DIFFERENTIAL-ALGEBRAIC EQUATIONS FROM A FUNCTIONAL-ANALYTIC VIEWPOINT: A SURVEY

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Abstract

The purpose of this paper is to provide an overview on the state of the art concerning functional-analytic properties associated with differential-algebraic equations (DAEs). We summarize the relevant literature and develop a basic theory of linear and nonlinear differential-algebraic operators. In particular, we consider Fredholm properties, normal solvability, generalized inverses, least squares solutions, splittings of regular linear differential-algebraic operators, bounded outer inverses, local solvability of equations with regular nonlinear differential-algebraic operators, Newton-Kantorovich iterations and regularizations of the ill-posed problems arising from higher-index operators.

Contents

1 Introduction

Functional analysis is a child of the twentieth century. It provides us with a new language that allows us to formulate apparently different topics in a unique way [75, p. ix]. The topic of differential-algebraic equations (DAEs) is another, much younger child of the same century. To a large extend, DAEs are merely recognized as special ordinary differential equations (ODEs), which dominates the hitherto existing analysis of DAEs. In contrast, the functional-analysis of DAEs remains still in its initial stage so far; even though there had been various quite early approaches in that score such as

- applying operator settings and aspects of functional-analytic discretization theory, e.g.,[51, 52, 18],
- treating higher-index DAEs consistently as ill-posed operator equations, e.g.,[40, 41, 35, 33],
- the work concerning abstract differential equations which are not solved for

the derivative in the context of optimization problems, e.g.,[48],
• diverse attempts in system and control theory to incorporate various types of solutions by appropriate function spaces, e.g.,[65, 70],
• particular results on degenerate differential equations in function spaces, e.g., [24], and on DAEs within the context of partial differential-algebraic equations (PDAEs) and abstract DAEs (cf.[50, Chapter 12]).

Nonetheless, as yet, an adequate sophisticated functional-analytic characterization of DAEs has not been accomplished.
The purpose of the present paper is to provide an overview on basic functional-analytic properties of linear and nonlinear differential-algebraic operators and equations, and, furthermore, to summarize the relevant literature to the best of the authors knowledge.

We investigate operators associated with linear and nonlinear DAEs in standard form

$$E(t)x'(t) + F(t)x(t) = q(t), \tag{1}$$

$$\mathfrak{f}(x'(t), x(t), t) = 0, \tag{2}$$

and linear and nonlinear DAEs showing a properly involded derivative

$$A(t)(Dx)'(t) + B(t)x(t) = q(t), \tag{3}$$

$$f((Dx)'(t), x(t), t) = 0. \tag{4}$$

Such a DAE comprises m unknown functions and k equations. When considering regular DAEs we suppose $k = m$.

We represent the DAEs as operator equations – we call the relevant operators *differential-algebraic operators* – and apply functional analytic tools for their characterization and further treatment.
In essence, here we focus on compact intervals $\mathcal{I} = [t_a, t_e]$, which allows us to apply Banach spaces equipped with maximum-norms. Modifications for open intervals would require Fréchet spaces and more technicalities.

At the beginning of the twentieth century, J.S.Hadamard formalized the classical concept of well-posedness for abstract equations

$$\mathcal{K}x = p, \tag{5}$$

where $\mathcal{K}$ is a mapping from some topological space X into a topological space Z. Equation (5) is said to be *well-posed*, e.g. [31, p. 3], if

(a) for each $p \in Z$, there is a solution $x \in X$ of (5);

(b) this solution x is unique in X;

(c) the dependence of x upon p is continuous.

An equation (5) which is not well-posed in this sense is said to be *ill-posed*. A method for solving approximately an ill-posed problem is called a *regularization method*.

Obviously, to answer the question whether a given equation is well-posed, one has to consider not only the operator $\mathcal{K}$ but also the spaces X and Z including

their topologies.
Here we prefer topological spaces whose topology is defined by a norm. This allows to measure global physical quantities, responsibilities, sensitivities.
In a well-posed problem, the operator $\mathcal{K}$ must be bijective and the inverse $\mathcal{K}^{-1}$ must be continuous. Often the so-called *well-posedness principle*, see [76, p. 180], is helpful: If $\mathcal{K}$ is a bounded linear operator mapping a Banach X into a Banach space Z, then the equation (5) is well-posed, exactly if $\mathcal{K}$ is bijective.

In our context, the operator $\mathcal{K}$ is composed from an operator K acting in normed spaces X and Y such that the equation $Kx = q$ represents a DAE and the equation (5), with $Z = Y \times \mathbb{R}^{\delta}$, contains the DAE and an additional finite-dimensional part representing boundary and initial conditions.
Having a linear operator K, for eventually obtaining a well-posed problem (5) we are specifically interested in a finite-dimensional nullspace $\ker K$, in boundedness or closedness of K and last but not least a closed range $\operatorname{im} K$. The closed range property plays its role in the theory of Fredholm operators; it is the crucial ingredient of *normal solvability.* The related operator equations can be ill-posed indeed, but they become well-posed in slightly modified settings.
If $\mathcal{K}$ is a bounded linear operator acting in Banach spaces X and Z, and if $\operatorname{im} \mathcal{K}$ fails to be closed in Z, then equation (5) is no longer well-posed, since items (2) and (3) do not hold. Those kind of problems are said to be *essentially ill-posed in the sense of A.N.Tikhonov.*

We emphasize that, in standard settings, equations given by regular higher-index differential-algebraic operators are essentially ill-posed in this sense, and their solutions basically show an ambivalent character; they behave smoothly with regard to appropriately stated initial values, but, as a function of perturbations on the right-hand side, they are no longer continuous. Merely index-1 and index-0 operators yield well-posed problems in standard settings, and index-2 operators in a reasonable and transparent enhanced setting. Index-0 and index-1 problems can be solved numerically nearly as safe as explicit ODEs. In contrast, concerning the direct treatment of general equations with higher index differential-algebraic operators as they are given, there remains a big gap between the practical needs and the theory available at this stage. Great future efforts are necessary to close this gap, also on the part of developing appropriate functional-analytic tools.

The material of this paper is organized in 4 sections who are almost independent of each other.
Serving as easy introduction into the topic, Section 2 deals with operators associated with regular matrix pencils and provides the constitutive characteristics of regular differential-algebraic operators in different settings.
Section 3 characterizes normally solvable differential-algebraic operators associated with linear DAEs (1) comprising m unknowns and k equations. Bounded inner inverse and least-squares-solutions are then constructed. Also nonlinear operators are considered. No special knowledge about DAEs is required in this section.
In Section 4 we consider basic properties of regular linear and Fréchet differentiable nonlinear differential-algebraic operators in their natural Banach space settings, in particular an operator splitting, Fredholm properties, solvability, bounded outer inverses of higher index operators, local solvability, Newton-Kantorovich iterations. This section relies on the projector based analysis ([50]).

It is recommended to take a look to the respective summary in the appendix.
Both Sections 3 and 4 are contain an extra subsection with notes and references to relate the existing literature to the material.
Section 5 collects known regularization methods from the viewpoint of functional-analysis.
The notation agreements, the functional-analytic background, and a short summary of the projector based DAE analysis can be found in the Appendices.

2 Constitutive characteristics of regular differential-algebraic operators – exemplified by means of regular matrix pencils

We consider the linear operator $\mathring{T} : X \to Y$,

$$\mathring{T}x := Ex' + Fx,\ x \in \operatorname{dom} \mathring{T} \subseteq X, \tag{6}$$

associated with the constant coefficient DAE

$$Ex'(t) + Fx(t) = q(t), \quad t \in \mathcal{I}, \tag{7}$$

and formed by the ordered pair $\{E, F\}$ of real $m \times m$ matrices E, F. $\mathcal{I} \subseteq \mathbb{R}$ is an interval. Diverse appropriate function spaces X and Y will be specified below, in particular we apply $\mathcal{C}(\mathcal{I}, \mathbb{R}^m)$ and $L^2(\mathcal{I}, \mathbb{R}^m)$.

2.1 Finite-dimensional nullspaces

The solutions of the homogeneous equation

$$Ex'(t) + Fx(t) = 0,\ \ t \in \mathcal{I}. \tag{8}$$

constitute the nullspace $\ker \mathring{T}$ of the operator $\mathring{T}$.
If E is nonsingular, the homogeneous equation (8) represents a regular implicit ODE, whose fundamental solution system forms an m-dimensional subspace in $\mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m)$; and then the associated operator $\mathring{T}$ has an m-dimensional nullspace — supposed the setting ensures the inclusion $\mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m) \subseteq \operatorname{dom} \mathring{T}$. However, what happens if E is singular? Is there a class of equations, such that equation (8) has a finite-dimensional solution space? The answer is closely related to regularity of the matrix pair $\{E, F\}$.

Definition 2.1 *Given the ordered pair $\{E, F\}$ of matrices $E, F \in \mathcal{L}(\mathbb{R}^m)$, the matrix pencil $\lambda E + F$ is said to be regular if the polynomial $\rho(\lambda) := \det(\lambda E + F)$ does not vanish identically. Otherwise the matrix pencil is said to be singular. Both the ordered pair $\{E, F\}$ and the DAE (7) are said to be regular if the accompanying matrix pencil is regular, and otherwise nonregular.*

The operator $\mathring{T}$ given by (6) is called *regular differential-algebraic operator* if the pair $\{E, F\}$ is regular.

If E is a nonsingular matrix, then the pair $\{E, F\}$ is always regular and the polynomial $p(\lambda)$ has degree m. If the first matrix E of a regular pair is singular,

then the polynomial degree is lower. To be more precise we quote some well-known facts due to Weierstraß and Kronecker, cf. [72, 45, 27].
For any regular pair $\{E, F\}$, $E, F \in \mathcal{L}(\mathbb{R}^m)$, there exist nonsingular matrices $L, K \in \mathcal{L}(\mathbb{R}^m)$ and integers $0 \leq l \leq m$, $0 \leq \mu \leq l$, such that

$$LEK = \begin{bmatrix} I & \\ & N \end{bmatrix} \begin{matrix} \}m-l \\ \}l \end{matrix} \quad , \quad LFK = \begin{bmatrix} W & \\ & I \end{bmatrix} \begin{matrix} \}m-l \\ \}l \end{matrix} \quad . \tag{9}$$

Thereby, N is absent if $l = 0$, and otherwise N is nilpotent of order μ, i.e., $N^\mu = 0$, $N^{\mu-1} \neq 0$. For $l = 0$ we set $\mu = 0$. The integers l and μ as well as the eigenstructure of the blocks N and W are uniquely determined by the pair $\{E, F\}$. If $l = m$ then the upper blocks are absent.
The real matrix N has the eigenvalue zero only and can be transformed into its Jordan form by means of a real similarity transformation. Therefore, the transformation matrices L and K can be chosen in such a way that N is in Jordan form.
Now it is evident that the relation

$$\operatorname{degree} \det(\lambda E + F) = m - l \leq \operatorname{rank} E \tag{10}$$

is given for each regular pencil.
The special pair given by (9) is said to be *Weierstraß–Kronecker form* of the original pair $\{E, F\}$.

Definition 2.2 *The Kronecker index of a regular matrix pair* $\{E, F\}$*,* $E, F \in \mathcal{L}(\mathbb{R}^m)$*,* E *singular, is defined to be the nilpotency order* μ *in the Weierstraß–Kronecker form (9). One writes* $\operatorname{ind}\{E, F\} = \mu$*.*
If E *is nonsingular, one states* $\operatorname{ind}\{E, F\} = \mu := 0$*.*
The Kronecker index of a regular DAE (7) is given by $\operatorname{ind}\{E, F\} = \mu$*.*

The Weierstraß–Kronecker form of a regular pair $\{E, F\}$ provides a broad insight into the structure of the associated DAE (7). Scaling of (7) by L, transforming $x = K \begin{bmatrix} y \\ z \end{bmatrix}$, and letting $Lq =: \begin{bmatrix} p \\ r \end{bmatrix}$ leads to the equivalent decoupled system

$$y'(t) + Wy(t) = p(t), \quad t \in \mathcal{I}, \tag{11}$$
$$Nz'(t) + z(t) = r(t), \quad t \in \mathcal{I}. \tag{12}$$

The first equation (11) represents an explicit ODE. The second one appears for $l > 0$, and it has the only solution

$$z(t) = \sum_{j=0}^{\mu-1} (-1)^j N^j r^{(j)}(t), \tag{13}$$

provided that r is smooth enough. The latter one becomes clear after recursive use of (12). In case of the homogeneous DAE (8), the functions $p(\cdot)$ and $r(\cdot)$ vanish identically, and so does the solution component $z(\cdot)$. The solutions of the explicit ODE (11), with $p(\cdot) = 0$, form an $(m - l)$-dimensional subspace in $\mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m)$.

We summarize what we already know about our operator $\mathring{T}$.

Proposition 2.3 *If the differential-algebraic operator $\mathring{T}$ (6) is given by a regular matrix pair $\{E, F\}$ and if the setting ensures the inclusion $\mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m) \subseteq \operatorname{dom} \mathring{T}$, then it holds that*

$$\dim \ker \mathring{T} = m - l, \quad \ker \mathring{T} \subseteq \mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m). \tag{14}$$

By means of an appropriate initial condition at a point $t_a \in \mathcal{I}$,

$$Cx(t_a) = d \in \mathbb{R}^{m-l}, \tag{15}$$

with $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m-l})$, $\operatorname{rank} C = m - l$, $\ker C = \ker([I\ 0]K^{-1})$, we obtain the composed operator $\mathring{\mathcal{T}} : X \to Y \times \mathbb{R}^{m-l}$,

$$\mathring{\mathcal{T}}x := (\mathring{T}x, Cx(t_a)),\ x \in \operatorname{dom} \mathring{T}, \tag{16}$$

which is then injective. We emphasize that, in contrast to regular ODEs, for describing an appropriate initial condition (15), the special structure of the DAE has to be attentively considered. The operator equation $\mathring{\mathcal{T}}x = (q, d)$ reflects the initial value problem (IVP) (7),(15).

As distinguished from the situation in Proposition 2.3, if the pair $\{E, F\}$ is nonregular, then $\ker \mathring{T}$ has no longer finite dimension, e.g. [50, Theorem 1.6]. We substantiate this fact by the following simple instance.

Example 2.4 ($\ker \mathring{T}$ fails to be finite-dimensional) *The matrix pair $\{E, F\}$ associated with the operator*

$$\mathring{T}x := \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E} x' + \underbrace{\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}}_{F} x, \quad x \in \operatorname{dom} \mathring{T},$$

is obviously singular. The nullspace $\ker \mathring{T} = \{x \in \operatorname{dom} \mathring{T} : x_1' = x_2\}$ *depends on the choice of X; however, with $\mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^2) \subseteq \operatorname{dom} \mathring{T} \subseteq X$, the inclusion*

$$\{ \begin{bmatrix} z \\ z' \end{bmatrix} : z \in \mathcal{C}^\infty(\mathcal{I}, \mathbb{R})\} \subseteq \ker \mathring{T}$$

attests that $\ker \mathring{T}$ *has no longer finite dimension.* □

2.2 Ill-posed behavior in higher-index cases

In the more interesting case if $\mu \geq 1$, the solution expression (13) elucidates the dependence of the solution x on derivatives of the right hand side q when indicated. The higher the index μ, the more differentiations are involved. Solely in the index-1 case do we have $N = 0$, hence $z(\cdot) = r(\cdot)$, and no derivatives are involved.
Since numerical differentiations in these circumstances may cause considerable trouble, it is very important to know the index μ as well as details of the structure responsible for the higher index when modeling and simulating in practice with DAEs. Not surprisingly, the typical solution behavior of so-called *ill-posed* problems can be observed in higher index DAEs: small perturbations of the right-hand side may cause enormous and somewhat discontinuous changes in the solution. We demonstrate this by the next example (cf.[50, Section 1.1]).

Example 2.5 (Ill-posed behavior) *The IVP for the regular DAE*

$$\underbrace{\begin{bmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}}_{E}x'(t)+\underbrace{\begin{bmatrix}-\alpha&-1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}}_{F}x(t)=q(t),\quad t\in\mathcal{I}=[0,1],$$

completed by the initial condition

$$Cx(0)=\begin{bmatrix}1&0&0&0&0\end{bmatrix}x(0)=d, \tag{17}$$

is uniquely solvable for each sufficiently smooth function q and each arbitrary $d\in\mathbb{R}$. The homogeneous IVP, with $q=0$ and $d=0$, has the identically zero solution only. The particular solution corresponding to the initial value $d\in\mathbb{R}$ and the excitation

$$q_k(t)=\begin{bmatrix}0\\0\\0\\0\\\gamma_k(t)\end{bmatrix},\quad \gamma_k(t)=\varepsilon\frac{1}{k}\sin kt,\quad k\in\mathbb{N},\ \varepsilon\neq 0\ \ \textit{small}, \tag{18}$$

reads as follows:

$$x_{k,1}(t)=\mathrm{e}^{\alpha t}d+\varepsilon\int_a^t k^2\mathrm{e}^{\alpha(t-s)}\cos ks\,\mathrm{d}s,\quad x_{k,2}(t)=\varepsilon k^2\cos kt,$$
$$x_{k,3}(t)=-\varepsilon k\sin kt,\quad x_{k,4}(t)=-\varepsilon\cos kt,\quad x_{k,5}(t)=\varepsilon\frac{1}{k}\sin kt.$$

No doubt, the solution depends smoothly on d, but what about the dependence on the excitation q_k? Put $d=0$. While the excitation q_k uniformly tends to zero for $k\to\infty$, the first three solution components grow unbounded, instead of also tending to zero. This is the typical ill-posed behavior. The solution value at $t=0$, that is,

$$x_{k,1}(0)=0,\ x_{k,2}(0)=\varepsilon n^2,\ x_{k,3}(0)=0,\ x_{k,4}(0)=-\varepsilon,\ x_{k,5}(0)=0,$$

moves away from the origin with increasing k. For the perturbed system, the origin is no longer a consistent value at $t=0$, as it is the case for the unperturbed system. Figures 1 and 2 show the excitation γ_k and the response $x_{k,2}$ for $\varepsilon=0.1$, $k=1$ and $k=100$.

Here the matrix pencil $\{E,F\}$ is regular with Kronecker index $\mu=4$ and $l=4$(cf.(9)).The associated operator $\mathring{T}$ has the nullspace

$$\ker\mathring{T}=\operatorname{im}\begin{bmatrix}w(\cdot)\\0\\0\\0\\0\end{bmatrix},\ w(t):=\mathrm{e}^{\alpha t},t\in\mathcal{I},$$

supposed the setting satisfies $\mathcal{C}^\infty(\mathcal{I},\mathbb{R}^5)\subseteq\operatorname{dom}\mathring{T}$. The corresponding composed operator $\mathring{\mathcal{T}}$ is injective. This is in full accordance with Proposition 2.3. □

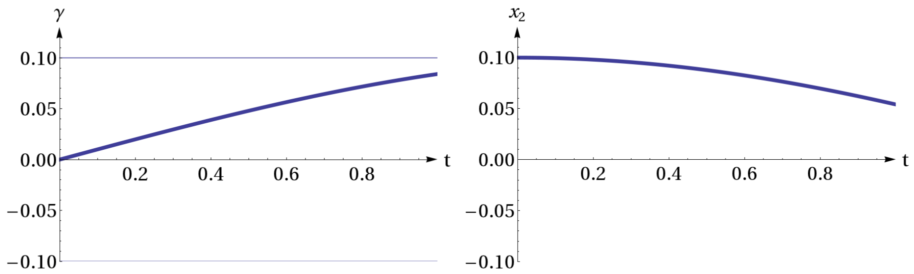

Figure 1: γ_k and $x_{k,2}$ for $k = 1$

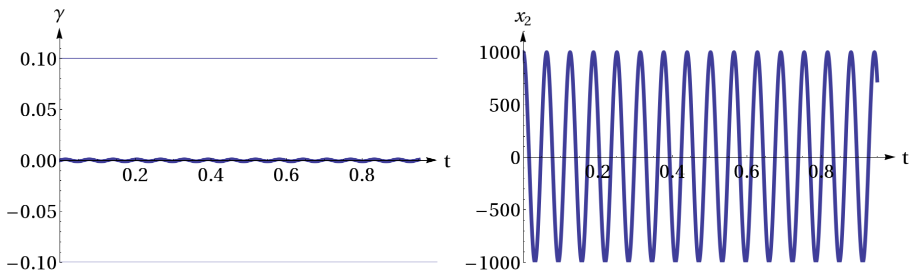

Figure 2: γ_k and $x_{k,2}$ for $k = 100$

This little constant coefficient example is alarming, but it is relatively harmless. In more general DAEs, time-dependent subspaces and nonlinear relations may considerably amplify the bad behavior. For this reason one should be careful in view of numerical simulations. It may well happen that an integration code seemingly works, however it generates wrong results.

2.3 Standard settings: looking for boundedness, closedness, and normal solvability

We inspect diverse standard settings of the operators $\mathring{T}$ and $\mathring{\mathcal{T}}$ given by (6) and (16), respectively. We ask if the operators and suitable extensions are bounded, closed and eventually continuously invertible. The interval $\mathcal{I}$ is supposed to be compact such that maximum norms can be applied, $\mathcal{I} := [t_a, t_e]$. We consider for X the Banach spaces $\mathcal{C}^1$, $\mathcal{C}$ and the Hilbert space L^2.

2.3.1 Space of continuously differentiable functions $X = \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m)$

Most authors favor $\mathcal{C}^1$-solutions when dealing with DAEs. This corresponds to the setting $\operatorname{dom} \mathring{T} = X = \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m)$, $Y = \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$. Applying usual norms, the operator $\mathring{T}$ becomes bounded.

The special case given by the simple index-1 matrix pair in Weierstraß–Kronecker form

$$\mathring{T}x = Ex' + Fx = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1' + Wx_1 \\ x_2 \end{bmatrix}$$

foreshadows the drawback of this setting: the derivative-free equation $x_2 = q_2$ which represents a part of the equation $\mathring{T}x = q$ has no solution unless one additionally supposes that q_2 is continuously differentiable. There is no natural reason for this demand. Formally, this setting yields

$$\operatorname{im} \mathring{T} = \{q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : q_2 \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^l)\}$$

which is a proper nonclosed subset in $Y = \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$ such that $\mathring{T}$ fails to be normally solvable. This motivates us to turn to spaces X being richer in content. It seems to be more reasonable to accept instead a solution component x_2 being merely continuous.

2.3.2 Space of continuous functions $X = \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$

Now we apply the function spaces $X = Y = \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$, with the maximum norm, and consider the operator

$$\mathring{T}x = Ex' + Fx, \quad x \in \operatorname{dom} \mathring{T} := \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m). \tag{19}$$

The definition domain $\operatorname{dom} \mathring{T} = \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m)$ is dense in X, i.e. $\mathring{T}$ is densely defined. However, except for the dull case $E = 0$, the operator $\mathring{T}$ is unbounded in this setting. To simplify matters, we verify this fact supposing the interval $\mathcal{I} = [0, 1]$. Since E is not the zero matrix, there is a $c \in \mathbb{R}^m$ such that $Ec \neq 0$ and $|c| = 1$. The functions defined by $x_k(t) := t^k c,\ t \in [0, 1]$, belong to the definition domain of T and $\|x_k\|_\infty = 1$ for all $k \in \mathbb{N}$. Derive

$$\|\mathring{T}x_k\|_\infty = \|Ex_k' + Fx_k\|_\infty \geq \|Ex_k'\|_\infty - \|Fx_k\|_\infty = k|Ec| - |Fc|.$$

If k increases, then $\|\mathring{T}x_k\|_\infty$ grows unboundedly, which shows that, in the given setting, the operator $\mathring{T}$ is no longer bounded. However, we may obtain a closure of the operator $\mathring{T}$, that is, a closed extension T of $\mathring{T}$.

First of all, the operator $\mathring{T}$ is closable. Namely, if $x_k \in \operatorname{dom} \mathring{T}$ for $k \in \mathbb{N}$ and $\|x_k\|_\infty \to 0$, $\|\mathring{T}x_k - y_*\|_\infty \to 0$, then it follows that $\|Ex_k\|_\infty \to 0$, $\|(Ex_k)' - y_*\|_\infty = \|Ex_k'\|_\infty = \|\mathring{T}x_k - y_* - Fx_k\|_\infty \to 0$, and hence $y_* = 0$.

Now we look for the closure of $\mathring{T}$ in the given setting. Preliminary, we factorize $E = A\tilde{D}$, with $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $\tilde{D} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $n \leq m$, in such a way that

$$\operatorname{im} E = \operatorname{im} A$$

is valid. We write

$$\mathring{T}x = Ex' + Fx = A\tilde{D}x' + Fx = A(\tilde{D}x)' + Fx, \quad x \in \operatorname{dom} \mathring{T} = \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m).$$

In particular, one can simply choose $A = E, \tilde{D} = I, n = m$ and $A = EE^+, \tilde{D} = E, n = m$. It should be emphasized that *the operator* $\mathring{T}$ itself *is independent of the description* by a special factorization.
Consider an arbitrary sequence of members $x_k \in \operatorname{dom} \mathring{T}$, with $\|x_k - x_*\|_\infty \to 0$ and $\|\mathring{T}x_k - y_*\|_\infty \to 0$, and limits $x_* \in X$, $y_* \in Y$. It holds that

$$A(\tilde{D}x_k)' = \mathring{T}x_k - Fx_k \to y_* - Fx_* \in \operatorname{im} A,$$

By means of a generalized inverse A^- of A (see Appendix 6.1.2) we express

$$(A^- A\tilde{D}x_k)' = A^- A(\tilde{D}x_k)' = A^-(\mathring{T}x_k - Fx_k) \to A^-(y_* - Fx_*).$$

We also have $A^- A\tilde{D}x_k \to A^- A\tilde{D}x_*$. Then, $A^- A\tilde{D}x_*$ is continuously differentiable and the relation $(A^- A\tilde{D}x_*)' = A^-(y_* - Fx_*)$ is given. Now it follows that $A(A^- A\tilde{D}x_*)' = AA^-(y_* - Fx_*) = y_* - Fx_*$, thus $A(A^- A\tilde{D}x_*)' + Fx_* = y_*$. It comes out that the operator T defined by

$$T := A(Dx)' + Fx, \quad x \in \operatorname{dom} T = \{w \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : Dw \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)\}, \quad (20)$$

with $D := A^- A\tilde{D}$, is the closure of the operator $\mathring{T}$ given by (19).

Yet another look at the resulting factorization $E = AD$ shows: We have $D := A^- A\tilde{D}$, $E = A\tilde{D}$ and $\operatorname{im} E = \operatorname{im} A$. It follows that $\operatorname{rank} AD = \operatorname{rank} A$, thus $\operatorname{rank} D \geq \operatorname{rank} A$. Since, on the other hand the inclusion $\operatorname{im} D \subseteq \operatorname{im} A^- A$ is valid we conclude the relations

$$\operatorname{im} D = \operatorname{im} A^- A, \quad \operatorname{rank} D = \operatorname{rank} A = \operatorname{rank} E,$$

and further the decomposition

$$\ker A \oplus \operatorname{im} D = \mathbb{R}^n. \quad (21)$$

In turn, (21) implies $\ker D = \ker AD = \ker E$.
Thereby it does not matter at all how the matrix $\tilde{D}$ and the generalized inverse A^- have been chosen. In particular, all resulting function spaces

$$\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m) := \{w \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : Dw \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)\} \quad (22)$$

coincide with

$$\{w \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : Ew \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m)\},$$

see Lemma 6.9, and for every $x \in \operatorname{dom} T$ it holds that

$$Tx = A(Dx)' + Fx = A(DE^+Ex)' + Fx = ADE^+(Ex)' + Fx = EE^+(Ex)' + Fx.$$

In other words, to obtain the closure T of the operator $\mathring{T}$ we are recommended to reformulate the operator by means of a factorization $E = AD$ with so-called *well-matched* factors $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $D \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $n \leq m$, satisfying the decomposition (21). We underline the fact that the special factorization does not at all matter for the operator T itself. It is merely a representation tool. However, it should be noticed that, for instance, when numerically integrating a DAE, a so-called full-rank factorization (with $n = \operatorname{rank} E$) appears to be quite preferable, see [50].

The function space $\operatorname{dom} T = \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ equipped with the respective graph-norm is a Banach space. For all possible factorizations satisfying the condition (21), the norms

$$\|x\|_{C^1_D} := \|x\|_\infty + \|(Dx)'\|_\infty, \quad x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m),$$

are equivalent to each other and also to the graph-norm, see Lemma 6.9.

Proposition 2.6 *Let a pair $\{E,F\}$ of matrices $E,F \in \mathcal{L}(\mathbb{R}^m)$ be given and $E \neq 0$. Then the following assertions are valid for the differential-algebraic operator $\mathring{T} \in \mathcal{L}(X)$ defined by (19) with $X = \mathcal{C}(\mathcal{I},\mathbb{R}^m)$:*

(1) *The operator $\mathring{T}$ is unbounded but closable. The closure T of $\mathring{T}$ can be expressed by (20), where $E = AD$ is any factorization satisfying the decomposition (21).*

(2) $\ker T$ *is closed in X.*

(3) *The closure T maps the Banach space $\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ continuously into the Banach space X.*

(4) *If $\{E,F\}$ is a regular pair with Kronecker index $\mu = 1$, then the closure T is a Fredholm operator. It holds that* $\operatorname{im} T = X$ *and* $\operatorname{ind}_{fred}(T) = \alpha(T) = m - l$, *whereby l is the structural size described in (9).*

(5) *If $\{E,F\}$ is a regular pair with Kronecker index $\mu > 1$, then T is densely solvable and the nullspace of T has finite dimension. The range* $\operatorname{im} T$ *is a nonclosed proper subset in X such that T is neither fredholm nor normally solvable..*

(6) *If $\{E,F\}$ is a regular pair with Kronecker index $\mu = 0$, that is, E is nonsingular, then $\mathring{T}$ is already closed, $T = \mathring{T}$,* $\operatorname{im} T = X$, $\dim\ker T = m$, *and T is a Fredholm operator with* $\operatorname{ind}_{fred}(T) = m$.

Proof: The statements (1) and (3) are already verified and (2) retrieves a general property of closed operators. Assertion (6) reflects well-known facts of the ODE theory. It remains to consider items (4) and (5). Applying the Weierstraß-Kronecker form (9) we factorize $E = AD,\ A := E,\ D := E^- E$ and express

$$E = L^{-1}\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} K^{-1}, \quad E^- := K \begin{bmatrix} I & 0 \\ 0 & N^+ \end{bmatrix} L, \quad E^- E = K \begin{bmatrix} I & 0 \\ 0 & N^+N \end{bmatrix} K^{-1},$$

and introduce $K^{-1}x =: \begin{bmatrix} y \\ z \end{bmatrix}$ such that

$$\begin{aligned} \operatorname{dom} T &= \{x \in \mathcal{C}(\mathcal{I},\mathbb{R}^m) : K \begin{bmatrix} I & 0 \\ 0 & N^+N \end{bmatrix} K^{-1}x \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^m)\} \\ &= \{x \in \mathcal{C}(\mathcal{I},\mathbb{R}^m) : \begin{bmatrix} I & 0 \\ 0 & N^+N \end{bmatrix} K^{-1}x \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^m)\} \\ &= \{x \in \mathcal{C}(\mathcal{I},\mathbb{R}^m) : y \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^{m-l}), N^+Nz \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^l)\}. \end{aligned}$$

The equation $Tx = q$ can be traced back to the decoupled system (cf.(11),(12))

$$y'(t) + Wy(t) = p(t), \quad N(N^+Nz)'(t) + z(t) = r(t). \tag{23}$$

If the index μ equals 1 then N is the zero matrix of size l, thus

$$\operatorname{dom} T = \{x \in \mathcal{C}(\mathcal{I},\mathbb{R}^m) : y \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^{m-l})\}.$$

In this case, for each arbitrary continuous right hand sides p and r, there exist a continuously differentiable solution y of the first equation and a continuous

solution z of the second one, which is now trivial. Altogether, for each continuous q there exists an element $x \in \operatorname{dom} T$ such that $Tx = q$. This confirmes assertion (4).
If $\mu \geq 2$, then N is a nontrivial nilpotent matrix. The first equation in (23) has again a continuously differentiable solution y for each continuous p. However, the demand $N^+Nz \in \mathcal{C}^1$ implies necessarily $Nz \in \mathcal{C}^1$, thus $N^{\mu-1}z \in \mathcal{C}^1$. Multiplying the second equation by $N^{\mu-1}$ yields $N^{\mu-1}z = N^{\mu-1}r$. In other words, for solvability of the second equation it is necessary that $N^{\mu-1}r$ is continuously differentiable. In consequence, the range $\operatorname{im} T$ contains only those continuous functions showing certain smoother components. In any case, at least one component has to be continuously differentiable. Such sets are not closed in the continuous function space. Regarding the inclusion $\mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m) \subseteq \operatorname{im} T$ we are done with assertion (5). □

Example 2.7 (Continuation 1 of Example 2.5) *A natural factorization is here*

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} =: AD,$$

and the closure T of the operator $\mathring{T}$ reads

$$Tx = A(Dx)' + Fx = \begin{bmatrix} x_1' - \alpha x_1 - x_2 \\ x_3' + x_2 \\ x_4' + x_3 \\ x_5' + x_4 \\ x_5 \end{bmatrix}, x \in \operatorname{dom} T = \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}),$$

$$\mathcal{C}_D^1(\mathcal{I}, \mathbb{R}) = \{x \in \mathcal{C}(\mathcal{I}, \mathbb{R}^5) : x_1, x_3, x_4, x_5 \in \mathcal{C}^1(\mathcal{I}, \mathbb{R})\}.$$

The pair $\{E, F\}$ is regular with Kronecker index $\mu = 4$ so that Proposition 2.6(5) applies. In more detail it results that

$$\operatorname{im} T = \{q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^5) : q_5,\ q_4 - q_5',\ q_3 - (q_4 - q_5')' \in \mathcal{C}^1(\mathcal{I}, \mathbb{R})\}.$$

The range $\operatorname{im} T$ is a proper nonclosed subset in $\mathcal{C}(\mathcal{I}, \mathbb{R}^5)$ which indicates essentially ill-posedness. The closure of the associated composed operator $\mathring{\mathcal{T}}$ is given by (cf.(17))

$$\mathcal{T}x = (Tx, x_1(0)), \quad x \in \operatorname{dom} T.$$

The operator $\mathcal{T}$ is injective, however, the inverse $\mathcal{T}^{-1}$ fails to be continuous. For instance, the sequence $(q_k, 0) \in \operatorname{im} T \times \mathbb{R}$ defined by (18) tends to zero for $k \to \infty$, that means $\|q_k\|_\infty \to 0$, whereas the sequence of the corresponding responses $x_k = \mathcal{T}^{-1}(q_k, 0)$ growths unboundedly,

$$\|x_k\|_\infty > \|x_{k,2}\|_\infty = \varepsilon k^2.$$

□

2.3.3 Space of integrable functions $X = L^2(\mathcal{I}, \mathbb{R}^m)$

We set $X = Y = L^2(\mathcal{I}, \mathbb{R}^m)$ and apply the Hilbert space $L^2(\mathcal{I}, \mathbb{R}^m)$ with the usual scalar product and norm. Consider the operator

$$\mathring{T}x = Ex' + Fx, \quad x \in \operatorname{dom} \mathring{T} := \mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m). \tag{24}$$

The definition domain is dense in X, i.e., $\mathring{T}$ is densely defined. Except for the dull case $E = 0$, the operator $\mathring{T}$ is unbounded in this setting. To simplify matters, we show this fact supposing the interval $\mathcal{I} = [0, 1]$. Since E is not the zero matrix, there is a $c \in \mathbb{R}^m$ such that $Ec \neq 0$ and $|c| = 1$. The functions defined by $x_k(t) := \sqrt{2k+1}\, t^k c,\ t \in [0, 1]$, belong to the definition domain of $\mathring{T}$, and we have $\|x_k\|_{L^2} = 1$ for all $k \in \mathbb{N}$. Derive

$$\|\mathring{T}x_k\|_{L^2} = \|Ex'_k + Fx_k\|_{L^2} \geq \|Ex'_k\|_{L^2} - \|Fx_k\|_{L^2} = k\frac{\sqrt{2k+1}}{\sqrt{2k-1}}|Ec| - |Fc|.$$

If k increases, then $\|\mathring{T}x_k\|_{L^2}$ grows unboundedly, which shows that, also in this setting, the operator $\mathring{T}$ is unbounded.

