our last task is to extract the coefficient of the monomial $\prod_{i=1}^n X_i^{k_i}$ in $\widetilde{G}_{0,n}(X_1, \ldots, X_n)$. For $k_i > 0$ this coefficient is the same in $G_{0,n}(X_1, \ldots, X_n)$ and was called $F_{0;k_1,\ldots,k_n}$ in (18). We return to the specialisation of Lemma 3.15 in genus 0. The arguments already at work in Section 3.7 together with the leading order (39) of the operators $\vec{\mathsf{U}}$ yield:

$$F_{0;k_1,\ldots,k_n} = \Big[\prod_{i=1}^{k} w_i^{k_i}\Big] \sum_{\substack{r_1,\ldots,r_n\geq 0 \\ \ell_1,\ldots,\ell_n\geq 0}} \prod_{i=1}^{n} \frac{k_i!}{(k_i-r_i)!} \frac{(G^\vee_{0,1}(w_i)-1)^{\ell_i+1}}{(\ell_i+1)!} \sum_{T\in\mathcal{G}_{0,n}(\mathbf{r}-\boldsymbol{\ell})} \prod_{I\in\mathcal{I}(T)} \widetilde{G}^\vee_{0,\#I}(w_I)\,.$$

For a given n-tuple $\boldsymbol{\ell}$, we may add $\ell_i + 1$ univalent black vertices to the i-th vertex. This absorbs the factor $(G^\vee_{0,1}(w_i) - 1)^{\ell_i+1}$ in the product over hyperedges; the $(\ell_i + 1)!$ in the denominator accounts for the automorphism of the tree at the i-th vertex, and we recognise Theorem 3.12.

3.9 *Dual formulations*

We can easily obtain dual functional relations expressing $G^\vee_n$ in terms of G_n. Writing $Z^\vee = \mathsf{D}^{-1}Z$, our starting point would be formula (30) with a minus sign. The only difference, apart from exchanging the role of X_i and w_i, is an opposite sign for v_i in the operator weight (21) for white vertices. Noticing that ς is even, the operator weights are now:

$$\begin{aligned}\vec{\mathsf{O}}(X) = &\sum_{m\geq 0}(P(X)X\partial_X)^m P(X) \\ &\cdot [v^m] \sum_{r\geq 0}\Big(\partial_y - \frac{v}{y}\Big)^r \exp\Big(-v\frac{\varsigma(\hbar v\partial_y)}{\varsigma(\hbar\partial_y)}\ln y + v\ln y\Big)\Big|_{y=G_{0,1}(X)} \\ &\cdot [u^r]\,\frac{\exp\big(\hbar u\varsigma(\hbar u X\partial_X)(G_1(X)-\hbar^{-1}) - u(G_{0,1}(X)-1)\big)}{\hbar u\varsigma(\hbar u)}\,,\end{aligned}$$

and the hyperedge weights are

$$\mathsf{c}(u_I, X_I) = \begin{cases} \big(\prod_{i\in I}\hbar u_i\varsigma(\hbar u_i X_i\partial_{X_i})\big)\widetilde{G}_{\#I}(X_I) & \text{if } I\neq\{j,j\}\,, \\ \big(\hbar u_j\varsigma(\hbar u_j X_j\partial_{X_j})\big)^2 G_2(X_j,X_j) & \text{if } I = \{j,j\}\,.\end{cases}$$

With the substitutions

$$w_i = \frac{X_i}{G_{0,1}(X_i)}\,,\qquad P(X_i) = \frac{\mathrm{d}\ln X_i}{\mathrm{d}\ln w_i} = \frac{1}{P^\vee(w_i)}\,,$$

the dual of Theorem 3.4 for $2g-2+n>0$ is

$$G^\vee_{g,n}(w_1,\ldots,w_n) = \delta_{n,1}\Delta_g(w_1) + [\hbar^{2g-2+n}]\sum_{\Gamma\in\mathcal{G}_n}\frac{1}{\#\mathrm{Aut}(\Gamma)}\prod_{i=1}^n\vec{\mathsf{O}}(X_i)\prod_{I\in\mathcal{I}(\Gamma)}\mathsf{c}(u_I,X_I)\,,$$

with the correction term for $n=1$ now given by

$$\begin{aligned}\Delta_g(w) = [\hbar^{2g}]&\sum_{m\geq 0}\big(P(X)X\partial_X\big)^m [v^{m+1}]\exp\Big(-v\,\frac{\varsigma(\hbar v\partial_y)}{\varsigma(\hbar\partial_y)}\ln y + v\ln y\Big)\Big|_{y=G_{0,1}(X)} \\ &\cdot P^\vee(X)X\partial_X G_{0,1}(X)\,.\end{aligned}$$

In genus 0, the relevant operator weight is

$$\vec{\mathsf{O}}_r(X) = \sum_{m\geq 0}(P(X)X\partial_X)^m P(X)\cdot[v^m]\,\Big(\partial_y - \frac{v}{y}\Big)^r\cdot 1\Big|_{y=G_{0,1}(X)}\,,$$

and the dual of Theorem 3.9 reads:

$$G^\vee_{0,n}(w_1,\ldots,w_n) = \sum_{r_1,\ldots,r_n\geq 0}\prod_{i=1}^n\vec{\mathsf{O}}_{r_i}(X_i)\sum_{T\in\mathcal{G}_{0,n}(\mathbf{r}+1)}\prod_{I\in\mathcal{I}(T)}{}' G_{0,\#I}(X_I)\,,$$

and $\prod'$ means that each occurrence of $G_{0,2}(X_i,X_j)$ for $i\neq j$ should be replaced with $\widetilde{G}_{0,2}(X_i,X_j)$.

To get the coefficient-wise relation dual to Theorem 3.12, we observe that the operator (39) is replaced at leading order with

$$\begin{aligned}\vec{\mathsf{U}}^\vee(w) &= \sum_{k\in\mathbb{Z}}w^k\cdot[X^k]\sum_{r\geq 0}\partial_y^r(1+y)^{-k}\big|_{y=0}\cdot[u^r]\frac{\exp\big(u(G_{0,1}(X)-1)\big)}{u} \\ &= \sum_{k\in\mathbb{Z}}w^k\cdot[X^k]\sum_{r\geq 0} = \frac{(-1)^r(r+k-1)!}{(k-1)!}\sum_{l\geq 0}\frac{(G_{0,1}(X)-1)^{l+1}}{(l+1)!}\cdot[u^{r-l}]\,.\end{aligned}$$

The only difference with (39) is the combinatorial factor, which is non-vanishing for all nonnegative k. This leads to the formula

$$F^\vee_{0;k_1,\ldots,k_n} = \Big[\prod_{i=1}^n X_i^{k_i}\Big]\sum_{r_1,\ldots,r_n\geq 0}\prod_{i=1}^n\frac{(-1)^{r_i}(r_i+k_1-1)!}{(k_i-1)!}\sum_{T\in\mathcal{T}_n(\mathbf{r}+1)}\frac{\prod''_{I\in\mathcal{I}(T)}G_{0,\#I}(X_I)}{\#\mathrm{Aut}(T)}\,,$$

where $\prod''$ means that one should replace each occurrence of $G_{0,1}(X_j)$ with $G_{0,1}(X_j) - 1$, and each occurrence of $G_{0,2}(X_i, X_j)$ for $i \neq j$ with $\widetilde{G}_{0,2}(X_i, X_j)$. The form is in fact similar to Theorem 3.12 when one observes that

$$\frac{k!}{(r-k)!} = r!\binom{k}{r}, \qquad \frac{(-1)^r(r+k-1)!}{(k-1)!} = r!\binom{-k}{r}.$$

4 Applications to free probability

We now present the interpretation of the results of Section 3 in the context of (higher order) free probabilities.

4.1 *Preliminary: decorated partitioned permutations*

If $\mathscr{A}$ is an associative algebra, we can consider partitioned permutations decorated by elements in $\mathscr{A}$. It means we want to study the set $PS(\mathscr{A}) := \bigcup_{d\geq 0} PS(d) \times \mathscr{A}^d$. If $f_1 \colon PS \to R$ and $f_2 \colon PS(\mathscr{A}) \to R$ are two functions, their convolution is defined by

$$\begin{aligned}(f_1 * f_2)(\mathcal{C},\gamma)[a_1,\ldots,a_d] &:= (f_1 * f_2[a_1,\ldots,a_d])(\mathcal{C},\gamma)\\ &= \sum_{(\mathcal{A},\alpha)\cdot(\mathcal{B},\beta)=(\mathcal{C},\gamma)} f_1(\mathcal{A},\alpha)\, f_2(\mathcal{B},\beta)[a_1,\ldots,a_d]\,,\end{aligned}$$

for $(\mathcal{C},\gamma) \in PS(d)$ and $a_1,\ldots,a_d \in \mathscr{A}$. A similar definition can be made for the extended convolution $f_1 \circledast f_2$. A function $\phi \colon PS(\mathscr{A}) \to R$ is *multiplicative* if for any $d \geq 0$ and $a_1,\ldots,a_d \in \mathscr{A}$:

- for any $\sigma,\pi \in S(d)$, we have $\phi(\mathbf{1}_d, \pi^{-1}\circ\sigma\circ\pi)[a_1,\ldots,a_d] = \phi(\mathbf{1}_d,\sigma)[a_{\pi(1)},\ldots,a_{\pi(d)}]$;
- for any $(\mathcal{A},\alpha) \in S(d)$, we have $\phi(\mathcal{A},\alpha)[a_1,\ldots,a_n] = \prod_{A\in\mathcal{A}} \phi(1_{\#A},\alpha_{|A})[(a_i)_{i\in A}]$, where bijections $[\#A] \to A$ have been chosen to make sense of the right-hand side, which is independent of this choice due to the first condition.

4.2 *Higher order free probability*

Definition 4.1. Let $\mathscr{A}$ an associative algebra. We say that an n-linear form $\tau \colon \mathscr{A}^n \to \mathbb{C}$ is tracial if

$$\forall b, a_1,\ldots,a_n \in \mathscr{A}, \quad \forall i \in [n], \qquad \tau(a_1,\ldots,a_ib,\ldots,a_n) = \tau(a_1,\ldots,ba_i,\ldots,a_n)\,.$$

Definition 4.2. A *higher order probability space* (HOPS) is the data $(\mathscr{A},\boldsymbol{\varphi})$ consisting of a unital associative (maybe non-commutative) algebra $\mathscr{A}$ over $\mathbb{C}$ and a family $\boldsymbol{\varphi} = (\varphi_n)_{n\geq 1}$ of tracial n-linear forms such that $\varphi_1(1) = 1$ and $\varphi_n(1,a_2,\ldots,a_n) = 0$ for any $n \geq 2$ and $a_2,\ldots,a_n \in \mathscr{A}$.

Given a HOPS $(\mathscr{A},\boldsymbol{\varphi})$, there is a natural multiplicative function $\phi \colon PS(\mathscr{A}) \to \mathbb{C}$ encoding the (higher order) moments, specified for $\lambda \vdash d$ of length ℓ by

$$\phi(\mathbf{1}_d,\pi_\lambda)[a_1,\ldots,a_d] = \varphi_\ell\Big(\prod_{j=1}^{\lambda_1} a_j\,, \prod_{j=1}^{\lambda_2} a_{\lambda_1+j}\,,\ldots, \prod_{j=1}^{\lambda_\ell} a_{\lambda_1+\cdots+\lambda_{\ell-1}+j}\Big)\,.$$

We then define another multiplicative function $\phi^\vee \colon PS(\mathscr{A}) \to \mathbb{C}$ by

$$\phi^\vee := \mu * \phi \qquad \Longleftrightarrow \qquad \phi = \zeta * \phi^\vee\,.$$

Definition 4.3. (Adapted[8] from [CMSS07]) The free cumulants at order n are encoded in the collection of d-linear forms $\kappa_{k_1,\ldots,k_\ell} \colon \mathscr{A}^d \to \mathbb{C}$ indexed by $k_1,\ldots,k_n > 0$ given by:

$$\kappa_{k_1,\ldots,k_n}(a_1,\ldots,a_d) = \phi^\vee(\mathbf{1}_d,\pi_{\lambda(\mathbf{k})})[a_1,\ldots,a_d], \qquad d = k_1+\cdots+k_n\,.$$

[8] In [CMSS07] the convolution of f_1 with f_2 is defined when f_1 is a function on $PS(\mathscr{A})$ and f_2 a function on PS, and the higher-order free cumulants are defined by right-convolution of ϕ with the zeta function on PS. In the present paper we rather introduced (Section 4.1) the convolution $f_1 * f_2$ when f_1 is a function on PS and f_2 a a function on $PS(\mathscr{A})$, and define the higher-order free cumulants by left-convolution of ϕ with ζ. As this only involves multiplicative functions and in light of Lemma 2.6, these definitions are equivalent.

If $a \in \mathscr{A}$, let us define the generating series of moments and cumulants at order n of a:

$$(40)\qquad \begin{aligned} M_n(X_1,\ldots,X_n) &= \delta_{n,1} + \sum_{k_1,\ldots,k_n>0} \varphi_n(a^{k_1},\ldots,a^{k_n}) \prod_{i=1}^{n} X_i^{k_i}\,, \\ C_n(w_1,\ldots,w_n) &= \delta_{n,1} + \sum_{k_1,\ldots,k_n>0} \kappa_{k_1,\ldots,k_n}(a,\ldots,a) \prod_{i=1}^{n} w_i^{k_i}\,. \end{aligned}$$

At first order, a fundamental result of Speicher [Spe94] states that

$$C_1(wM(w)) = w\,.$$

This can also be formulated in terms of the Voiculescu R-transform, defined by $C(w) = 1 + wR(w)$, as the functional relation $\frac{x}{M(1/x)} + R\big(\frac{M(1/x)}{x}\big) = x$. Collins, Mingo, Speicher and Śniady established in [CMSS07] the formula at second order:

$$\frac{M_2(1/x_1,1/x_2)}{x_1x_2} = \frac{\mathrm{d}w_1}{\mathrm{d}x_1}\,\frac{\mathrm{d}w_2}{\mathrm{d}x_2}\left(\frac{C_2(w_1,w_2)}{w_1w_2} + \frac{1}{(w_1-w_2)^2}\right) - \frac{1}{(x_1-x_2)^2}\,, \qquad w_i = \frac{M(1/x_i)}{x_i}\,,$$

and asked for the generalisation of these functional relations to order $n \geq 3$. Converting the notations into $G_{0,n} = M_n$, $G^{\vee}_{0,n} = C_n$, Theorem 3.9 answers the question (the formulas for $n = 1$ and $n = 2$ match). Even knowing the formulas, it seems rather complicated to prove them directly from the combinatorics of partitioned permutations — even in the $n = 3$ case (27). We had to make a detour via the master relation (Theorem 2.13) and algebraic manipulations in the bosonic Fock space in order to derive the results.

4.3 *Surfaced permutations*

Another way to work with the extended product is to add the data of a genus function to partitioned permutations. It corresponds to the notion of surfaced permutations proposed in [CMSS07, Appendix]. We are in fact going to allow half-integer genus, so that infinitesimal freeness cumulants will find a natural place in the framework (see end of Section 4.5).

Definition 4.4. A *surfaced permutation* of $[d]$ is a triple $(\mathcal{A},\alpha,g)$ where $(\mathcal{A},\alpha) \in PS(d)$ and $g\colon \mathcal{A} \to \frac{1}{2}\mathbb{Z}_{\geq 0}$ is a function. We denote $\mathbb{PS}(d)$ the set of surfaced permutations of $[d]$, with the convention $\mathbb{PS}(0) = \{\emptyset\}$, and $\mathbb{PS} = \bigcup_{d\geq 0} \mathbb{PS}(d)$.

The colength of $(\mathcal{A},\alpha,g) \in \mathbb{PS}(d)$ is defined by:

$$|(\mathcal{A},\alpha,g)| := |(\mathcal{A},\alpha)| + \sum_{A\in\mathcal{A}} 2g(A)\,.$$

Definition 4.5. The extended product of $(\mathcal{A},\alpha,g)$, $(\mathcal{B},\beta,h) \in \mathbb{PS}(d)$ is defined as $(\mathcal{A},\alpha,g) \odot (\mathcal{B},\beta,h) = (\mathcal{A}\vee\mathcal{B},\alpha\circ\beta,k)$ in which the genus function at $C \in \mathcal{A}\vee\mathcal{B}$ takes the value:

$$(41)\qquad k(C) := \frac{|(\mathcal{A}_{|C},\alpha_{|C},g_{|C})| + |(\mathcal{B}_{|C},\beta_{|C},h_{|C})| - |(\mathcal{A}\vee\mathcal{B}_{\,|C},\alpha\circ\beta_{\,|C})|}{2}\,.$$

Here, if $\mathcal{D} = \{D_1,\ldots,D_l\} \in P(d)$ and $C \subseteq [d]$, the notation $\mathcal{D}_{|C}$ stands for $\{D_1\cap C,\ldots,D_l\cap C\}$ from which one removes the elements which are empty sets. If $\mathscr{A}$ is an associative algebra, the convolution of two functions $f_1, f_2\colon \mathbb{PS}(d) \to R$ is

$$(f_1 \circledast f_2)(\mathcal{C},\gamma,k) = \sum_{(\mathcal{A},\alpha,g)\odot(\mathcal{B},\beta,h)=(\mathcal{C},\gamma,k)} f_1(\mathcal{A},\alpha,g)\ f_2(\mathcal{B},\beta,h)\,.$$

It is easy to check that $\circledast$ is associative.