After the idea of the previous subsection we introduce the function space

$$H^1_D(\mathcal{I}, \mathbb{R}^m) := \{w \in L^2(\mathcal{I}, \mathbb{R}^m) : Dw \in H^1(\mathcal{I}, \mathbb{R}^n)\}$$

and the operator T by

$$T := A(Dx)' + Fx,\ x \in \operatorname{dom} T = H^1_D(\mathcal{I}, \mathbb{R}^m), \tag{25}$$

whereby we apply a factorization $E = AD$ such that the decomposition (21) is valid, i.e., $\ker A \oplus \operatorname{im} D = \mathbb{R}^n$. Eventually, the operator T will be proved to be the closure of $\mathring{T}$. Again, the special choice of the factorization does not matter, neither for the function space $H^1_D(\mathcal{I}, \mathbb{R}^n)$ nor for the operator T, see Lemma 6.9. We follow the lines of [35] applying Hilbert space basics (see the Appendix).

Lemma 2.8 *The operator $\mathring{T}$ given by (24) possesses an adjoint and a biadjoint. The adjoint and biadjoint operators $\mathring{T}^*$ and $\mathring{T}^{**}$ of $\mathring{T}$ are given on the domains*

$$\begin{aligned}
\operatorname{dom} \mathring{T}^* \quad &= \{w \in L^2(\mathcal{I}, \mathbb{R}^m) : A^*w \in H^1(\mathcal{I}, \mathbb{R}^n), A^*w(t_a) = 0, A^*w(t_e) = 0\},\\
\operatorname{dom} \mathring{T}^{**} \quad &= \{w \in L^2(\mathcal{I}, \mathbb{R}^m) : Dw \in H^1(\mathcal{I}, \mathbb{R}^n)\},
\end{aligned}$$

respectively. These domains do not depend on the chosen factorization $E = AD$ with (21).

Proof: For arbitrary $x \in \mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m)$ and suitable functions y by partial integration we obtain

$$\begin{aligned}
(\mathring{T}x, y) &= \int_{t_a}^{t_e} \langle Ex'(t) + Fx(t), y(t)\rangle \mathrm{d}t = \int_{t_a}^{t_e} \langle A(Dx)'(t) + Fx(t), y(t)\rangle \mathrm{d}t\\
&= \int_{t_a}^{t_e} \langle x(t), -D^*(A^*y)'(t) + F^*y(t)\rangle \mathrm{d}t\\
&\quad + \langle Dx(t_e), A^*y(t_e)\rangle - \langle Dx(t_a), A^*y(t_a)\rangle.
\end{aligned}$$

Therefore, the adjoint operator is explicitly given on the set

$$\{w \in L^2(\mathcal{I},\mathbb{R}^m) : A^*w \in H^1(\mathcal{I},\mathbb{R}^n), A^*w(t_a) = 0, A^*w(t_e) = 0\} =: \operatorname{dom} \mathring{T}^*$$

by means of

$$\mathring{T}^*y = -D^*(A^*y)' + F^*y, \quad y \in \operatorname{dom} \mathring{T}^*.$$

In fact, we have $(\mathring{T}x, y) = (x, \mathring{T}^*y)$ for all $x \in \operatorname{dom} \mathring{T}$ and $y \in \operatorname{dom} \mathring{T}^*$. Since $\operatorname{dom} \mathring{T}^*$ is dense in $L^2(\mathcal{I},\mathbb{R}^m)$, the adjoint of the operator $\mathring{T}^*$ exists. Compute

$$\begin{aligned}(\mathring{T}^*y, x) &= \int_{t_a}^{t_e} \langle -D^*(A^*y)'(t) + F^*y(t), x(t)\rangle \mathrm{dt} \\ &= \int_{t_a}^{t_e} \langle y(t), A(Dx)'(t) + Fx(t)\rangle \mathrm{dt} \\ &=: (y, \mathring{T}^{**}x)\end{aligned}$$

for all $y \in \operatorname{dom} \mathring{T}^*$ and all $x \in \{w \in L^2(\mathcal{I},\mathbb{R}^m) : Dw \in H^1(\mathcal{I},\mathbb{R}^n)\} =: \operatorname{dom} \mathring{T}^{**}$ which proves the first part of the assertion.
It remains to verify the invariance with respect to the factorization $E = AD$. Consider a further factorization $E = \bar{A}\bar{D}$, with $\bar{A} \in \mathcal{L}(\mathbb{R}^{\bar{n}},\mathbb{R}^m), \bar{D} \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^{\bar{n}})$ and $\ker \bar{A} \oplus \operatorname{im} \bar{D} = \mathbb{R}^{\bar{n}}$, $\bar{n} \leq m$. Because of $\ker D = \ker \bar{D}$ it holds that $D^+D = \bar{D}^+\bar{D}$. Therefore, $Dw \in H^1(\mathcal{I},\mathbb{R}^n)$ implies $\bar{D}w = \bar{D}\bar{D}^+\bar{D}w = \bar{D}D^+Dw \in H^1(\mathcal{I},\mathbb{R}^{\bar{n}})$, and hence $\operatorname{dom} \mathring{T}^{**}$ does not depend on the factorization.
Analogously, due to $\operatorname{im} A = \operatorname{im} \bar{A}$, $AA^+ = \bar{A}\bar{A}^+$, we derive from $A^*w \in H^1(\mathcal{I},\mathbb{R}^n)$ that $\bar{A}^*w = \bar{A}^*\bar{A}^{*+}\bar{A}^*w = \bar{A}^*A^{*+}A^*w \in H^1(\mathcal{I},\mathbb{R}^{\bar{n}})$.
Additionally, for $\tau = t_a, t_e$, we compute $\bar{A}^*w(\tau) = \bar{A}^*A^{*+}A^*w(\tau)$ and $A^*w(\tau) = A^*\bar{A}^{*+}\bar{A}^*w(\tau)$. This proves $\operatorname{dom} \mathring{T}^*$ to be independent of the special factorization. □

The next assertion is the counterpart of Proposition 2.6.

Proposition 2.9 *Let the pair $\{E, F\}$ of matrices $E, F \in \mathcal{L}(\mathbb{R}^m)$ be given and $E \neq 0$. Then the following assertions are valid for the differential-algebraic operator $\mathring{T} \in \mathcal{L}(X)$, defined by (24) with $X = L^2(\mathcal{I},\mathbb{R}^m)$:*

(1) *The operator $\mathring{T}$ is unbounded but closable. The closure T of $\mathring{T}$ can be expressed by (25), where $E = AD$ is any factorization satisfying the decomposition (21).*

(2) $\ker \hat{T}$ *is closed in X.*

(3) *The closure T maps the Hilbert space $H^1_D(\mathcal{I},\mathbb{R}^m)$ continuously into the Hilbert space X.*

(4) *If $\{E, F\}$ is a regular pair with Kronecker index $\mu = 1$, then the closure T is a Fredholm operator. It holds that* $\operatorname{im} T = X$ *and* $\operatorname{ind}_{fred}(T) = \alpha(T) = m - l$, *whereby l is the structural size described in (9).*

(5) *If $\{E, F\}$ is a regular pair with Kronecker index $\mu > 1$, then T is densely solvable and the nullspace of T is finite-dimensional. The range* $\operatorname{im} T$ *is a nonclosed proper subset in X such that T is neither fredholm nor normally solvable.*

(6) *If $\{E,F\}$ is a regular pair with Kronecker index $\mu = 0$, that is, E is nonsingular, then $\mathring{T}$ is already closed, $T = \mathring{T}$, $\operatorname{im} T = X$, $\dim\ker T = m$, and T is a Fredholm operator with $\operatorname{ind}_{fred}(T) = m$.*

Proof: Owing to the existence of the biadjoint $\mathring{T}^{**}$ one has $T = \mathring{T}^{**}$, and the assertion (1) is a consequence of Lemma 2.8. Assertion (2) reflects a general property of closed operators. Assertion (3) is due to the inequality

$$\|Tx\|^2_{L^2} = \|A(Dx)' + Fx\|^2_{L^2} \leq c\,(\|(Dx)'\|^2_{L^2} + \|x\|^2_{L^2}) = c\|x\|^2_{H^1_D},$$

which is valid for all $x \in H^1_D(\mathcal{I}, \mathbb{R}^m)$. The statements (4)-(6) can be proved along the lines of Proposition 2.6 by replacing the function spaces $\mathcal{C}$ and $\mathcal{C}^1$ by L^2 and H^1, correspondingly. □

2.4 Peculiar approaches

2.4.1 Enforcing surjectivity by image space adaption

Suppose that the pair $\{E,F\}$ is regular and the factorization $E = AD$ satisfies the condition (21). The operator

$$T \in \mathcal{L}(\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m), \mathcal{C}(\mathcal{I}, \mathbb{R}^m)), \quad Tx := A(Dx)' + Fx,\; x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m),$$

is bounded. By Proposition 2.6, if $\operatorname{ind}\{E,F\} \geq 2$, the range $\operatorname{im} T$ is a proper, nonclosed subset in $\mathcal{C}(\mathcal{I}, \mathbb{R}^m)$. The resulting composed operator (cf.(16))

$$\mathcal{T} \in \mathcal{L}(\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m), \mathcal{C}(\mathcal{I}, \mathbb{R}^m) \times \mathbb{R}^{m-l}),$$

is injective but inherits the nonclosed range. In the consequence, the equation

$$\mathcal{T}x = (q, d) \tag{26}$$

is essentially ill-posed in this setting. We refer once again to Examples 2.5 and 2.7 for the ill-posed feature of the solution which actually justifies that notion.

Applying the Weierstraß–Kronecker form or the projector based decoupling procedure described in [50, Chapter 1] one can specify in full detail how the range of the operator T looks like. Being aware of the precise description of $\operatorname{im} T$ we can apply the new function space $Y_{new} := \operatorname{im} T$ as well as a suitable norm so that Y_{new} becomes a Banach space and $T \in \mathcal{L}(\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m), Y_{new})$ remains bounded. Then, in this peculiar setting, the operator T is fredholm and the associated composed operator $\mathcal{T}$ becomes a homeomorphism as a continuous bijection in Banach spaces. This way, in the new setting, the equation (26) becomes well-posed. This idea seems fine; however, it rather obscures the actual solution behavior and it is essentially factitious as our example demonstrates.

Example 2.10 (Continuation 2 of Example 2.5) *The operator (see also Example 2.7)*

$$Tx := \begin{bmatrix} x_1' - \alpha x_1 - x_2 \\ x_3' + x_2 \\ x_4' + x_3 \\ x_5' + x_4 \\ x_5 \end{bmatrix}, \quad x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^5),$$

is bounded, where $\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^5) = \{x \in \mathcal{C}(\mathcal{I},\mathbb{R}^5) : x_1, x_3, x_4, x_5 \in \mathcal{C}^1(\mathcal{I},\mathbb{R})\}$*. Its image*

$$\operatorname{im} T = \{q \in \mathcal{C}(\mathcal{I},\mathbb{R}^5) : q_5, q_4 - q_5', q_3 - (q_4 - q_5')' \in \mathcal{C}^1(\mathcal{I},\mathbb{R})\}$$

is a proper nonclosed subset in $\mathcal{C}(\mathcal{I},\mathbb{R}^5)$*. The resulting composed operator* $\mathcal{T}$ *has a discontinuous inverse. Set* $Y_{new} = \operatorname{im} T$ *and define for each* $q \in Y_{new}$ *the norm*

$$\|q\|_{Y_{new}} := \|q\|_\infty + \|q_5'\|_\infty + \|(q_4 - q_5')'\|_\infty + \|(q_3 - (q_4 - q_5')')'\|_\infty,$$

which yields a Banach space. In the new setting $T \in \mathcal{L}(\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^5), Y_{new})$*, the operator* T *is surjective by construction and again bounded. Namely, it holds that*

$$\begin{aligned}\|Tx\|_{Y_{new}} &= \|Tx\|_\infty + \|x_5'\|_\infty + \|x_4\|_\infty + \|x_3\|_\infty \\ &\le c_{new}\|x\|_{\mathcal{C}^1_D}, \quad x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^5).\end{aligned}$$

The associated composed operator $\mathcal{T} \in \mathcal{L}(\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^5), Y_{new} \times \mathbb{R})$ *becomes actually a homeomorphism. In the new setting, the continuity of* $\mathcal{T}^{-1}$ *is enforced by the stronger norm. Now, the sequence* q_k *given by (18) does no longer converge to zero, instead, one has*

$$\|q_k\|_{Y_{new}} > \|\gamma_k^{(3)}\|_\infty = \varepsilon k^2.$$

We emphasize that this problem manipulation does not at all change the actual bad solution behavior documented by Figures 1 and 2. □

The resulting this way operator $T \in \mathcal{L}(\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m), Y_{new})$ is surjective and has a finite-dimensional nullspace (cf.Proposition 2.3), and hence, it is fredholm, with $\operatorname{ind}_{fred}(T) = \alpha(T) = m - l$.
The last result has quite limited importance: The information needed for this kind of manipulation is indispensable in general. Both, the set Y_{new} and the norm $\|\cdot\|_{Y_{new}}$ strongly depend on the given matrix pair $\{E, F\}$; in higher index cases they are strongly individual ones for each pair.

As mentioned before, most authors favor $\mathcal{C}^1$-solutions of DAEs. One can manipulate correspondingly the operator

$$T \in \mathcal{L}(\mathcal{C}^1(\mathcal{I},\mathbb{R}^m), \mathcal{C}(\mathcal{I},\mathbb{R}^m)), \quad Tx := Ex' + Fx,\ x \in X = \mathcal{C}^1(\mathcal{I},\mathbb{R}^m),$$

being of interest when insisting on $\mathcal{C}^1$-solutions. At this place we point out that all statements in [10, 11, 14] concerning the Fredholm index (there called Noetherian index) of differential-algebraic operators hold good for such a context only.

Example 2.11 (Continuation 3 of Example 2.5) *For the operator (see also Examples 2.7, 2.10)*

$$Tx := \begin{bmatrix} x_1' - \alpha x_1 - x_2 \\ x_3' + x_2 \\ x_4' + x_3 \\ x_5' + x_4 \\ x_5 \end{bmatrix}, \quad x \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^5),$$

one immediately derives

$Y_{new} = \operatorname{im} T =$
$\{q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^5) : q_5, q_4 - q_5', q_3 - (q_4 - q_5')', q_2 - q_2 - (q_3 - (q_4 - q_5')')' \in \mathcal{C}^1(\mathcal{I}, \mathbb{R})\},$

and

$$\begin{aligned}\|q\|_{Y_{new}} :=& \|q\|_\infty + \|q_5'\|_\infty + \|(q_4 - q_5')'\|_\infty \\ &+ \|(q_3 - (q_4 - q_5')')'\|_\infty + \|(q_2 - (q_3 - (q_4 - q_5')')')'\|_\infty.\end{aligned}$$

□

We finish this subsubsection by revisiting once again the case of a regular index-1 pair $\{E.F\}$. There are different Banach space settings and bounded operators available. From Subsubsection 2.3.1 we know that then $T \in \mathcal{L}(\mathcal{C}^1(\mathcal{I}, \mathbb{R}^m), \mathcal{C}(\mathcal{I}, \mathbb{R}^m))$ fails to be normally solvable since $\operatorname{im} T$ is a nonclosed subset in $\mathcal{C}(\mathcal{I}, \mathbb{R}^m))$.
The setting $T \in \mathcal{L}(\mathcal{C}^1(\mathcal{I}, \mathbb{R}^m), Y_{new})$ yields a surjective operator, but needs special structural information also in case of regular index-1 pairs $\{E.F\}$. Namely, applying the corresponding Weierstraß-Kronecker form (9), with $N = 0$, we derive

$$Y_{new} = \operatorname{im} T = \{q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : [0\ I]Lq \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^l)\}.$$

and

$$\|q\|_{Y_{new}} = \|q\|_\infty + \|([0\ I]Lq)'\|_\infty.$$

In comparison, in the setting $T \in \mathcal{L}(\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m), \mathcal{C}(\mathcal{I}, \mathbb{R}^m))$, the operator T is also surjective. It seems to be an advantage of the latter setting that it uses original given information or easily available information concerning merely the coefficient E.

2.4.2 Topological vector-space $X = \mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m)$

The Weierstraß–Kronecker form (11),(12) tells us that, letting

$$X = Y = \mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m), \quad Tx = Ex' + Fx, \quad x \in \operatorname{dom} T = X,$$

the operator T induced by a regular matrix pair is surjective so that the equation $Tx = q$ is solvable in X for every $q \in Y$. Then the corresponding composed operator $\mathcal{T}$ acts bijectively between X and $Y \times \mathbb{R}^{m-l}$. This makes this topological vector-space setting quite popular, also for DAEs with real-analytic coefficients E and F.
The family of semi-norms

$$\eta_k(x) := \|x^{(k)}\|_\infty,\ x \in \mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m), \quad k = 0, 1, 2, \ldots,$$

generates (see [42, Section X.64]) a locally convex topology on $\mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m)$; and convergence $x_j \xrightarrow{j\to\infty} x_*$ actually means that $\|x_j^{(k)} - x_*^{(k)}\|_\infty \xrightarrow{j\to\infty} 0$ is valid for all $k = 0, 1, 2, \ldots$. Unfortunately, this vector-space setting does not offer valuable clues to the question whether the solution of the equation

$$\mathcal{T}x = (q, d).$$

depends continuously on the data q and d. Our alarming Example 2.5 suits to this setting, however, now the problem is circumvented since the sequence (18) does no longer converge to zero here.

2.4.3 A too lean Banach space $X = \check{\mathcal{C}}(\mathcal{I},\mathbb{R}^m) \subset \mathcal{C}^\infty(\mathcal{I},\mathbb{R}^m)$

A bounded linear operator acting bijectively in Banach spaces has also a bounded inverse. To practice this well-known fact we refer briefly the function space

$$\check{\mathcal{C}}(\mathcal{I},\mathbb{R}^m) := \{x \in \mathcal{C}^\infty(\mathcal{I},\mathbb{R}^m) : \sup_{j\geq 0} \|x^{(j)}\|_\infty < \infty\},$$

endowed with the norm

$$\|x\| := \sup_{j\geq 0} \|x^{(j)}\|_\infty, \quad x \in \check{\mathcal{C}}(\mathcal{I},\mathbb{R}^m),$$

which is a Banach space. In the setting $\operatorname{dom} T = X = Y = \check{\mathcal{C}}(\mathcal{I},\mathbb{R}^m)$, the composed operator $\mathcal{T}$ associated with a regular matrix pair becomes a bounded bijection between X and $Y \times \mathbb{R}^{m-\check{l}}$, with an appropriate $\check{l} \geq l$. This simulates continuous invertibility, which seems to be in contradiction to Example 2.5. However, the function space $\check{\mathcal{C}}(\mathcal{I},\mathbb{R}^m)$ is much too lean in capacity. The basic inclusion $\mathcal{C}^\infty(\mathcal{I},\mathbb{R}^m) \subseteq \operatorname{dom} T$ is no longer satisfied. In Example 2.5, to fit into this special setting, one has to assume $|\alpha| \leq 1$ and the considered there sequence of excitations (18) is not at all admissible. By this, the danger is factitious out of bounds. The same would happen when applying equivalent weighted norms. Therefore, this special Banach space setting is much too restrictive, and hence inappropriate already for constant coefficient DAEs.

3 Normally solvable differential-algebraic operators

As it is well-known, if a densely defined, closed operator acting in Hilbert spaces has a closed image, then it is normally solvable, and there exist bounded inner inverses. In particular, the Moore-Penrose inverse is then bounded, and it makes good sense seeking least-squares solutions, see [57], also Appendix 6.1.2. For a normally solvable operator, the problem of calculating a pseudosolution becomes well-posed in Hadamard's sense. For this reason, if K is an operator with closed image, the equation $Kx = q$ is also said to be *well-posed in the sense of G. Fichera.*

We ask if the operator associated with the linear time-varying DAE

$$E(t)x'(t) + F(t)x(t) = q(t), \quad t \in \mathcal{I} = [t_a, t_e], \tag{27}$$

and the composed operator associated with the IVP

$$E(t)x'(t) + F(t)x(t) = q(t), \quad t \in \mathcal{I}, \quad Cx(t_a) = d, \tag{28}$$

have closed images. The matrix coefficients $E(t), F(t) \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^k)$ depend continuously on t on the compact interval $\mathcal{I}$, and the nullspace of E is a $\mathcal{C}^1$-subspace. As distinguished from the regular DAEs in Section 2, the DAE (27) comprises k equations but m unknown functions.

Inspecting once again the settings discussed in Section 2, for the preimage space X we favor the two function spaces $\mathcal{C}(\mathcal{I},\mathbb{R}^m)$, $L^2(\mathcal{I},\mathbb{R}^m)$ being fully independent of DAE data, and, additionally, the two spaces $\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$, $H^1_D(\mathcal{I},\mathbb{R}^m)$

that incorporate certain problem data via a factorization $E = AD$.
Supposing a nonvanishing coefficient E, the differential-algebraic operators becomes unbounded, but densely defined and closable in the first two cases. We have to deal with their closures. The latter two function spaces already involve problem data, in essence, these spaces are the definition domains of the closures obtained in the first two cases, and endowed with the graph-norms. Then the differential-algebraic operators are bounded.

We establish practically useful sufficient criteria for normal solvability in terms of the original data, in Subsection 3.1 for closed operators and in Subsection 3.2 for bounded ones. Subsection 3.3 provides bounded generalized inverses and least-squares solutions.

In Subsection 3.4 we provide a large class of nonlinear DAEs

$$\mathfrak{f}(x'(t), x(t), t) = 0,$$

with m unknowns and k equations, yielding normally solvable linearizations in a natural Banach space setting. Again we establish the criteria for normal solvability in terms of the original data, and this serves as basis for Newton-Kantorovich iteration procedures etc.

3.1 Settings with closed differential-algebraic operators

We associate the linear DAE (27) with the operator

$$\mathring{T} \in \mathcal{L}(X, Y), \quad \mathring{T}x := Ex' + Fx, \; x \in \operatorname{dom} \mathring{T} = \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m) \subseteq X. \tag{29}$$

We apply the function spaces $X = \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$, $Y = \mathcal{C}(\mathcal{I}, \mathbb{R}^k)$ when seeking classical DAE solutions which satisfy the DAE at all $t \in \mathcal{I}$. Instead, the spaces $X = L^2(\mathcal{I}, \mathbb{R}^m)$, $Y = L^2(\mathcal{I}, \mathbb{R}^k)$ are applied, when we are interested in generalized solutions satisfying the DAE for almost all $t \in \mathcal{I}$.

We suppose that E does not vanish such that the DAE is nontrivial and $\mathring{T}$ is unbounded in both instances (cf. Subsubsections 2.3.2, 2.3.3).

Theorem 3.1 *If $E, F \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^k))$ and $\ker E$ is a $\mathcal{C}^1$-subspace, then the following statements are valid:*

(1) *There exist factorizations $E = AD$ such that $A \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k))$, $D \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$, $\ker A$ and $\operatorname{im} D$ are $\mathcal{C}^1$-subspaces, and the transversality condition*

$$\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{R}^n, \; t \in \mathcal{I}, \tag{30}$$

is valid. The projector $R(t) \in \mathcal{L}(\mathbb{R}^n)$ onto $\operatorname{im} D(t)$ along $\ker A(t)$ depends continuously differentiably on t.
$B := F - AD'$ is continuous.

(2) *For $X = \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$, $Y = \mathcal{C}(\mathcal{I}, \mathbb{R}^k)$, and each arbitrary factorization from (1), the operator*

$$T \in \mathcal{L}(X, Y), \quad Tx := A(Dx)' + Bx, \; x \in \operatorname{dom} T = \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m) \subseteq X, \tag{31}$$

represents the closure of the operator $\mathring{T} \in \mathcal{L}(X, Y)$ given by (29).

(3) *For* $X = L^2(\mathcal{I}, \mathbb{R}^m)$*,* $Y = L^2(\mathcal{I}, \mathbb{R}^k)$*, and each arbitrary factorization from (1), the operator*

$$T \in \mathcal{L}(X, Y), \quad Tx := A(Dx)' + Bx,\ x \in \operatorname{dom} T = H^1_D(\mathcal{I}, \mathbb{R}^m) \subseteq X, \tag{32}$$

represents the closure of the operator $\mathring{T} \in \mathcal{L}(X, Y)$ *given by (29).*

To shorten the wording, a factorization according to (1) is said to be a *proper factorization.*

Proof: (1) Since $\ker E$ is a $\mathcal{C}^1$-subspace, the orthoprojector function E^+E onto $(\ker E)^\perp$ is continuously differentiable. Put $A := E$ and $D := E^+E$. Then $E = EE^+E = AD$ and $R = E^+E$, which makes the statement evident.
(2) The transversality condition 30 generalizes condition (21). The proof follows the arguments given in Subsubsection 2.3.2.
(3) We proceed as in Lemma 2.8 and [35] to verify the existence of the biadjoint $\mathring{T}^{**}$. Then the closure equals the biadjoint, $T = \mathring{T}^{**}$.
We derive for each arbitrary $x \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m)$ and suitable functions y by partial integration we obtain

$$\begin{aligned}(\mathring{T}x, y) &= \int_{t_a}^{t_e} \langle E(t)x'(t) + F(t)x(t), y(t)\rangle \mathrm{dt}\\ &= \int_{t_a}^{t_e} \langle A(t)(Dx)'(t) + B(t)x(t), y(t)\rangle \mathrm{dt}\\ &= \int_{t_a}^{t_e} \langle x(t), -D(t)^*(A^*y)'(t) + B(t)^*y(t)\rangle \mathrm{dt}\\ &\quad + \langle D(t_e)x(t_e), A(t_e)^*y(t_e)\rangle - \langle D(t_a)x(t_a), A(t_a)^*y(t_a)\rangle.\end{aligned}$$

Therefore, the adjoint operator is given on the set

$$\{w \in L^2(\mathcal{I}, \mathbb{R}^m) : A^*w \in H^1(\mathcal{I}, \mathbb{R}^n), A(t_a)^*w(t_a) = 0, A(t_e)^*w(t_e) = 0\} =: \operatorname{dom} \mathring{T}^*$$

by means of

$$\mathring{T}^*y = -D^*(A^*y)' + B^*y, \quad y \in \operatorname{dom} \mathring{T}^*.$$

In fact, we have $(\mathring{T}x, y) = (x, \mathring{T}^*y)$ for all $x \in \operatorname{dom} \mathring{T}$ and $y \in \operatorname{dom} \mathring{T}^*$. Since $\operatorname{dom} \mathring{T}^*$ is dense in $L^2(\mathcal{I}, \mathbb{R}^m)$, the adjoint of the operator $\mathring{T}^*$ also exists. Compute further

$$\begin{aligned}(\mathring{T}^*y, x) &= \int_{t_a}^{t_e} \langle -D^*(A^*y)'(t) + F^*y(t), x(t)\rangle \mathrm{dt}\\ &= \int_{t_a}^{t_e} \langle y(t), A(Dx)'(t) + Fx(t)\rangle \mathrm{dt}\\ &=: (y, \mathring{T}^{**}x)\end{aligned}$$

for all $y \in \operatorname{dom} \mathring{T}^*$ and $x \in \{w \in L^2(\mathcal{I}, \mathbb{R}^m) : Dw \in H^1(\mathcal{I}, \mathbb{R}^n)\} =: \operatorname{dom} \mathring{T}^{**}$. □

As densely defined closed operator, T has a closed nullspace, however, now $\ker T$ is not necessarily finite-dimensional as it is the case for regular differential-algebraic operators (e.g. Propositions 2.6, 2.9).

For time-invariant regular index-0 and index-1 pairs $\{E, F\}$, the operator T is surjective, thus $\operatorname{im} T$ is closed. For regular higher index pairs $\{E, F\}$ the closed-image property gets lost, see Propositions 2.6, 2.9. Next we characterize a further large class of linear time-varying DAEs (27) yielding closed images of the associated operators T, too.

Applying a proper factorization as described in Theorem 3.1, we introduce the following denotations to be used all through Section 3.

$$
\begin{aligned}
G_0 &:= AD = E,\ B = F - AD', \\
P_0 &:= G_0^+ G_0 = E^+ E, \quad Q_0 := I - P_0, \\
W_0 &:= I - G_0 G_0^+, && (33)\\
G_1 &:= G_0 + BQ_0 = E + (F - AD')Q_0 = E + FQ_0 + EQ_0' && (34)\\
P_1 &:= G_1^+ G_1, \quad Q_1 := I - P_1, \\
W_1 &:= I - G_1 G_1^+. && (35)
\end{aligned}
$$

Since $E(t)$ has constant rank on the interval $\mathcal{I}$, $\operatorname{rank} E(t) =: r$, the orthoprojector function W_0 is continuous. The orthoprojector functions P_0 and Q_0 are even continuously differentiable, since $\ker E$ is a $\mathcal{C}^1$-subspace. The matrix function G_1 is continuous. If $G_1(t)$ has constant rank, then W_1 is also continuous. We have further

$$
\begin{aligned}
\operatorname{im} G_0 &= \operatorname{im} A = \operatorname{im} E, \\
W_0 F &= W_0 B, \\
\operatorname{im} G_1 &= \operatorname{im} G_0 \oplus \operatorname{im} W_0 B Q_0 = \operatorname{im} E \oplus \operatorname{im} W_0 F Q_0.
\end{aligned}
$$

The matrix functions G_1 and $W_0 F Q_0$ have constant rank at the same time. This fact will frequently be exploited later on.
Let D^- denote the pointwise reflexive generalized inverse such that

$$
DD^-D = D,\ D^-DD^- = D^-,\ D^-D = P_0,\ DD^- = R.
$$

D^- is continuously differentiable (e.g. [50, Proposition A.17]). The factorization according to Theorem 3.1(1) can be chosen so that the sum (30) is orthogonal and it results that $D^- = D^+$.

The continuous projector-valued function $W_1 \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{R}^k))$ is uniformly bounded on the compact interval $\mathcal{I}$. The assignment

$$
y \in \mathcal{C}(\mathcal{I}, \mathbb{R}^k) \to W_1(t)y(t) =: (W_1 y)(t), \ \text{ for all } t \in \mathcal{I}, \tag{36}
$$

defines a bounded projector acting on $\mathcal{C}(\mathcal{I}, \mathbb{R}^k)$, which we also denote by W_1, more precisely $W_1 \in \mathcal{L}_b(\mathcal{C}(\mathcal{I}, \mathbb{R}^k))$, and we write

$$
\{y \in \mathcal{C}(\mathcal{I}, \mathbb{R}^k) : W_1 y = 0\} = \ker W_1 \subseteq \mathcal{C}(\mathcal{I}, \mathbb{R}^k). \tag{37}
$$

Similarly, the assignment

$$
y \in L^2(\mathcal{I}, \mathbb{R}^k) \to W_1(t)y(t) =: (W_1 y)(t), \ \text{ for almost all } t \in \mathcal{I}, \tag{38}
$$

defines a bounded projector acting on $L^2(\mathcal{I}, \mathbb{R}^k)$, which we denote by W_1, too, more precisely $W_1 \in \mathcal{L}_b(L^2(\mathcal{I}, \mathbb{R}^k))$, and we write

$$
\{y \in L^2(\mathcal{I}, \mathbb{R}^k) : W_1 y = 0\} = \ker W_1 \subseteq L^2(\mathcal{I}, \mathbb{R}^k). \tag{39}
$$

No confusion should arise from this within the respective context. We proceed analogously with further continuous projector-valued functions, for instance P_1. Owing to the uniformly boundedness of the continuous projector-valued functions, the resulting projectors acting on the function spaces are bounded, and hence, their nullspaces and images are closed.