Remark 4.6. There are two alternative ways to think about the formula for the genus. On the one hand, we observe that (41) is such that we have a block-additivity of the colength under products of surfaced permutations:

$$|(\mathcal{A}_{|C},\alpha_{|C},g_{|C})| + |(\mathcal{B}_{|C},\beta_{|C},h_{|C})| = |(\mathcal{A}\vee\mathcal{B}_{\,|C},\alpha\circ\beta_{\,|C},k_{|C})|\,.$$

On the other hand, we also have

$$\begin{aligned} k(C) &= \sum_{\substack{A\in\mathcal{A}\\ A\subseteq C}} g(A) + \sum_{\substack{B\in\mathcal{B}\\ B\subseteq C}} h(B) + \frac{1}{2}\Big(|(\mathcal{A}_{|C},\alpha_{|C})| + |(\mathcal{B}_{|C},\beta_{|C})| - |(\mathcal{A}\vee\mathcal{B}_{\,|C},\alpha\circ\beta_{\,|C})|\Big) \\ &\geq \sum_{\substack{A\in\mathcal{A}\\ A\subseteq C}} g(A) + \sum_{\substack{B\in\mathcal{B}\\ B\subseteq C}} h(B)\,. \end{aligned} \tag{42}$$

Since $|\alpha\circ\beta| - |\alpha| - |\beta|$ is nonnegative and even, the last term in the first line is a nonnegative integer. We interpret this equation by saying that the product of surfaced permutation can create genus in integer units.

The relation with the setting of Section 2.2 (justifying that we keep the same notations $\odot$ and $\circledast$) is that, if we associate to the multiplicative functions $f_i\colon \mathbb{PS}\to R$, for $i=1,2$, the multiplicative functions $\hat{f}_i\colon PS\to R[\![\hbar]\!]$ given for $(\mathcal{A},\alpha)$ by

$$\widehat{f}_i(\mathcal{A},\alpha) = \sum_{g\colon\mathcal{A}\to\frac{1}{2}\mathbb{Z}_{\geq 0}} \hbar^{|(\mathcal{A},\alpha,g)|}\, f_i(\mathcal{A},\alpha,g)\,, \tag{43}$$

then we have

$$\widehat{f_1\circledast f_2} = \widehat{f}_1\circledast\widehat{f}_2\,,$$

where on the left-hand side (resp. on the right-hand side) this is the extended convolution on $\mathbb{PS}$ (resp. PS). This is due to the block-additivity of the colength of surfaced permutations mentioned in Remark 4.6.

We have an injection $\iota\colon PS(d)\to\mathbb{PS}(d)$ consisting in completing a partitioned permutation by taking zero as genus function. All functions f on $PS(d)$ can be considered as functions on $\mathbb{PS}(d)$ by extending them by 0 outside $\iota(PS(d))$; we denote it $\iota_* f$. Clearly, $\mathbb{\delta}=\iota_*\delta$ is the unit for $\circledast$, while $\mathbb{\zeta}=\iota_*\zeta$ is the zeta function, also characterised by $\hat{\mathbb{\zeta}}=\zeta_\hbar$. By the previous discussion, it admits as inverse for $\circledast$ the function $\mathbb{\mu}$ characterised by $\hat{\mathbb{\mu}}=\mu_\hbar$, whose existence comes from Lemma 2.4. An alternative way to prove the existence of the Möbius function $\mathbb{\mu}$ is to realize that $\mathbb{PS}(d)$ admits a poset structure, by declaring that

$$(\mathcal{A},\alpha,g)\preceq(\mathcal{C},\gamma,k)\qquad\Leftrightarrow\qquad \exists(\mathcal{B},\beta,0)\in\mathbb{PS}(d),\quad (\mathcal{A},\alpha,g)\odot(\mathcal{B},\beta,0)=(\mathcal{C},\gamma,k)\,,$$

and invoking the general fact that posets admit Möbius functions [Rot64].

Definition 4.7. A function $f\colon\mathbb{PS}\to R$ is *multiplicative* if for any $d,h\in\mathbb{Z}_{\geq 0}$ and $\sigma\in S(d)$, $f(\mathbf{1}_d,\sigma,h)$ depends only on the conjugacy class of σ, and for any $(\mathcal{A},\alpha,g)\in\mathbb{PS}$:

$$f(\mathcal{A},\alpha,g)=\prod_{A\in\mathcal{A}} f(\mathbf{1}_{\#A},\alpha_{|A},g_{|A})\,.$$

$\mathbb{\delta}$, $\mathbb{\zeta}$ and $\mathbb{\mu}$ are examples of multiplicative functions.

Remark 4.8. We say that $f\colon\mathbb{PS}\to R$ is *even* when $f(\mathcal{A},\alpha,g)=0$ for any $(\mathcal{A},\alpha)\in PS$ such that there exists a block $A\in\mathcal{A}$ having $g(A)\notin\mathbb{Z}_{\geq 0}$. Due to the last point in Remark 4.6, the extended convolution of two even functions is even. In the previous sections we only considered functions with integer genus: it fits with the present setting by restricting to even functions on $\mathbb{PS}$.

We may also consider the analog of the product $\cdot$ between surfaced permutations, by keeping only the cases in $\odot$ with no genus creation.

Definition 4.9. If $(\mathcal{A},\alpha,g),(\mathcal{B},\beta,h)$ are two surfaced permutations of $[d]$, we define their product:

$$(\mathcal{A},\alpha,g)\cdot(\mathcal{B},\beta,h)=\begin{cases}(\mathcal{A}\vee\mathcal{B},\alpha\circ\beta,k) & \text{if } |(\mathcal{A},\alpha)|+|(\mathcal{B},\beta)|=|(\mathcal{A}\vee\mathcal{B},\alpha\circ\beta)|,\\ 0 & \text{otherwise.}\end{cases}$$

Note that in the first case we have

$$k(C)=\sum_{\substack{A\in\mathcal{A}\\ A\subseteq C}} g(A)+\sum_{\substack{B\in\mathcal{B}\\ B\subseteq C}} h(B)\,.$$

The convolution between functions $f_1, f_2 \colon \mathbb{PS}(d) \to R$ is defined as

$$(f_1 * f_2)(\mathcal{C}, \gamma, k) = \sum_{(\mathcal{A},\alpha,g)\cdot(\mathcal{B},\beta,h)=(\mathcal{C},\gamma,k)} f_1(\mathcal{A}, \alpha, g)\, f_2(\mathcal{B}, \beta, h)\,.$$

The extraction of leading order in Lemma 2.7 can be upgraded to include the first subleading order (encoded in genus $\frac{1}{2}$). To describe this, we need more notations.

Definition 4.10. We say that two multiplicative functions $\phi_1, \phi_2 \colon \mathbb{PS} \to R$ agree infinitesimally if their value coincides on $(\mathcal{A}, \alpha, g)$ for any $g \colon \mathcal{A} \to \frac{1}{2}\mathbb{Z}_{\geq 0}$ such that $\sum_{A \in \mathcal{A}} g(A) \leq \frac{1}{2}$. In that case we denote $\phi_1 \approx \phi_2$.

Lemma 4.11. *Let $\phi_1, \phi_2 \colon \mathbb{PS} \to R$ be two multiplicative functions. The relation $\phi_1 = \textcolor{black}{\zeta} \circledast \phi_2$ implies the infinitesimal agreement $\phi_1 \approx \zeta * \phi_2$.*

Proof. Same as in Lemma 2.7, taking into account that the creation of genus occurs by integer units only (Remark 4.6). ☐

Remark 4.12. This has an equivalent presentation via the ring of dual numbers $R' = R[\![\hbar]\!]/(\hbar^2)$. Namely, let us write:

$$\widehat{\phi}_i(\mathcal{A}, \alpha) = \hbar^{|(\mathcal{A},\alpha)|}\big(\phi_i(\mathcal{A}, \alpha, 0) + \hbar \phi_i'(\mathcal{A}, \alpha) + o(\hbar)\big)\,,$$

and define multiplicative functions

$${}^\flat\phi_i \colon PS \to R', \qquad \text{by} \quad {}^\flat\phi_i(\mathcal{A}, \alpha) = \phi_i(\mathcal{A}, \alpha, 0) + \hbar \phi_i'(\mathcal{A}, \alpha)\,.$$

Then, the relation $\phi_1 = \zeta \circledast \phi_2$ between R-valued functions on surfaced permutations implies the relation ${}^\flat\phi_1 = \zeta * {}^\flat\phi_2$ between R'-valued functions on partitioned permutations. Observe that $\phi_i(-,-,0)$ is a multiplicative function on PS, but ϕ_i' is not. Instead, we have

$$\phi_i'(\mathcal{A}, \alpha) = \sum_{A \in \mathcal{A}} \phi_i'(\mathbf{1}_{\#A}, \alpha_{|A}) \prod_{\substack{A' \in \mathcal{A} \\ A' \neq A}} \phi_i(1_{\#A'}, \alpha_{|A'}, 0)\,.$$

As in Section 4.1, we may also work with the set $\mathbb{PS}(\mathscr{A})$ of surfaced permutations decorated by elements of an associative algebra $\mathscr{A}$.