Theorem 3.2 *Let $E, F \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^k))$ and $\ker E$ be a $\mathcal{C}^1$-subspace. Let the matrix function $W_0FQ_0 = W_0BQ_0$ have constant rank on $\mathcal{I}$ and let the condition*

$$W_1FP_0 = 0 \tag{40}$$

be satisfied. Then the following statements are valid:

(1) *The operator $T \in \mathcal{L}_c(\mathcal{C}(\mathcal{I}, \mathbb{R}^m), \mathcal{C}(\mathcal{I}, \mathbb{R}^k))$ from Theorem 3.1(2) has a closed image, namely $\operatorname{im} T = \ker W_1 \subseteq \mathcal{C}(\mathcal{I}, \mathbb{R}^k)$.*

(2) *The operator $T \in \mathcal{L}_c(L^2(\mathcal{I}, \mathbb{R}^m), L^2(\mathcal{I}, \mathbb{R}^k))$ from Theorem 3.1(3) has a closed image, namely $\operatorname{im} T = \ker W_1 \subseteq L^2(\mathcal{I}, \mathbb{R}^k)$.*

Proof: Since E and W_0FQ_0 are continuous, constant-rank matrix functions, so are G_1 and W_1, and the subspaces $\ker W_1 \subseteq \mathcal{C}(\mathcal{I}, \mathbb{R}^k)$ and $\ker W_1 \subseteq L^2(\mathcal{I}, \mathbb{R}^k)$ are closed in fact.
We derive for each arbitrary $x \in \operatorname{dom} T$:

$$\begin{aligned} Tx &= A(Dx)' + Bx = G_1(D^-(Dx)' + Q_0x) + BP_0x \\ &= G_1(D^-(Dx)' + Q_0x) + G_1G_1^+BP_0x + W_1BP_0x. \end{aligned}$$

Condition (40) yields $W_1BP_0x = W_1FP_0x = 0$, which immediately implies $W_1(Tx) = 0$. It follows that $\operatorname{im} T \subseteq \ker W_1$. Next we verify that, for each arbitrary $q \in \ker W_1$, there exists at least one $x \in \operatorname{dom} T$ satisfying $Tx = q$.
We fix a $t_a \in \mathcal{I}$. The IVP

$$u' - R'u + DG_1^+BD^-u = DG_1^+q, \quad u(t_a) = 0$$

has a unique solution $u \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)$ and $u \in H^1(\mathcal{I}, \mathbb{R}^n)$, accordingly. It holds that $u = Ru$, thus $Ru' = -DG_1^+BD^-u + DG_1^+q$.
Put $x := D^-u - Q_0G_1^+BD^-u + Q_0G_1^+q$ such that $Dx = DD^-u = u$, thus $x \in \operatorname{dom} T$, $P_0x = D^-u$, $Q_0x = -Q_0G_1^+BD^-u + Q_0G_1^+q$, and further

$$\begin{aligned} Tx &= A(Dx)' + Bx = G_1(D^-(Dx)' + Q_0x) + BP_0x \\ &= G_1(D^-(Dx)' + Q_0x) + G_1G_1^+BP_0x \\ &= G_1\{D^-u' + Q_0x + G_1^+BP_0x\} \\ &= G_1\{D^-(-DG_1^+BD^-u + DG_1^+q) + Q_0x + Q_0G_1^+BP_0x + P_0G_1^+BP_0x\} \\ &= G_1\{-P_0G_1^+BP_0x + P_0G_1^+q + Q_0x + Q_0G_1^+BP_0x + P_0G_1^+BP_0x\} \\ &= G_1\{P_0G_1^+q + Q_0x + Q_0G_1^+BP_0x\} = G_1\{P_0G_1^+q + Q_0G_1^+q\} = G_1G_1^+q = q. \end{aligned}$$

In consequence, q belongs to $\operatorname{im} T$, and hence $\operatorname{im} T = \ker W_1$. □

Though the operator T has here a closed image, it is not necessarily fredholm. The nullspace may fail to be finite-dimensional. Furthermore, if $W_1 \neq 0$, then there is no finite codimension.

The DAE described in Theorem 3.2 is tractable with index 0, if $W_0FQ_0 = 0$, and tractable with index 1 otherwise, see [50].
For $W_0 = 0$, $m = k$, the DAE is even regular with index 0. For $W_0 \neq 0$, $W_1 = 0$, and $m = k$, the DAE is regular with tractability index 1.

The matrix pair $\{E, F\}$ in Example 2.4 obviously satisfies the condition $W_0F = 0$. The associated DAE is tractable with index 1. The same happens for [77, Example, p.485]. In both instances, the differential-algebraic operators are normally solvable in fact. A large class of differential-algebraic operators which satisfy the conditions of Theorem 3.2 is associated with so-called *strangeness-free DAEs*, as we show by the following example.

Example 3.3 (Strangeness-free DAEs) *Many papers start supposing a so-called strangeness-free DAE. We recall the respective description of this class, see [46, Definition 2.4]: The DAE (27) is said to be strangeness-free, if there are pointwise orthogonal matrix functions $L \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{R}^k))$ and $K \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{R}^m))$ such that*

$$LEK = \tilde{E} = \begin{bmatrix} \tilde{E}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad LFK + LEK' = \tilde{F} = \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{12} & \tilde{F}_{13} \\ \tilde{F}_{21} & \tilde{F}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with nonsingular blocks $\tilde{E}_{11}, \tilde{F}_{22}$, and all blocksizes are allowed to be zero.
For strangeness-free DAEs all conditions of Theorem 3.2 are satisfied so that the associated operator T is normally solvable. Namely, we have $\operatorname{im} \tilde{E} = L \operatorname{im} E$, $\ker E = K \ker \tilde{E}$. *We set and compute*

$$\tilde{Q}_0 = \tilde{W}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \ \tilde{A} = \tilde{E}, \ \tilde{D} = \tilde{P}_0, \ \tilde{B} = \tilde{F},$$

$$\tilde{W}_0\tilde{F}\tilde{Q}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tilde{F}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \tilde{G}_1 = \begin{bmatrix} \tilde{E}_{11} & \tilde{F}_{12} & \tilde{F}_{13} \\ 0 & \tilde{F}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \tilde{W}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix},$$

thus $\tilde{W}_1\tilde{F} = 0$. Then we set $P_0 = K\tilde{P}_0K^{-1}$ and factorize $E = AD$, $A = E$, $D = P_0$. P_0 is the orthoprojector function onto $\ker E$, *and the decomposition* $\ker E \oplus \operatorname{im} P_0 = \mathbb{R}^m$ *is valid.*
$W_0 = L^{-1}\tilde{W}_0L$ is the orthoprojector function with $\ker W_0 = \operatorname{im} E$. *Regarding that $\tilde{W}_0LE = 0$ we derive*

$$\begin{aligned} W_0FQ_0 &= L^{-1}\tilde{W}_0LFK\tilde{Q}_0K^{-1} \\ &= L^{-1}\tilde{W}_0(\tilde{F} - LEK')\tilde{Q}_0K^{-1} = L^{-1}\tilde{W}_0\tilde{F}\tilde{Q}_0K^{-1}, \end{aligned}$$

thus $\operatorname{rank} W_0FQ_0 = \operatorname{rank} \tilde{W}_0\tilde{F}\tilde{Q}_0 = \operatorname{rank} \tilde{F}_{22}$ *is constant. Compute further*

$$\begin{aligned} G_1 &= E + BQ_0 = E + (F - EP_0')Q_0, \\ LG_1K &= LEK + L(F - EP_0')KK^{-1}Q_0K = \tilde{E} + LFK\tilde{Q}_0 - LEP_0'K\tilde{Q}_0 \\ &= \tilde{E} + (\tilde{F} - LEK')\tilde{Q}_0 - LEP_0'K\tilde{Q}_0 \\ &= \tilde{E} + \tilde{F}\tilde{Q}_0 - LE(P_0K' + P_0'K)\tilde{Q}_0 = \tilde{G}_1 - LE(P_0K)'\tilde{Q}_0. \end{aligned}$$

Next, $W_1 = L^{-1}\tilde{W}_1 L$ *is the orthoprojector function with* $\ker W_1 = \operatorname{im} G_1$*. Regarding that* $\tilde{W}_1 LE = 0$ *we finally derive*

$$\begin{aligned} W_1 F P_0 &= L^{-1}\tilde{W}_1 LFK\tilde{P}_0 K^{-1} \\ &= L^{-1}\tilde{W}_1(\tilde{F} - LEK')\tilde{P}_0 K^{-1} = L^{-1}\tilde{W}_1\tilde{F}\tilde{P}_0 K^{-1} = 0. \end{aligned}$$

We mention that such a strangeness-free DAE has tractability index 0, if $\tilde{F}_{22}$ *has size zero, and otherwise tractability index 1, see[50].* □

Next we deal with the linear IVP

$$E(t)x'(t) + F(t)x(t) = q(t),\ t \in \mathcal{I}, \quad Cx(t_a) = d. \tag{41}$$

As before, the coefficients E and F are continuous, $\ker E$ is a $\mathcal{C}^1$-subspace, and E has constant rank $r > 0$. Let the matrix $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^r)$ have full row-rank r, and

$$\ker C = \ker E(t_a). \tag{42}$$

We consider the previous differential-algebraic operator $\mathring{T} \in \mathcal{L}(X, Y)$ corresponding to the DAE as well as its composed associate $\mathring{\mathcal{T}} \in \mathcal{L}(X, Y \times \mathbb{R}^r)$ defined by

$$\mathring{\mathcal{T}}x := (\mathring{T}x, Cx(t_a)),\ x \in \operatorname{dom}\mathring{\mathcal{T}} = \operatorname{dom}\mathring{T}.$$

Since the composed operator is also unbounded but closable, we immediately turn to its closure $\mathcal{T} \in \mathcal{L}_c(X, Y \times \mathbb{R}^r)$,

$$\mathcal{T}x := (Tx, Cx(t_a)),\ x \in \operatorname{dom}\mathcal{T} = \operatorname{dom} T.$$

We continue using the previous notations, e.g. (34), (35). Additionally, we introduce the matrix function $U \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{R}^r))$ to be the unique solution of the IVP

$$U' - R'U + DG_1^+ BD^- U = 0, \quad U(t_a) = I. \tag{43}$$

U is the fundamental solution matrix of ODE in (43) normalized at t_a. It satisfies the conditions

$$U(t)R(t_a) = R(t)U(t)R(t_a), \quad U(t)^{-1}R(t) = R(t_a)U(t)^{-1}R(t), \quad t \in \mathcal{I}.$$

Theorem 3.4 *Let* E *and* F *be continuous,* $\ker E$ *be a* $\mathcal{C}^1$*-subspace, and let the matrix function* $W_0 F Q_0$ *have constant rank. Let the matrix* $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^r)$ *have full row-rank* $r = \operatorname{rank} E$ *and satisfy the condition* $\ker C = \ker E(t_a)$*. Additionally, let the inclusion*

$$\ker(E + FQ_0 + EQ_0') \subseteq \ker E \tag{44}$$

be satisfied pointwise on $\mathcal{I}$*. Then the following statements are true for the two choices concerning the function spaces*

$$X = \mathcal{C}(\mathcal{I}, \mathbb{R}^m),\ Y = \mathcal{C}(\mathcal{I}, \mathbb{R}^k),\ X^1 = \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m),\ X_D^1 = \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m), \tag{45}$$

and

$$X = L^2(\mathcal{I}, \mathbb{R}^m),\ Y = L^2(\mathcal{I}, \mathbb{R}^k),\ X^1 = H^1(\mathcal{I}, \mathbb{R}^m),\ X_D^1 = H_D^1(\mathcal{I}, \mathbb{R}^m), \tag{46}$$

respectively, and for the closure $\mathcal{T} \in \mathcal{L}_c(X, Y \times \mathbb{R}^r)$*,* $\operatorname{dom}\mathcal{T} = X_D^1$*, of the operator* $\mathring{\mathcal{T}} \in \mathcal{L}(X, Y \times \mathbb{R}^r)$ *associated with the IVP (41):*

(1) *The operator $\mathcal{T}$ is normally solvable and*

$$\operatorname{im}\mathcal{T} = \{(q,d) \in Y \times \mathbb{R}^r :$$
$$W_1 q = W_1 B D^- U(D(t_a)C^+ d + \int_{t_a} U(s)^{-1} D(s) G_1(s)^+ q(s) \mathrm{ds}\,)\}.$$

(2) $\ker\mathcal{T} = \{x \in X : P_1 x = 0\} \subset \{x \in X : Dx = 0\} \subset X_D^1$.

(3) *If $E + FQ_0$ has full column-rank, then $\mathcal{T}$ is injective.*

(4) *If $(q,d) \in \operatorname{im}\mathcal{T}$, then there exists a unique $x_* \in X_D^1$ satisfying $\mathcal{T}x_* = (q,d)$ and $Q_1 x_* = 0$, as well as the inequality*

$$\|x_*\|_{L^2} < \|x\|_{L^2} \tag{47}$$

for all other solutions x of the equation $\mathcal{T}x = (q,d)$,

Proof: (1) Since E and $W_0 F Q_0$ are continuous constant-rank matrix functions, so are G_1 and W_1. The inclusion (44) means actually $\ker G_1 \subseteq \ker G_0$.
For given $x \in X_D^1$ and $d := Cx(t_a)$, $q := Tx$ we derive

$$\begin{aligned} q = Tx &= A(Dx)' + Bx = ADD^-(Dx)' + BQ_0 x + BP_0 x \\ &= G_1(D^-(Dx)' + Q_0 x) + G_1 G_1^+ B P_0 x + W_1 B P_0 x \\ &= G_1(D^-(Dx)' + Q_0 x + G_1^+ B P_0 x) + W_1 B P_0 x, \end{aligned}$$

thus

$$G_1 G_1^+ q = G_1(D^-(Dx)' + Q_0 x + G_1^+ B P_0 x) \text{ and } W_1 q = W_1 B P_0 x. \tag{48}$$

From the first part we obtain the relation

$$D^-(Dx) + Q_0 x + G_1^+ B P_0 x - G_1^+ q = \xi, \tag{49}$$

whereby ξ is an arbitrary function belonging to $\ker P_1 \subseteq X$. Owing to the properties $\ker G_1 \subseteq \ker G_0$ and $\ker D = \ker G_0$ it follows that $D\xi = 0$, $\xi = Q_0\xi$, $\xi \in X_D^1$. Now (49) decomposes into the system

$$\begin{aligned} (Dx)' - R'Dx + DG_1^+ B P_0 x - DG_1^+ q = D\xi = 0, \\ Q_0 x + Q_0 G_1^+ B P_0 x - Q_0 G_1^+ q = Q_0 \xi = \xi. \end{aligned}$$

From $d = Cx(t_a)$ we obtain $D(t_a)x(t_a) = D(t_a)C^+ d$. This yields the representation

$$\begin{aligned} x &= D^- Dx + Q_0 x \\ &= (I - Q_0 G_1^+ B) D^- U\{D(t_a)C^+ d + \int_{t_a} U(s)^{-1} D(s) G_1(s)^+ q(s) \mathrm{ds}\} \\ &\quad + Q_0 G_1^+ q + \xi, \end{aligned} \tag{50}$$

further

$$Dx = U\{D(t_a)C^+ d + \int_{t_a} U(s)^{-1} D(s) G_1(s)^+ q(s) \mathrm{ds}\}. \tag{51}$$

Now it is evident that q must satisfy the consistency condition

$$W_1 q = W_1 B D^- U\{D(t_a)C^+ d + \int_{t_a} U(s)^{-1} D(s) G_1(s)^+ q(s)\mathrm{d}s\}. \qquad (52)$$

Conversely, the equation $\mathcal{T}x = (q, d)$ is solvable for each arbitrary $d \in \mathbb{R}^r$, $G_1 G_1^+ q \in Y$, and the corresponding $W_1 q$ defined by (52). Namely, for such d and q, we put

$$\tilde{x} := (I - Q_0 G_1^+ B) D^- U\{D(t_a)C^+ d + \int_{t_a} U(s)^{-1} D(s) G_1(s)^+ q(s)\mathrm{d}s\} + Q_0 G_1^+ q$$

and obtain

$$\begin{aligned} D\tilde{x} &= DD^- U\{D(t_a)C^+ d + \int_{t_a} U(s)^{-1} D(s) G_1(s)^+ q(s)\mathrm{d}s\} \\ &= U\{D(t_a)C^+ d + \int_{t_a} U(s)^{-1} D(s) G_1(s)^+ q(s)\mathrm{d}s\}. \end{aligned}$$

It becomes clear that $\tilde{x}$ belongs to $\operatorname{dom} T = X_D^1$ and

$$C\tilde{x}(t_a) = CD(t_a)^- D(t_a)\tilde{x}(t_a) = CD(t_a)^- D(t_a) C^+ d = CC^+ d = d.$$

Finally, one finds that $T\tilde{x} = q$. This proves that

$$\begin{aligned} \operatorname{im} \mathcal{T} = \{(q, d) \in Y \times \mathbb{R}^r : \\ W_1 q = W_1 B D^- U(D(t_a)C^+ d + \int_{t_a} U(s)^{-1} (D G_1^+ q)(s)\mathrm{d}s)\}. \end{aligned}$$

in fact. This set is obviously closed.
(2) For $d = 0$ and $q = 0$, we obtain the general solution (cf.(50)) $x = \xi$, such that $ker P_1 \supseteq \ker \mathcal{T}$. Conversely, we compute for $x \in \ker P_1$, that is, for $x = (I - P_1)x = Q_0(I - P_1)x$:

$$Tx = A(Dx)' + Bx = G_1(D^-(Dx)' + Q_0 x) + BP_0 x = G_1 Q_0 x = G_1 x = 0.$$

(3) Regarding the relation $\operatorname{rank} G_1 = \operatorname{rank}(E + FQ_0)$, this statement is a simple consequence of (2).
(4) Since P_1, Q_1 are complementary orthoprojector functions we have

$$\|x\|_{L^2}^2 = \|P_1 x\|_{L^2}^2 + \|Q_1 x\|_{L^2}^2.$$

For all solutions x_*, the component $P_1 x_*$ is completely fixed by d and q. The only free component is $\xi_* = Q_1 \xi_*$. Letting

$$\begin{aligned} \xi_* = &- Q_1(I - Q_0 G_1^+ B) D^- U\{D(t_a)C^+ d + \int_{t_a} U(s)^{-1} D(s) G_1(s)^+ q(s)\mathrm{d}s\} \\ &- Q_1 Q_0 G_1^+ q, \end{aligned}$$

one arrives at the only solution with $Q_1 x_* = 0$. □

The DAEs captured by Theorem 3.4 comprise a number of equation k that can be less, equal and larger than the number of unknown functions m. The DAEs are tractable with index 0, if $W_0 F Q_0 = 0$, and otherwise tractable with index 1. The corollary below specifies the index-0 case.

Corollary 3.5 *Let E and F be continuous,* $\ker E$ *be a* $\mathcal{C}^1$*-subspace. Let the matrix* $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^r)$ *have full row-rank* $r = \operatorname{rank} E$ *and satisfy* $\ker C = \ker E(t_a)$*.*
Additionally, let the condition

$$FQ_0 + EQ_0' = 0 \tag{53}$$

be fulfilled. Then all statements of Theorem 3.4 are valid with W_0*,* P_0*,* G_0^+ *instead of* W_1*,* P_1*,* G_1^+*.*
Moreover, $\operatorname{rank} E = \operatorname{rank}(E + FQ_0)$*.*

Proof: Condition (53) is a special case of the inclusion (44), which leads to $W_0FQ_0 = 0$, $G_1 = G_0$, $E + FQ_0 = E(I - P_0Q_0')$, and hence $W_1 = W_0$, $P_1 = P_0$, $\operatorname{rank}(E + FQ_0) = \operatorname{rank} E$. □

Example 3.6 (Overdetermined index-0 DAE) *The IVP for the overdetermined DAE (27) with* $m = 2$*,* $k = 3$*, and the coefficients*

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

leads to the closure $T = \mathring{T}$*, with the trivial factorization* $E = AD$*,* $A = E$*,* $D = I$*,* $n = 2$*. The operator T is injective. The corresponding DAE is tractable with index 0. We have* $R = I$*,* $D^- = I$*,* $B = F$ *and*

$$W_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Q_0 = 0,\ P_0 = I,\ G_1 = G_0,\ W_1 = W_0,\ G_0^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and, trivially, $W_0FQ_0 = 0$ *so that Corollary 3.5 applies. Derive further*

$$(DG_0^+BD^-)(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad U(t) = \begin{bmatrix} 1 & 0 \\ -(t - t_a) & 1 \end{bmatrix}.$$

The consistency condition for q and d (cf.(52)) reads here

$$q_3 = -(t - t_a)(d_1 + \int_{t_a} q_1(s)\mathrm{d}s) + d_2 + \int_{t_a} ((s - t_a)q_1(s) + q_2(s))\mathrm{d}s.$$

□

The following proposition generalizes the previous results. The structural conditions (40) and (44) are special instances of the condition (54) below. This is obvious for (40). Since (44) implies the inclusion $\ker G_1 \subseteq \ker G_0 = \ker D$, we have also $P_0(I - P_1) = 0$. Also condition (57) below is given in these particular instances. The differential-algebraic operator in Example 3.8 below meets the assumptions of the next proposition, but it satisfies neither condition (40) nor condition (44).

Proposition 3.7 *Let E and F be continuous,* $\ker E$ *be a* $\mathcal{C}^1$*-subspace, and let the matrix function* W_0FQ_0 *have constant rank. Let the matrix* $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^r)$

have full row-rank $r = \operatorname{rank} E$, and satisfy the condition $\ker C = \ker E(t_a)$. *Additionally, let the condition*

$$W_1 F P_0 (I - P_1) = 0 \tag{54}$$

be given. Then the following statements are true for the two choices concerning the function spaces

$$X = \mathcal{C}(\mathcal{I}, \mathbb{R}^m),\ Y = \mathcal{C}(\mathcal{I}, \mathbb{R}^k),\ X^1 = \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m),\ X^1_D = \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m), \tag{55}$$

and

$$X = L^2(\mathcal{I}, \mathbb{R}^m),\ Y = L^2(\mathcal{I}, \mathbb{R}^k),\ X^1 = H^1(\mathcal{I}, \mathbb{R}^m),\ X^1_D = H^1_D(\mathcal{I}, \mathbb{R}^m), \tag{56}$$

respectively, and for the closure $\mathcal{T} \in \mathcal{L}_c(X, Y \times \mathbb{R}^r)$, $\operatorname{dom} \mathcal{T} = X^1_D$, of the operator $\mathring{\mathcal{T}} \in \mathcal{L}(X, Y \times \mathbb{R}^r)$ associated with the IVP (41):

(1) *The map $\mathcal{H} \in \mathcal{L}(X)$ defined by*

$$\mathcal{H}p = Q_0 p + (I - Q_0 G_1^+ B) D^- U \int_{t_a} U(s)^{-1} D(s) p(s) \mathrm{d}s, \quad p \in X,$$

is bounded and

$$\operatorname{im} \mathcal{H} = \mathring{X}^1_D(\mathcal{I}, \mathbb{R}^m) := \{x \in X^1_D(\mathcal{I}, \mathbb{R}^m) : D(t_a) x(t_a) = 0\}.$$

(2) $\|\mathcal{H}p\|_{X^1_D} \leq C \|p\|_X$ *for all $p \in X$.*

(3) *If, additionally,*

$$\xi \in \ker P_1 \subseteq X \text{ implies } W_1 B P_0 \mathcal{H} \xi = 0, \tag{57}$$

then $\ker \mathcal{T} = \mathcal{H}(\ker P_1)$ and $\mathcal{T}$ has a closed image, namely

$$\operatorname{im} \mathcal{T} = \{(q, d) \in Y \times \mathbb{R}^r : W_1 q = W_1 B D^- (D \mathcal{H}(G_1^+ q) + U D(t_a) C^+ d)\}.$$

(4) *Supposed condition (57) is valid, for each arbitrary $(q, d) \in \operatorname{im} \mathcal{T}$, the solutions $x_* \in \operatorname{dom} \mathcal{T}$ of the equation $\mathcal{T} x = (q, d)$ have the form*

$$x_* = (I - Q_0 G_1^+ B) D^- U D(t_a) C^+ d + \mathcal{H}(G_1^+ q + \xi),$$

with an arbitrary part $\xi \in \ker P_1$.

Proof: (1) The boundedness of $\mathcal{H}$ is evident. We investigate the image of the map $\mathcal{H}$. $p \in X$ implies $\mathcal{H}p \in X$ and $D\mathcal{H}p = DD^- U \int_{t_a} U(s)^{-1} D(s) p(s) \mathrm{d}s \in X^1(\mathcal{I}, \mathbb{R}^n)$, and hence $\mathcal{H}p \in \mathring{X}^1_D(\mathcal{I}, \mathbb{R}^m)$.
Next we show that the equation $\mathcal{H}p = x$ is solvable for each arbitrary $x \in \mathring{X}^1_D(\mathcal{I}, \mathbb{R}^m)$. Let $x \in \mathring{H}^1_D(\mathcal{I}, \mathbb{R}^m)$ be given. We set $\tilde{p} = D^- D \tilde{p} + Q_0 \tilde{p}$,

$$\begin{aligned} D\tilde{p} &:= (Dx)' - R' D x + D G_1^+ B D^- D x, \\ Q_0 \tilde{p} &:= Q_0 x + Q_0 G_1^+ B D^- U \int_{t_a} U(s)^{-1} D(s) \tilde{p}(s) \mathrm{d}s. \end{aligned}$$

It follows that

$$
\begin{aligned}
Dx &= U\int_{t_a} U(s)^{-1}D(s)\tilde{p}(s)\mathrm{ds},\\
\mathcal{H}\tilde{p} &= Q_0\tilde{p} + (I - Q_0G_1^+B)D^-U\int_{t_a} U(s)^{-1}D(s)\tilde{p}(s)\mathrm{ds}\\
&= Q_0x + D^-U\int_{t_a} U(s)^{-1}D(s)\tilde{p}(s)\mathrm{ds},\\
&= Q_0x + D^-Dx = x.
\end{aligned}
$$

This proves that $\operatorname{im}\mathcal{H} = \mathring{X}^1_D(\mathcal{I},\mathbb{R}^m)$.
(2) Regarding the relation

$$
(D\mathcal{H}p)' = (RU)'\int_{t_a} U(s)^{-1}D(s)p(s)\mathrm{ds} + RUU^{-1}Dp, \quad p \in X,
$$

the inequality follows immediately. (4) We decompose $Tx - q = 0$ into $G_1G_1^+(Tx - q) = 0$ and $W_1(Tx - q) = 0$, that is,

$$
G_1(D^-(Dx)' + Q_0x + G_1^+BP_0x - G_1^+q) = 0, \quad W_1BP_0x - W_1q = 0,
$$

and, equivalently,

$$
D^-(Dx)' + Q_0x + G_1^+BP_0x - G_1^+q = \xi \in \ker P_1, \quad W_1q = W_1BP_0x. \tag{58}
$$

For their part, the first equation decomposes into

$$
\begin{aligned}
DD^-(Dx)' + DG_1^+BP_0x - DG_1^+q &= D\xi,\\
Q_0x + Q_0G_1^+BP_0x - Q_0G_1^+q &= Q_0\xi.
\end{aligned}
$$

We reformulate the last systems once more as

$$
\begin{aligned}
(Dx)' - R'Dx + DG_1^+BD^-Dx &= D(G_1^+q + \xi), \qquad (59)\\
Q_0x + Q_0G_1^+BD^-Dx &= Q_0(G_1^+q + \xi).
\end{aligned}
$$

The initial condition $Cx(t_a) = d$ means $D(t_a)x(t_a) = D(t_a)C^+d$. Together with (59) this yields the representations

$$
Dx = UD(t_a)C^+d + U\int_{t_a} U(s)^{-1}D(s)(G_1(s)^+q(s) + \xi(s))\mathrm{ds},
$$

as well as

$$
\begin{aligned}
x = D^-Dx + Q_0x &= D^-Dx - Q_0G_1^+BD^-Dx + Q_0(G_1^+q + \xi)\\
&= (I - Q_0G_1^+B)D^-Dx + Q_0(G_1^+q + \xi)\\
&= \mathcal{H}(G_1^+q + \xi) + (I - Q_0G_1^+B)D^-UD(t_a)C^+d.
\end{aligned}
$$

The second equation of (58) reformulates now as

$$
\begin{aligned}
W_1q &= W_1BD^-Dx\\
&= W_1BD^-\{UD(t_a)C^+d + U\int_{t_a} U(s)^{-1}D(s)(G_1(s)^+q(s) + \xi(s))\mathrm{ds}\}\\
&= W_1BD^-\{UD(t_a)C^+d + D\mathcal{H}(G_1^+q + \xi)\}\\
&= W_1BD^-\{UD(t_a)C^+d + D\mathcal{H}(G_1^+q) + D\mathcal{H}(\xi)\}.
\end{aligned}
$$

Regarding condition (57) we find the relation

$$W_1 q = W_1 B D^- \{U D(t_a) C^+ d + D\mathcal{H}(G_1^+ q)\}. \tag{60}$$

Therefore, if $d \in \mathbb{R}^r$ and $q \in Y$ are given, and (60) is satisfied, then $\tilde{x} := \mathcal{H}(G_1^+ q + \xi) + (I - Q_0 G_1^+ B) D^- U D(t_a) C^+ d$, with arbitrary $\xi \in \ker P_1$ satisfies $\mathcal{T}\tilde{x} = (q, d)$. This proves the statement.
(3) If $x \in \operatorname{dom} T$ is given, $q := Tx$, $d := Cx(t_a)$, then condition (60) must be valid. The resulting set $\operatorname{im}\mathcal{T}$ is obviously closed in $Y \times \mathbb{R}^r$.
For $q = 0$, $d = 0$ we obtain the solution representation $x = \mathcal{H}\xi$. Therefore, the nullspace of $\mathcal{T}$ is formed by the functions $x = \mathcal{H}\xi$ with $\xi \in \ker P_1$. □

Condition (40) trivially implies both (54) and (57). In this case, $\operatorname{im}\mathcal{T}$ has a very simple structure, see Theorem 3.2.
Condition (44) also implies both (54) and (57). Because of $D\xi = 0$ we have $\mathcal{H}\xi = Q_0\xi = \xi$, and hence $\ker\mathcal{T} = \ker P_1$.
In the following example, neither (40) nor (44) are satisfied, but (54) and (57) are valid.

Example 3.8 (Supplementary DAE for Proposition (3.7)) *The DAE with constant coefficients*

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; F = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

has (nonregular) tractability index 0, see [50]. We apply the factorization $E = AD$, $A = E$, $D = E^+E$ *such that* $D^- = D^+$. *We have then*

$$P_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; G_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; G_1^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \; P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

further

$$W_1 = W_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; W_1 F P_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \neq 0, \quad W_1 F P_0 (I - P_1) = 0.$$

$\ker E$ *does not include* $\ker G_1$, *and neither condition (40) nor (44) is satisfied, but condition (54) is given. Taking into account that*

$$D G_1^+ F D^- = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \quad U(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2}(t - t_a) & 0 & 1 \end{bmatrix},$$

we verify that $W_1 F P_0 \mathcal{H}\xi = 0$ *for all functions* $\xi \in \ker P_1$, *thus condition (57) is valid in fact.* □

3.2 Settings with bounded differential-algebraic operators

The available theory of bounded operators in Banach spaces and Hilbert spaces is rich in content. Bounded operators are favorable in many situations; for instance, when aiming to investigate smooth nonlinear problems (cf. Section 4) and when looking for least-squares solutions (cf. Theorem 6.5.

Fixing a proper factorization $E = AD$ according to Theorem 3.1(3), we turn from the standard form DAE (27) to the DAE with properly stated leading term

$$A(t)(Dx)'(t) + B(t)x(t) = q(t), \quad t \in \mathcal{I} = [t_a,\ t_e], \tag{61}$$

whereby $A \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k))$, $D \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$, $B \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^k))$, and $\ker A$, $\operatorname{im} D$ are $\mathcal{C}^1$-subspaces satisfying the transversality condition (30). The function spaces $\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ and $H^1_D(\mathcal{I}, \mathbb{R}^m)$ endowed with the norm $\|\cdot\|_{\mathcal{C}^1_D}$ and the inner product $(\cdot,\cdot)_{H^1_D}$, become a Banach space and a Hilbert space, respectively, see Lemma 6.9.

The differential-algebraic operator $T \in \mathcal{L}(\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m), \mathcal{C}(\mathcal{I}, \mathbb{R}^m))$ defined by

$$Tx = A(Dx)' + Bx, \quad x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m), \tag{62}$$

is bounded. We continue using the denotations (33)-(35).

$$\begin{aligned} &G_0 = AD,\ P_0 = G_0^+ G_0,\ Q_0 = I - P_0,\ W_0 = I - G_0 G_0^+, \\ &G_1 = G_0 + BQ_0,\ W_1 = I - G_1 G_1^+. \end{aligned}$$

The next theorem states sufficient conditions for T to be normally solvable in terms of the original data.It can be seen as natural counterpart of Theorem 3.2 (1).

Theorem 3.9 *Assume $A \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k))$, $D \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$, $\ker A$ is a $\mathcal{C}^1$-subspace, and $B \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^k))$, and let the transversality condition (30) be given.*
Let the matrix function W_0BQ_0 have constant rank on the given interval $\mathcal{I}$ and let the condition

$$W_1 B P_0 = 0 \tag{63}$$

be satisfied. Then the operator T defined by (62) is bounded and normally solvable, and

$$\operatorname{im} T = \ker W_1 \subseteq \mathcal{C}(\mathcal{I}, \mathbb{R}^k). \tag{64}$$

Proof: The arguments used in the proof of Theorem 3.2(1) apply again. □

The particular case if G_1 has pointwise full row-rank is quite important for minimization with differential-algebraic constraints, see [56]. Then, it holds that $W_1 = 0$ and the corresponding differential-algebraic operator is surjective.
Strangeness-free DAEs yield a special class of bounded and normally solvable operators T, see Example 3.3.

Analogously, the operator $T \in \mathcal{L}(H^1_D(\mathcal{I}, \mathbb{R}^m), L^2(\mathcal{I}, \mathbb{R}^m))$ defined by

$$Tx = A(Dx)' + Bx, \quad x \in H^1_D(\mathcal{I}, \mathbb{R}^m), \tag{65}$$

is bounded. The first statement of the next theorem represents the counterpart of Theorem 3.2 (2).

Theorem 3.10 *Let the assumptions of Theorem 3.9 be satisfied.*

(1) *Then the operator T defined by (65) is bounded and normally solvable, with*

$$\operatorname{im} T = \ker W_1 \subseteq L^2(\mathcal{I}, \mathbb{R}^k). \tag{66}$$

(2) *For each $q \in L^2(\mathcal{I}, \mathbb{R}^k)$,*

$$\inf\{\|Tx - q\|_{H_D^1} : x \in H_D^1\} \tag{67}$$

is attained, and the orthogonal generalized (Moore-Penrose) inverse T^+ is bounded.

Proof: The arguments used in the proof of Theorem 3.2(2) apply once more, and verify part (1). Part (2) is then a consequence of Theorem 6.5. □

Naturally, also Theorem 3.4 and Proposition 3.7 can be adapted to settings with bounded differential-algebraic operators.

3.3 Least-squares solutions of IVPs and bounded generalized inverses of the composed operator

Let X and Y be Hilbert spaces and $K \in \mathcal{L}(X, Y)$ be a bounded or closed and densely defined operator. Then the equation $Kx = y$ possesses a least-squares solution (LSS) $x_* \in \operatorname{dom} K$ exactly if $y \in \operatorname{im} K + (\operatorname{im} K)^\perp$. Denote by $LSS_y \subseteq \operatorname{dom} K$ the set of all LSS corresponding to y. Since the nullspace of K is closed, the set LSS_y is either empty or linear affine and closed. Therefore, for each $y \in \operatorname{im} K + (\operatorname{im} K)^\perp$, there exists a unique minimum-norm LSS or pseudosolution. So far so good! However, when computing LSS in practice, one is confronted with the question whether a small residuum $\|K\tilde{x} - y\|$ ensures that $\tilde{x}$ is actually closed to a LSS or even to the pseudosolution. Unfortunately, for differential-algebraic operators in standard settings, even if y is consistent and the pseudosolution x_* exists, a minimizing sequence $\{x_l\}$ such that $\delta_l = \|Kx_l - y\| \xrightarrow{n\to\infty} 0$ does not necessarily converge; instead, $\|x_l - x_*\|$ may grow unboundedly, see Example 2.5 and its continuation by Example 2.7. This essentially ill-posed behavior is caused by the nonclosedness of the image of the operator. *The calculation of a LSS and the pseudosolution is practically safe only for operators with closed image*, that means, for normally solvable operators, otherwise regularization techniques should be applied, see Section 5.

We continue investigating the linear IVP (41), that is,

$$E(t)x'(t) + F(t)x(t) = q(t),\ t \in \mathcal{I}, \quad Cx(t_a) = d.$$

The coefficients E and F are continuous, $\ker E$ is a $\mathcal{C}^1$-subspace, and E has constant rank $r > 0$. The matrix $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^r)$ has full row-rank r, and condition 42 is valid, that is,

$$\ker C = \ker E(t_a).$$

We choose now the Hilbert spaces

$$X = L^2(\mathcal{I}, \mathbb{R}^m), \quad Y = L^2(\mathcal{I}, \mathbb{R}^k) \tag{68}$$

as pre-image and image spaces. The differential-algebraic operator $\mathring{T} \in \mathcal{L}(X,Y)$ as well as its composed associate $\mathring{\mathcal{T}} \in \mathcal{L}(X, Y \times \mathbb{R}^r)$ defined by

$$\mathring{T}x := Ex' + Fx, \ x \in \operatorname{dom} \mathring{T} = \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m) \subseteq X,$$
$$\mathring{\mathcal{T}}x := (\mathring{T}x, Cx(t_a)), \ x \in \operatorname{dom} \mathring{\mathcal{T}} = \mathring{T}.$$

are unbounded but closable, we immediately turn to their closures (cf. Subsection 3.1) given by $T \in \mathcal{L}_c(X,Y)$ and $\mathcal{T} \in \mathcal{L}_c(X, Y \times \mathbb{R}^r)$,

$$Tx := A(Dx)' + Bx, \ x \in \operatorname{dom} T = H^1_D(\mathcal{I}, \mathbb{R}^m) \subseteq X,$$
$$\mathcal{T}x := (Tx, Cx(t_a)), \ x \in \operatorname{dom} \mathcal{T} = \operatorname{dom} T.$$

whereby we apply a proper factorization $E = AD$ being consistent with Theorem 3.1. Owing to the property $\ker C = \ker E(t_a) = \ker D(t_a)$ the expression $Cx(t_a) = CD(t_a)^- D(t_a)x(t_a)$ makes sense for each arbitrary $x \in H^1_D(\mathcal{I}, \mathbb{R}^m)$. We continue using the notations introduced in Subsection 3.1.

Definition 3.11 *For given $q \in Y$, $d \in \mathbb{R}^r$, the function $x_* \in H^1_D(\mathcal{I}, \mathbb{R}^m)$ is called least-squares solution of the IVP (41), if it represents a LSS of the operator equation $\mathcal{T}x = (q,d)$, that means*

$$\begin{aligned} &\|Tx_* - q\|^2_{L^2} + |Cx_*(t_a) - d|^2 \\ &\qquad = \inf\{\|Tx - q\|^2_{L^2} + |Cx(t_a) - d|^2 : x \in H^1_D(\mathcal{I}, \mathbb{R}^m)\}. \end{aligned} \tag{69}$$

A LSS x_* is called pseudosolution of the IVP (41), if the inequality $\|x_*\|_{L^2} \leq \|\tilde{x}_*\|_{L^2}$ is valid for all further LSS $\tilde{x}_*$.

Proposition 3.7 ensures the normal solvability of the operator $\mathcal{T} \in \mathcal{L}_c(X, Y \times \mathbb{R}^r)$. In particular, since $\ker \mathcal{T}$ and $\operatorname{im} \mathcal{T}$ are closed, the orthogonal direct sum decompositions

$$X = \ker \mathcal{T} \oplus (\ker \mathcal{T})^\perp, \quad Y \times \mathbb{R}^r = \operatorname{im} \mathcal{T} \oplus (\operatorname{im} \mathcal{T})^\perp$$

are valid, and there exist symmetric bounded projectors $\mathfrak{P}$ and $\mathfrak{R}$ acting on X and $Y \times \mathbb{R}^r$, respectively, such that

$$\operatorname{im} \mathfrak{P} = (\ker \mathcal{T})^\perp, \quad \operatorname{im} \mathfrak{R} = \operatorname{im} \mathcal{T}.$$

For each arbitrary $(q,d) \in Y \times \mathbb{R}^r$, Proposition 3.7(4) provides the solutions x_* of the equation $\mathcal{T}x = \mathfrak{R}(q,d) =: (q_{\mathfrak{R}}, d_{\mathfrak{R}})$ as

$$x_* = (I - Q_0 G_1^+ B) D^- U D(t_a) C^+ d_{\mathfrak{R}} + \mathcal{H}(G_1^+ q_{\mathfrak{R}} + \xi), \tag{70}$$

whereby the component $\xi \in \ker P_1$ can be chosen arbitrarily. Since the inequality

$$\begin{aligned} \|\mathcal{T}x - (q,d)\|^2 &= \|\mathcal{T}x - \mathfrak{R}(q,d)\|^2 + \|(I - \mathfrak{R})(q,d)\|^2 \\ &\geq \|(I - \mathfrak{R})(q,d)\|^2 = \|\mathcal{T}x_* - (q,d)\|^2 \end{aligned}$$

is valid for all $x \in \operatorname{dom} T$, each of those x_* is a LSS of the given IVP (41).

By omitting the free component $\xi \in \ker P_1$, a *special bounded generalized inverse* $\mathcal{T}^-$ of $\mathcal{T}$ results, as the following theorem based on Proposition 3.7 shows.

Theorem 3.12 *Let E and F be continuous,* $\ker E$ *be a $\mathcal{C}^1$-subspace, and let the matrix function W_0FQ_0 have constant rank. Let the matrix $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^r)$ have full row-rank $r = \operatorname{rank} E$, and satisfy the condition* $\ker C = \ker E(t_a)$. *Additionally, assume that $W_1FP_0(I - P_1) = 0$ and*

$$W_1BP_0\mathcal{H}\xi = 0 \textit{ for } \xi \in \ker P_1 \subseteq X.$$

(1) *Then the map $\mathcal{T}^- \in \mathcal{L}(Y \times \mathbb{R}^r, X)$ defined by*

$$\mathcal{T}^-(q,d) := (I - Q_0G_1^+B)D^-UD(t_a)C^+d_{\mathfrak{R}} + \mathcal{H}(G_1^+q_{\mathfrak{R}}),$$
$$\textit{for } (q,d) \in Y \times \mathbb{R}^r, \quad (q_{\mathfrak{R}}, d_{\mathfrak{R}}) := \mathfrak{R}(q,d).$$

is a bounded generalized inverse of the operator $\mathcal{T}$ such that $\mathcal{T}\mathcal{T}^- = \mathfrak{R}$.

(2) *If even $W_1FP_0 = 0$, then it results that* $\operatorname{im} \mathcal{T} = \ker W_1 \times \mathbb{R}^r$, $\mathfrak{R} = \operatorname{diag}(I - W_1, I)$, *and*

$$\mathcal{T}^-(q,d) := (I - Q_0G_1^+B)D^-UD(t_a)C^+d + \mathcal{H}(G_1^+q),$$
$$\textit{for } (q,d) \in Y \times \mathbb{R}^r.$$

Proof: (1) The map $\mathcal{T}^-$ is bounded by construction.
For each arbitrary $(q,d) \in Y \times \mathbb{R}^r$, the value $x_{(q,d)} := \mathcal{T}^-(q,d)$ fulfills the relation $\mathcal{T}x_{(q,d)} = \mathfrak{R}(q,d)$, and hence $\mathcal{T}\mathcal{T}^-(q,d) = \mathfrak{R}(q,d)$ as well as $\mathcal{T}^-\mathcal{T}\mathcal{T}^-(q,d) = \mathcal{T}^-\mathfrak{R}(q,d) = \mathcal{T}^-(q,d)$.
For each arbitrary $x \in \operatorname{dom} T$ and $q_x := Tx$, $d_x := Cx(t_a)$, it holds that $(q_x, d_x) = \mathfrak{R}(q_x, d_x)$, thus $\mathcal{T}x = (q_x, d_x) = \mathcal{T}\mathcal{T}^-(q_x, d_x) = \mathcal{T}\mathcal{T}^-\mathcal{T}x$.
(2) is a direct consequence of Theorem 3.2. □

The second statement applies in particular to the class of strangeness-free DAEs, see Example 3.3.

One can compute the value $\mathcal{T}^-(q,d)$ by solving first the standard IVP

$$u' = R'u - DG_1^+BD^-u + DG_1^+q_{\mathfrak{R}}, \quad u(t_a) = D(t_a)C^+d_{\mathfrak{R}}, \tag{71}$$

and letting

$$\mathcal{T}^-(q,d) = (I - Q_0G_1^+B)D^-u + Q_0G_1^+q_{\mathfrak{R}}. \tag{72}$$

At this place, we mention that, for regular DAEs with tractability index 0 and 1, the operator $\mathcal{T}$ is invertible and $\mathfrak{R} = I$. Then, of course, $\mathcal{T}^-$ and $\mathcal{T}^{-1}$ coincide. In the index-0 case, it results that $x = u$, and the IVP (71) reads simply $u' = -E^{-1}Fu + E^{-1}q, \quad u(t_a) = d$.

In general, $\mathcal{T}^-(q,d)$ is a special LSS, but it is not necessarily a pseudosolution. For reaching a pseudosolution, an extra minimization has to be carried out among the LSS. Nevertheless, the Theorem 3.4 allows a direct choice of the pseudosolution. Namely, owing to the assumptions of Theorem 3.4 the map $\mathcal{H}$ simplifies so that

$$\mathcal{H}\xi = Q_0\xi = \xi \;\text{ for }\; \xi \in \ker P_1,$$

and we have further

$$\ker \mathcal{T} = \ker P_1, \quad \mathfrak{P} = P_1.$$

Then the minimum norm LSS x_{**} is attained by applying (cf. Theorem 3.4)

$$\xi_* = -Q_1(I - Q_0G_1^+B)D^-U(D(t_a)C^+d_{\mathfrak{R}} - Q_1\mathcal{H}(G_1^+q_{\mathfrak{R}})$$

with $Q_1 := I - P_1$, yielding

$$x_{**} = P_1(I - Q_0G_1^+B)D^-UD(t_a)C^+d_{\mathfrak{R}} + P_1\mathcal{H}(G_1^+q_{\mathfrak{R}}).$$

Equivalently, we can compute the solution u of the IVP (71) and set

$$\xi_* = -Q_1(I - Q_0G_1^+B)D^-u - Q_1Q_0G_1^+q_{\mathfrak{R}}.$$

Again, the resulting

$$x_{**} = P_1(I - Q_0G_1^+B)D^-u - P_1Q_0G_1^+q_{\mathfrak{R}}$$

is the pseudosolution of the original IVP (41). Namely, Q_1, P_1 are complementary orthoprojectors and the part P_1x_* of each LSS x_* (cf. (70)) is completely independent of ξ. The choice $\xi = \xi_*$ leads to $Q_1x_{**} = 0$, thus

$$\|x_{**}\|_{L^2}^2 = \|\mathfrak{P}x_{**}\|_{L^2}^2 \leq (\|\mathfrak{P}x_{**}\|_{L^2}^2 + \|(I - \mathfrak{P})x_*\|_{L^2}^2) = \|x_*\|_{L^2}^2$$

is valid for all LSS x_*.
If, moreover, $G_1 = G_0$, then the DAE is tractable with index 0, and the pseudosolution is simply

$$x_{**} = D^-u.$$

Example 3.13 (Pseudosolution) *The DAE with the coefficients*

$$E(t) = \begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix}, \quad F(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

is tractable with index 0. The associated closed differential-algebraic operator has the coefficients

$$A(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}, D(t) = \begin{bmatrix} -1 & t \end{bmatrix}, B(t) = \begin{bmatrix} 1 & -t \\ 0 & 0 \end{bmatrix}, D(t)^- = \frac{1}{1+t^2}\begin{bmatrix} -1 \\ t \end{bmatrix}.$$

Put $t_a = 0$ and $C = [1\,0]$. The related initial condition reads

$$Cx(0) = x_1(0) = d.$$

Compute $G_0 = AD = E$, $\ker B = \ker G_0$, thus $BQ_0 = 0$, $G_1 = G_0$. The DAE is tractable with index 0, and Corollary 3.5 applies.
Compute further

$$Q_0(t) = \frac{1}{1+t^2}\begin{bmatrix} t^2 & t \\ t & 1 \end{bmatrix}, W_0(t) = \frac{1}{1+t^2}\begin{bmatrix} 1 & -t \\ -t & t^2 \end{bmatrix}, G_1(t)^+ = \frac{1}{(1+t^2)^2}\begin{bmatrix} -t & -1 \\ t^2 & t \end{bmatrix},$$

$R = 1$, $P_1 = P_0 = W_0$, $DG_1^+BD^- = -\frac{t}{1+t^2}$.
The solution of the IVP $U' + DG_1^+BD^-U = 0, U(0) = 1$ is $U(t) = (1+t^2)^{\frac{1}{2}}$.
(q, d) belongs to $\operatorname{im}\mathcal{T}$, exactly if the consistency condition

$$W_0q = \frac{1}{(1+t^2)^{\frac{1}{2}}}\begin{bmatrix} -1 \\ t \end{bmatrix}\{-d + \int_0 \frac{1}{(1+s^2)^{\frac{3}{2}}}(sq_1(s) + q_2(s))\mathrm{d}s\}. \qquad (73)$$

is valid.
For consistent (q,d) we derive the pseudosolution

$$x_{**} = D^- u = \frac{1}{(1+t^2)^{\frac{1}{2}}} \begin{bmatrix} -1 \\ t \end{bmatrix} \{-d + \int_0 \frac{1}{(1+s^2)^{\frac{3}{2}}} (sq_1(s) + q_2(s))\mathrm{ds}\}.$$

Regarding the consistency condition (73) we find the further expression

$$x_{**}(t) = W_0(t)q(t) = \frac{1}{(1+t^2)} \begin{bmatrix} 1 \\ -t \end{bmatrix} (q_1(t) - tq_2(t)).$$

In contrast, in [46], this DAE has strangeness index 1, and hence, it is first reduced to strangeness-free form via the derivative array approach (which needs derivatives also of q) and not till then the least-squares calculus is applied. Thereby, operator images are not at all considered. The function

$$x_{\circledast}(t) = \frac{1}{(1+t^2)} \begin{bmatrix} 1 \\ -t \end{bmatrix} (q_1(t) - tq_2(t) - x_{0\,1} + tx_{0\,2}) + x_0$$

is offered as the pseudosolution of the IVP

$$E(t)x'(t) + F(t)x(t) = q(t), \quad x(0) = x_0.$$

Observe that $x_{\circledast}$ formally coincides with x_{**} for $x_0 = 0$. □

3.4 Nonlinear differential-algebraic operators

There are reach resources in the functional-analytic literature concerning implicit function theorems, the Kantorovich-type analysis for Newton-like iteration methods, and related topics for operator equations and least-squares problems in Banach spaces, e.g. [16, 58, 21, 59]. Usually those investigations rely on Fréchet-differentiable operators. A Fréchet-derivative is linear and bounded by definition (cf. Appendix 6.1.3.

The standard form DAE

$$\mathfrak{f}(x'(t), x(t), t) = 0 \tag{74}$$

is described by the continuous function $\mathfrak{f} : \mathbb{R}^m \times \mathcal{D}_\mathfrak{f} \times \mathcal{I}_\mathfrak{f} \to \mathbb{R}^k$ which has continuous partial derivatives $\mathfrak{f}_{x^1}$ and $\mathfrak{f}_x$. The set $\mathcal{D}_\mathfrak{f} \times \mathcal{I}_\mathfrak{f} \subseteq \mathbb{R}^m \times \mathbb{R}$ is open. The nullspace $\ker \mathfrak{f}_{x^1}$ is supposed to be a $\mathcal{C}^1$-subspace. We associate with the DAE (74) the operator

$$\begin{aligned} \mathring{F} &: \operatorname{dom} \mathring{F} \subseteq X \to Y, \\ \operatorname{dom} \mathring{F} &:= \{x \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m) : x(t) \in \mathcal{D},\ t \in \mathcal{I}\}, \\ (\mathring{F}x)(t) &:= \mathfrak{f}(x'(t), x(t), t), \quad t \in \mathcal{I}, \quad x \in \operatorname{dom} \mathring{F}, \end{aligned} \tag{75}$$

whereby $\mathcal{D} \subseteq \mathcal{D}_\mathfrak{f}$ is an open, connected set and $\mathcal{I} = [t_a, t_e] \subset \mathcal{I}_\mathfrak{f}$ is a compact interval.
We put $Y = \mathcal{C}(\mathcal{I}, \mathbb{R}^k)$ and look for an appropriate Banach space X to serve as pre-image space. The operator equation $\mathring{F}x = 0$ represents the DAE (74) in the classical sense.

As pointed out in Subsubsection 2.3.1, it is not reasonable to choose $\mathcal{C}^1(\mathcal{I},\mathbb{R}^m)$ as the pre-image space. We momentarily attempt $X = C(\mathcal{I},\mathbb{R}^m)$. Then, for each fixed $x_* \in \operatorname{dom} \mathring{F}$ and arbitrary $x \in C^1(\mathcal{I},\mathbb{R}^m)$, the directional derivative

$$\lim_{\alpha\to 0} \frac{1}{\alpha}(\mathring{F}(x_* + \alpha x) - \mathring{F}x_*) = E_*x' + F_*x =: \mathring{F}'(x_*)x,$$

is well-defined and linear, with

$$E_*(t) := \mathfrak{f}_{x^1}(x_*'(t), x_*(t), t),\ F_*(t) := \mathfrak{f}_x(x_*'(t), x_*(t), t),\ t \in \mathcal{I}.$$

The linear operator equation $\mathring{F}'(x_*)x = q$ represents the linearized DAE

$$E_*x' + F_*x = q. \tag{76}$$

In this setting, the operator $\mathring{F}'(x_*)$ is closely defined, but, unfortunately, it is unbounded. By means of the proper factorization (cf. Theorem 3.1) $E_* = A_*D_*$, $A_* := E_*$, $D_* := E_*^+E_*$, we obtain the closure of $\mathring{F}'(x_*)$,

$$\begin{aligned} F'(x_*)x &= A_*(D_*x)' + B_*x, \quad x \in \operatorname{dom} F'(x_*),\\ \operatorname{dom} F'(x_*) &= \{x \in \mathcal{C}(\mathcal{I},\mathbb{R}^m) : D_*x \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^m)\}, \end{aligned}$$

with $B_* := F_* - A_*D_*'$. The closure $F'(x_*)$ has a definition domain individually for each x_*. This configuration does not meet the usual requirements of the functional-analytic procedures.

The following *structural restriction* of the DAE makes the situation much more comfortable: it ensures a common domain for the derivatives $F'(x_*)$ and allows then to turn to bounded derivatives in a Banach space setting. Fortunately, most applications meet this structural restriction.

Let the *nullspace of the leading Jacobian* $\mathfrak{f}_{x^1}(x^1,x,t)$ *be independent of* x^1 *and* x, such that

$$\ker \mathfrak{f}_{x^1}(x^1,x,t) =: N_0(t), \quad (x^1,x,t) \in \mathbb{R}^m \times \mathcal{D} \times \mathcal{I}. \tag{77}$$

Let the matrix $D(t) \in \mathcal{L}(\mathbb{R}^m)$ represent the orthoprojector along $N_0(t)$, $t \in \mathcal{I}$. The matrix function D is continuously differentiable since N_0 is a $\mathcal{C}^1$-subspace. The identity

$$\mathfrak{f}(x^1,x,t) \equiv \mathfrak{f}(D(t)x^1,x,t) \tag{78}$$

follows from

$$\mathfrak{f}(x^1,x,t) - \mathfrak{f}(D(t)x^1,x,t) = \int_a^1 \mathfrak{f}_{x^1}(sx^1 + (1-s)D(t)x^1,x,t)(I - D(t))x^1 \mathrm{ds} = 0.$$

It results that $D_* = E_*^+E_* = D$ uniformly for all $x_* \in \operatorname{dom} \mathring{F}$. Moreover, we have

$$\begin{aligned} (\mathring{F}x)(t) &= \mathfrak{f}(x'(t), x(t), t) = \mathfrak{f}(D(t)x'(t), x(t), t)\\ &= \mathfrak{f}((Dx)'(t) - D'(t)x(t), x(t), t), \quad t \in \mathcal{I}, \quad x \in \operatorname{dom} \mathring{F}, \end{aligned}$$

which suggests to turn to the extension F of $\mathring{F}$,

$$(Fx)(t) := \mathfrak{f}((Dx)'(t) - D'(t)x(t), x(t), t), \quad t \in \mathcal{I}, \quad x \in \operatorname{dom} F. \tag{79}$$

$$\operatorname{dom} F := \{x \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m) : x(t) \in \mathcal{D},\ t \in \mathcal{I}\}. \tag{80}$$

Motivated also by the experience in the previous sections, we *apply now the enhanced setting*

$$F : \operatorname{dom} F \subseteq \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m) \to \mathcal{C}(\mathcal{I}, \mathbb{R}^k) \tag{81}$$

for the extended differential-algebraic operator F determined by (79). In this enhanced setting, F is Fréchet differentiable,

$$\begin{aligned}
F'(x_*)x &= A_*(Dx)' + B_*x, \quad x \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m), \quad x_* \in \operatorname{dom} F,\\
A_*(t) &:= \mathfrak{f}_{x^1}((Dx_*)'(t) - D'(t)x_*(t), x_*(t), t),\\
B_*(t) &:= \mathfrak{f}_x((Dx_*)'(t) - D'(t)x_*(t), x_*(t), t) - A_*(t)D'(t), \quad t \in \mathcal{I}.
\end{aligned}$$

The operator equation $Fx = 0$ represents the DAE (74) still in classical sense. Together with the differential-algebraic operator F we investigate the composed operator

$$\begin{aligned}
\mathcal{F} &: \operatorname{dom} F \subset \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m) \to \mathcal{C}(\mathcal{I}, \mathbb{R}^k) \times \mathbb{R}^r,\\
\mathcal{F}x &: = (Fx, Cx(t_a)), \quad x \in \operatorname{dom} F.
\end{aligned}$$

which represents the IVP. The $m \times r$ matrix C will be specified later on.

Next we define the matrix functions

$$\begin{aligned}
G_0(x^1, x, t) &:= \mathfrak{f}_{x^1}(x^1, x, t) = \mathfrak{f}_{x^1}(x^1, x, t)D(t),\\
B_0(x^1, x, t) &:= \mathfrak{f}_x(x^1, x, t) - \mathfrak{f}_{x^1}(x^1, x, t)D'(t),\\
G_1(x^1, x, t) &:= G_0(x^1, x, t) + B_0(x^1, x, t)Q_0(t),
\end{aligned}$$

pointwise for $x^1 \in \mathbb{R}^m, x \in \mathcal{D}, t \in \mathcal{I}$, and introduce the projector-valued functions

$$P_0 := D,\ Q_0 := I - P_0, \quad W_0 := I - G_0G_0^+,\ W_1 := I - G_1G_1^+.$$

For a given reference function $x_* \in \operatorname{dom} F$ we abbreviate

$$\begin{aligned}
G_{*\,0}(t) &:= G_0((Dx_*)'(t) - D'(t)x_*(t), x_*(t), t) = A_*(t)D(t),\\
G_{*\,1}(t) &:= G_1((Dx_*)'(t) - D'(t)x_*(t), x_*(t), t), \quad t \in \mathcal{I},
\end{aligned}$$

and so on.

Now we are prepared to characterize nonlinear differential-algebraic operators via their derivatives. For instance, the following statement is a nonlinear counterpart of Theorem 3.9.

Theorem 3.14 *Let the function $\mathfrak{f}$ be continuous, with continuous partial derivatives $\mathfrak{f}_{x^1}$ and $\mathfrak{f}_x$. Let the nullspace of $\mathfrak{f}_{x^1}$ be a $\mathcal{C}^1$-subspace who varies with t only,*

and let the matrix function $W_0B_0Q_0$ have constant rank.
If, additionally,

$$W_1B_0P_0 = W_1\mathfrak{f}_xP_0 = 0, \tag{82}$$

then the derivative $F'(x_*)$ of the differential-algebraic operator F is normally solvable for each arbitrary $x_* \in \operatorname{dom} F$ and

$$\operatorname{im} F'(x_*) = \{x \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m : W_{*\,1}x = 0\}. \tag{83}$$

Proof: For each arbitrary reference function $x_* \in \operatorname{dom} F$ we obtain the linearization

$$F'(x_*)x = A_*(Dx)' + B_*x, \quad x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m), \tag{84}$$

which satisfies all conditions of Theorem 3.9 (cf. also (33), (34), (35)). In detail, condition (82) implies relation (63), that is, for $t \in \mathcal{I}$,

$$W_{*\,1}(t)B_*(t)P_0(t) = (W_1B_0)((Dx_*)'(t) - D'(t)x_*(t), x_*(t), t)P_0(t) = 0.$$

□

Example 3.15 (Strangeness-free reduced DAEs) *A particular instance of normally solvable nonlinear differential-algebraic operators is given by the so-called strangeness-free reduced DAEs in [47],*

$$\begin{aligned} x_1'(t) + \mathcal{L}(x_1(t), x_2(t), x_3(t), t) = 0,\\ x_2(t) + \mathcal{R}(x_1(t), x_3(t), t) = 0, \end{aligned}$$

which represent overdetermined DAEs with free component x_3. One has simply

$$G_0 = \begin{bmatrix} I & 0 & 0\\ 0 & 0 & 0 \end{bmatrix},\ W_0B_0Q_0 = \begin{bmatrix} 0 & 0 & 0\\ 0 & I & * \end{bmatrix},\ G_1 = \begin{bmatrix} I & * & *\\ 0 & I & * \end{bmatrix},\ W_1 = 0.$$

□

Theorem 3.16 *Let the function $\mathfrak{f}$ be continuous, with continuous partial derivatives $\mathfrak{f}_{x^1}$ and $\mathfrak{f}_x$. Let the nullspace of $\mathfrak{f}_{x^1}$ be a $\mathcal{C}^1$-subspace who varies with t only and let the matrix function $W_0B_0Q_0$ have constant rank.*
Let the matrix $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^r)$ have full row-rank $r := \operatorname{rank} D(t_a)$ and let $\ker C = \ker D(t_a)$. If, additionally,

$$\ker G_1 \subseteq \ker G_0. \tag{85}$$

then the derivative $\mathcal{F}'(x_*)$ of the composed operator $\mathcal{F}$ is normally solvable for each arbitrary $x_* \in \operatorname{dom} F$ and

$$\begin{aligned} \operatorname{im} \mathcal{F}'(x_*) = \{&(q,d) \in \mathcal{C}(\mathcal{I}, \mathbb{R}^k) \times \mathbb{R}^r : \\ &W_{*\,1}q = W_{*\,1}B_*D^+U_*\,(D(t_a)C^+d + \int_{t_a} U_*(s)^{-1}D(s)G^+_{*\,1}q(s)\mathrm{ds}\,)\}, \end{aligned} \tag{86}$$

whereby U_* is the fundamental solution matrix uniquely determined by

$$U' - P_0'U + DG^+_{*\,1}B_*D^+U = 0, \quad U(t_a) = I.$$

Further, it holds that

$$\ker \mathcal{F}'(x_*) = \{x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m) : G_{*\,1}x = 0\},$$

and $\mathcal{F}'(x_*)$ is injective, if the matrix function G_1 shows full column-rank. Moreover, if $(q,d) \in \operatorname{im} \mathcal{F}'(x_*)$, then there is a unique $z_* \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ such that

$$\|z_*\|_{L^2} = \min\{\|z\|_{L^2} : z \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m), \mathcal{F}'(x_*)z = (q,d)\}.$$

Proof: For each arbitrary reference function $x_* \in \operatorname{dom} F$ we obtain the linearization

$$\mathcal{F}'(x_*)x = (F(x_*)x,\, Cx(t_a)) = (A_*(Dx)' + B_*x,\, Cx(t_a)), \quad x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m).$$

Theorem 3.4 can be adapted to a setting with bounded operators analogously as Theorem 3.9 has been obtained from Theorem 3.2. Our linearization satisfies all conditions of the adapted version of Theorem 3.2 which proves the statements here. Note that here the special choice of D leads to $R = P_0$ and $D^- = D^+$. □

Regular index-0 DAEs are characterized by an equal number of equations and unknowns, $m = k$, and a nonsingular matrix function G_0, so that $W_1 = W_0 = 0$. Regular index-1 DAEs are given, if $m = k$, the matrix function G_0 is singular, but has constant rank, and the matrix function G_1 is nonsingular (e.g.[29, 50]). Obviously, Theorem 3.16 applies in both instances.
General underdetermined index-0 DAEs are characterized by $m > k$ and a matrix function G_0 that has full row-rank k. General underdetermined index-1 DAEs are characterized by $m > k$, a matrix function G_0 that has constant rank smaller than k, and a matrix function G_1 with full row-rank k, see [50]. This leads also to $W_1 = 0$, and Theorem 3.16 applies again.

We emphasize once again that our criteria of normal solvability are given in terms of the original data. No transformation in a special reduced form is required.

3.5 Notes and references

Remark 3.17 *In optimal control one often prefers the spaces of essentially bounded functions. Denote by $L^\infty(\mathcal{I}, \mathbb{R}^m)$ and $W^{1,\infty}(\mathcal{I}, \mathbb{R}^m)$ the space of essentially bounded functions and the space of essentially bounded functions with essentially bounded first derivatives, respectively, further*

$$W^{1,\infty}_D(\mathcal{I}, \mathbb{R}^m) := \{x \in L^\infty(\mathcal{I}, \mathbb{R}^m) : Dx \in W^{1,\infty}(\mathcal{I}, \mathbb{R}^n)\}.$$

The differential-algebraic operator

$$T : \operatorname{dom} T := W^{1,\infty}_D(\mathcal{I}, \mathbb{R}^m) \subseteq X \to L^\infty(\mathcal{I}, \mathbb{R}^k)$$

can then be treated analogously to Section 3, with $X = W^{1,\infty}_D(\mathcal{I}, \mathbb{R}^m)$ as bounded operator and with $X = L^\infty(\mathcal{I}, \mathbb{R}^m)$ as closed operator, accordingly.

Remark 3.18 *To our knowledge, [35] was the very first paper providing the closure of an operator associated with a DAE, and formulating conditions ensuring normal solvability. The function space applied in [35] is $L^2(\mathcal{I}, \mathbb{R}^m)$. In contrast to the present paper, in [35] also the coefficients E and F are integrable functions. In [77] a constant leading coefficient E is supposed for obtaining the closure of the differential-algebraic operator acting in L^2-spaces, and, in essence, also F is constant when regarding normal solvability. In [77] different numbers*

of unknowns m and equations k are allowed, whereas $m = k$ is supposed in [35]. Theorem 3.4 substantially generalizes [77, Theorem 3] which is proved there for $X = L^2$ by means of a quite involved regularization procedure.
In essence, the DAEs described in Proposition 3.7 are tractable with index 0 and index 1. We conjecture that exactly the DAEs with tractability index 0 and 1 (cf. [50, Chapter 10]) yield normally solvable operators. Example 3.8 supports this idea.