4.4 *Extension of the main formulas*

Section 2.5 and Section 3 extend without effort to topological partition functions allowing $g \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, while the monotone Hurwitz numbers are unchanged and have integer genus. In particular, the functional relations of Theorem 3.4 for the n-point functions of multiplicative functions $\Phi, \Phi^\vee \colon \mathbb{PS} \to R$ satisfying $\Phi = \zeta \circledast \Phi^\vee$ hold in the same form. Taking into account Lemma 4.11, the proofs of Theorem 3.9 and Theorem 3.12 can easily be adapted to obtain functional relations in the genus 0 and $\frac{1}{2}$.

Definition 4.13. Let $\mathcal{G}_{0,n}'$ be the set of bicoloured trees as in Definition 3.7 except that they must contain one special black vertex, whose corresponding hyperedge I' may be univalent. Let $\mathcal{T}_n'$ be the set of bicoloured trees obtained by connecting to a $T' \in \mathcal{G}_{0,n}'$ (see Definition 3.11) finitely many univalent black vertices. In $\mathcal{G}_{0,n}'(\mathbf{r}+1)$ and $\mathcal{T}_n'(\mathbf{r}+1)$ we require the i-th vertex to have valency $r_i + 1$.

Theorem 4.14. *Let $\phi, \phi', \phi^\vee, \phi'^\vee \colon PS \to R$ be functions so that*

$${}^\flat\phi = \phi + \hbar\phi' \colon PS \to R', \qquad {}^\flat\phi^\vee = \phi^\vee + \hbar\phi'^\vee \colon PS \to R'$$

are multiplicative. Introduce the n-point functions

$$(44)\qquad \begin{aligned} G_{0,n}(X_1,\ldots,X_n) &= \delta_{n,1} + \sum_{k_1,\ldots,k_n>0} \phi(\mathbf{1}_{k_1+\cdots+k_n},\pi_{\lambda(\mathbf{k})}) \prod_{i=1}^{n} X_i^{k_i}\,, \\ G_{\frac{1}{2},n}(X_1,\ldots,X_n) &= \sum_{k_1,\ldots,k_n>0} \phi'(\mathbf{1}_{k_1+\cdots+k_n},\pi_{\lambda(\mathbf{k})}) \prod_{i=1}^{n} X_i^{k_i}\,, \end{aligned}$$

and likewise $G_{0,n}^{\vee}$ and $G_{\frac{1}{2},n}^{\vee}$.

Suppose that we have ${}^{\flat}\phi = \zeta * {}^{\flat}\phi^{\vee}$. Then, the genus 0 functional relations given in Theorem 3.9 and 3.12 hold, and with the same substitution and notations we have for any $n \geq 1$:

$$(45)\qquad G_{\frac{1}{2},n}(X_1,\ldots,X_n) = \sum_{r_1,\ldots,r_n\geq 0} \prod_{i=1}^{n} \vec{\mathsf{O}}_{r_i}^{\vee}(w_i) \sum_{T\in\mathcal{G}'_{0,n}(\mathbf{r}+1)} G_{\frac{1}{2},\#I'}(w_{I'}) \prod_{I\neq I'}' G_{0,\#I}^{\vee}(w_I)\,.$$

Equivalently, for any $n \geq 1$, and $\lambda \vdash d$ of length n we have:

$$(46)\qquad \phi'(\mathbf{1}_d,\pi_\lambda) = \Big[\prod_{i=1}^{n} w_i^{\lambda_i}\Big] \sum_{\substack{0\leq r_i\leq\lambda_i \\ i\in[n]}} \prod_{i=1}^{n} \frac{\lambda_i!}{(\lambda_i-r_i)!} \sum_{T\in\mathcal{T}'_n(\mathbf{r}+1)} \frac{G_{\frac{1}{2},\#I'}(w_{I'}) \prod''_{I\neq I'} G_{0,\#I}(w_I)}{\#\mathrm{Aut}(T)}\,.$$

Corollary 4.15. *We have*

$$(47)\qquad G_{\frac{1}{2},1}(X) = P^{\vee}(w)\, G_{\frac{1}{2},1}^{\vee}(w)\,, \qquad X = \frac{w}{G_{0,1}^{\vee}(w)}\,, \qquad P^{\vee}(w) = \frac{\mathrm{d}\ln w}{\mathrm{d}\ln X}\,.$$

Equivalently:

$$G_{\frac{1}{2},1}(X)\frac{\mathrm{d}X}{X} = G_{\frac{1}{2},1}^{\vee}(w)\frac{\mathrm{d}w}{w}\,.$$

Proof of Theorem 4.14. Let $\phi_\hbar^{\vee}\colon PS \to R[\![\hbar]\!]$ be the unique multiplicative function which for any $d \in \mathbb{Z}_{\geq 0}$ and $\alpha \in S(d)$ satisfies

$$\phi_\hbar^{\vee}(\mathbf{1}_d,\alpha) = \hbar^{|(\mathbf{1}_d,\alpha)|}\big(\phi(\mathbf{1}_d,\alpha) + \hbar\phi'(\mathbf{1}_d,\alpha)\big)\,.$$

We then introduce the multiplicative function $\phi_\hbar = \zeta_\hbar \circledast \phi_\hbar^{\vee}$. Let G_n and $G_n^{\vee}$ be the n-point functions corresponding to $\phi_\hbar$ and $\phi_\hbar^{\vee}$: they have an $\hbar$ expansion of the form (19) with half-integer g allowed, and Theorem 3.4 applies. As we are in the situation of Lemma 4.11, $G_{0,n}$ and $G_{\frac{1}{2},n}$ must coincide with the right-hand sides of (44) defined in terms of ϕ and ϕ'. We shall focus on them but rewrite Theorem 3.4 differently. Namely, we decide to replace $G_1^{\vee}(w_i)$ in the operator weight $\vec{\mathsf{O}}^{\vee}(w_i)$ with

$$G_{\mathrm{even},1}^{\vee}(w) = \sum_{g\in\mathbb{Z}_{\geq 0}} \hbar^{2g-1}\, G_{g,1}^{\vee}(w)\,.$$

Following the proof of Theorem 3.4 in Section 3.6, we see that the functional relation (23) still holds with this modified operator weight provided we sum over the set $\mathcal{G}'_n$ of bicoloured graphs like in Definition 3.2 but now allowing univalent black vertices. When connecting to the i-th white vertex the latter receive the weight

$$\mathsf{c}^{\vee}(u_i,w_i) = \hbar u_i \varsigma(\hbar u_i w_i \partial_{w_i}) G_{\mathrm{odd},1}^{\vee}(w_i)\,,$$

where:

$$G_{\mathrm{odd},1}^{\vee}(w) = \sum_{g\in\frac{1}{2}+\mathbb{Z}_{\geq 0}} \hbar^{2g-1} G_{g,1}^{\vee}(w)\,.$$

When extracting the genus $\frac{1}{2}$ part of the formula, only the leading power in $\hbar$ of each of the weights and only the trees in $\mathcal{G}'_n$ in which exactly one factor of $G_{\frac{1}{2},\#I}$ is picked will contribute. This is because the series ς where $\hbar$ occurs is even, corresponding to the fact that monotone Hurwitz numbers have integer genus. The outcome is then (45), and adapting the proof of Section 3.8 gives (46). ☐

Proof of Corollary 4.15. We specialise (46) to $n = 1$. The set $\mathcal{T}_1'(r+1)$ contains a single tree, namely the white vertex connected to the special vertex and r other univalent black vertices. We therefore find for $k > 0$:

$$\phi'(\mathbf{1}_k, (12\cdots k)) = [X^k]\ G_{\frac{1}{2},1}(X) = [w^k]\ \sum_{r=0}^{k} \frac{k!}{(k-r)!}\, G^{\vee}_{\frac{1}{2},1}(w)\, \frac{(G^{\vee}_{0,1}(w) - 1)^r}{r!}$$
$$= [w^k]\ G^{\vee}_{\frac{1}{2},1}(w) G^{\vee}_{0,1}(w)^k\,,$$

which leads to (47) thanks to Lagrange inversion formula. ☐

4.5 *Surfaced free probability*

Section 3 suggests a natural generalisation of the notion of higher order free cumulants that includes information about higher genera, using the extended convolution instead of the convolution. As much as free probability applied to random ensembles of hermitian matrices at leading order when the size N goes to ∞, the surfaced version we propose captures information about the corrections of order N^{-2g} beyond the leading order (we will make this precise in Section 4.6). With Sections 4.3-4.4 and Remark 4.8 under our belt, it does not cost anything to allow half-integer g.

Definition 4.16. A surfaced probability space (SPS) is the data $(\mathscr{A}, \boldsymbol{\varphi})$ consisting of a unital associative algebra $\mathscr{A}$ over $\mathbb{C}$ and a family $\boldsymbol{\varphi} = \big(\varphi_{g,n} \ :\ g \in \frac{1}{2}\mathbb{Z}_{\geq 0},\ n \in \mathbb{Z}_{>0}\big)$ of tracial n-linear forms on $\mathscr{A}$ such that $\varphi_{0,1}(1) = 1$ and $\varphi_{g,n}(1, a_2, \ldots, a_n) = 0$ for any $(g,n) \neq (0,1)$ and $a_2, \ldots, a_n \in \mathscr{A}$.

Given a SPS, we encode the moments into the multiplicative function on $\mathbb{PS}(\mathscr{A})$ valued in $R = \mathbb{C}$ and given, for $\lambda \vdash d$ of length ℓ, genus $g \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $a_1, \ldots, a_n \in \mathscr{A}$, by

$$\phi(\mathbf{1}_d, \pi_\lambda, g)[a_1, \ldots, a_d] = \varphi_{g,n}\Big(\prod_{j=1}^{\lambda_1} a_j\,,\ \prod_{j=1}^{\lambda_2} a_{\lambda_1+j}\,, \ldots,\ \prod_{j=1}^{\lambda_\ell} a_{\lambda_1+\cdots+\lambda_{\ell-1}+j}\Big)\,.$$

Then we consider the multiplicative function $\phi^{\vee}$ on $\mathbb{PS}(\mathscr{A})$ by

$$\phi^{\vee} = \mu \circledast \phi \qquad \Longleftrightarrow \qquad \phi = \zeta \circledast \phi^{\vee}\,, \tag{48}$$

and we define the genus g, order n free cumulants to be the n-linear forms:

$$\kappa_{g;k_1,\ldots,k_n} : \mathscr{A}^n \to \mathbb{C}\,, \qquad \kappa_{g;k_1,\ldots,k_n}(a_1, \ldots, a_d) = \phi^{\vee}(\mathbf{1}_d, \pi_{\lambda(\mathbf{k})}, g)[a_1, \ldots, a_d]\,,$$

where $d = k_1 + \cdots + k_n$. For a given $a \in \mathscr{A}$, the generating series of "surfaced moments" and "surfaced free cumulants" of a:

$$\begin{aligned} G_{g,n}(X_1, \ldots, X_n) &= \delta_{g,0}\delta_{n,1} + \sum_{k_1,\ldots,k_n > 0} \varphi_{g,n}(a^{k_1}, \ldots, a^{k_n}) \prod_{i=1}^{n} X_i^{k_i}\,, \\ G^{\vee}_{g,n}(w_1, \ldots, w_n) &= \delta_{g,0}\delta_{n,1} + \sum_{k_1,\ldots,k_n > 0} \kappa_{g;k_1,\ldots,k_n}(a, \ldots, a) \prod_{i=1}^{n} w_i^{k_i}\,, \end{aligned} \tag{49}$$

are then related by Theorem 3.4 where half-integer genera are allowed. In particular the $g = 0$ case is covered by Theorem 3.9 and the $g = \frac{1}{2}$ by Theorem 4.14.

Definition 4.17. We define a partial order[9] on $\mathsf{Typ} = \frac{1}{2}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$ by declaring $(g,n) \preceq (g',n')$ when $g \leq g'$ and $g' + n' \leq g + n$. We also define

$$\overline{\mathsf{Typ}} = \mathsf{Typ} \cup \Big(\bigcup_{g \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \{(g,\infty)\}\Big) \cup \{(\infty,\infty)\}\,,$$

on which the partial order relation extends in a natural way, for instance, if $g, g' \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ with $g < g'$ and $n \in \mathbb{Z}_{>0}$

$$(g,n) \prec (g,\infty) \prec (g',\infty)\,, \qquad (g,\infty) \prec (g',n)\,.$$

[9]This is not a total order. For instance $(1,1)$ and $(0,3)$ are not comparable. More generally, two distinct elements $(g,n), (g',n')$ having $2g - 2 + n = 2g' - 2 + n'$ are not comparable.

The following lemma shows that this order reflects the structure of the relations (48) or of Theorem 3.4.

Lemma 4.18. *Let $(g_0, n_0) \in \overline{\mathsf{Typ}}$. The knowledge of $\varphi_{g,n}$ for all $(g,n) \preceq (g_0,n_0)$ is equivalent to the knowledge of $\kappa_{g;k_1,\ldots,k_n}$ for all $(g,n) \preceq (g_0,n_0)$ and $k_1,\ldots,k_n > 0$.*

Proof. This can be in principle extracted by elementary means from the definition (48), but we propose here to read it from Theorem 3.4. By multilinearity, it is enough to prove the thesis for the evaluations of $\varphi_{g,n}$ and $\kappa_{g;k_1,\ldots,k_n}$ on tuples of the form $(a,\ldots,a)$. The claim clearly holds for $(g,n) = (0,1)$, and for $(\frac{1}{2},1)$ (see Corollary 4.15). Now take $a \in \mathscr{A}$ and $(g,n) \in \mathsf{Typ}$ with $2g-2+n > 0$: we consider (23) expressing $G_{g,n}$ in terms of $G^{\vee}_{g',n'}$, both defined as in (49). The summand associated to a graph $\Gamma \in \mathcal{G}_n$ contains – besides $G^{\vee}_{0,1}$ – contributions from the hyperedges involving $G^{\vee}_{g_I,\#I}$ for some $g_I \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, and contributions from the i-th vertex which give either a 1 or a product $\prod_{p=1}^{q_i} G^{\vee}_{g_{i,p},1}$ with $q_i > 0$ and $g_{i,p} \in \frac{1}{2}\mathbb{Z}_{>0}$. The extraction of the right power of $\hbar$ in (23) shows that

$$2g-2+n = \sum_{i=1}^{n}\Big(-1+\sum_{p=1}^{q_i} 2g_{i,p}\Big) + \sum_{I \in \mathcal{I}(\Gamma)} 2(g_I - 1 + \#I)\,.$$

In other words:

$$g-1+n = \sum_{i=1}^{n}\sum_{p=1}^{q_i} g_{i,p} + \sum_{I\in\mathcal{I}(\Gamma)} (g_I - 1 + \#I)\,. \tag{50}$$

Besides, as Γ is connected, we must have $\sum_{I\in\mathcal{I}(\Gamma)}(-1+\#I) \geq n-1$ and thus

$$g \geq \sum_{i=1}^{n}\sum_{p=1}^{q_i} g_{i,p} + \sum_{I\in\mathcal{I}(\Gamma)} g_I\,. \tag{51}$$

As in the right-hand side of (50)-(51) all terms of the sums are nonnegative, we deduce that only $G^{\vee}_{g',n'}$ with $g'+n' \leq g+n$ and $g' \leq g$ are involved in the sum over graphs, that is $(g',n') \preceq (g,n)$. Noticing that the correction term $\Delta^{\vee}_g$ in (24) only involves $G^{\vee}_{g',1}$ for $g' \leq g$, we deduce that $G_{g,n}$ is expressed as a function of $G^{\vee}_{g',n'}$ with $(g',n') \preceq (g,n)$. ☐

The raison d'être of cumulants is to formulate a notion of freeness for elements (or more generally, subalgebras) of $\mathscr{A}$, by the vanishing of mixed free cumulants. We can propose a similar definition here.

Definition 4.19. Let $(g_0,n_0) \in \overline{\mathsf{Typ}}$. A family $(\mathscr{X}_i)_{i\in I}$ of subsets of $\mathscr{A}$ is called (g_0,n_0)-*free* if for any $(g,n) \preceq (g_0,n_0)$, for any $d \geq 0$, any $(a_1,\ldots,a_d) \in \prod_{p=1}^{d}\mathscr{X}_{i(p)}$ and $k_1,\ldots,k_n > 0$ so that $k_1+\cdots+k_n = d$, we have $\kappa_{g;k_1,\ldots,k_n}(a_1,\ldots,a_d) = 0$ whenever there exists $p,q \in [d]$ and $i(p) \neq i(q)$.