Remark 3.19 *Least-squares solutions are in discussion since the beginning of the DAE research. Already in the early contribution [9], linear DAEs are treated as least-squares problems, with function spaces H^1 and L^2, and by a gradient method. It is reported, that satisfactory numerical results are obtained only for matrix pencils having a simple structure, that is, in the absence of higher order nilpotent blocks. This fits naturally in our theory.*
An updated analysis is reported in [13]. For a certain integer κ, the cost functional

$$J_\kappa(x) := \sum_{j=0}^{\kappa} \|(Tx - q)^{(j)}\|^2_{L^2}$$

is to be minimized subject to the fixed initial condition $x(t_a) = x_a$. Special interest is spent to $\kappa = 0$. A simple gradient descent method and discretization by polynomials are discussed.

Remark 3.20 *For bounded operators $K_b(X, Y)$ acting in Hilbert spaces, possibly with nonclosed image $\operatorname{im} K$, there is a reach spectrum of iteration methods relying on Gaussian symmetrization, i.e., on the normal equation (e.g.,[71, 21])*

$$K^*Kx = K^*q, \tag{87}$$

among others, Tikhonov's method. To the author's knowledge, those iteration procedures are not developed for DAE problems as yet.

Remark 3.21 *Generalized inverses of a differential-algebraic operator associated with strangeness-free DAEs are already constructed in [46, 68] by strongly exploiting the special reduced form (cf. Example 3.3). Theorems 3.2 and 3.12, in particular condition (40), apply to the class of strangeness-free DAEs. In essence, regardless several inconsistencies in [46], the corresponding generalized inverses $\mathcal{T}^-$ reproduce those in [46] as special cases. The DAE*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x'(t) + \begin{bmatrix} 1 - \frac{t}{2} & \frac{t}{2} & 1 \\ -1 & 1 & 0 \end{bmatrix} x(t) = q(t),$$

which serves as test example for numerical experiments in [68], fulfills all conditions of Theorem 3.2, with $W_1 = 0$.
Section 3 provides generalized inverses for a larger class of DAEs than that of strangeness-free ones. In contrast to [46], it is not required to preliminary perform a reduction procedure via a derivative array system. Note that this approach requires higher derivatives. Example 3.3 demonstrates that the pseudosolution obtained that way does not necessarily coincide with the pseudosolution in our context.

Remark 3.22 *The structural condition (77) has been introduced in [29]. Nonlinear DAEs in standard form (74) who satisfy this condition are associated with operator equations in Banach spaces already e.g. in [52, 53]. There, solvability results for regular index-1 DAEs are obtained via the classical abstract implicit function theorem, and numerical methods for IVPs and BVPs are treated as discretizations of operator equations. It is shown how to accomplish well-posed problems by appropriately choosing initial and boundary conditions. It is further pointed out, that, in natural settings as described in Subsections 3.2 and 3.4, DAEs with higher index yield necessarily operators with nonclosed images, and hence essentially ill-posed problems.*

Remark 3.23 *Relying on the structural condition (77) for the nonlinear DAE (74), well-posed BVPs in regular index-1 DAEs are established and treated by Newton-Kantorovich iterations in natural Banach spaces e.g. in [54, 63], see also Section 4.*

4 Regular differential-algebraic operators in their natural Banach spaces

4.1 Notations and basic assumptions

In this chapter we treat nonlinear DAEs as operator equations for Fréchet differentiable operators. Motivated by the expressions obtained for the closures of the unbounded operators in Theorem 3.1 and by the arguments in Subsection 3.4 we immediately turn to DAEs with properly stated leading terms. More precisely. we investigate the nonlinear DAE

$$f((Dx)'(t), x(t), t) = 0, \tag{88}$$

who exhibits the involved derivative by means of an extra matrix valued function D.

The function $f : \mathbb{R}^n \times \mathcal{D}_f \times \mathcal{I}_f \longrightarrow \mathbb{R}^m$, $\mathcal{D}_f \times \mathcal{I}_f \subseteq \mathbb{R}^m \times \mathbb{R}$ open, is continuous and has continuous partial derivatives f_y and f_x with respect to the first two variables $y \in \mathbb{R}^n$, $x \in \mathcal{D}_f$. The partial Jacobian $f_y(y, x, t)$ is everywhere singular. The matrix function $D : \mathcal{I}_f \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ is continuously differentiable and $D(t)$ has constant rank r on the given interval $\mathcal{I}_f$. Let the transversality condition

$$\ker f_y(y, x, t) \oplus \operatorname{im} D(t) = \mathbb{R}^n, \quad (y, x, t) \in \mathbb{R}^n \times \mathcal{D}_f \times \mathcal{I}_f, \tag{89}$$

be valid and $\ker f_y$ be a $\mathcal{C}^1$-subspace. We say that the DAE (88) has a *properly stated leading term*, also a *properly involved derivative*.

Let $\mathcal{I} \subseteq \mathcal{I}_f$ be a compact interval, $\mathcal{I} =: [t_a, t_e]$, and $\mathcal{D}_F \subseteq \mathcal{D}_f$ be open. We associate with the DAE (88) the nonlinear operator

$$\begin{aligned} F : \operatorname{dom} F \subseteq \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m) &\to \mathcal{C}(\mathcal{I}, \mathbb{R}^m), \\ \operatorname{dom} F :&= \{x \in C^1_D(\mathcal{I}, \mathbb{R}^m) : x(t) \in \mathcal{D}_F \text{ for all } t \in \mathcal{I}\}, \\ (Fx)(t) :&= f((Dx)'(t), x(t), t), \quad t \in \mathcal{I}, \quad x \in \operatorname{dom} F. \end{aligned} \tag{90}$$

Since D is continuously differentiable, the inclusions

$$\mathcal{C}^\infty(\mathcal{I},\mathbb{R}^m) \subseteq \mathcal{C}^\nu(\mathcal{I},\mathbb{R}^m) \subseteq \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m) \tag{91}$$

are valid for all $\nu \in \mathbb{N}$. Endowed with the norm

$$\|x\|_{\mathcal{C}^1_D} := \|x\|_\infty + \|(Dx)'\|_\infty\ , \quad x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m), \tag{92}$$

the function space $\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ is a Banach space (see Lemma 6.9) and the DAE (88) is represented as the operator equation

$$Fx = 0. \tag{93}$$

The operator F is said to be a *differential-algebraic operator*. The operator equation (93) reflects the classical view on a DAE: the solutions belong to $\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ and satisfy the DAE pointwise for all $t \in \mathcal{I}$. The arguments in Subsection 2.3 enable us to speak of the *natural* Banach space setting.

For given $x_* \in \operatorname{dom} F$ we denote

$$\begin{aligned} A_*(t) &:= f_y((Dx_*)'(t), x_*(t), t), \\ B_*(t) &:= f_x((Dx_*)'(t), x_*(t), t),\ t \in \mathcal{I}. \end{aligned}$$

The directional derivative

$$F'(x_*)x := \lim_{\tau\to 0} \frac{1}{\tau}(F(x_* + \tau x) - F(x_*)) = A_*(Dx)' + B_*x$$

is well defined for each arbitrary $x_* \in \operatorname{dom} F$ and $x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$. The resulting map

$$F'(x_*) : \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m) \to \mathcal{C}(\mathcal{I},\mathbb{R}^m) \tag{94}$$

is linear and bounded. Moreover, $F'(x_*)$ varies continuously with respect to x_*. This means that the operator F *is Fréchet differentiable* and the map $F'(x_*)$ defined by

$$F'(x_*)x = A_*(Dx)' + B_*x, \quad x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m),$$

is the Fréchet derivative of F at x_*. The linear operator equation

$$F'(x_*)x = q$$

stands now for the *linearization* of the original DAE at x_*, that is, for the linear DAE

$$A_*(Dx)' + B_*x = q. \tag{95}$$

We complete the DAE (88) by the boundary condition

$$b(x(t_a), x(t_e)) = 0. \tag{96}$$

The continuously differentiable function $b : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m-l}$ will be specified later. Often we apply the particular case of an initial condition

$$Cx(t_a) = 0, \tag{97}$$

by letting $b(x,\bar{x}) := Cx$. The composed operator

$$\begin{aligned} \mathcal{F} &: \operatorname{dom} F \subseteq \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m) \to \mathcal{C}(\mathcal{I},\mathbb{R}^m)\times\mathbb{R}^{m-l}, \\ \mathcal{F}x &:= (Fx,\ b(x(t_a),x(t_e))), \quad x\in\operatorname{dom} F, \end{aligned} \tag{98}$$

is Fréchet differentiable since F is so. The equation $\mathcal{F}x = 0$ represents the BVP (88), (96), whereas the equation $\mathcal{F}x = (q,d)$ is the operator form of the perturbed BVP

$$f((D(t)x(t))', x(t), t) = q(t),\ t\in\mathcal{I}, \quad b(x(t_a),x(t_e)) = d. \tag{99}$$

4.2 Regular linear differential-algebraic operators

First we study the linear bounded operator $T\in\mathcal{L}(\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m),\mathcal{C}(\mathcal{I},\mathbb{R}^m))$ given by

$$Tx := A(Dx)' + Bx, \quad x\in\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m), \tag{100}$$

with coefficients

$$A\in\mathcal{C}(\mathcal{I},\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)),\ D\in\mathcal{C}^1(\mathcal{I},\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)),\ B\in\mathcal{C}(\mathcal{I},\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)),$$

in some detail. Let $\ker A$ and $\operatorname{im} D$ be $\mathcal{C}^1$-subspaces and let the transversality condition

$$\ker A(t)\oplus\operatorname{im} D(t) = \mathbb{R}^n, \quad t\in\mathcal{I}, \tag{101}$$

be satisfied. Denote by $R(t)$ the projector matrix onto $\operatorname{im} D(t)$ along $\ker A(t)$, $t\in\mathcal{I}$. The resulting function R is continuously differentiable.
Let $Q_0, P_0\in\mathcal{C}(\mathcal{I},\mathcal{L}(\mathbb{R}^m))$ be projector-valued functions such that $\operatorname{im} Q_0(t) = \ker D(t)$ for $t\in\mathcal{I}$, and $P_0 = I - Q_0$[1]. Moreover, denote by $D^-\in\mathcal{C}(\mathcal{I},\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m))$ the pointwise generalized inverse of D which is uniquely determined by

$$DD^-D = D, \quad D^-DD^- = D^-, \quad DD^- = R, \quad D^-D = P_0.$$

Definition 4.1 *For given coefficients A, D and B, and any level $\kappa\in\mathbb{N}$, the sequence $G_0,\dots,G_\kappa$ is said to be an admissible matrix function sequence, if it is built pointwise for all $t\in\mathcal{I}$ by the rule:*
set $G_0 := AD$, $B_0 := B$, $N_0 := \ker G_0$,
and for $i\geq 1$:

$$G_i := G_{i-1} + B_{i-1}Q_{i-1}, \tag{102}$$

$$N_i := \ker G_i, \quad \widehat{N_i} := (N_0+\cdots+N_{i-1})\cap N_i,$$

find a complement X_i such that $N_0+\cdots+N_{i-1} = \widehat{N_i}\oplus X_i$,
choose a projector Q_i such that $\operatorname{im} Q_i = N_i$ and $X_i\subseteq\ker Q_i$,
set $P_i := I - Q_i$, $\Pi_i := \Pi_{i-1}P_i$,

$$B_i := B_{i-1}P_{i-1} - G_iD^-(D\Pi_iD^-)'D\Pi_{i-1}, \tag{103}$$

and, if additionally,

(a) *the matrix function G_i has constant rank r_i, $i = 0,\dots,\kappa$,*

[1] In contrast to Section 3, here we do not fix these projectors to be orthogonal.

(b) *the intersection* $\widehat{N_i}$ *has constant dimension* $u_i := \dim \widehat{N_i}$,

(c) *the product function* Π_i *is continuous and* $D\Pi_i D^-$ *is continuously differentiable,* $i = 0, \ldots, \kappa$.

The projector functions $Q_0, \ldots, Q_\kappa$ *associated with an admissible matrix function sequence are said to be admissible themselves.*
An admissible matrix function sequence $G_0, \ldots, G_\kappa$ *is said to be regular admissible, if*

$$\widehat{N_i} = \{0\} \quad \textit{for all} \quad i = 1, \ldots, \kappa.$$

Then, also the projector functions $Q_0, \ldots, Q_\kappa$ *are called regular admissible.*
The numbers $r_0 = \operatorname{rank} G_0, \ldots, r_\kappa = \operatorname{rank} G_\kappa$ *and* $u_1, \ldots, u_\kappa$ *are named characteristic values of the DAE on* $\mathcal{G}$.

We refer to [50] for many useful properties of the admissible matrix function sequences. By construction, it holds that

$$r_0 \leq r_1 \leq \ldots \leq r_\kappa.$$

Now we are prepared to generalize the traditional notion of regular differential-algebraic operators given in Subsection 2.1 for time-invariant coefficients accordingly.

Definition 4.2 *The differential-algebraic operator (100) is said to be*

(1) *regular with tractability index 0, if* $r_0 = m$.

(2) *regular with tractability index* μ, *if there is an admissible matrix function sequence such that*

$$r_0 \leq r_1 \leq \ldots \leq r_{\mu-1} < r_\mu = m. \tag{104}$$

(3) *regular, if it is regular with any index.*

The numbers (104) and μ *are said to be the characteristic values and the tractability index of the regular differential-algebraic operator* T.

In case of a constant coefficients A, D, B, the matrix function sequence simplifies to a sequence of matrices. In particular, the second term in the definition of B_i disappears. It is aging ([30]) that the pair $\{AD, B\}$ of $m \times m$ matrices AD, B is regular with Kronecker index μ (cf. Subsection 2.1) exactly if an admissible sequence of matrices starting with $G_0 = AD$, $B_0 := B$ yields

$$r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m. \tag{105}$$

Thereby, neither the factorization nor the special choice of admissible projectors do matter. The characteristic values describe the structure of the Weierstraß-Kronecker form (9): we have $l = \sum_{j=0}^{\mu-1}(m - r_j)$ and the nilpotent part N contains altogether $s = m - r_0$ Jordan blocks, among them $r_i - r_{i-1}$ Jordan blocks of order i, $i = 1, \ldots, \mu$.

Also in the general regular time-varying case, the ingredients of an admissible matrix function sequence allow a decoupling which is quite similar to the

Weierstraß-Kronecker form (e.g.[50, Section 2.4], also Appendix 6.3). Thereby, the matrix function

$$\mathcal{K} := (I - \Pi_{\mu-1})G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1} + \sum_{l=1}^{\mu-1}(I - \Pi_{l-1})(P_l - Q_l)(D\Pi_l D^-)' D\Pi_{\mu-1},$$

plays its role. The decoupling of the two basic parts yielding the inherent explicit ODE and the differentiation problems is *complete* as in the Weierstraß-Kronecker form, exactly if $\mathcal{K}$ vanishes identically. Supposed the original data A, D, B show some additional smoothness, the admissible projector functions can be chosen in such a way that $\mathcal{K}$ disappears (e.g.[50, Section 2.4]). In this case, we speak of *completely decoupling projector functions* $Q_0, \dots, Q_{\mu-1}$.

Definition 4.3 *The differential-algebraic operator (100) is said to be fine, if it is regular and the coefficients A, D, B are as smooth as required for the existence of completely decoupling projectors $Q_0, \dots, Q_{\mu-1}$.*

Example 4.4 (Continuation 4 of Example 2.5) *The following matrix sequence is admissible for the pair $\{E, F\} = \{AD, B\}$ from Example 2.5 which is regular with index 4:*

$$G_0 = \begin{bmatrix} 1&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0 \end{bmatrix}, Q_0 = \begin{bmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{bmatrix}, B_0 = \begin{bmatrix} -\alpha&-1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 1&-1&0&0&0\\0&1&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0 \end{bmatrix}, Q_1 = \begin{bmatrix} 0&0&-1&0&0\\0&0&-1&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{bmatrix}, \Pi_0 Q_1 = \begin{bmatrix} 0&0&-1&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 1&-1&\alpha&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&0&1\\0&0&0&0&0 \end{bmatrix}, Q_2 = \begin{bmatrix} 0&0&0&1+\alpha&0\\0&0&0&1&0\\0&0&0&-1&0\\0&0&0&1&0\\0&0&0&0&0 \end{bmatrix}, \Pi_1 Q_2 = \begin{bmatrix} 0&0&0&\alpha&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0 \end{bmatrix},$$

$$G_3 = \begin{bmatrix} 1&-1&\alpha&-\alpha^2&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&0 \end{bmatrix}, Q_3 = \begin{bmatrix} 0&0&0&0&-1-\alpha-\alpha^2\\0&0&0&0&-1\\0&0&0&0&1\\0&0&0&0&-1\\0&0&0&0&1 \end{bmatrix}, \Pi_2 Q_3 = \begin{bmatrix} 0&0&0&0&-\alpha^2\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1 \end{bmatrix},$$

$$G_4 = \begin{bmatrix} 1&-1&\alpha&-\alpha^2&\alpha^3\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1 \end{bmatrix}, \Pi_3 = \begin{bmatrix} 1&0&1&-\alpha&\alpha^2\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{bmatrix},$$

and the characteristic values are $r_0 = r_1 = r_2 = r_3 = 4$, $r_4 = 5$ and $\mu = 4$. Additionally, it follows that

$$Q_3 G_4^{-1} B_0 \Pi_3 = 0, \qquad Q_2 P_3 G_4^{-1} B_0 \Pi_3 = 0,$$
$$Q_1 P_2 P_3 G_4^{-1} B_0 \Pi_3 = 0, \qquad Q_0 P_1 P_2 P_3 G_4^{-1} B_0 \Pi_3 = 0,$$

and

$$\Pi_3 G_4^{-1} B_0 \Pi_3 = -\alpha \Pi_3. \tag{106}$$

The projectors Q_0, Q_1, Q_2, Q_3 provide a complete decoupling of the given DAE $A(Dx)'(t) + Bx(t) = q(t)$. The projectors Q_0, $\Pi_0 Q_1$, $\Pi_1 Q_2$ and $\Pi_2 Q_3$ represent the variables x_2, x_3, x_4 and x_5, respectively. The projector Π_3 and the coefficient (106) determine the inherent regular ODE, namely (the zero rows are dropped)

$$\begin{aligned}(x_1 + x_3 - \alpha x_4 + \alpha^2 x_5)' - \alpha(x_1 + x_3 - \alpha x_4 + \alpha^2 x_5)& \\ = q_1 + q_2 - \alpha q_3 + \alpha^2 q_4 - \alpha^3 q_5.&\end{aligned} \tag{107}$$

It is noteworthy that no derivatives of the excitation q encroach in this part. Here Π_3 is the spectral projector of the pair $\{E, F\}$. The decoupling projector of the basic parts is

$$G_4 \Pi_3 G_4^{-1} = \begin{bmatrix} 1 & 1 & -\alpha & \alpha^2 & -\alpha^3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the equation $G_4 \Pi_3 G_4^{-1} T x = G_4 \Pi_3 G_4^{-1} q$ results in the ODE (107). □

Parts of the following theorem can be seen as counterparts and generalizations of Proposition 2.6.

Theorem 4.5 *Let the linear differential-algebraic operator (100) be fine with tractability index $\mu \geq 1$ and characteristic values $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$. Let the projector valued function $Q_0, \ldots, Q_{\mu-1} \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{R}^m))$ be associated with a complete decoupling. Then the following statements are true:*

(1) *The topological direct sum decomposition*

$$\begin{aligned}&\mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m) \\ &\quad = \{x \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m) : \Pi_{\mu-1} x = 0\} \oplus \{x \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m) : (I - \Pi_{\mu-1}) x = 0\} \\ &\quad =: \ker \Pi_{\mu-1} \oplus \operatorname{im} \Pi_{\mu-1}\end{aligned}$$

is valid.[2]

(2) *The operator splits into the sum of two bounded operators*

$$T = \underbrace{G_\mu \Pi_{\mu-1} G_\mu^{-1} T}_{T_{dyn}} + \underbrace{(I - G_\mu \Pi_{\mu-1} G_\mu^{-1}) T}_{T_{alg}} =: T_{dyn} + T_{alg},$$

in which the operator T_{dyn} is normally solvable, and

$$\operatorname{im} T_{dyn} = \operatorname{im} G_\mu \Pi_{\mu-1} G_\mu^{-1}, \quad \ker T_{alg} = \operatorname{im} \Pi_{\mu-1}.$$

[2] $\ker \Pi_{\mu-1}$ and $\operatorname{im} \Pi_{\mu-1}$ are used twofold: in $\mathbb{R}^m$ and in $\mathcal{C}_D^1$, but no confusion should arise.

(3) $\ker T = \ker T_{dyn} \cap \ker T_{alg} = \ker T_{dyn} \cap \operatorname{im} \Pi_{\mu-1}$ *has the finite dimension* $\delta := \operatorname{rank} \Pi_{\mu-1} = m - \sum_{i=0}^{\mu-1}(m - r_i) = m - l$.

(4) T *is normally solvable exactly if* T_{alg} *is so;*

$$\operatorname{im} T = \operatorname{im} T_{dyn} \dot{+} \operatorname{im} T_{alg} = \operatorname{im} G_\mu \Pi_{\mu-1} G_\mu^{-1} \dot{+} \operatorname{im} T_{alg}. \tag{108}$$

(5) *If* $\mu = 1$*, then* T_{alg} *is normally solvable and* T *is surjective, thus fredholm.*

(6) *If* $\mu > 1$*, then* $\operatorname{im} T_{alg}$ *and* $\operatorname{im} T$ *are nonclosed proper subsets of* $\mathcal{C}(\mathcal{I}.\mathbb{R}^m)$)*, and the equation* $Tx = q$ *is essentially ill-posed.*

Proof: (1) Since $D\Pi_{\mu-1}(I - D^+D) = 0$ and $D\Pi_{\mu-1}D^+ = D\Pi_{\mu-1}D^-DD^+$ is continuously differentiable, this statement is a consequence of Lemma 6.10.
(2) The boundedness of both operators T_{dyn} and T_{alg} is evident. Rearranging several terms as described in [50, p.93-96] we express

$$\begin{aligned} T_{dyn}x &= G_\mu \Pi_{\mu-1} G_\mu^{-1}(A(Dx)' + Bx) \\ &= G_\mu\{D^- D\Pi_{\mu-1}D^-(D\Pi_{\mu-1}x)' + \Pi_{\mu-1}G_\mu^{-1}B_\mu D^- D\Pi_{\mu-1}x\}, \end{aligned} \tag{109}$$

for $x \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$, and

$$\begin{aligned} T_{alg}x &= (I - G_\mu \Pi_{\mu-1} G_\mu^{-1})(A(Dx)' + Bx) \\ &= G_\mu\{\sum_{l=0}^{\mu-1} Q_l x - \sum_{l=0}^{\mu-2}(I - \Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}x)' + \sum_{l=0}^{\mu-2} \mathcal{M}_{l+1} D\Pi_l Q_{l+1}x\}, \end{aligned} \tag{110}$$

with coefficients, see [50, p.95], also Appendix 6.3.2,

$$\mathcal{M}_j := \sum_{k=0}^{j-1}(I - \Pi_k)\{P_k D^-(D\Pi_k D^-)' - Q_{k+1}D^-(D\Pi_{k+1}D^-)'\}D\Pi_{j-1}Q_l D^-,$$

for $l = 1, \ldots, \mu - 1$. The additional coefficient $\mathcal{K}$ arising in [50, p.95] here disappears owing to the complete decoupling, cf. Appendix 6.3.2.
Expression (110) shows that $T_{alg}x = 0$ if $x = \Pi_{\mu-1}x$. Conversely, if x satisfies the equation $T_{alg}x = 0$, then

$$\sum_{l=0}^{\mu-1} Q_l x - \sum_{l=0}^{\mu-2}(I - \Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}x)' + \sum_{l=0}^{\mu-2} \mathcal{M}_{l+1} D\Pi_l Q_{l+1}x = 0$$

follows. Multiplying by $\Pi_{\mu-2}Q_{\mu-1}$ yields $\Pi_{\mu-2}Q_{\mu-1}x = 0$, $Q_{\mu-1}x = 0$, and

$$\sum_{l=0}^{\mu-2} Q_l x - \sum_{l=0}^{\mu-3}(I - \Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}x)' + \sum_{l=0}^{\mu-3} \mathcal{M}_{l+1} D\Pi_l Q_{l+1}x = 0.$$

Multiplying successively by $\Pi_{\mu-3}Q_{\mu-2}$,...$\Pi_0 Q_1$, we obtain $Q_{\mu-2}x = 0$,...,$Q_1 x = 0$, respectively, and finally $Q_0 x = 0$. Therefore, each element x of the nullspace of T_{alg} has the form $x = \Pi_{\mu-1}x$, such that actually $kerT_{alg} = \operatorname{im} \Pi_{\mu-1}$.

Next we turn to the operator T_{dyn}. For each arbitrary $q \in \operatorname{im} G_\mu \Pi_{\mu-1} G_\mu^{-1}$, the equation $T_{dyn}x = q = G_\mu \Pi_{\mu-1} G_\mu^{-1} q$ is equivalent with

$$D^- D\Pi_{\mu-1} D^-(D\Pi_{\mu-1}x)' + \Pi_{\mu-1} G_\mu^{-1} B_\mu D^- D\Pi_{\mu-1} x = \Pi_{\mu-1} G_\mu^{-1} q,$$

and further with

$$D\Pi_{\mu-1} D^-(D\Pi_{\mu-1}x)' + D\Pi_{\mu-1} G_\mu^{-1} B_\mu D^- D\Pi_{\mu-1} x = D\Pi_{\mu-1} G_\mu^{-1} q,$$

The standard IVP

$$u' - (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1} G_\mu^{-1} B_\mu D^- u = D\Pi_{\mu-1} G_\mu^{-1} q, \quad u(t_a) = 0, \tag{111}$$

possesses the unique solution $u_* \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)$, and the relation $u_* = D\Pi_{\mu-1}D^- u_*$ is given. The function $x_* := D^- u_* = \Pi_{\mu-1} D^- u_*$ belongs to $\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ and satisfies $T_{dyn}x_* = G_\mu \Pi_{\mu-1} G_\mu^{-1} q = q$. This verifies the property $\operatorname{im} T_{dyn} = \operatorname{im} G_\mu \Pi_{\mu-1} G_\mu^{-1}$. This subspace is closed in $\mathcal{C}(\mathcal{I}, \mathbb{R}^m)$ and T_{dyn} is normally solvable.
(3) By construction, $Tx = 0$ means $T_{dyn}x = 0$ and $T_{alg}x = 0$, and equivalently,

$$D\Pi_{\mu-1} D^-(D\Pi_{\mu-1}x)' + D\Pi_{\mu-1} G_\mu^{-1} B_\mu D^- D\Pi_{\mu-1} x = 0, \quad x = \Pi_{\mu-1} x.$$

Therefore, if $x_* \in \ker T$, then $Dx_* = D\Pi_{\mu-1}x_*$ is a solution of the ODE from (111) with $q = 0$.
Denote by $U \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{R}^\delta))$ the fundamental solution matrix normalized at t_a of the ODE from (111). We have then

$$(I - D\Pi_{\mu-1}D^-)UD(t_a)\Pi_{\mu-1}(t_a) = 0.$$

Each function $x_* := D^- UD(t_a)\Pi_{\mu-1}(t_a)c$, $c \in \mathbb{R}^m$ belongs to $\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$, since $Dx_* = DD^- UD(t_a)\Pi_{\mu-1}(t_a)c = DD^- D\Pi_{\mu-1}D^- UD(t_a)\Pi_{\mu-1}(t_a)c = UD(t_a)\Pi_{\mu-1}(t_a)c$ is continuously differentiable. Moreover, Dx_* satisfies the homogeneous ODE from (111), with $q = 0$, and it holds that $x_* = \Pi_{\mu-1}x_*$, and hence $x_* \in \ker T$. It follows that

$$\ker T = \{x = D^- U\eta : \eta \in \operatorname{im} D(t_a)\Pi_{\mu-1}(t_a)\},$$

and $\dim \ker T = \operatorname{rank} D(t_a)\Pi_{\mu-1}(t_a) = \Pi_{\mu-1}(t_a) = \delta$.
(4) is a direct consequence of (2).
(5) If $\mu = 1$ then (110) simplifies to

$$T_{alg}x = G_1\{Q_0 x\}.$$

For each $q = (I - G_1 \Pi_0 G_1^{-1})q = G_1 Q_0 G_1^{-1} q$, the equation $T_{alg}x = q$ has the solutions $x = Q_0 G_1^{-1} q + h$, with a free component $h = P_0 h$. This proves that $\operatorname{im} T_{alg} = \ker G_1 \Pi_0 G_1^{-1}$ is closed and T_{alg} is normally solvable. Moreover, T is surjective, thus fredholm.
(6) Suppose $\mu \geq 2$. Applying once more the representation (110) we find for given $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ and $q := T_{alg}x$ the relation

$$q = G_\mu\{\sum_{l=0}^{\mu-1} Q_l x - \sum_{l=0}^{\mu-2}(I - \Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}x)' + \sum_{l=0}^{\mu-2} \mathcal{M}_{l+1} D\Pi_l Q_{l+1} x\},$$

which implies

$$D\Pi_{\mu-2}Q_{\mu-1}q = D\Pi_{\mu-2}Q_{\mu-1}T_{alg}x = D\Pi_{\mu-2}Q_{\mu-1}x.$$

The component $D\Pi_{\mu-2}Q_{\mu-1}x = D\Pi_{\mu-2}Q_{\mu-1}D^{-}Dx$ is continuously differentiable, hence $D\Pi_{\mu-2}Q_{\mu-1}q$ is also necessarily continuously differentiable. Therefore, $\operatorname{im} T_{alg}$ contains continuous functions having certain smoother components, which means that $\operatorname{im} T_{alg}$ is a nonclosed subset within the continuous function space. □

Ill-posed problems are known to need so-called *regularizations*. Therefore, also regular higher-index DAEs need those regularizations, which sounds in a way confusing. Note once more that regularity in the traditional DAE theory is tied to regular matrix pairs and their generalizations.

Corollary 4.6 *Under the assumptions of Theorem 4.5, the solutions of the equation*

$$T_{dyn}x = G_\mu \Pi_{\mu-1} G_\mu^{-1} q \tag{112}$$

have the form $x = v + D^{-}u$, whereby v is an arbitrary function from $\ker \Pi_{\mu-1}$ *and u is a solution of the ODE*

$$u' - (D\Pi_{\mu-1}D^{-})'u + D\Pi_{\mu-1}G_\mu^{-1}B_\mu D^{-}u = D\Pi_{\mu-1}G_\mu^{-1}q, \tag{113}$$

with $u(t_a) = D(t_a)\Pi_{\mu-1}(t_a)c,\ c \in \mathbb{R}^m$.

In the DAE analysis, the ODE (113) plays a central role, it is said to be the *inherent explicit regular ODE* (IERODE) of the DAE. The operator T_{dyn} represents the dynamical part of the DAE, which motivates the subscript *dyn*.
It may happen that the projector function $\Pi_{\mu-1}$ vanishes identically. Then T_{dyn} is the zero operator and $T = T_{alg}$ is injective.

Example 4.7 (T_{dyn}=0) *The operator*

$$Tx = \begin{bmatrix}1\\0\end{bmatrix}(\begin{bmatrix}1 & 0\end{bmatrix}x)' + \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}x \tag{114}$$

is fine with tractability index 2. It leads to $\Pi_1 = 0$, thus $T = T_{alg}$. We observe that $\operatorname{im} T_{alg} = \{q \in \mathcal{C}(\mathcal{I},\mathbb{R}^2) : q_2 \in \mathcal{C}^1(\mathcal{I},\mathbb{R})\}$ is a proper subset in $\mathcal{C}(\mathcal{I},\mathbb{R}^2)$. Moreover, the inverse

$$T_{alg}^{-1}q = \begin{bmatrix}q_2\\ q_1 - q_2'\end{bmatrix}, \quad q \in \operatorname{im} T, \tag{115}$$

is again a differential-algebraic operator. □

Theorem 4.5 is meaningful also for the composed operator

$$\mathcal{T} \in \mathcal{L}(\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m), C(\mathcal{I},\mathbb{R}^m) \times \mathbb{R}^\delta),$$
$$\mathcal{T}x = (Tx, Cx(t_a)), \quad x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m).$$

The operator equation $\mathcal{T}x = (q,d)$ reflects the IVP

$$A(Dx)' + BX = q, \quad Cx(t_a) = d. \tag{116}$$

The composed operator $\mathcal{T}$ is bounded together with T. $\mathcal{T}$ is injective, supposed T is regular and the matrix $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\delta)$ is such that

$$\ker C = \ker \Pi_{\mu-1}, \quad \delta = \operatorname{rank} \Pi_{\mu-1}. \tag{117}$$

If $\mu = 1$, then $\mathcal{T}$ is even surjective. Then, as a bijective bounded operator acting in Banach spaces, it has a bounded inverse, and hence the IVP (116) is well-posed. In contrast, if $\mu \geq 2$, then the surjectivity gets lost and $\operatorname{im} \mathcal{T}$ is a nonclosed subset in $\mathcal{C}(\mathcal{I}, \mathbb{R}^m) \times \mathbb{R}^\delta$. Then the IVP (116) is no longer well-posed, but essentially ill-posed.

Owing to Theorem 4.5, in all higher index cases, the composed operator $\mathcal{T}$ is bounded and injective, but the inverse $\mathcal{T}^{-1}$ is unbounded. Nevertheless, there are nontrivial bounded linear outer inverses as the next theorem states.