Remark 4.20. Voiculescu's freeness is $(0,1)$-freeness, the second-order freeness of [MS06] is $(0,2)$-freeness, the all-order freeness of [CMSS07] is $(0,\infty)$-freeness. Due to Lemma 4.11 along with Remark 4.12 or looking at the definition of the convolution $*$ on $\mathbb{PS}$ (Definition 4.9), $(\frac{1}{2},1)$-freeness coincides with the notion of infinitesimal freeness of [FN09]. We note that it involves only the free cumulants of type $(g,n) = (0,1)$ and $(\frac{1}{2},1)$. More precisely, in terms of generating series we have from Corollary 4.15 the functional relations

$$G_{0,1}(X) = G^{\vee}_{0,1}(w)\,, \qquad G_{\frac{1}{2},1}(X) = \frac{\mathrm{d}\ln w}{\mathrm{d}\ln X}\,G^{\vee}_{\frac{1}{2},1}(w)\,, \qquad X = \frac{w}{G^{\vee}_{0,1}(w)}\,,$$

which are indeed the ones known for infinitesimal free cumulants. Besides, the order k infinitesimal freeness of [Fév12] corresponds to $(1,0)$-freeness using multiplicative functions valued in the ring R of upper triangular Töplitz matrices of size $(k+1)$ matrices (instead of $\mathbb{C}[\![\hbar]\!]/(\hbar^2)$ that corresponds to $k=1$).

An immediate consequence is that, if $a_1 \in \mathscr{X}_1$ and $a_2 \in \mathscr{X}_2$ but $(\mathscr{X}_1, \mathscr{X}_2)$ is (g_0, n_0)-free, the free cumulants of $a_1 + a_2$ are additive for orders $(g, n) \preceq (g_0, n_0)$, that is:

$$\kappa_{g;k_1,\ldots,k_n}(a_1 + a_2, \ldots, a_1 + a_2) = \kappa_{g;k_1,\ldots,k_n}(a_1, \ldots, a_1) + \kappa_{g;k_1,\ldots,k_n}(a_2, \ldots, a_2)\,.$$

A good notion of freeness for sets should pass to subalgebras. It was shown to be the case for $(0, n)$-freeness in [CMSS07, Section 7.3]. We now extend those arguments to higher genera, only stressing what is new compared to [CMSS07].

Definition 4.21. The type of a surfaced permutation $(\mathcal{A}, \alpha, g)$ is $(\mathsf{g}, \mathsf{n}) \in \mathsf{Typ}$ where n is the number of cycles of α and $\mathsf{g} = \sum_{A \in \mathcal{A}} g(A)$.

Lemma 4.22. *Let $(\mathscr{A}, \varphi)$ be a SPS and $\mathscr{X}_1, \mathscr{X}_2 \subset \mathscr{A}$, and $(g_0, n_0) \in \overline{\mathsf{Typ}}$. We denote $\mathscr{X}_i^+ = \mathscr{X}_i \cup \{1\}$. The following statements are equivalent:*

(i) *$\mathscr{X}_1$ and $\mathscr{X}_2$ are (g_0, n_0)-free.*
(ii) *$\mathscr{X}_1^+$ and $\mathscr{X}_2^+$ are (g_0, n_0)-free.*
(iii) *For any $d \in \mathbb{Z}_{\geq 0}$, any $(\mathcal{C}, \gamma, k) \in \mathbb{PS}(d)$ of type $(\mathsf{g}, \mathsf{n}) \preceq (g_0, n_0)$, any $a_1, \ldots, a_d \in \mathscr{X}_1^+$ and $b_1, \ldots, b_d \in \mathscr{X}_2^+$, we have*

$$\phi(\mathcal{C}, \gamma, k)[a_1 b_1, \ldots, a_d b_d] = \sum_{(\mathcal{A},\alpha,g) \odot (\mathcal{B},\beta,h) = (\mathcal{C},\gamma,k)} \phi^{\vee}(\mathcal{A}, \alpha, g)[a_1, \ldots, a_d]\, \phi(\mathcal{B}, \beta, h)[b_1, \ldots, b_d]\,.$$

(iv) *For any $d \in \mathbb{Z}_{\geq 0}$, any $(\mathcal{C}, \gamma, k) \in \mathbb{PS}(d)$ of type $(\mathsf{g}, \mathsf{n}) \preceq (g_0, n_0)$, any $a_1, \ldots, a_d \in \mathscr{X}_1^+$ and $b_1, \ldots, b_d \in \mathscr{X}_2^+$, we have*

$$\phi^{\vee}(\mathcal{C}, \gamma, k)[a_1 b_1, \ldots, a_d b_d] = \sum_{(\mathcal{A},\alpha,g) \odot (\mathcal{B},\beta,h) = (\mathcal{C},\gamma,k)} \phi^{\vee}(\mathcal{A}, \alpha, g)[a_1, \ldots, a_d]\, \phi^{\vee}(\mathcal{B}, \beta, h)[b_1, \ldots, b_d]\,.$$

Proposition 4.23. *Let $(\mathscr{X}_i)_{i \in I}$ be a family of subsets of $\mathscr{A}$, and $\mathscr{A}_i$ the subalgebra generated by $\mathscr{X}_i$. Let $(g_0, n_0) \in \overline{\mathsf{Typ}}$. The (g_0, n_0)-freeness of $(\mathscr{X}_i)_{i \in I}$ is equivalent to the (g_0, n_0)-freeness of $(\mathscr{A}_i)_{i \in I}$.*

Proof of Lemma 4.22. This is the higher genus generalisation of Theorem [CMSS07, Theorem 7.9]. Here we only explain (i) $\Rightarrow$ (iii). The converse direction is then proved as in [CMSS07]. The implication (iii) $\Rightarrow$ (iv) comes by extended convolution with the Möbius function and (iv) $\Rightarrow$ (iii) by extended convolution with the zeta function. The equivalence between (i) and (ii) comes from a direct adaptation of [CMSS07, Proposition 7.8].

Let $(\mathcal{C}, \gamma, k) \in \mathbb{PS}(d)$ of type (g, n). We take a second copy $[\bar{d}]$ of the set $[d]$ and interleave their elements

$$[d, \bar{d}] \coloneqq \{1,\ \bar{1},\ 2,\ \bar{2},\ 3,\ \bar{3}, \ldots,\ d,\ \bar{d}\} \cong [2d]\,.$$

We call $\psi \colon [d] \to [\bar{d}]$ the canonical identification. For a set $C = \{i_1, \ldots, i_\ell\} \subset [d]$, we denote by $\bar{C} = \{\bar{i_1}, \ldots, \bar{i_\ell}\} \subset [\bar{d}]$. We define the surfaced permutation $(\hat{\mathcal{C}}, \hat{\gamma}, \hat{k}) \in \mathbb{PS}(2d)$ such that the blocks of $\hat{\mathcal{C}}$ are of the form $\hat{C} \coloneqq C \cup \bar{C}$ where $C \in \hat{\mathcal{C}}$, the permutation $\hat{\gamma}$ is characterised by $\hat{\gamma}|_{[d]} = \psi \circ \gamma$ and $\hat{\gamma}_{|[\bar{d}]} = \gamma \circ \psi^{-1}$ and the genus function is $\hat{k}(C \cup \bar{C}) = k(C)$. We have

$$\begin{aligned}\phi(\mathcal{C}, \gamma, k)[a_1 b_1, \ldots, a_d b_d] &= \phi(\hat{\mathcal{C}}, \hat{\gamma}, \hat{k})[a_1, b_1, \ldots, a_d, b_d] \\ &= \sum_{(\mathbf{0}_\alpha,\alpha,0) \odot (\mathcal{B},\beta,h) = (\hat{\mathcal{C}},\hat{\gamma},\hat{k})} \phi^{\vee}(\mathcal{B}, \beta, h)[a_1, b_1, \ldots, a_d, b_d]\,.\end{aligned}$$

Assume that $(\mathscr{X}_1, \mathscr{X}_2)$ is (g_0, n_0)-free. The vanishing of the mixed surfaced free cumulants up to order (g_0, n_0) means that the only terms remaining in the right-hand side of (iii) come surfaced permutations $(\mathcal{B}, \beta, h)$ where blocks $B \in \mathcal{B}$ are included either in $[d]$ or in $[\bar{d}]$. Since $\mathbf{0}_\beta \leq \mathcal{B}$, the permutation β leaves $[d]$ and $[\bar{d}]$ stable and we introduce:

$$\beta_1 = \beta_{|[d]}\,, \qquad \beta_2 = \psi^{-1} \circ \beta_{|[\bar{d}]} \circ \psi\,, \qquad \text{both in } S(d)\,.$$

By multiplicativity of $\phi^\vee$, we obtain:

$$\begin{aligned}
&\phi(\mathcal{C},\gamma,k)[a_1b_1,\dots,a_db_d]\\
&= \phi(\hat{\mathcal{C}},\hat{\gamma},\hat{k})[a_1,b_1,\dots,a_d,b_d]\\
&= \sum_{(\mathbf{0}_\alpha,\alpha,0)\odot(\mathcal{B},\beta,h)=(\hat{\mathcal{C}},\hat{\gamma},\hat{k})} \phi^\vee(\mathcal{B}_1,\beta_1,h_1)[a_1,\dots,a_d]\,\phi^\vee(\mathcal{B}_2,\beta_2,h_2)[b_1,\dots,b_d]\,.
\end{aligned}\tag{52}$$