Theorem 4.8 *Let the linear differential-algebraic operator T be fine with tractability index $\mu \geq 2$. Let the projector valued functions $Q_0 \cdots Q_{\mu-1} \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{R}^m))$ be associated with a complete decoupling. Let the matrix $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\delta)$ satisfy condition (117). Then the following statements are valid:*

(1) *For each $y \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$ and $d \in \mathbb{R}^\delta$, the initial value problem*

$$Tx = G_\mu P_1 \cdots P_{\mu-1} G_\mu^{-1} y, \quad Cx(t_a) = d, \tag{118}$$

is uniquely solvable in $\mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$ and the solution satisfies the inequality

$$\|x\|_{\mathcal{C}_D^1} \leq c\,(|d| + \|G_\mu P_1 \cdots P_{\mu-1} G_\mu^{-1} y\|_\infty) \leq \tilde{c}\,(|d| + \|y\|_\infty). \tag{119}$$

(2) *The operator $\mathcal{T}^- \in \mathcal{L}(\mathcal{C}(\mathcal{I}, \mathbb{R}^m) \times \mathbb{R}^\delta, \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m))$ defined by*

$$\mathcal{T}^-(y, z) := \text{solution of (118) for } (y, z) \in \operatorname{dom} \mathcal{T}^- = \mathcal{C}(\mathcal{I}, \mathbb{R}^m) \times \mathbb{R}^\delta, \tag{120}$$

is a bounded outer inverse of the composed operator $\mathcal{T}$. The topological direct sum decomposition $\ker \mathcal{T}^- \oplus \operatorname{im} \mathcal{T}\mathcal{T}^- = \mathcal{C}(\mathcal{I}, \mathbb{R}^m) \times \mathbb{R}^\delta$ is valid with

$$\ker \mathcal{T}^- = \{(y, z) \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) \times \mathbb{R}^\delta : G_\mu P_1 \cdots P_{\mu-1} G_\mu^{-1} y = 0,\ z = 0\},$$
$$\operatorname{im} \mathcal{T}\mathcal{T}^- = \{(y, z) \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) \times \mathbb{R}^\delta : (I - G_\mu P_1 \cdots P_{\mu-1} G_\mu^{-1}) y = 0\}.$$

Proof: (1) Because of $\Pi_{\mu-1} P_1 \cdots P_{\mu-1} = \Pi_{\mu-1}$ and $P_1 \cdots P_{\mu-1} - \Pi_{\mu-1} = Q_0 P_1 \cdots P_{\mu-1}$, the equation $Tx = G_\mu P_1 \cdots P_{\mu-1} G_\mu^{-1} y$ decomposes into the system

$$T_{dyn} x = G_\mu \Pi_{\mu-1} G_\mu^{-1} y, \quad T_{alg} x = G_\mu Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1} y.$$

Equation $T_{alg} x = G_\mu Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1} y$ means in detail, see (110),

$$G_\mu \{\sum_{l=0}^{\mu-1} Q_l x - \sum_{l=0}^{\mu-2} (I - \Pi_l) Q_{l+1} D^- (D \Pi_l Q_{l+1} x)' + \sum_{l=0}^{\mu-2} \mathcal{M}_{l+1} D \Pi_l Q_{l+1} x\}$$
$$= G_\mu Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1} y.$$

Multiplying successively by $\Pi_{\mu-2} Q_{\mu-1} G_\mu^{-1}, \ldots, \Pi_0 Q_1 G_\mu^{-1}$, we obtain the relations $\Pi_{\mu-2} Q_{\mu-1} x = 0, \ldots, \Pi_0 Q_1 x = 0$. It results that $(I - \Pi_{\mu-1}) x = Q_0 x =$

$Q_0P_1\cdots P_{\mu-1}G_\mu^{-1}y$.
Regarding condition (117) we obtain

$$D(t_a)\Pi_{\mu-1}(t_a) = D(t_a)C^+C\Pi_{\mu-1}(t_a) = D(t_a)C^+C.$$

The IVP $T_{dyn}x = G_\mu\Pi_{\mu-1}G_\mu^{-1}y,\ D(t_a)\Pi_{\mu-1}x(t_a) = D(t_a)C^+z$ has the solutions $x = D^-u + v$, where $v \in \ker \Pi_{\mu-1}$ is arbitrary, and u is the unique solution of the ODE (113) (with y replacing q) satisfying the initial condition $u(t_a) = D(t_a)C^+z$, see Corollary 4.6.
Summarizing we know that $v = Q_0P_1\cdots P_{\mu-1}G_\mu^{-1}y$ and

$$Cx(t_a) = CD(t_a)^-u(t_a) = CD(t_a)^-D(t_a)C^+z = CC^+z = z,$$

and hence,

$$x = D^-u + Q_0P_1\cdots P_{\mu-1}G_\mu^{-1}y \in \mathcal{C}_D^1(\mathcal{I},\mathbb{R}^m)$$

is the unique solution of the IVP (118). Because of $Dx = u$, the inequality (119) is evident.
(2) The inequality (119) actually means that $\mathcal{T}^-$ is bounded. The relations $\mathcal{T}\mathcal{T}^-(y,z) = (G_\mu P_1\cdots P_{\mu-1}G_\mu^{-1}y, z)$ and $\mathcal{T}^-\mathcal{T}\mathcal{T}^- = \mathcal{T}^-$ follow immediately, and hence $\mathcal{T}^-$ is an outer inverse. Furthermore, $G_\mu P_1\cdots P_{\mu-1}G_\mu^{-1}y = 0,\ z = 0$ yield $\mathcal{T}^-(y,z) = 0$, and vice versa. □

4.3 Regular nonlinear differential-algebraic operators

As we have seen in Subsection 4.1, the nonlinear differential-algebraic operator (90) is Fréchet differentiable under natural assumptions.

Definition 4.9 *The nonlinear differential-algebraic operator F defined by (90) is said to be regular, if the derivative $F'(x_*) \in \mathcal{L}(\mathcal{C}_D^1(\mathcal{I},\mathbb{R}^m), \mathcal{C}(\mathcal{I},\mathbb{R}^m))$ is a fine regular differential-algebraic operator at least for each arbitrary reference function $x_* \in \operatorname{dom} F \cap \mathcal{C}^m(\mathcal{I},\mathbb{R}^m)$.*

Owing to Corollary 6.16 the derivative $F'(x_*)$ of a regular operator must have characteristics $r_0,\ldots,r_\mu$ and tractability index μ being uniform for all reference functions x_*. We *assign these characteristic values and the tractability index also to the nonlinear operator F.*

Theorem 4.10 *Let the nonlinear differential-algebraic operator F defined by (90) be regular. Then F is fredholm, if and only if it has tractability index $\mu \in \{0,1\}$. When indicated, the Fredholm index equals $\operatorname{ind}_{fred} = \operatorname{rank} D = r_0$.*

Proof: If F is regular with index 0 and 1, the linearization $F'(x_*)$ is fine not only for $x_* \in \operatorname{dom} F \cap \mathcal{C}^m(\mathcal{I},\mathbb{R}^m)$ but for all $x_* \in \operatorname{dom} F$. Then the statement concerning $\mu = 1$ follows from Theorem 4.5. In the less interesting case $\mu = 0$ the matrix function D remains nonsingular, which makes the statement evident. If F is regular with tractability index $\mu > 1$, applying Theorem 4.5 once again, we know that F fails to be fredholm. □

In higher-index cases, the linearization $F'(x_*)$ has a nonclosed image $\operatorname{im} F'(x_*)$ in $\mathcal{C}(\mathcal{I},\mathbb{R}^m)$ and, what makes matters worse, the nonclosed proper subset $\operatorname{im} F'(x_*)$ may actually depend on the reference function x_*. We demonstrate this fact by the next example.

Example 4.11 (Image depends on the reference function) *The operator F defined by $m = 4, n = 1$,*

$$f(y,x,t) = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} y + \begin{bmatrix}x_4 - \gamma(t)\\ x_1 + x_2x_3\\ x_2\\ x_3\end{bmatrix}, \; y \in \mathbb{R}, x \in \mathbb{R}^4, t \in \mathcal{I}, \quad D = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix},$$

with $\gamma \in \mathcal{C}(\mathcal{I},\mathbb{R})$, $\operatorname{dom} F = \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^4) = \{x \in \mathcal{C}(\mathcal{I},\mathbb{R}^4) : x_1 \in \mathcal{C}^1(\mathcal{I},\mathbb{R})\}$, is associated with the simple DAE taken from [67, p. 41]

$$\begin{aligned} x_1'(t) + x_4(t) - \gamma(t) &= 0,\\ x_1(t) + x_2(t)x_3(t) &= 0.\\ x_2(t) &= 0,\\ x_3(t) &= 0. \end{aligned}$$

For any reference function $x_* \in \operatorname{dom} F$ the linearization reads

$$F'(x_*)x = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \left(\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix} x\right)' + \begin{bmatrix}0 & 0 & 0 & 1\\ 1 & x_{*\,3} & x_{*\,2} & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix} x, \; x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^4),$$

the linear operator $F'(x_*)$ is regular with tractability index 2, and its image is

$$\operatorname{im} F'(x_*) = \{q \in \mathcal{C}(\mathcal{I},\mathbb{R}^4) : q_2 - x_{*\,3}q_3 - x_{*\,2}q_4 \in \mathcal{C}^1(\mathcal{I},\mathbb{R})\}.$$

For instance, if we choose the reference functions

$$x_* = \begin{bmatrix}0\\0\\0\\\gamma\end{bmatrix} \quad \textit{and} \quad x_{**} = \begin{bmatrix}0\\0\\\epsilon\\\gamma\end{bmatrix},$$

then we are confronted with the different sets

$$\begin{aligned} \operatorname{im} F'(x_*) &= \{q \in \mathcal{C}(\mathcal{I},\mathbb{R}^4) : q_2 \in \mathcal{C}^1(\mathcal{I},\mathbb{R})\},\\ \operatorname{im} F'(x_{**}) &= \{q \in \mathcal{C}(\mathcal{I},\mathbb{R}^4) : q_2 - \epsilon q_3 \in \mathcal{C}^1(\mathcal{I},\mathbb{R})\}. \end{aligned}$$

□

4.3.1 Local solvability by outer inverses

We apply a generalized implicit function theorem from [16] which does not base upon a bounded inverse of the derivative. Instead, a suitably chosen *approximate outer inverse* is used (see Appendix 6.1.2, Theorem 6.8).

If F is regular with tractability index $\mu \in \{0,1\}$, then the linear IVP

$$F'(x_*)x = y, \quad Cx(t_a) = 0, \tag{121}$$

with $C \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^{m-l})$, $l = m - r_0$, and $\ker C = \ker D(t_a)$, is uniquely solvable for each arbitrary $y \in \mathcal{C}(\mathcal{I},\mathbb{R}^m)$. The operator

$$\begin{aligned} &T_*^- \in \mathcal{L}(\mathcal{C}(\mathcal{I},\mathbb{R}^m), \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)),\\ &T_*^- y := \text{ solution of the IVP (121)}, \quad y \in \mathcal{C}(\mathcal{I},\mathbb{R}^m), \end{aligned}$$

is actually an injective bounded outer inverse of $F'(x_*)$. Namely, it holds that $\ker T_*^- = \{0\}$ and

$$F'(x_*)T_*^- y = y, \quad T_*^- F'(x_*)T_*^- y = T_*^- y, \quad y \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m).$$

Theorem 4.12 *let F be the differential-algebraic operator described in Subsection 4.1, $x_* \in \operatorname{dom} F$, and $F(x_*) = 0$. Let F be regular with tractability index $\mu \in \{0, 1\}$.*
Then, whenever $z_* \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$ satisfies $\|z_*\|_{\mathcal{C}_D^1} = 1$ and $F(x_*)z_* = 0$, there exists a solution $x(s) = x_* + sz_* + o(s)$ to $Fx = 0$, with $s > 0$ sufficiently small.

Proof: Since the outer inverse T_*^- is injective, the equation $Fx = 0$ is equivalent with $T_*^- Fx = 0$; and hence the statement follows from [16, Theorem 3]. □

Regarding that here $\ker F'(x_*)$ has dimension $m - l = r_0$ we obtain to $Fx = 0$ a local solution set of dimension r_0.

The situation in higher-index cases is more involved. If F is regular with tractability index $\mu \geq 2$ and $x_* \in \operatorname{dom} F$ is sufficiently smooth, then there are completely decoupling projector functions $Q_{*0}, \dots Q_{*\mu-1}$ associated with the linearization $F'(x_*)$. The linear IVPs

$$F'(x_*)x = \underbrace{G_{*\mu} P_{*1} \cdots P_{*\mu-1} G_{*\mu}^{-1}}_{=:\mathcal{P}_*} y, \quad C_* x(t_a) = 0, \tag{122}$$

with $C_* \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m-l})$, $\ker C_* = \ker \Pi_{*\mu-1}(t_a)$, $l = \sum_{j=0}^{\mu-1}(m - r_j)$, are uniquely solvable for each arbitrary $y \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$. The operator

$$T_*^- \in \mathcal{L}(\mathcal{C}(\mathcal{I}, \mathbb{R}^m), \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)),$$
$$T_*^- y := \text{ solution of the IVP (122)}, \quad y \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m),$$

is actually a bounded outer inverse of $F'(x_*)$. It holds that

$$F'(x_*)T_*^- y = y, \quad T_*^- F'(x_*)T_*^- y = T_*^- y, \quad y \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m).$$

Now, in the higher-index case, T_*^- is no longer injective, but

$$\ker T_*^- = \{y \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : \mathcal{P}_* y = 0\}.$$

If $x_* \in \operatorname{dom} F$ lacks in smoothness, $x_* \notin \mathcal{C}^m(\mathcal{I}, \mathbb{R}^m)$, then $F'(x_*)$ is at least approximately outer invertible (cf.Appendix6.1.2) as the following lemma. shows.

Lemma 4.13 *Let F be the differential-algebraic operator described in Subsection 4.1, Let F be regular with tractability index $\mu \geq 2$, $x_* \in \operatorname{dom} F$. Then the derivative $F'(x_*)$ is approximately outer invertible with a constant bound function $\Gamma(\rho) = M$ and approximate outer inverses $T_\rho^{\simeq}$ such that (cf.(123))*

$$\ker T_\rho^{\simeq} = \ker \mathcal{P}_\rho.$$

Proof: Because of the continuity of $F'(\cdot)$, to $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\|x - x_*\|_{\mathcal{C}_D^1} \leq \delta(\varepsilon) \quad \Rightarrow \quad \|F'(x) - F'(x_*)\| \leq \varepsilon.$$

Let $\delta_0 > 0$ be sufficiently small. We consider an arbitrary element $x_\rho \in B(x_*, \delta_0) \cap \mathcal{C}^m(\mathcal{I}, \mathbb{R}^m)$ and turn to the linearization $F'(x_\rho)$ who is fine. There are completely decoupling projector functions $Q_{\rho\,0}, \dots Q_{\rho\,\mu-1}$ associated with the linearization $F'(x_\rho)$. The linear IVPs

$$F'(x_\rho)x = \underbrace{G_{\rho\,\mu} P_{\rho\,1} \cdots P_{\rho\,\mu-1} G_{\rho\,\mu}^{-1}}_{=:\mathcal{P}_\rho} y, \quad C_\rho x(t_a) = 0, \tag{123}$$

with $C_\rho \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m-l})$, $\ker C_\rho = \ker \Pi_{\rho\,\mu-1}(t_a)$, $l = \sum_{j=0}^{\mu-1}(m - r_j)$, are uniquely solvable for each arbitrary $y \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$. The operator

$$\begin{aligned} &T_\rho^- \in \mathcal{L}(\mathcal{C}(\mathcal{I}, \mathbb{R}^m), \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)), \\ &T_\rho^- y := \text{ solution of the IVP (123)}, \quad y \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m), \end{aligned}$$

is a bounded outer inverse of $F'(x_\rho)$. It holds that

$$F'(x_\rho)T_\rho^- y = \mathcal{P}_\rho y, \quad T_\rho^- F'(x_\rho)T_\rho^- y = T_\rho^- y, \quad y \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m).$$

Owing to the construction of the matrix function sequences and the representation of the coefficients of the linearizations (cf.[50], also Appendix 6.1), there is a uniform upper bound $\|T_\rho^-\| \leq M$ for all those reference functions x_ρ.
Next, for each $\rho \in (0, 1)$ we ensure $\delta(\rho/M) \leq \delta_0$, and fix an element $x_\rho \in B(x_*, \delta(\rho/M)) \cap \mathcal{C}^m(\mathcal{I}, \mathbb{R}^m)$. We have then

$$\|(T_\rho^- F'(x_*)T_\rho^- - T_\rho^-)y\|_{\mathcal{C}_D^1} = \|(T_\rho^-(F'(x_*) - F'(x_\rho))T_\rho^- y\|_{\mathcal{C}_D^1} \leq \rho \|T_\rho^- y\|.$$

This means that $T_\rho^{\simeq} := T_\rho^-$ represents the required approximate outer inverse of $F'(x_*)$, with the bound function $\Gamma(\rho) = M$. □

Theorem 4.14 *Let F be the differential-algebraic operator described in Subsection 4.1, Let F be regular with tractability index $\mu \geq 2$, $x_* \in \operatorname{dom} F$, and $F(x_*) = 0$.*
Then, whenever $z_* \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m))$ satisfies $\|z_*\|_{\mathcal{C}_D^1} = 1$ and $F'(x_*)z_* = 0$, there exists a solution $x(s) = x_* + sz_* + o(s)$ to the equation $T_\rho^- Fx = 0$, with $s > 0$ sufficiently small, with an appropriate choice of $\rho = \rho(s) \downarrow 0$ as $s \downarrow 0$, that means

$$F(x(s)) \in \ker \mathcal{P}_{\rho(s)}, \quad s > 0 \quad \textit{sufficiently small.}$$

If, moreover, x_* is sufficiently smooth so that the derivative $F'(x_*)$ is fine, then $x(s)$ solves the equation $T_*^- Fx = 0$. It holds that

$$F(x(s)) \in \ker \mathcal{P}_*, \quad s > 0 \quad \textit{sufficiently small.}$$

Proof: The statement follows from [16, Theorem 3], cf. Theorem 6.8. □
Regarding that $\ker F'(x_\rho)$ has dimension $m - l$ one obtains a solution set of dimension $m - l$ to equation $T_*^- Fx = 0$.

4.3.2 Well-posed IVPs and BVPs with regular index-1 operators

Any regular differential-algebraic operator F with tractability index $\mu \in \{0, 1\}$ is fredholm, its Fréchet derivative $F'(x)$ is surjective and $\ker F'(x)$ has dimension $m - l = r_0 = \operatorname{rank} D$, thus the Fredholm index is $\operatorname{ind}_{fred} F'(x) = r_0$. As indicated by the Fredholm index, aiming at a well-posed equation $\mathcal{F}x = 0$, one has to complete the equation $Fx = 0$ by exactly r_0 appropriate boundary and initial conditions. If one adds more or less boundary conditions, the composed equation fails to be well-posed.

We concentrate on the index-1 case. Index-0 operators can be treated analogously, however, then we have $r_0 = m$, which makes this case less interesting and close to the well-known ODE theory.

The operator equation (see 98)

$$\mathcal{F}x = (Fx, b(x(t_a), x(t_e))) = (q, d) \tag{124}$$

describes the BVP

$$f((Dx)'(t), x(t), t) = q(t),\ t \in \mathcal{I} = [t_a, t_e], \quad b(x(t_a), x(t_e)) = d. \tag{125}$$

Let F be regular with tractability index 1, $x_* \in \operatorname{dom} F$. Owing to Theorem 4.5 and Corollary 4.6, if $\mathbf{e}_i \in \mathbb{R}^m$ denotes the ith unit vector, there exists exactly one $\xi_i \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ such that

$$F'(x_*)\xi_i = 0, \quad D(t_a)\xi_i(t_a) = D(t_a)\mathbf{e}_i, \quad i = 1, \dots, m.$$

Introduce the matrix-valued function

$$X_* := [\xi_1, \dots, \xi_m] \tag{126}$$

such that

$$A_*(DX_*)' + B_*X_* = 0, \quad D(t_a)(X_*(t_a) - I) = 0.$$

This yields the representation

$$\ker F'(x_*) = \{x = X_*c : c \in \mathbb{R}^m\}.$$

Notice that X_* is the so-called *maximal fundamental solution matrix normalized at* t_a, e.g.,[50]. The matrix function X_* has constant rank r_0, and it holds that $\ker X_* = \ker D$.

The following theorem provides conditions for the map $\mathcal{F}$ to be a local diffeomorphism (cf.Theorem 6.7)

Theorem 4.15 *Let $\mathcal{F}$ be the composed operator (98) described in Subsection 4.1, with differential-algebraic operator F being regular with tractability index 1. Suppose further $x_* \in \operatorname{dom} F$, $\mathcal{F}(x_*) = 0$, and let the matrix*

$$S_* := b'_a(x_*(t_a), x_*(t_e))X_*(t_a) + b'_e(x_*(t_a), x_*(t_e))X_*(t_e) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{r_0})$$

satisfy the conditions

$$\operatorname{im} S_* = \mathbb{R}^{r_0}, \quad \ker S_* = \ker D(t_a). \tag{127}$$

Then, the equation $\mathcal{F}x = 0$ is well-posed around x_*; to each $(q, d) \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) \times \mathbb{R}^{r_0}$ being sufficiently small, the perturbed equation $\mathcal{F}x = (q, d)$ possesses exactly one solution $x(q, d)$ in the neighborhood of x_*, and the inequality

$$\|x(q, d) - x_*\|_{\mathcal{C}^1_D} \le \kappa(\|q\|_\infty + |d|)$$

is valid with a constant κ.

Proof: We introduce the differentiable map $\mathcal{H}(x,q,d) := \mathcal{F}x - (q,d)$, $\mathcal{H}'_x(x,q,d) = \mathcal{F}'(x)$, $\mathcal{H}(x_*,0,0) = 0$. If $\mathcal{F}'(x_*)$ is a bijection, then the statement results from the standard implicit function theorem in Banach spaces. It remains to show the injectivity and surjectivity of the linearization $\mathcal{F}'(x_*)$.
If $\mathcal{F}'(x_*)z = 0$ then $F'(x_*)z = 0$, thus $z = X_*c$, with a certain $c \in \mathbb{R}^m$. On the other hand, then we have $b'_a(x_*(t_a), x_*(t_e))z(t_a) + b'_e(x_*(t_a), x_*(t_e))z(t_e) = 0$, therefore $S_*c = 0$. Condition (127) implies $c \in \ker D(t_a)$, and hence $z = 0$.
We already know that F is surjective. It remains to prove that the BVP

$$F'(x_*)z = 0, \quad b'_a(x_*(t_a), x_*(t_e))z(t_a) + b'_e(x_*(t_a), x_*(t_e))z(t_e) = d,$$

is solvable for each arbitrary $d \in \mathbb{R}^{r_0}$. The function $z_d := X_*S_*^+d$ satisfies the the first equation and the boundary condition reads $b'_a(x_*(t_a), x_*(t_e))z_d(t_a) + b'_e(x_*(t_a), x_*(t_e))z_d(t_e) = S_*S_*^+ = d$. □

Often one deals with the simpler IVPs.

Corollary 4.16 *Let $\mathcal{F}$ be the composed operator (98) described in Subsection 4.1, with differential-algebraic operator F being regular with tractability index 1. Let the boundary condition simplify to the initial condition*

$$b(x(t_a), x(t_e)) := Cx(t_a) - h = 0, \tag{128}$$

whereby $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{r_0})$, $h \in \mathbb{R}^{r_0}$.
Let $x_* \in \operatorname{dom} F$, $\mathcal{F}(x_*) = 0$, and

$$\ker C \cap \{w \in \mathbb{R}^m : B_*(t_a)w \in A_*(t_a)\} = \{0\}. \tag{129}$$

Then the IVP $\mathcal{F}x = 0$ is well-posed around x_*.

Proof: $\Pi_{*\,can}(t_a) \in \mathcal{L}(\mathbb{R}^m)$ represents the projector given by

$$\operatorname{im} \Pi_{*\,can}(t_a) = \{w \in \mathbb{R}^m : B_*(t_a)w \in A_*(t_a)\},$$
$$\ker \Pi_{*\,can}(t_a) = \ker D(t_a).$$

$\Pi_{*\,can}(t_a)$ has rank r_0. Regarding that $X_*(t_a) = \Pi_{*\,can}(t_a)$ we arrive at $S_* = C\Pi_{*\,can}(t_a)$. It remains to check condition (127).
$S_*z = 0$ means $\Pi_{*\,can}(t_a)z \in \ker C$. Because of (129) it follows that $\Pi_{*\,can}(t_a)z = 0$, thus $D(t_a)z = D(t_a)\Pi_{*\,can}(t_a)z = 0$. Conversely, $z \in \ker D(t_a)$ implies $S_*z = C\Pi_{*\,can}(t_a)z = C\Pi_{*\,can}(t_a)D(t_a)^+D(t_a)z = 0$. This proves the relation $\ker S_* = \ker D(t_a)$. Finally, for reasons of dimensions, it holds that $\operatorname{im} S_* = \mathbb{R}^{r_0}$.
□

The decomposition

$$\{w \in \mathbb{R}^m : B_*(t_a)w \in A_*(t_a)\} \oplus \ker D(t_a) = \mathbb{R}^m, \tag{130}$$

which is associated with the projector $\Pi_{*\,can}(t_a)$, makes evident that, choosing the matrix C so that

$$\ker C = \ker D(t_a). \tag{131}$$

condition (129) is always valid. Otherwise, additional structural restrictions are required. The choice (131) is natural in the sense that it directly applies to the inherent dynamical part. However, in practice, it can be required to fix other components for different reasons.

Example 4.17 (Additional structural restriction) *The differential-algebraic operator F associated with the semi-explicit system of $m_1 + m_2 = m$ equations*

$$x_1'(t) - g_1(x_1(t), x_2(t), t) = 0,$$
$$g_2(x_1(t), x_2(t), t) = 0,$$

is regular with tractability index 1, if the partial Jacobian g_{2,x_2} is everywhere nonsingular. Then $r_0 = m_1$ results. Assume, for simplicity, $m_1 = m_2$ and set

$$C = \begin{bmatrix} 0 & I \end{bmatrix}.$$

Condition (129) means now that

$$\begin{aligned} \ker C \cap \{z \in \mathbb{R}^m : & g_{*\,2,x_1}(t_a)z_1 + g_{*\,2,x_i}(t_a)z_2 = 0\} \\ &= \{z \in \mathbb{R}^m : z_2 = 0,\ g_{*\,2,x_1}(t_a)z_1 = 0\} = \{0\} \end{aligned}$$

is valid, with $g_{*\,2,x_i}(t_a) := g_{2,x_i}(x_{*\,1}(t_a), x_{*\,2}(t_a), t_a)$. Evidently, here condition (129) requires that also the matrix $g_{*\,2,x_1}(t_a)$ must be nonsingular. □

4.3.3 Regular index-2 operators and enhanced setting

Let the linear differential-algebraic operator $T \in \mathcal{L}_b(\mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m), \mathcal{C}^(\mathcal{I}, \mathbb{R}^m))$,

$$Tx = A(Dx)' + Bx, \quad x \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m), \tag{132}$$

be regular with tractability index 2. Then, by Theorem 4.5, the image $\operatorname{im} T$ is a nonclosed proper subset in $\mathcal{C}(\mathcal{I}, \mathbb{R}^m)$. We are interested in revealing the detailled structur of this subset, and, eventually, in modifying the image space to enforce surjectivity and the Fredholm property as in Subsection 2.4.1.
Choose $Q_0 = D^+D$, and let Q_1 denote the projector function onto $N_1 = \ker G_1$ along $S_1 = \{z \in \mathbb{R}^m : Bz \in \operatorname{im} G_1\}$. This provides a fine decoupling, and, in particular, $Q_1 = Q_1 G_2^{-1} B P_0$ (e.g.,[50, Subsection 2.4.3]). Compute for each arbitrary $x \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$

$$\begin{aligned} D\Pi_0 Q_1 G_2^{-1} Tx &= D\Pi_0 Q_1 G_2^{-1}\{G_1(D^-(Dx)' + Q_0 x) + BP_0 x\} \\ &= D\Pi_0 Q_1 G_2^{-1} BP_0 x = D\Pi_0 Q_1 x = \underbrace{D\Pi_0 Q_1 D^-}_{\in \mathcal{C}^1} Dx \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n), \end{aligned}$$

such that $\operatorname{im} T \subseteq \mathcal{C}^{ind\,2}(\mathcal{I}, \mathbb{R}^m)$,

$$\mathcal{C}^{ind\,2}(\mathcal{I}, \mathbb{R}^m) := \{q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : D\Pi_0 Q_1 G_2^{-1} q \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)\}. \tag{133}$$

By applying the decoupling procedure to equation $Tx = q$ one shows solvability for each arbitrary $q \in \mathcal{C}^{ind\,2}(\mathcal{I}, \mathbb{R}^m)$, and hence

$$\operatorname{im} T = \mathcal{C}^{ind\,2}(\mathcal{I}, \mathbb{R}^m). \tag{134}$$

We equip the linear space $\mathcal{C}^{ind\,2}(\mathcal{I}, \mathbb{R}^m)$ with the norm

$$\|q\|_{ind\,2} := \|q\|_\infty + \|(D\Pi_0 Q_1 G_2^{-1} q)'\|_\infty, \quad x \in \mathcal{C}^{ind\,2}(\mathcal{I}, \mathbb{R}^m), \tag{135}$$

which yields a Banach space. We derive further that

$$\begin{aligned} \|Tx\|_{ind\,2} &= \|Tx\|_\infty + \|(D\Pi_0 Q_1 G_2^{-1} Tx)'\|_\infty \\ &\leq \kappa \|x\|_{\mathcal{C}_D^1} + \|(D\Pi_0 Q_1 D^- Dx)'\|_\infty \leq \kappa_{new} \|x\|_{\mathcal{C}_D^1} \end{aligned}$$

for each arbitrary $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$. This means that our differential-algebraic operator is bounded and surjective, thus normally solvable in the new setting $T \in \mathcal{L}(\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m), \mathcal{C}^{ind\,2}(\mathcal{I}, \mathbb{R}^m))$. By Theorem 4.5, $\ker T$ has dimension $m-l = r_0 - (m - r_1)$ such that T is a Fredholm operator in the adapted setting, and $\mathrm{ind}_{fred}(T) = m - l$.

The composed operator $\mathcal{T} \in \mathcal{L}(\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m), \mathcal{C}^{ind\,2}(\mathcal{I}, \mathbb{R}^m) \times \mathbb{R}^{m-l})$ associated to a respective IVP,

$$\mathcal{T}x = (Tx, Cx(t_a)), \quad x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m),$$

with $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m-l})$, $\ker C = N_0(t_a) \oplus N_1(t_a)$, acts bijectively in Banach spaces. Therefore, the IVP $\mathcal{T}x = (q, d)$ is well-posed in the adapted setting. So far so good.

We turn to the nonlinear differential-algebraic operator F and ask whether we can modify Theorem 4.15 and Corollary 4.16 accordingly.

Suppose $x_* \in \mathrm{dom}\, F$, $Fx_* = 0$ and apply the matrix function sequence and the associated projector functions to the linearization $F(x_*)$. Below, the subscript $*$ indicates the resulting dependence of x_*.
We modify first the matrix function X_*. Choose a matrix $C_* \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m-l})$, $l = m - r_0 + m - r_1$, so that the condition $\ker C_* = N_0(t_a) \oplus N_{*\,1}(t_a)$ is satisfied. Then there exists exactly one $\xi_i \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ such that

$$F'(x_*)\xi_i = 0, \quad C_*\xi_i(t_a) = C_*\mathbf{e}_i, \quad i = 1, \dots, m.$$

The matrix-valued function

$$X_* := [\xi_1, \dots, \xi_m] \tag{136}$$

satisfies the IVP

$$A_*(DX_*)' + B_*X_* = 0, \quad C_*(X_*(t_a) - I) = 0,$$

and the representation

$$\ker F'(x_*) = \{x = X_*c : c \in \mathbb{R}^m\}.$$

Proposition 4.18 *Let $\mathcal{F}$ be the composed operator (98) described in Subsection 4.1, with differential-algebraic operator F being regular with tractability index 2, $l = m - r_0 + m - r_1$.*
Let $x_* \in \mathrm{dom}\, F$, $\mathcal{F}(x_*) = 0$. Let there exist an open neighborhood $\mathcal{U}_{x_*}$ of x_* such that

$$Fx \in \mathrm{im}\, F'(x_*), \quad x \in \mathcal{U}_{x_*}, \tag{137}$$

and let the matrix

$$S_* := b'_a(x_*(t_a), x_*(t_e))X_*(t_a) + b'_e(x_*(t_a), x_*(t_e))X_*(t_e) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m-l})$$

satisfy the conditions

$$\mathrm{im}\, S_* = \mathbb{R}^{m-l}, \quad \ker S_* = N_0(t_a) \oplus N_{*\,1}(t_a). \tag{138}$$

Then, the equation $\mathcal{F}x = 0$ is well-posed around x_* in the adapted setting; to each $(q, d) \in \mathcal{C}^{ind\,2}_*(\mathcal{I}, \mathbb{R}^m) \times \mathbb{R}^{m-l}$ being sufficiently small, the perturbed equation

$\mathcal{F}x = (q,d)$ possesses exactly one solution $x(q,d)$ in the neighborhood of x_*, and the inequality

$$\|x(q,d) - x_*\|_{\mathcal{C}^1_D} \le \kappa(\|q\|_\infty + \|(D\Pi_0 Q_{*1} G_{*2}^{-1} q)'\|_\infty + |d|)$$

is valid with a constant κ.

Proof: Regarding the advanced setting the statement proves in the same way as Theorem 4.15. □

Corollary 4.19 *Let $\mathcal{F}$ be the composed operator (98) described in Proposition 4.18. Let the projector functions Q_{*0}, Q_{*1} provide a complete decoupling of the linearization $F'(x_*)$, $\Pi_{*can} := P_{*0}P_{*1}$. Let the boundary condition simplify to the initial condition*

$$b(x(t_a), x(t_e)) := Cx(t_a) - h = 0, \tag{139}$$

whereby $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m-l})$, $h \in \mathbb{R}^{m-l}$ and

$$\ker C \cap \operatorname{im} \Pi_{*can}(t_a) = \{0\}. \tag{140}$$

Then the IVP $\mathcal{F}x = 0$ is well-posed around x_* in the adapted setting.

Proof: $\Pi_{*can}(t_a)$ has rank $m-l$. Regarding that $X_*(t_a) = \Pi_{*can}(t_a)$ we arrive at $S_* = C\Pi_{*can}(t_a)$. It remains to check condition (138).
$S_* z = 0$ means $\Pi_{*can}(t_a)z \in \ker C$. Because of (136) it follows that $\Pi_{*can}(t_a)z = 0$, thus $z \in \ker \Pi_{*can}(t_a) = N_0(t_a) \oplus N_{*1}(t_a)$. Conversely, $z \in (N_0(t_a) \oplus N_{*1}(t_a))$ implies $S_* z = C\Pi_{*can}(t_a)z = 0$. This proves the relation $\ker S_* = N_0(t_a) \oplus N_{*1}(t_a)$. For reasons of dimensions, it holds that $\operatorname{im} S_* = \mathbb{R}^{m-l}$. □

In the light of the fact that $\operatorname{im} F'(\tilde{x}_*)$ may vary with the reference function $\tilde{x}_*$, see Example 4.11, Condition (137) limits the class of relevant DAEs. The following structural restriction ensures (137). Let x_* be the reference solution and let the relation

$$f(y,x,t) - f(0, P_0(t)x, t) \in \operatorname{im} G_{*1}(t), \quad y \in \mathbb{R}^n, x \in \mathcal{D}_F, t \in \mathcal{I}, \tag{141}$$

be valid. Then it follows for $x \in \operatorname{dom} F$ that

$$\begin{aligned} &D(t)\Pi_0(t)Q_{*1}(t)G_{*2}(t)^{-1} f((Dx)'(t), x(t), t) \\ &\quad = D(t)\Pi_0(t)Q_{*1}(t)G_{*2}(t)^{-1} f(0, P_0(t)x(t), t)), \quad t \in \mathcal{I}. \end{aligned}$$

Assuming that, additionally, f has also a continuous partial derivative f_t, and regarding that $P_0 x = D^- Dx$ is continuously differentiable, we conclude

$$\begin{aligned} D\Pi_0 Q_{*1} G_{*2}^{-1} Fx &= D\Pi_0 Q_{*1} G_{*2}^{-1} f(0, (P_0 x)(\cdot), \cdot) \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n), \\ Fx &\in \operatorname{im} F'(x_*). \end{aligned}$$

Condition (141) does not apply to Example 4.11. Fortunately, it applies to the widely used Hessenberg size-2 systems as we demonstrate by the next example.

Example 4.20 (Hessenberg size-2 DAE) *Given is the special system of* $m = m_1 + m_2$ *equations*

$$\begin{aligned} x_1'(t) + g_1(x_1(t), x_2(t), t) &= 0, \\ g_1(x_1(t), t) &= 0, \end{aligned}$$

yielding

$$D = \begin{bmatrix} I & 0 \end{bmatrix} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m_1}), \quad f(y, x, t) = \begin{bmatrix} I \\ 0 \end{bmatrix} y + \begin{bmatrix} g_1(x_1, x_2, t) \\ g_2(x_1, t) \end{bmatrix},$$

$$P_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} I & g_{1\,x_2} \\ 0 & 0 \end{bmatrix},$$

and hence

$$f(y, x, t) - f(0, P_0 x, t) = \begin{bmatrix} y + g_1(x_1, x_2, t) - g_1(x_1, 0, t) \\ 0 \end{bmatrix}.$$

$\operatorname{im} G_1$ *is independent of its arguments and condition (141) is fulfilled.* □

4.3.4 Newton-Kantorovich-like iterations

Supposed that the composed operator $\mathcal{F}$ associated with the BVP (99) is a local diffeomorphism at $x_* \in \operatorname{dom} \mathcal{F}$ and $\mathcal{F}(x_*) = 0$, the well-known Newton-Kantorovich iteration

$$x_{k+1} = x_k - \mathcal{F}'(x_k)^{-1} \mathcal{F}(x_k), \quad k \geq 0, \tag{142}$$

can be applied to approximate x_*. If the initial guess x_0 is sufficiently close to x_*, then these iterations are well-defined and x_k tends to x_*. Practically, one solves the linear equations

$$\mathcal{F}'(x_k) z = -\mathcal{F}(x_k), \quad k \geq 0, \tag{143}$$

and, having the solution z_{k+1} of the linear problem (143), one puts

$$x_{k+1} = x_k + z_{k+1}. \tag{144}$$

The linear problem (143) represents the linear BVP

$$\begin{aligned} f_y(\xi_k(t))(Dz)'(t) + f_x(\xi_k(t)) z(t) &= -f(\xi_k(t)), \quad t \in \mathcal{I}, \\ b_a(x_k(t_a), x_k(t_e)) z(t_a) + b_e(x_k(t_a), x_k(t_e)) z(t_e) &= -b(x_k(t_a), x_k(t_e)), \end{aligned}$$

with $(\xi_k(t)) := ((Dx_k)'(t), x_k(t), t)$ and the first partial derivatives b_a, b_e of the function b with respect to its first and second arguments.
Mostly, a damping parameter is incorporated, and instead of (144) one applies

$$x_{k+1} = x_k + \alpha_{k+1} z_{k+1}, \quad \text{with} \quad \alpha_{k+1} \in (0, 1]. \tag{145}$$

Usually the damping parameter is choosen so that the residuum $\mathcal{F}(x_{k+1})$ becomes smaller in some sense, that is

$$\|\mathcal{F}(x_{k+1})\|_{res} < \|\mathcal{F}(x_k)\|_{res},$$

with a suitable measure of the residuum, for instance,

$$\|\mathcal{F}(x)\|_{res} := \|\mathcal{F}(x)\| = \|F(x)\|_\infty + |b(x(t_a), x(t_e))|$$
$$\text{and} \quad \|\mathcal{F}(x)\|_{res}^2 := \|F(x)\|_{L^2}^2 + |b(x(t_a), x(t_e))|^2.$$

Sufficient conditions for the composed operator $\mathcal{F}$ to be a local diffeomorphism are described in Subsubsection 4.3.2 for the index-1 case and in Subsubsection 4.3.3 for a class of index-2 problems.

Next we take a look at the differentiable functional

$$J(x) := \frac{1}{2}\|F(x)\|_{L^2}^2 + \frac{1}{2}|b(x(t_a), x(t_e))|^2, \quad x \in \operatorname{dom}\mathcal{F}. \tag{146}$$

Of course, the problem to solve the equation $\mathcal{F}(x) = 0$ can be regarded as the problem to minimize this functional.
For $x \in \operatorname{dom}\mathcal{F}$ and $z \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$, the directional derivative reads

$$\begin{aligned} J'(x)z &= (F'(x)z, F(x))_{L^2} \\ &\quad + \langle\, b_a(x(t_a), x(t_e))z(t_a) + b_e(x(t_a), x(t_e))z(t_e),\ b(x(t_a), x(t_e))\,\rangle. \end{aligned}$$

If $x^0 \in \operatorname{dom}\mathcal{F}$ is fixed, $\mathcal{F}(x^0) \neq 0$, and if there exists a solution z_N of the linear equation,

$$\mathcal{F}'(x^0)z = -\mathcal{F}(x^0), \quad k \geq 0, \tag{147}$$

then it results that

$$J'(x^0)z_N = -\|F(x^0)\|_{L^2}^2 - |b(x^0(t_a), x^0(t_e))|^2 < 0$$

thus $J(x^0 + \alpha z_N) < J(x^0)$ for all sufficiently small $\alpha > 0$. Therefore, the so-called Newton direction z_N serves as descent direction. Constructing a descent method by applying Newton directions is essentially the same as the damped Newton-Kantorovich iteration. This works supposed the conditions described in Subsubsections 4.3.2 and 4.3.3 are given, that is, for index-1 and a restricted class of index-2 problems.

For equations $\mathcal{F}(x) = 0$ involving higher index differential-algebraic operators F, there are two principal difficulties concerning Newton descent and Newton-Kantorovich iteration as well:
1. The linear equation (143) resp. (147) is essentially ill-posed and might not be solvable. Changing to least-squares solutions does not make great sense, since the linearizations $\mathcal{F}'(x)$ are not normally solvable.
2. For an essentially ill-posed problem a small residuum $\mathcal{F}(x_k)$ does not at all mean that x_k is close to a solution, see Example 2.5.

Among the methods for ill-posed problems one finds generalizations of Newton-like methods using outer inverses, see [59]. Instead of the unbounded inverse $\mathcal{F}(x_k)^{-1}$ in (142) one uses a bounded outer inverse. Theorem 4.8 provides such an outer inverse.

Let $\mathcal{A}(x) \in \mathcal{L}_b(X, Y)$ be an approximation of $\mathcal{F}'(x) \in \mathcal{L}_b(X, Y)$, with $X = \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$, $Y = \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$. Further, let $x_0 \in \operatorname{dom}\mathcal{F}$ and $\Gamma \in \mathcal{L}_b(Y, X)$ be an outer inverse of $\mathcal{A}(x_0)$. Then, supposed that x_k is sufficiently close to x_0 (cf. Lemma 6.6), by

$$\mathcal{A}(x_k)^- := (I + \Gamma(\mathcal{A}(x_k) - \mathcal{A}(x_0))^{-1}\Gamma$$

we obtain a bounded outer inverse of $\mathcal{A}(x_k)$ such that

$$\ker \mathcal{A}(x_k)^- = \ker \Gamma, \quad \operatorname{im} \mathcal{A}(x_k)^- = \operatorname{im} \Gamma.$$

We refer to [59, Theorem 3.1] for conditions ensuring the sequence

$$x_{k+1} = x_k - \mathcal{A}(x_k)^- \mathcal{F}(x_k), \quad k \geq 0, \tag{148}$$

with initial guess x_0, to be well-defined and to converge to a solution of the equation

$$\Gamma \mathcal{F}(x) = 0. \tag{149}$$

Then the equation (149) possesses a unique solution in $\{x_0 + \operatorname{im} \Gamma\} \cap B(x_0, \tau)$, τ sufficiently small.

4.4 Different views on constant-rank conditions

Regularity of differential-algebraic operators in Definitions 4.2 and 4.9 is supported by several constant-rank conditions. These definitions are compatible with the regularity notion for DAEs within the projector based analysis (e.g.,[50], also Appendix 6.3). In the DAE literature several different opinions concerning regularity of time-varying and nonlinear DAEs can be found, which all reflect and generalize regularity of matrix pencils, see [50] for a comprehensive discussion.
Much work concerning DAEs (e.g.,[9, 6]) is focused on problems

$$E(t)x'(t) + F(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{150}$$

being smoothly transformable into the so-called standard canonical form (SCF)

$$\begin{bmatrix} I & 0 \\ 0 & N(t) \end{bmatrix} x'(t) + \begin{bmatrix} W(t) & 0 \\ 0 & I \end{bmatrix} x(t) = g(t), \quad t \in \mathcal{I},$$

which generalizes the Weierstraß–Kronecker form of matrix pencils, (cf.,(9)). The matrix function N is strictly lower or upper diagonal, but there is no restriction concerning the rank of $N(t)$. In the easier cases, if N is absent or vanishes identically, the matrix $E(t)$ has constant rank, and this is, in essence, in agreement with our notion of regular index-0 and index-1 problems. However, concerning nontrivial N, there is an ongoing controversy.
Supposing sufficiently smooth or even real analytic N and g, DAEs being transformable into SCF are solvable in $\mathcal{C}^1$, and the flow does not show critical behavior.

In contrast, the regularity concept within the framework of the tractability index consequently indicates all corresponding rank changes as critical points. This way it detects serious singularities of the flow such as bifurcations, which are ruled out by the SCF approach a priori, but also so-called *harmless critical points*, which do not affect the flow in smooth environments,see [50, 20].
The next example explains the difficulty arising from harmless critical points in a rigorous functional-analytic setting.

Example 4.21 (DAE in SCF with harmless critical point) *The DAE*

$$\underbrace{\begin{bmatrix} 0 & 1 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{=E} x'(t) + x(t) = q(t), \quad t \in \mathcal{I} := [-1, 1],$$

is already in SCF and quasi-regular in the sense of [50]. Let the coefficient functions $\alpha \in \mathcal{C}^1(\mathcal{I}, \mathbb{R})$ *and* $\beta \in \mathcal{C}(\mathcal{I}, \mathbb{R})$ *be given so that*

$$t \in \mathcal{I}_- := [-1, 0] \Rightarrow \alpha(t) + \beta(t) = 0, \quad t \in \mathcal{I}_+ := (0, 1] \Rightarrow \alpha(t) + \beta(t) > 0.$$

The DAE is associated with the differential-algebraic operator $\mathring{T}x = Ex' + x$, $x \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^4)$*, and the closure of* $\mathring{T} \in \mathcal{L}(\mathcal{C}(\mathcal{I}, \mathbb{R}^4), \mathcal{C}(\mathcal{I}, \mathbb{R}^4))$ *is given as*

$$Tx = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \beta \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{=A} (\underbrace{\begin{bmatrix} 0 & 1 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{=D} x)' + \underbrace{\begin{bmatrix} 1 & 0 & -\alpha' & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{=B} x, \ x \in \operatorname{dom} T = \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^4).$$

Compute

$$Q_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1 & 1 & \alpha - \alpha' & 0 \\ 0 & 0 & -\alpha & \beta \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have here $r_0 = \operatorname{rank} G_0 = \operatorname{rank} E = 2$*. The matrix* $G_1(t)$ *changes the rank at* $t_c = 0$*, we have* $r_0 = 2$ *on* $\mathcal{I}_-$ *and* $r_0 = 3$ *on* $\mathcal{I}_+$*. The operator* T *fails to be regular on* $\mathcal{I}$ *since* t_c *is a critical point, more precisely, a harmless critical point. On the other hand, considering the restrictions* $T|_{\mathcal{I}_-}$ *and* $T|_{\mathcal{I}_+}$ *corresponding to the restrictions of the functions* x *onto the subintervals* $\mathcal{I}_-, \mathcal{I}_+$*, we may check that both operators are regular, however with different characteristics. Namely,* $T|_{\mathcal{I}_-}$ *has tractability index* $\mu = 2$ *and characteristics* $r_0 = 2, r_1 = 2, r_2 = 4$*, whereas* $T|_{\mathcal{I}_+}$ *has tractability index* $\mu = 3$ *and characteristics* $r_0 = 2, r_1 = 3, r_2 = 3, r_3 = 4$*. We observe qualitatively different operator images:*

$$\begin{aligned} \operatorname{im} T|_{\mathcal{I}_-} &= \{q \in \mathcal{C}(\mathcal{I}_-, \mathbb{R}^4) : q_2, q_4 \in \mathcal{C}^1(\mathcal{I}_-, \mathbb{R})\}, \\ \operatorname{im} T|_{\mathcal{I}_+} &= \{q \in \mathcal{C}(\mathcal{I}_+, \mathbb{R}^4) : q_2 - (\alpha + \beta)q_4', \ q_4 \in \mathcal{C}^1(\mathcal{I}_+, \mathbb{R})\}. \end{aligned}$$

In essence, on $\mathcal{I}_+$*, one needs the additional derivative* q_4''*. This play its role in rigorous input-output relations. It does not matter if one is only interested in the flow for smooth data.* □

In the framework of the projector based analysis (cf.[50]), the harmless critical points are compensated within so-called quasi-regular DAEs. However, neither for the concept of quasi-regular DAEs nor for the concepts associated with the SCF and derivative arrays a functional-analytic interpretation seems to be available.

4.5 Notes and references

Remark 4.22 *A DAE in so-called standard form,*

$$\mathfrak{f}(x'(t), x(t), t) = 0$$

can be brought to the form (88) by introducing the additional function $\chi = x'$ *and regarding the extended system*

$$\begin{aligned} x'(t) - \chi(t) &= 0, \\ \mathfrak{f}(\chi(t), x(t), t) &= 0. \end{aligned}$$

However, we do not extra recommend this way, which would constrain us to $\mathcal{C}^1$*-solutions (cf. the discussion in Subsection 2.3), and already for regular linear constant coefficient DAEs the index would be increased by 1. Fortunately, there are more appropriate reformulations for large classes of DAEs (e.g.[50]), and, what is more important, the most frequently applied classes of DAEs such as the semi-explicit ones are originally in the form (88).*
Furthermore, as applied e.g. in [29, 52, 53], if $\ker \mathfrak{f}_{x^1}(x^1, x, t)$ *is a* $\mathcal{C}^1$*-subspace who is independent of* x^1 *and* x*, then there is a continuously differentiable projector valued function* $D : \mathcal{I}_\mathfrak{f} \to \mathcal{L}(\mathbb{R}^m)$ *such that*

$$\begin{aligned} \mathfrak{f}(x^1, x, t) &\equiv \mathfrak{f}(D(t)x^1, x, t), \quad \textit{and} \\ \mathfrak{f}(x'(t), x(t), t) &= \mathfrak{f}((Dx)'(t) - D'(t)x(t), x(t), t) \\ &=: f((Dx)'(t), x(t), t), \quad \textit{for } x \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m). \end{aligned}$$

Moreover, DAEs

$$f((d(x(t), t))', x(t), t) = 0 \tag{151}$$

with properly involved nonlinear derivative term and border projector function $R \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{R}^n))$ *(cf. [50, 55]) can be treated as DAEs of the form (88) via the enlarged system*

$$\begin{aligned} f((Ry)'(t), x(t), t) &= 0, \\ y(t) - d(x(t), t) &= 0. \end{aligned}$$

This is of particular interest for quasi-linear DAEs

$$A(x(t), t)(d(x(t), t))' + b(x(t), t) = 0$$

which arise e.g., in circuit simulation. We refer to [55] for detailed relations between (151) and its enlarged form.

Remark 4.23 *Regularity of nonlinear differential-algebraic operators in the sense of Definition 4.9 is consistent with the definition of regular DAEs in [50] via Corollary 6.16, and hence, justified this way, see Appendix 6.3. In essence,* F *is a regular differential-algebraic operator, if the set* $\mathcal{D}_F \times \mathcal{I}$ *totally belongs to a regularity region of the associated DAE.*

Remark 4.24 *It was pointed out e.g. in [52, 40] that operator equations representing higher-index DAEs are essentially ill-posed in their natural settings. [50, Section 3.9] addresses DAEs as operator equations and presents some parts*

of Section 4 concerning IVPs. It is pointed out that DAE solutions depend smoothly on a well-defined part of their initial value.
Let the differential-algebraic operator F be regular with index 1, $x_* \in \operatorname{dom} F$, $F(x_*) = 0$, $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{r_0})$, $\operatorname{im} C = \mathbb{R}^{r_0}$, $\ker C = \ker D(t_a)$. Then the IVP

$$F(x) = 0, \quad Cx(t_a) = Cx_*(t_a) + z \tag{152}$$

is uniquely solvable for all sufficiently small $z \in \mathbb{R}^{r_0}$ (cf. Theorem 4.15). The solution is continuously differentiable with respect to z and the sensitivity matrix $X(t; z) := x'_z(t; z)$ satisfies the variational system

$$f_y(\eta(t; z))(DX)'(t; z) + f_x(\eta(t; z))X(t; z) = 0,\ t \in \mathcal{I}, \quad CX(t_a; z) = I_{r_0},$$

with $(\eta(t; z)) := ((Dx)'(t; z), x(t; z), t)$, that is,

$$F'(x(z))X(z) = 0, \quad CX(t_a; z) = I_{r_0}.$$

We emphasize that the initial condition, that is, the requirement for C, is in accordance with the Fredholm index of F. This statement would not longer hold for $C = I$.
For regular higher-index DAEs the situation is much more subtle. Since $F'(x_*)$ is no longer surjective and fredholm, for applying the implicit function theorem in the enhanced setting, we are obliged to assume the inclusion $F(x) \in \operatorname{im} F'(x_*)$ for all x in a neighborhood of the reference solution x_*. Furthermore, less initial conditions are allowed and the correct formulation of C depends on the reference solution.
For index-2 problems then the above statement concerning the IVP (152) and the variational system is valid with a full row-rank matrix C such that

$$\ker C = N_0(t_a) + N_{*\,1}(t_a).$$

The index $*$ indicates the dependence on the reference solution x_*. This is not too bad for $\mu = 2$, but worse for higher index, cf. Remark 4.26.

Remark 4.25 *Analogous results concerning linear differential-algebraic operators as described in Subsection 4.2 can also be accomplished for different settings (cf. Remark 3.17), e.g.,*

$$\begin{aligned}
T \in&\mathcal{L}_b(H^1_D(\mathcal{I}, \mathbb{R}^m), L^2(\mathcal{I}, \mathbb{R}^m)),\\
T \in&\mathcal{L}_c(\mathcal{C}(\mathcal{I}, \mathbb{R}^m), \mathcal{C}(\mathcal{I}, \mathbb{R}^m)),\\
T \in&\mathcal{L}_c(L^2(\mathcal{I}, \mathbb{R}^m), L^2(\mathcal{I}, \mathbb{R}^m)),\\
T \in&\mathcal{L}_b(W^{1,\infty}_D(\mathcal{I}, \mathbb{R}^m), L^\infty(\mathcal{I}, \mathbb{R}^m)),
\end{aligned}$$

Hilbert spaces are favorable when looking for least-squares solutions, Moore-Penrose inverses and pseudo-solutions. In any case, if T is regular with a higher index $\mu \geq 2$, then the equation $Tx = q$ is essentially ill-posed. In particular, then the Moore-Penrose inverse is unbounded, and a small residuum does not at all indicate a good approximation of a least-squares solution, see Example 2.5.

Remark 4.26 *The well-posed BVPs and IVPs in Subsubsections 4.3.2 and 4.3.3 are associated with composed maps $\mathcal{F}$ being local $\mathcal{C}^1$- diffeomorphisms at the solution x_*. Thereby, $Y = \mathcal{C}(\mathcal{I}, \mathbb{R}^m) \times \mathbb{R}^{r_0}$ serves as image space in the index-1 case.*
If the index is $\mu = 2$, we are already confronted with the more difficult adapted setting

$$Y_{*\,new} = \mathcal{C}_*^{index\mu}(\mathcal{I}, \mathbb{R}^m) \times \mathbb{R}^{m-l}, \quad l = \sum_{i=0}^{\mu-1}(m - r_i).$$

Also in higher index cases $\mu > 2$ one can precisely describe the operator image $\operatorname{im} \mathcal{F}'(x_*)$ and introduce an appropriate norm $\|\cdot\|_{*\,index\mu}$ for obtaining the adapted Banach space $Y_{*\,new}$ (cf. Subsection 2.4, also [50, Section 2.6.4]). Assuming again condition (137), that is,

$$Fx \in \operatorname{im} F'(x_*), \quad x \in \mathcal{U}(x_*), \tag{153}$$

the composed operator $\mathcal{F}$ is a local $\mathcal{C}^1$- diffeomorphisms at the solution x_* and the equation $\mathcal{F}x = 0$ is well-posed in this adapted setting. This sounds fine, however, there are two principal concerns with this:
1. Though for $\mu = 2$ the adapted space $Y_{*\,new}$ is transparent and can be seen as reasonable compromise, in higher index cases the topology defined by the new norm $\|\cdot\|_{*\,index\mu}$ is far from meeting practical needs, since it is too strong by all means as Examples 2.5, 2.10 demonstrate.
2. Condition (153) is valid for a quite large class of index-2 DAEs including Hessenberg size-2 systems, which is very useful in optimal control. However, this condition is not given in many index-2 DAEs arising in circuit simulation, see [67] and Example 4.11. Moreover, for $\mu > 2$, no practical conditions ensuring (153) are in sight such as (141) ensuring (153) for $\mu = 2$.

Remark 4.27 *Let the coefficients of the differential-algebraic operator $T \in \mathcal{L}(\mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m), C(\mathcal{I}, \mathbb{R}^m))$ are sufficiently smooth and T be regular with tractability index μ. Then the composed operator $\mathcal{T}$ associated with the IVP $Tx = q$, $Cx(t_a) = d$, with $\operatorname{im} C = \mathbb{R}^{m-l}$, $\ker C = N_0(t_a) + \cdots + N_{\mu-1}(t_a)$, is injective and the inclusion $\mathcal{C}^{\mu-1}(\mathcal{I}, \mathbb{R}^m) \subset \operatorname{im} T$ is valid.*
Set $Y_{new} := \operatorname{im} T$ and introduce on Y_{new} an appropriate norm $\|\cdot\|_{index\,\mu}$ to attain a Banach space (cf. [50, Section 3.9], also Subsection 2.4). Then the inequality

$$\|\mathcal{T}^{-1}(q,d)\|_\infty \le \|\mathcal{T}^{-1}(q,d)\|_{\mathcal{C}_D^1} \le \kappa(\,|d| + \|q\|_{index\,\mu}) \le \tilde{\kappa}(\,|d| + \sum_{i=0}^{\mu-1} \|q^{(i)}\|_\infty),$$

is true for all $d \in \mathbb{R}^{m-l}$ and $q \in \mathcal{C}^{\mu-1}(\mathcal{I}, \mathbb{R}^m)$. This implies that a DAE associated with a regular differential-algebraic operator T has perturbation index μ (cf. [32]).

Let the nonlinear differential-algebraic operator F be regular with tractability index μ, $x_* \in \operatorname{dom} F$, $Fx_* = 0$, and let the function f defining F as well as x_* be sufficiently smooth so that the linearization $F'(x_*)$ has sufficiently smooth coefficients for the inclusion $\mathcal{C}^{\mu-1}(\mathcal{I}, \mathbb{R}^m) \subset \operatorname{im} F'(x_*)$ to hold. If, additionally, the condition (153) is satisfied, then the IVP $\mathcal{F}x = (q, Cx_*(t_a))$, with suitable *matrix C, is uniquely solvable for all sufficiently small $q \in \mathcal{C}^{\mu-1}(\mathcal{I},\mathbb{R}^m)$, and the inequality*

$$\|x(q)-x_*\|_\infty \le \|x(q)-x_*\|_{\mathcal{C}^1_D} \le \kappa\|q\|_{*\,index\mu} \le \tilde{\kappa}\sum_{i=0}^{\mu-1}\|q^{(i)}\|_\infty$$

follows, and hence, the DAE has perturbation index μ.
Since the unpleasing condition (153) is applied, this fails to confirm the conjecture in [50, p. 290] in its general form. It remains open whether this general conjecture can be verified.

Remark 4.28 *Well-posed BVPs for regular index-1 DAEs and their discretizations are treated as operator equation e.g., in [51, 52, 18, 19, 54, 53].*
The statements of Theorem 4.15 and Proposition 4.18 related to IVPs for quasilinear index-1 and index-2 DAEs of the form (151) are proved in [55].

Remark 4.29 *Least-squares collocation methods are known to belong to so-called regularizing algorithms for ill-posed problems (e.g. [31]). A first attempt to treat boundary value problems*

$$E(t)x'(t)+F(t)x(t)=q(t),\ t\in[t_a,t_e],\quad C_ax(t_a)+C_ex(t_e)=d,$$

for higher-index time-varying linear DAEs by a least-squares collocation method is already reported in [33]. Recent experiments give rise to expect some further progress on that score.
Let $\pi\subset[t_a,t_e]$ denote a finite set of points. A function $x_\pi:[t_a,t_e]\to\mathbb{R}^m$ is called least-squares collocation solution of the BVP, if

$$E(t)x_\pi'(t)+F(t)x_\pi(t)=q(t),\ t\in\pi,\quad C_ax_\pi(t_a)+C_ex_\pi(t_e)=d, \tag{154}$$

and x_π minimizes some scalar product norm among all solutions of (154). Reproducing kernels ([2]) serve as essential tool for constructing and analyzing least-square collocation methods. The operator $\mathcal{T}$ associated with the BVP is given on the Sobolev space $H^k(\mathcal{I},\mathbb{R}^m)$, $k\ge 2$. In order to reduce the computational expense, [33] concentrates on $k=2$. $H^2(\mathcal{I},\mathbb{R}^m)$ is endowed with the scalar product

$$(x,y)_{H^2}:=\langle x(t_a),x(t_a)\rangle+\langle x(t_e),x(t_e)\rangle+(x'',y'')_{L^2}.$$

The so-called normal spline method in [28] repeats this approach for integro-differential equations,

$$E(t)x'(t)+F(t)x(t)-\int_{t_a}^{t_e}K(t,s)x(s)\mathrm{ds}=q(t),\ t\in\mathcal{I},\ C_ax(t_a)+C_ex(t_e)=d.$$

As basic Sobolev space in [28] serves $H^k(\mathcal{I},\mathbb{R}^m)$ equipped with the scalar product

$$(x,y)_{H^k}:=\sum_{j=0}^{k-1}\langle x^{(j)}(t_a),x^{(j)}(t_a)\rangle+(x^{(k)},y^{(k)})_{L^2}.$$

Remark 4.30 *Convergence conditions for the Newton-Kantorovich iteration applied to well-posed BVPs for regular index-1 DAEs are derived in [54, 63]. Moreover, also for the class of regular index-2 DAEs described in Subsubsection 4.3.3, well-posedness is ensured by adapting the image-space and convergence conditions are provided. Practical computational experiments are reported in [63].*
Further, [14, Subsubsection 2.2.10] is also devoted to Newton-Kantorovich iterations via adapting the image spaces. However, it has been overlooked there that the images of the linearizations as well as the new advanced norms depend on the reference functions, see Remark 4.26.

Remark 4.31 *The bounded outer inverse Γ in Theorem 4.8 was constructed first in [66, Chapter 4] for the index-3 case, aiming for a solvability result of the corresponding equation $\Gamma(\mathcal{F}x - (q,d)) = 0$ by applying [59, Theorem 3].*
What concerns the computational treatment of DAEs by Newton-like iteration methods using outer inverses, as yet, there seems to be no practical experience and no efforts are reported.

Remark 4.32 *By means of differentiating certain derivative-free equations of the DAE*

$$A(Dx)' + Bx = q, \tag{155}$$

with m unknowns and $k \geq m$ equations, and adding the differentiated part to the given one, one can reduce the tractability index (see [50, Section 10.2] for the definition of the tractability index of nonregular DAEs). This allows to modify the essentially ill-posed problem $Tx = q$ to an enlarged system $\bar{T}x = \bar{q}$ having a normally solvable differential-algebraic operator $\bar{T}$ and $\operatorname{dom}\bar{T} = \operatorname{dom}T$.
More precisely, if the DAE (155) has index $\mu \geq 2$ and G_μ has full column-rank, then the enlarged DAE

$$\begin{bmatrix} A \\ W_{\mu-1}BD^- \end{bmatrix} (Dx)' + \begin{bmatrix} B \\ (W_{\mu-1}BD^-)'D = \end{bmatrix} \begin{bmatrix} q \\ (W_{\mu-1}q)' \end{bmatrix}, \tag{156}$$

with the same m unknowns, but $k + m$ equations, has tractability index $\mu - 1$ owing to [50, Proposition 10.8].
In particular, starting from a regular DAE with tractability index $\mu \geq 2$, one can successively reduce the index and eventually arrives at an overdetermined DAE with tractability index 1. The latter is associated with a normally solvable differential-algebraic operator. Possibly, this explains the advantage of overdetermined discretizations as used e.g., in [25]) for regular index-3 DAEs in Hessenberg form.

Remark 4.33 *The present paper intends to provide a basic functional analysis for linear DAEs with continuous coefficients and nonlinear DAEs given by continuously differentiable data.*
Appropriate modification of the linear theory for linear DAEs with integrable coefficients can be accomplished. A first approach can be found in [35], see also [50, Section 12.3]. By now, there is a lack of a comprehensive theory in this respect.
Furthermore, in [61, 62, 60, 73], quasi-linear DAEs

$$A(t)(Px)'(t) + B(t)x(t) + g(x(t), t) = 0, \quad \text{a.e. in } [t_a, t_e],$$

with a measurable and bounded function g, are treated as inclusions

$$A(t)(Px)'(t) + B(t)x(t) \in G(x(t),t), \quad a.e.\ in\ [t_a, t_e],$$

with Filippov functions G. Regularity with index 1 (transferability in [61, 62]) is adapted to the inclusion. Then, solvability in Fillippov's sense is proved. Linear DAEs with piecewise-smooth distributional coefficients are considered in [70] in order to manage DAEs whose coefficient matrices have jumps in view of system theoretical aspects. This approach is limited to linear DAE.

5 Regularization of ill-posed DAEs

Regularization in the context of ill-posed problems is essentially the approximation by means of a certain family of well-posed problems. For instance, the well-known Tikhonov regularization utilizes the functional

$$J_\alpha(x) := \|Kx - q\|_Y^2 + \alpha\|x\|_X^2 \tag{157}$$

for the ill-posed equation $Kx = q$ stated in Hilbert spaces X and Y. The functional J_α is to be minimized for special sequences of values of $\alpha > 0$, e.g., [21, 71, 23].
The traditional nomenclature used in the theory of matrix pencils and subsequently in the DAE theory occupies the notion *regular* for special pencils and DAEs. However, those regular pencils and DAEs may induce ill-posed operator equations. This entails that also regular DAEs may need a further regularization in the sense of ill-posed problems. We describe in Subsection 5.2 the respective efforts.

Most common in DAE analysis and applications are index reduction procedures and transformations into special form so that the latter allows a safer numerical treatment. Actually, this way, an ill-posed problem is modified into a well-posed one, too. This justifies also to speak of a *regularization.* For instance, Remark 4.32 describes a modification of an essentially ill-posed DAE to a normally solvable overdetermined DAE.
Usually one forms a derivative array, also known as *prolongated system*, so that all eventually required derivatives are carried out analytically and then one looks for an explicit ODE and derivative-free equations. This sounds simpler as it is! Of course, there are various ways to do this. We refer to [64, 14, 47, 8] for different overviews. As yet, to the author's knowledge, a systematical comparison of all these reduction procedures is missing. Below we describe the version of Chistyakov ([10, 11] since it is basically a functional-analytic approach.

To the author's knowledge, in all those index reduction procedures the derivative array must be provided analytically. No errors are allowed in the prolongated system, since an appropriate sensitivity analysis is not available.