The condition $\alpha\circ\beta=\hat{\gamma}$ implies that α sends $[d]$ to $[\bar{d}]$ and vice versa. Introducing

$$\alpha_1=\psi^{-1}\circ\alpha_{|[d]}\,,\qquad \alpha_2=\alpha_{|[\bar{d}]}\circ\psi\,,\qquad \text{both in } S(d)\,.$$

we get $\alpha_i\circ\beta_i=\gamma$ for $i=1,2$. Introducing $\tilde{\alpha}=\alpha_2\circ\beta_1^{-1}$, we therefore have $\tilde{\alpha}\circ\beta_1\circ\beta_2=\gamma$. We also observe that for any $C\in\mathcal{C}$

$$|\tilde{\alpha}_{|C}|=|\gamma\circ\beta_2^{-1}\circ\beta_1^{-1}{}_{|C}|=|\gamma\circ\beta_{|C}|=|\alpha|\,.\tag{53}$$

We denote $(\mathcal{B}_1,\beta_1,h_1),(\mathcal{B}_2,\beta_2,h_2)\in\mathbb{PS}(d)$ which are obtained by restricting $(\mathcal{B},\beta,h)$ to $[d]$ and $[\bar{d}]$, using again the canonical identification $[d]\cong[\bar{d}]$. We claim that

$$(\mathcal{A},\mathbf{0}_\alpha,0)\odot(\mathcal{B},\beta,h)=(\hat{\mathcal{C}},\gamma,\hat{k})\tag{54}$$

implies

$$(\mathbf{0}_{\tilde{\alpha}},\tilde{\alpha},0)\odot(\mathcal{B}_1,\beta_1,h_1)\odot(\mathcal{B}_2,\beta_2,h_2)=(\mathcal{C},\gamma,k)\,.\tag{55}$$

This amounts to check that $\mathbf{0}_{\tilde{\alpha}}\vee\mathcal{B}_1\vee\mathcal{B}_2=\mathcal{C}$ and match the genus functions. The former is already in [CMSS07] and we focus on the latter which is new to our setting. By Remark 4.6, we have to check block-additivity of colengths. Given a block $C\in\mathcal{C}$, we have using (53)

$$\begin{aligned}
&|(\mathbf{0}_{\tilde{\alpha}|C},\tilde{\alpha}_{|C})|+|(\mathcal{B}_{1|C},\beta_{1|C},h_{1|C})|+|(\mathcal{B}_{2|C},\beta_{2|C},h_{2|C})|\\
&=|\tilde{\alpha}_{|C}|+|(\mathcal{B}_{|C},\beta_{|C},h_{|C})|+|(\mathcal{B}_{|\bar{C}},\beta_{|\bar{C}},h_{|\bar{C}})|\\
&=|\alpha_{|\hat{C}}|+|(\mathcal{B}_{|\hat{C}},\beta_{|\hat{C}},h_{|\hat{C}})|\\
&=2\hat{k}(\hat{C})=2k(C)\,,
\end{aligned}$$

where we used (6) in the second line, (53) in the third line, and the definition (41) of the genus function of the genus function for $\odot$ in the last line. This justifies the claim, and a closer look at this argument shows that (54) and (55) are in fact equivalent.

Observing that the first factor in (55) is exactly the kind of surfaced permutations in the support of the zeta function, this allows us transforming (52) into

$$\begin{aligned}
&\phi(\mathcal{C},\gamma,k)[a_1b_1,\dots,a_db_d]\\
&=\sum_{(\tilde{\mathcal{A}},\tilde{\alpha},\tilde{g})\odot(\mathcal{B}_1,\beta_1,h_1)\odot(\mathcal{B}_2,\beta_2,h_2)=(\mathcal{C},\gamma,k)} \zeta(\tilde{\mathcal{A}},\tilde{\alpha},\tilde{g})\,\phi^\vee(\mathcal{B}_1,\beta_1,h_1)[a_1,\dots,a_d]\,\phi^\vee(\mathcal{B}_2,\beta_2,h_2)[b_1,\dots,b_d]\\
&=\sum_{(\tilde{\mathcal{B}}_1,\tilde{\beta}_1,\tilde{h}_1)\odot(\mathcal{B}_2,\beta_2,h_2)=(\mathcal{C},\gamma,k)} \phi(\tilde{\mathcal{B}}_1,\tilde{\beta}_1,\tilde{h}_1)[a_1,\dots,a_d]\,\phi^\vee(\mathcal{B}_2,\beta_2,h_2)[b_1,\dots,b_d]\,,
\end{aligned}$$

where we have recognised convolution by the zeta function to get the second line. □

Proof of Corollary 4.23. By multilinearity, the linear spans of two free sets is free. The only thing that deserves a check is that freeness of $\mathscr{X}_1$ and $\mathscr{X}_2$ implies freeness of $\mathscr{X}_1$ and $\mathscr{X}_1\mathscr{X}_2$. Given Proposition 4.22, the proof is identical to [CMSS07, Theorem 7.12]. □

4.6 *Application in random matrix theory*

The formalism of surfaced free cumulants and freeness up to order (g_0,n_0) can be directly applied in random matrix theory, generalising the known cases of order $(0,1)$ [Voi91], (0,2) [MŚS07].

Definition 4.24. If Ω is a matrix of size N and $\lambda \vdash d$ of length n, with $d \leq N$, we denote

$$p_\lambda(\Omega) = \prod_{i=1}^{n} \mathrm{Tr}(\Omega^{\lambda_i}), \qquad \mathcal{P}_\lambda(\Omega) = \prod_{c=1}^{d} \Omega_{c,\pi_\lambda(c)}. \tag{56}$$

We recall the following result, which comes from Weingarten calculus.

Theorem 4.25. *([CMSS07, Theorem 4.4], [BGF20, Theorem 8.8]) Let Ω be a random hermitian matrix of size N, whose law is invariant under unitary conjugation. Then for any $\lambda \vdash d$:*

$$\mathbb{E}[p_\lambda(\Omega)] = z(\lambda) \sum_{\nu \vdash d} N^d\, H^{<}(\lambda,\nu)\big|_{\hbar=1/N}\, \mathbb{E}[\mathcal{P}_\nu(\Omega)],$$
$$\mathbb{E}[\mathcal{P}_\lambda(\Omega)] = z(\lambda) \sum_{\nu \vdash d} N^{-d} H^{\leq}(\lambda,\nu)\big|_{\hbar=1/N}\, \mathbb{E}[p_\lambda(\Omega)].$$

Definition 4.26. Let $(A_N)_N$ be a sequence of random hermitian matrices of size N. We say that it admits a limit distribution up to order (g_0, n_0) if there exists $F_{g;k_1,\ldots,k_n}$ indexed by $(g,n) \preceq (g_0,n_0)$ and $k_1,\ldots,k_n > 0$, independent of N, such that for any $n \in [\lfloor g_0 \rfloor + n_0]$ and any $k_1,\ldots,k_n > 0$, we have when $N \to \infty$

$$\mathbb{E}^\circ\big[\mathrm{Tr}(A_N^{k_1}),\ldots,\mathrm{Tr}(A_N^{k_n})\big] = \sum_{\substack{g \in \frac{1}{2}\mathbb{Z}_{\geq 0} \\ g \leq g_0 + \min(0,n_0-n)}} N^{2-2g-n}\, F_{g;k_1,\ldots,k_n} + o(N^{2-2g_0-n_0+|n_0-n|}),$$

where $\mathbb{E}^\circ$ denote the cumulant expectation value (with the meaning already explained in Section 3.1). In this expression, the order of the $o(\cdots)$ is adjusted to be the next subleading term compared to the sum. When $g_0 = \infty$, we ask for the existence of such an asymptotic expansion to an arbitrary order $o(N^{-K})$ for all $n \leq n_0$.

Remark 4.27. Several important classes of ensembles of random hermitian matrices do exhibit limit distributions to all orders [APS01; BG13; BGK15], and it is typical that only integer g appear.

From Theorem 4.25 it can be observed that $(\Omega_N)_N$ has a limit distribution up to order (g_0, n_0), then for any partition $\lambda \vdash d$ of length $n \leq \lfloor g_0 \rfloor + n_0$ we have when $N \to \infty$:

$$\mathbb{E}[\mathcal{P}_\lambda(A_N)] = \sum_{\substack{g \in \frac{1}{2}\mathbb{Z}_{\geq 0} \\ g \leq g_0 + \min(0,n_0-n)}} N^{2-2g-n-d}\, \kappa_{g;\lambda_1,\ldots,\lambda_n} + o(N^{2-2g_0-n_0+|n_0-n|-d}). \tag{57}$$

We obtain the structure of a SPS on the algebra $\mathscr{A} = \mathbb{C}[a]$ by taking as moments:

$$\forall (g,n) \preceq (g_0,n_0), \qquad \varphi_{g,n}(a^{k_1},\ldots,a^{k_n}) = F_{g;k_1,\ldots,k_n}.$$

Combining Theorem 4.25 and the expansion (57) with Theorem 2.13 indicates that $\kappa_{g;k_1,\ldots,k_n}$ are the free cumulants at order $(g,n) \preceq (g_0,n_0)$.

Theorem 4.28. *Let $(A_N)_N$ and $(B_N)_N$ be two sequences of ensembles of random matrices of size N, at least one of them being unitarily invariant, and such that for each N, A_N is independent from B_N. Assume that both ensembles have a limit distribution up to order (g_0,n_0) (possibly ∞), and consider the algebra $\mathscr{A} = \mathbb{C}\langle a,b\rangle$ of non-commutative polynomials in two letters. Then, for any $Q \in \mathscr{A}$, $Q(A_N,B_N)$ admits a limit distribution up to order (g_0,n_0), so $\mathscr{A}$ can be upgraded to a SPS. Besides, the subalgebras $\mathbb{C}[a]$ and $\mathbb{C}[b]$ are (g_0,n_0)-free.*

Proof. Let $(A_N)_N$ and $(B_N)_N$ as in the theorem. For any $k,k' > 0$, $(A_N^k)_N$ and $(B_N^{k'})$ clearly have a limit distribution up to order (g_0,n_0). Examining the finite N formula in [CMSS07, Theorem 4.4, (2)], the products $(A_N^k B_N^{k'})_N$ also have a limit distribution up to order (g_0,n_0). Take $\sigma \in S(d)$, $N \geq d$ and a map $\Omega\colon [d] \to \{A_N, B_N\}$. Due to independence of A_N and B_N, and unitary invariance of the law of one of the matrices (say A_N), we have

$$
\begin{aligned}
&\mathbb{E}\Bigg[\prod_{c=1}^{d}(\Omega(c))_{c,\sigma(c)}\Bigg] = \mathbb{E}\Bigg[\prod_{c\in\Omega^{-1}(A_N)}(A_N)_{c,\sigma(c)}\Bigg]\,\mathbb{E}\Bigg[\prod_{c\in\Omega^{-1}(B_N)}(B_N)_{c,\sigma(c)}\Bigg] \\
&= \int_{U(N)}\mathrm{d}U\,\mathbb{E}\Bigg[\prod_{c\in\Omega^{-1}(A_N)}(UA_NU^{-1})_{c,\sigma(c)}\Bigg]\,\mathbb{E}\Bigg[\prod_{c\in\Omega^{-1}(B_N)}(B_N)_{c,\sigma(c)}\Bigg] \\
&= \sum_{\substack{i_c,j_c\in[N]\\ c\in\Omega^{-1}(A_N)}}\Bigg(\int_{U(N)}\mathrm{d}U\prod_{c\in\Omega^{-1}(A_N)}U_{c,i_c}U^{-1}_{j_c,\sigma(c)}\Bigg)\mathbb{E}\Bigg[\prod_{c\in\Omega^{-1}(A)}(A_N)_{i_c,j_c}\Bigg]\,\mathbb{E}\Bigg[\prod_{c\in\Omega^{-1}(B)}(B_N)_{i_c,j_c}\Bigg].
\end{aligned}
\tag{58}
$$

By Weingarten calculus [Col03] the integral over $U(N)$ vanishes unless there exists two permutations $\alpha,\beta\in S(\Omega^{-1}(A_N))$ such that $c=j_{\beta(c)}$ and $i_c=\sigma(\alpha(c))$ for all $c\in\Omega^{-1}(A_N)$. This cannot happen when Ω takes at least once the values A_N and B_N, as we can find $c_0\in[d]$ such that $\Omega(c_0)=A_N$, and $\Omega(\sigma(c_0))=B_N$ or $\Omega(\sigma^{-1}(c_0))=B_N$. Since the surfaced free cumulants evaluated on $(\Omega(c))_{c\in[d]}$ are extracted from the asymptotic expansion of (58) when $N\to\infty$ (recall (57)), all mixed surfaced free cumulants between $\mathbb{C}[a]$ and $\mathbb{C}[b]$ vanish up to order (g_0,n_0) (which from the assumption is the order up to which the asymptotic expansion exist), *i.e.* these two algebras are (g_0,n_0)-free in the surfaced probability space $\mathbb{C}\langle a,b\rangle$. ☐

Remark 4.29. The combinatorics underlying infinitesimal free cumulants is the truncation keeping the leading (genus 0) and the first subleading (genus $\frac{1}{2}$) of the master relation involving monotone Hurwitz numbers, whose apparition can be traced back to Weingarten calculus for the unitary group. Typically, for topological expansions in unitarily invariant random hermitian matrices the genus $\frac{1}{2}$ order (corresponding to a term of order N^{-1} compared to the leading term) vanish. An example of situation where it does not vanish is the 1-hermitian matrix model with an N-dependent potential of the form $V_0+N^{-1}V_1$. Although the non-vanishing of the genus $\frac{1}{2}$ order is typical in the topological expansion for orthogonally invariant random symmetric matrices, the observation of Mingo [Min19] that infinitesimal freeness cannot describe the relations between moments and cumulants in such models is not a surprise, as they should be rather governed by the Weingarten calculus for the orthogonal group.

A Gaussian case

We consider the SPS on $\mathscr{A}=\mathbb{C}[a]$ given by the topological expansion of a GUE matrix. In this case $\varphi_{g,n}(a^{k_1},\ldots,a^{k_n})$ is the weighted count of the number of maps of genus g, with n rooted boundary faces labeled from 1 to n and having respective degree $k_1,\ldots,k_n$, and no internal face. Due to [BGF20], $\kappa_{g;k_1,\ldots,k_n}(a,\ldots,a)$ is the number of fully simple maps without internal face. The fully simple condition forbids any self-glueing among boundary faces, except in the case $(g,n)=(0,1)$ and $k_1=2$. Thus, for any $g\geq 0$ and $n\geq 1$:

$$G^{\vee}_{g,n}(w_1,\ldots,w_n)=\delta_{g,0}\delta_{n,1}(1+w_1^2)\,,$$

and Theorem 3.4 then gives a formula for the GUE moments. There are only contributions from the graphs $\Gamma\in\mathcal{G}_n$ in which black vertices have valency 2 and cannot be connected to the same white vertex. Erasing the black vertices, this amounts to consider the set Gr_n of graphs having n vertices labeled from 1 to n without loops. We observe $\varsigma(\hbar u w\partial_w)w^k=\varsigma(k\hbar u)w^k$ for $k>0$. The weight of an edge between i-th and j-th vertex becomes

$$\mathsf{c}^{\vee}(u_i,w_i,u_j,w_j)=\ln\left(\frac{(e^{\hbar u_i/2}w_i-e^{\hbar u_j/2}w_j)(e^{-\hbar u_i/2}w_i-e^{-\hbar u_j/2}w_j)}{(e^{\hbar u_i/2}w_i-e^{-\hbar u_j/2}w_j)(e^{-\hbar u_i/2}w_i-e^{\hbar u_j/2}w_j)}\right),$$

while the operator weight on an edge is

$$\vec{\mathsf{O}}^\vee(w) = \sum_{m\geq 0}(P^\vee(w)w\partial_w)^m P^\vee(w)\cdot[v^m] \sum_{r\geq 0}\left(\partial_y + \frac{v}{y}\right)^r \exp\left(v\,\frac{\varsigma(\hbar v\partial_y)}{\varsigma(\hbar\partial_y)}\ln y - v\ln y\right)\Big|_{y=1+w^2}$$
$$\cdot\,[u^r]\,\frac{\exp\left((\varsigma(2\hbar u)-1)w^2\right)}{\hbar u\varsigma(\hbar u)}\,.$$

The formula is then

$$G_{g,n}(X_1,\ldots,X_n) = [\hbar^{2g-2+n}]\sum_{\Gamma\in\mathrm{Gr}_n}\frac{1}{\#\mathrm{Aut}(\Gamma)}\prod_{i=1}^n \vec{\mathsf{O}}^\vee(w_i)\prod_{\{i,j\}=\mathrm{edge}}\mathsf{c}^\vee(u_i,w_i,u_j,w_j)\,,$$

with the substitutions:

$$X_i = \frac{w_i}{1+w_i^2}\,,\qquad P^\vee(w_i) = \frac{1+w_i^2}{2w_i^2} = \frac{X_i}{2w_i}\,.$$

For $n=1$, as loops are forbidden, the vertex with no edge is the only graph:

$$G_{g,1}(X_1) = [\hbar^{2g-1}]\,\vec{\mathsf{O}}^\vee(w_1)\,.$$

The resulting formula is essentially the Harer–Zagier formula, as shown in [Lew19] where the proof is a special case of the general method we have used in Section 3.

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