5.1 Chistyakov's approach

An operator version of the reduction concept is developed in [10, 11, 14]. By means of prolongated systems, special left regularizers are arranged. Thereby several rank conditions play their role.
Without any doubt, this is closely related to other reduction concepts aiming at

regular (explicit or implicit) ODEs or index-1 DAEs such as those in [4, 64, 47]. However, the present treatise is not the right place to analyze the interrelations between the various reduction concepts which are developed to a large extent in parallel and without having notice of each other.

Given is the operator

$$T \in \mathcal{L}(X,Y), \quad Tx := Ex' + Fx, \quad x \in X, \tag{158}$$

with matrix coefficients $E(t), F(t) \in \mathcal{L}(\mathbb{R}^m)$ sufficiently smooth on a neighborhood $\mathcal{I}_0$ of the interval $\mathcal{I} := [t_a, t_e]$. $E(t)$ is everywhere singular, which excludes regular ODEs. Denote

$$\rho := \max\{\operatorname{rank} E(t) : t \in \mathcal{I}\} < m.$$

The role of the pre-image space X is assigned either to the function space $\mathcal{C}^\infty(\mathcal{I},\mathbb{R}^m)$ or to $\mathcal{C}^1(\mathcal{I},\mathbb{R}^m)$, whereas the role of the image space Y is assigned to $\mathcal{C}^\infty(\mathcal{I},\mathbb{R}^m)$ and a factitious function space $\mathcal{C}^s_*(\mathcal{I},\mathbb{R}^m)$, respectively. The latter one will be specified below, which requires full information about $T(\mathcal{C}^1(\mathcal{I},\mathbb{R}^m))$. It is investigated in [10, 11, 14] whether the nullspace of the operator T has finite dimension and whether T represents a Fredholm operator (called Noetherian there). The operator T is associated with the DAE

$$E(t)x'(t) + F(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{159}$$

as well as with the IVP

$$E(t)x'(t) + F(t)x(t) = q(t), \quad t \in \mathcal{I}, \quad x(t_a) = x_a \in \mathbb{R}^m. \tag{160}$$

We emphasize that $E(t)$ may change its rank here, cf. Subsection 4.4.

The DAE (159) as well as the coefficient pair $\{E,F\}$ are said to be *transformable into standard canonical form* (SCF), if there exist a coordinate change $L \in \mathcal{C}^1(\mathcal{I},\mathcal{L}(\mathbb{R}^m))$ and a nonsingular scaling $K \in \mathcal{C}(\mathcal{I},\mathcal{L}(\mathbb{R}^m))$ converting the DAE coefficients to (e.g.,[4, Subsection 2.4.2])

$$LEK = \begin{bmatrix} I & \\ & N \end{bmatrix} \begin{matrix} \}m-l =: \delta \\ \}l \end{matrix} \quad , \; LFK + LEK' = \begin{bmatrix} W & \\ & I \end{bmatrix} \begin{matrix} \}\delta \\ \}l \end{matrix} \quad .$$

Thereby, N is strictly upper (or lower) triangular. It may happen that $l = m$, $\delta = 0$, and then the so-called dynamic part is absent such that $LEK = N$, $LFK + LEK' = I$.

Evidently, if the coefficient pair $\{E,F\}$ is transformable into SCF, then the nullspace $\ker T$ has finite dimension, namely $\dim\ker T = \delta$. The opposite is true in a limited version ([14, p.68]):

Proposition 5.1 *Suppose that* $E, F \in \mathcal{C}^m(\mathcal{I}_0, \mathcal{L}(\mathbb{R}^m))$ *and* $\dim\ker T < \infty$. *Then there is a subinterval* $[\bar{t}_a, \bar{t}_e] \subseteq \mathcal{I}$ *on which the DAE is transformable into SCF.*

The central ideas in [10, 11, 14] are *regularizers* and *solution representations of Cauchy type*. We quote the respective notions and results from [14] and illustrate it by an example.

Definition 5.2 *The differential operator*

$$Lz := \sum_{j=0}^{s} L_j z^{(j)}, \quad z \in \mathcal{C}^s(\mathcal{I}, \mathbb{R}^m), \tag{161}$$

with coefficient functions $L_j \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{R}^m))$, $j = 0, \ldots s$, *is called a left regularizer (LR) of the operator* T, *if*

$$(L \circ T)x = L(Tx) = x' + \tilde{F}x, \quad \text{for all } x \in \mathcal{C}^{s+1}(\mathcal{I}, \mathbb{R}^m), \tag{162}$$

with a continuous matrix function $\tilde{F}$.
The minimal such index s *is refered to as left index (in [10]: unsolvability index) of the operator* T *on the interval* $\mathcal{I}$.

Example 5.3 (Left regularizer) *Consider the differential-algebraic operator* $Tx := Ex' + Fx$ *with constant coefficients*

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The matrix pair $\{E, F\}$ *is regular with Kronecker index 2. At the beginning we show that there is no left regularizer with* $s = 1$. *In contrary, if* $Lz := L_1 z' + L_0 z$ *is a left regularizer, then it holds that* $L_1 E = 0$ *and* $L_0 E + L_1 F = I$, *which implies* $L_0 E^2 + L_1(FE - E) = E$. *The latter relation reads in detail*

$$L_0 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + L_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

but this can never be valid.
Next we put $s = 2$ *and* $Lz := L_2 z'' + L_1 z' + L_0 z$. *We have*

$$L(Tx) = L_2 E x''' + (L_1 E + L_2 F)x'' + (L_0 E + L_1 F)x' + L_0 F x.$$

For arbitrary $a, b, c \in \mathbb{R}$, *the matrices*

$$L_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{bmatrix}, L_0 = \begin{bmatrix} 0 & a & 0 \\ 1-c & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{163}$$

yield

$$L_2 E = 0, \; L_1 E + L_2 F = 0, \; L_0 E + L_1 F = I, \; L_0 F = \begin{bmatrix} 0 & a & 0 \\ 1-c & b & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and hence, L *is a left regularizer and the DAE has left index 2. more precisely, we are confronted with a family of left regularizers. Each of these left regularizers replaces the original DAE* $Tx = q$ *by the explicit ODE* $(L \circ T)x = x' + L_0 F x = Lq$, *in more detail, the DAE*

$$\begin{aligned} x_2' + x_1 &= q_1 \\ x_2 &= q_2 \\ x_3' + x_2 &= q_3 \end{aligned}$$

is replaced by the explicit ODE

$$\begin{aligned} x_1' + ax_2 &= aq_2 + q_1' - q_2'' \\ x_2' + (1-c)x_1 + bx_2 &= (1-c)q_1 + bq_2 + cq_2' \\ x_3' + x_2 &= q_3. \end{aligned}$$

The matrix L_0F *has the eigenvalues* 0 *and* $\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + a(1-c)}$*. Therefore, for different values* a, b, c*, one obtains explicit ODEs which might feature quite different solution quality. This complicates the recognition of the original DAE solution in practice.* □

Definition 5.4 *One says that the DAE (159) possesses a general solution of Cauchy type on* $\mathcal{I}$*, if the DAE (159) is solvable for each* $q \in \mathcal{C}^{\rho+1}(\mathcal{I}, \mathbb{R}^m)$ *and if there are a smooth matrix function* X_δ *with constant rank* δ *and a vector valued function* φ *such that any function* x *given by*

$$x(t,c) = X_\delta(t)c + \varphi(t), \quad t \in \mathcal{I},\ c \in \mathbb{R}^m, \tag{164}$$

represents a solution of the DAE, and, moreover, on any subinterval $[\bar{t}_a, \bar{t}_e] \subseteq \mathcal{I}$*, there are no solutions other than restrictions of (164) onto* $[\bar{t}_a, \bar{t}_e]$*.*

Proposition 5.5 *Let the operator* T *with coefficients* $E, F \in \mathcal{C}^{2m}(\mathcal{I}_0, \mathcal{L}(\mathbb{R}^m))$ *have a left regularizer with coefficients from* $\mathcal{C}^{2s}(\mathcal{I}, \mathcal{L}(\mathbb{R}^m))$*, whereby* s *is the left index of the operator* T*.*
Then, for $q \in \mathcal{C}^s(\mathcal{I}, \mathbb{R}^m)$*, the equation* $Tx = q$ *has the general solution*

$$x(t,c) = X_\delta(t)c + \int_{t_a}^{t} K(t,s)q(s)\mathrm{d}s + \sum_{j=0}^{s-1} C_j(t)q^j(t), \quad t \in \mathcal{I},\ c \in \mathbb{R}^m, \tag{165}$$

which is a Cauchy type solution.
Moreover, supposed x_a *is consistent, the IVP (160) is uniquely solvable, and the solution satisfies the inequalities*

$$\begin{aligned} \|x\|_\infty &\le \kappa_1\|x_a\| + \kappa_2\|q\|_{\mathcal{C}^{s-1}}, \\ \|x\|_{L^2} &\le \kappa_3\|x_a\| + \kappa_4\|q\|_{W_2^{s-1}}. \end{aligned}$$

Theorem 5.6 *If the coefficients* E*,* F *of the operator* T *are real analytic, then the following statements are equivalent:*

(1) *The operator* T *has on the interval* $\mathcal{I}$ *a left regularizer.*

(2) *The DAE (159) has a general solution of Cauchy type on the interval* $\mathcal{I}$*.*

(3) *The DAE (159) can be transformed on the interval* $\mathcal{I}$ *into canonical form by real analytic transforms* L *and* K*.*

By means of successive elimination and differentiation steps, left regularizers L can be stepwise constructed such that ([14, Section 2.1.5]

$$L = E_{[s]}^{-1}\Omega_s \cdots \Omega_1,$$

where $E_{[s]}$ is a nonsingular matrix function, s is the left index, and each factor Ω_i represents a special first order differential-algebraic operator.
Denote by $\mathcal{C}^s_*(\mathcal{I},\mathbb{R}^m)$ the completion of $\mathcal{C}^s(\mathcal{I},\mathbb{R}^m)$ with respect to the norm

$$\|q\|_* := \|q\|_\infty + \|\Omega_1 q\|_\infty + \cdots + \|\Omega_s \cdots \Omega_1 q\|_\infty, \quad q \in \mathcal{C}^s(\mathcal{I},\mathbb{R}^m),$$

We emphasize that the resulting Banach space strongly depends on the special problem, thus it is rather factitious, see Subsections 2.3 and 2.4.

Example 5.7 (Continuation of Example 5.3) *Consider once more the matrix pair*

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

First we compute

$$K_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; K_1 E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Omega_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\mathrm{d}}{\mathrm{dt}} \end{bmatrix} K_1,$$

$$\Omega_1 z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} z' + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} z,$$

and further

$$\Omega_1(Tx) := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x' + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x =: E_{[1]} x' + F_{[1]} x.$$

The equation $\Omega_1(Tx) = \Omega_1 q$ *reads in detail*

$$\begin{aligned} x_2' + x_1 &= q_1, \\ x_3' + x_2 &= q_3, \\ x_2' &= q_2', \end{aligned}$$

which confirms a first index reduction; the pair $\{E_{[1]}, F_{[1]}\}$ *is regular with Kronecker index 1. Next we perform*

$$K_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \; K_2 E_{[1]} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Omega_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{dt}} & 0 \\ 0 & 0 & 1 \end{bmatrix} K_2,$$

$$\Omega_2 z = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} z' + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} z,$$

$$\Omega_2\Omega_1 z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} z'' + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} z' + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} z,$$

and further

$$\Omega_2\Omega_1(Tx) := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x' + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x =: E_{[2]} x' + F_{[2]} x.$$

The equation $\Omega_2\Omega_1(Tx) = \Omega_2\Omega_1 q$ reads in detail

$$\begin{aligned} x_3' + x_2 &= q_3, \\ x_1' &= q_1' - q_2'', \\ x_2' &= q_2', \end{aligned}$$

which confirms another index reduction. Obviously, $L := E_{[2]}^{-1}\Omega_2\Omega_1$ is a left regularizer of the operator T. It coincides with the left regularizer given by (163) for $a = b = 0$, $c = 1$.
The norm

$$\|q\|_* := \|q\|_\infty + \|\Omega_1 q\|_\infty + \|\Omega_2\Omega_1 q\|_\infty = \|q\|_\infty + \| \begin{bmatrix} q_1 \\ q_3 \\ q_2' \end{bmatrix} \|_\infty + \| \begin{bmatrix} q_3 \\ q_1' - q_2'' \\ q_2' \end{bmatrix} \|_\infty$$

defined on $\mathcal{C}^2(\mathcal{I}, \mathbb{R}^3)$ is equivalent to the norm

$$\|q\|_{**} := \|q\|_\infty + \|q_2'\|_\infty + \|q_1' - q_2''\|_\infty,$$

and the function space

$$\mathcal{C}_*^2(\mathcal{I}, \mathbb{R}^3) = \{q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^3) : q_1 - q_2', q_2 \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^1)\} = T(\mathcal{C}^1(\mathcal{I}, \mathbb{R}^3))$$

results as completion of $\mathcal{C}^2(\mathcal{I}, \mathbb{R}^3)$ by this norm. □

If the operator T has a real-analytic coefficient pair $\{E, F\}$ then, for $X = \mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m)$, it holds that $\operatorname{im} T \subseteq \mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m)$. In contrast, if $X = \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m)$ then $\operatorname{im} T \subseteq \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$. Concerning both these settings, similar argument apply as in the case of constant coefficients in Section 2.

Theorem 5.8 *Let the coefficients E, F of the operator T are real-analytic and let the associated operator $T \in \mathcal{L}(X, Y)$ have a left regularizer. Suppose either $X = Y = \mathcal{C}^\infty(\mathcal{I}, \mathbb{R}^m)$ or $X = \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m)$, $Y = \mathcal{C}_*^s(\mathcal{I}, \mathbb{R}^m)$.*

(1) *Then the operator T is fredholm with $\alpha(T) = \delta$, $\beta(T) = 0$, $\operatorname{ind}_{fred}(T) = \delta$.*

(2) *The composed operator $\mathcal{T} \in \mathcal{L}(X, Y \times \mathbb{R}^m)$ associated with the IVP (160) is fredholm with $\alpha(T) = 0$, $\beta(T) = m - \delta$, $\operatorname{ind}_{fred}(T) = -(m - \delta)$.*

Once again, as already exposed in Subsections 2.2 and 2.4, in higher index cases, surjectivity is exacted by a special and too factitious setting.

As demonstrated in Example 5.7, the construction of left regularizers can be done stepwise by index reduction. One can also earlier terminate the procedure when either a regular index-1 DAE or a regular implicit ODE is achieved. This leads to the notion of *left semi-regularizers* ([14, p.105] which incorporates this idea.

Definition 5.9 *The differential operator*

$$Lz := \sum_{j=0}^{s} L_j z^{(j)}, \quad z \in \mathcal{C}^s(\mathcal{I}, \mathbb{R}^m), \tag{166}$$

with coefficient functions $L_j \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{R}^m))$, $j = 0, \dots s$, *is called a left semi-regularizer (LSR) of the operator* T, *if*

$$(L \circ T)x = \tilde{E}x' + \tilde{F}x, \quad \textit{for all } x \in \mathcal{C}^{s+1}(\mathcal{I}, \mathbb{R}^m), \tag{167}$$

and $\tilde{E}, \tilde{F}$ *are continuous matrix functions such that*

$$\deg \det(\lambda \tilde{E}(t) + \tilde{F}(t)) = \operatorname{rank} \tilde{E}(t) = \delta = const, \quad t \in \mathcal{I}. \tag{168}$$

The condition (168) is valid, exactly if the pair $\{\tilde{E}(t), \tilde{F}(t)\}$ is regular with Kronecker index 0 or 1, uniformly for all $t \in \mathcal{I}$. A DAE having coefficients which satisfy (168) is well-known to be a DAE with index 0 or 1. From this viewpoint the next assertion ([14, p.105]) is self-explanatory.

Theorem 5.10 *If* $E, F \in \mathcal{C}^{2s+3}(\mathcal{I}_0, \mathcal{L}(\mathbb{R}^m))$, *then the operator* T *has a left regularizer if and only if it has a left semi-regularizer.*

5.2 Singular perturbation and Tikhonov regularization

Aiming for a regularization of regular higher-index DAEs, several parametrizations are investigated mainly in the early literature concerning DAEs, which was strongly affected by the singular perturbation theory.

The so-called *pencil regularization* ([3, 15, 5, 7]) which approximates the standard form DAE

$$Ex' + Fx = q \tag{169}$$

by the regular implicit ODE

$$(E + \alpha F)x' + Fx = q,$$

with a small parameter $\alpha > 0$, is the earliest general such approach. Thereby, one has to assume regular local matrix pairs $\{E(t), F(t)\}$. This property is not given in general, but it is given e.g., for DAEs in Hessenberg form. A review of convergence results is presented in [37]. [12] provides some additional discussion. Roughly speaking, if it works, then the pencil regularization leads to singular singularly perturbed equations.

Alternative approaches aiming at a regular index-1 DAE are studied e.g., in [40, 41, 34, 44, 36, 37, 38, 39]. For instance, the parametrizations

$$(E + \alpha FP)(Px)' + (F - EP')x = q \tag{170}$$

and

$$(E + \alpha WFP)(Px)' + (F - EP')x = q, \tag{171}$$

with P and W being projector functions along $\ker E$ and $\operatorname{im} E$, respectively, are used to approximate the equation

$$E(Px)' + (F - EP')x = q. \tag{172}$$

The DAE (172) has a properly stated leading term; this version corresponds to the closure of the operator representing the DAE (169), cf. Section 2. The latter two parametrizations have appropriate physical interpretations for DAEs describing electrical networks ([44, 38]).They lead to singularly perturbed index-1

DAEs, and they are less severe than the ODEs resulting from pencil regularization.
For special autonomous nonlinear DAEs

$$\mathfrak{f}(x', x) = 0, \tag{173}$$

with constant $\ker \mathfrak{f}_{x'}$, such as Hessenberg form DAEs, in [37, 39] the parametrizations

$$\mathfrak{f}(x', x + \alpha x') = 0 \quad \text{and} \quad \mathfrak{f}(x', x + \alpha P x') = 0 \tag{174}$$

are compared. Owing to the identity $\mathfrak{f}(x', x) \equiv \mathfrak{f}(Px', x))$, the second version is more favorable. Deep results concerning convergence as $\alpha \to 0$ and asymptotic expansions are obtained for regular index-2 and index-3 DAEs (169) and also for special classes of nonlinear DAEs in Hessenberg form, e.g., in ([37, 38, 39]). In appropriate Hilbert space settings, these parametrizations yield regularizations in the sense of Tikhonov (see [69, 23]). We quote a typical result for regular index-2 DAEs. Consider the operator $T \in \mathcal{L}(H^1_P(\mathcal{I}, \mathbb{R}^m), L^2(\mathcal{I}, \mathbb{R}^m))$ defined by

$$Tx := E(Px)' + (F - EP')x, \quad x \in H^1_P(\mathcal{I}, \mathbb{R}^m),$$

the composed operator $\mathcal{T} \in \mathcal{L}(H^1_P(\mathcal{I}, \mathbb{R}^m), L^2(\mathcal{I}, \mathbb{R}^m) \times \mathbb{R}^m)$,

$$\mathcal{T}x := (Tx,\, Cx(t_a)), \quad x \in H^1_P(\mathcal{I}, \mathbb{R}^m),$$

and further the operators

$$T_\alpha x := (E + \alpha FP)(Px)' + (F - EP')x, \quad x \in H^1_P(\mathcal{I}, \mathbb{R}^m),$$

$$\mathcal{T}_\alpha x := \begin{bmatrix} T_\alpha x \\ Cx(t_a) \\ (\Pi_0 Q_1)(t_a)x(t_a) \end{bmatrix}, \quad x \in H^1_P(\mathcal{I}, \mathbb{R}^m),$$

with $C \in \mathcal{L}(\mathbb{R}^m)$. We refer to Section 4)for the meaning of $\Pi_0 Q_1, \Pi_1$. Denote $M := \operatorname{im} C$ and $L := \operatorname{im}(\Pi_0 Q_1)(t_a)$.

Proposition 5.11 *Let T be regular with tractability index 2 and* $\ker C = \ker \Pi_1(t_a)$. *Then the following statements hold:*

(1) *The operator $\mathcal{T}$ is injective.*

(2) *The operator T_α is regular with tractability index 1 for each sufficiently small $\alpha > 0$.*

(3) *The operator $\mathcal{T}_\alpha \in \mathcal{L}(H^1_P(\mathcal{I}, \mathbb{R}^m), L^2(\mathcal{I}, \mathbb{R}^m) \times M \times L)$ is a bijection for all sufficiently small $\alpha > 0$.*

(4) *For $(q, d) \in \operatorname{im} \mathcal{T}$ and*

$$x_* := \mathcal{T}^{-1}(q, d) \quad x_\alpha := \mathcal{T}_\alpha^{-1}(q, d, (\Pi_1 Q_1 G_2^{-1} q)(t_a)),$$

$\alpha \to 0$ *implies* $\|x_\alpha - x_*\|_{H^1_P} \to 0$ *and* $\|Px_\alpha - Px_*\|_{H^1} = O(\alpha)$.

(4) *If $(q, d) \in L^2(\mathcal{I}, \mathbb{R}^m) \times M$, $q \notin \operatorname{im} T$, then*

$$\|x_\alpha\|_{H^1_P} \to \infty, \quad \text{as} \quad \alpha \to 0.$$

Under additional smoothness one obtains $\|x_\alpha - x_*\|_{H^1_P} = O(\alpha^{\frac{1}{2}})$ and even $\|x_\alpha - x_*\|_{H^1_P} = O(\alpha)$.

The convergence behavior is similar to Tikhonov regularization for integral equations of the first kind ([31]). Motivated by this experience, in [22] nonlinear semi-explicit DAEs are treated by Tikhonov regularization in several settings.

In spite of all these contributions, as yet, there are no sufficiently matured procedures to solve BVPs and IVPs in higher-index DAEs relying on Tikhonov regularization.

6 Appendices

6.1 Functional-analytic basics and notations

We are mainly interested in bounded operators acting in real Banach spaces and in closed operators acting in real Hilbert spaces. We refer to [26, 17, 75, 76, 42] for details regarding the material below.

6.1.1 Linear Operators in normed spaces

Let X and Y be normed linear spaces over $\mathbb{R}$. $\mathcal{L}(X,Y)$ denotes the set of all linear operators K mapping an individual *definition domain* $\operatorname{dom} K \subseteq X$ into Y. We also shorten $\mathcal{L}(X) := \mathcal{L}(X,X)$. For each operator $K \in \mathcal{L}(X,Y)$ we introduce its *range* (also image) and *nullspace* (also kernel) as

$$\operatorname{im} K := \{y \in Y : \exists x \in X,\, y = Kx\}, \quad \ker K := \{x \in X : Kx = 0\}.$$

The operator K is said to be *densely defined* if $\operatorname{dom} K$ is dense in X and *densely solvable* if $\operatorname{im} K$ is dense in Y.
The *graph* of K is determined by

$$\mathit{graph}\, K := \{(x, Kx) : x \in \operatorname{dom} K\} \subseteq X \times Y.$$

The sets $\mathcal{L}_b(X,Y)$ and $\mathcal{L}_c(X,Y)$ consist of all linear bounded and closed operators, respectively, such that the inclusions

$$\mathcal{L}_b(X,Y) \subseteq \mathcal{L}_c(X,Y) \subseteq \mathcal{L}(X,Y)$$

are given.
The operator $K \in \mathcal{L}(X,Y)$ is said to be *closed*, if for each sequence $x_n \to x_* \in X$, $x_n \in \operatorname{dom} K$ for all $n \in \mathbb{N}$, and $Kx_n \to y_* \in Y$ it results that $x_* \in \operatorname{dom} K$ and $Kx_* = y_*$.
The operator $K \in \mathcal{L}(X,Y)$ is closed, exactly if *graph* K is a closed subspace in $X \times Y$. The graph-theorem says that a closed linear operator K who maps all of a Banach space X (i.e. $\operatorname{dom} K = X$) into a Banach space Y is bounded. In contrast, often one is confronted with the fact that $\operatorname{dom} K$ is merely a proper subset of X.
The operator $K \in \mathcal{L}(X,Y)$ is called *closable*, if it admits a closed extension. The minimal closed extension is said to be the *closure* of K.
We equip the linear space $\operatorname{dom} K =: X_K$ with the so-called *graph-norm*

$$\|x\|_K := \|x\| + \|Kx\|, \quad \|x\| \leq \|x\|_K \text{ for all } x \in \operatorname{dom} K.$$

Because of the evident inequality $\|Kx\| \leq \|x\|_K$ given for all $x \in X_K$, the operator K is bounded in this new setting (we keep the notation K), therefore

$$K \in \mathcal{L}_c(X,Y) \quad \text{implies} \quad K \in \mathcal{L}_b(X_K,Y).$$

If X and Y are Banach spaces then also X_K is a Banach space.

The operator $K \in \mathcal{L}_c(X,Y)$, where X and Y are normed spaces, is said to be a *Fredholm operator*, if it has a closed range $\operatorname{im} K$ and the dimensions $\dim\ker K =: \alpha(K)$ and $\operatorname{codim}\operatorname{im} K =: \beta(K)$ are finite. Then, the difference

$$\alpha(K) - \beta(K) =: \operatorname{ind}_{fred}(K)$$

is called *Fredholm index.* The operator $K \in \mathcal{L}_c(X,Y)$ is said to be *semifredholm*, if it has a closed range $\operatorname{im} K$ and either $\dim\ker K =: \alpha(K)$ or $\operatorname{codim}\operatorname{im} K =: \beta(K)$ is finite.
Often, Fredholm operators are ab initio supposed to act in Banach spaces and to be bounded, and then the closed-range property is not explicitly listet. Owing to Kato's theorem (e.g.[42, page 310]), if then $\operatorname{im} K$ possesses a finite codimension it is necessarily closed. Although similar arguments apply also to closed operators acting in Banach spaces, we impose an explicit listing of the closed-range property.
Sometimes one uses the name *Noetherian index* instead of Fredholm index, e.g., [10, 14].

A closed, densely defined operator K acting from the Banach space X into the Banach space Y is said to be *normally solvable* if $\operatorname{im} K$ is closed in Y (e.g. [43, p. 234]). Normally solvable operators comprise useful properties similar to Fredholm operators (cf. Theorem 6.2), and sometimes then the problem $Kx = q$ is said to be *well-posed in the sense of Fichera.* Well-posedness in Fichera' sense does not necessarily suppose neither injectivity nor surjectivity.

If K is a bounded bijection acting on Banach spaces, then K is a Fredholm operator with $\operatorname{ind}_{fred}(K) = \alpha(K) = 0$ and the equation $Kx = q$ is well-posed in the sense of Hadamard.

We denote by $X^* := \mathcal{L}_b(X,\mathbb{R})$ the *dual* of the real normed space X. Equipped with its natural norm, X^* becomes a Banach space.
We set $\langle \eta, x\rangle := \eta(x)$ for every $x \in X$ and $\eta \in X^*$. The resulting bilinear form $\langle\cdot,\cdot\rangle$ indicates the duality pairing between X^* and X, also called *scalar product* between X^* and X.

For each densely defined operator $K \in \mathcal{L}(X,Y)$ there is the *dual operator* $K^* \in \mathcal{L}(Y^*,X^*)$ given by

$$\begin{aligned}
&\langle K^*g, x\rangle = \langle g, Kx\rangle \quad \text{for all} \quad x \in \operatorname{dom} K,\ g \in \operatorname{dom} K^*,\\
&\operatorname{dom} K^* = \{g \in Y^* : \exists \eta_g \in X^* \text{ such that } \langle g, Kx\rangle = \langle \eta_g, x\rangle\ \forall x \in \operatorname{dom} K\},\\
&\quad K^*g = \eta_g \quad \text{for} \ \ g \in \operatorname{dom} K^*.
\end{aligned}$$

The dual operator of a bounded operator is also bounded and it holds that $\|K^*\| = \|K\|$.
We advert to the fact that the dual operator K^* is also called transposed and adjoint operator in the literature.
To each element x of the normed space X there exists an element z of the

bidual space $X^{**} := (X^*)^*$ such that $\langle z, g\rangle = \langle g, x\rangle$ for all $g \in X^*$ and further $\|x\| = \|z\|$. This allows to assume the inclusion $X \subseteq X^{**}$. If $X = X^{**}$, then X is called a *reflexive* Banach space.

Proposition 6.1 *Let X and Y be Banach spaces, and $K \in \mathcal{L}(X, Y)$ be densely defined. Then the following holds:*

(1) *The dual operator K^* is closed, $K^* \in \mathcal{L}_c(Y^*, X^*)$.*

(2) *If X and Y are reflexive, then the dual operator K^* is likewise densely defined.*

(3) *If X and Y are reflexive and the operator K is closable, then K^{**} represents the closure of K.*

Proofs can be found, e.g., in [43, Chapter III].

We quote the closed image theorem, e.g., [17, p. 348],[74, p. 205]:

Theorem 6.2 *Let X and Y be Banach spaces, $K \in \mathcal{L}_c(X, Y)$ be densely defined. Then $\operatorname{im} K$ is closed in Y if and only if $\operatorname{im} K^*$ is closed in Y^*.*
In addition, under this hypothesis,

$$\begin{aligned}
(\operatorname{im} K)^\perp &= \{y \in Y : \langle y, g\rangle = 0 \ \forall g \in \ker K^*\} = \ker K^*, \\
\operatorname{im} K^* &= \{\eta \in X^* : \langle x, \eta\rangle = 0 \ \forall x \in \ker K\} = (\ker K)^\perp, \\
\alpha(K) &= \beta(K^*), \\
\alpha(K^*) &= \beta(K).
\end{aligned}$$

A Hilbert space X is comfortable owing to the scalar product $(\cdot,\cdot)$ which is defined on $X \times X$. As before, the symbol $\langle\cdot,\cdot\rangle$ indicates the scalar product between X^* and X. If $\mathcal{J} \in \mathcal{L}_b(X, X^*)$ denotes the so-called duality map, then it holds that

$$\langle \mathcal{J}x, \xi\rangle = (x, \xi) \ \text{ for all } x, \xi \in X, \quad \text{and } \ \|\mathcal{J}x\| = \|x\| \text{ for all } x \in X.$$

This feature allows to identify the Hilbert space and its dual, that is $X = X^*$. This implies reflexivity.

If X and Y are Hilbert spaces and the operator $K \in \mathcal{L}(X, Y)$ is densely defined, then K has the *adjoint operator* $K^* \in \mathcal{L}(Y, X)$ given by

$$\begin{aligned}
(K^*y, x) &= (y, Kx) \quad \text{for all} \quad x \in \operatorname{dom} K,\ y \in \operatorname{dom} K^*, \\
\operatorname{dom} K^* &= \{y \in Y : \exists \eta_y \in X \text{ such that } (y, Kx) = (\eta_y, x) \ \forall x \in \operatorname{dom} K\}, \\
K^*y &= \eta_y \quad \text{for } \ y \in \operatorname{dom} K^*.
\end{aligned}$$

The adjoint operator is likewise densely defined, and it is closely related to the dual operator. If $\mathcal{J}_X$ and $\mathcal{J}_Y$ are the duality maps associated to X and Y, respectively, and if $K^* \in \mathcal{L}(Y^*, X^*)$ is the dual operator of the densely defined operator $K \in \mathcal{L}(X, Y)$, then the adjoint operator is given by $\mathcal{J}_X^{-1} K^* \mathcal{J}_Y$. We apply the same symbol for the dual and the adjoint operators. No confusion will arise.

Moreover, for a linear densely defined operator K acting in Hilbert spaces X and Y, the *biadjoint* $K^{**} := (K^*)^* \in \mathcal{L}(X,Y)$ exists and is given by

$$
\begin{aligned}
&(K^{**}x, y) = (x, K^*y) \quad \text{for all} \quad y \in \operatorname{dom} K^*,\ x \in \operatorname{dom} K^{**},\\
&\operatorname{dom} K^{**} = \{x \in X : \exists \eta_x \in X \text{ such that } (x, K^*y) = (\eta_x, y)\ \forall y \in \operatorname{dom} K^*\},\\
&\qquad K^*y = \eta_y \quad \text{for } \ y \in \operatorname{dom} K^*.
\end{aligned}
$$

Owing to Proposition 6.1, the biadjoint K^{**} of a closable, densely defined operator $K \in \mathcal{L}(X,Y)$ acting in Hilbert spaces represents the closure of K.

6.1.2 Inner inverses, outer inverses, generalized inverses, and least-squares solutions

In this part we adapt the terminology applied in [57, 16, 59] and collect few of the results reported therein.

Let X and Y be Banach spaces, $K \in \mathcal{L}(X,Y)$, and $K^- \in \mathcal{L}(Y,X)$. If $KK^-K = K$ holds on $\operatorname{dom} K$, then the map K^- is said to be an *inner inverse* of K. If $K^-KK^- = K^-$ holds on $\operatorname{dom} K^-$, then the map K^- is said to be an *outer inverse* of K. If K^- is both an inner and an outer inverse of K, then it is called an *algebraic generalized inverse* of K.

If K^- is either an inner or an outer inverse of K, then both KK^- and K^-K are linear idempotents, that is, *algebraic projectors*.

From the viewpoint of analysis, however, these algebraic constructions are of little use unless the resulting operators are continuous; hence our special interest is directed to results on bounded projectors and inverses.

The linear mapping K is called *approximately outer invertible*, if, for each $\rho \in (0,1)$, there exists an operator $K_\rho^{\simeq} \in \mathcal{L}_b(Y,X)$ and a bound $\Gamma(\rho)$ such that

$$
\|(K_\rho^{\simeq} K K_\rho^{\simeq} - K_\rho^{\simeq})y\| \le \rho \|K_\rho^{\simeq} y\| \ \text{ and } \ \|K_\rho^{\simeq} y\| \le \Gamma(\rho)\|y\|, \ \text{ for all } \ y \in Y.
$$

Then each $K_\rho^{\simeq}$ is called *approximate outer inverse of K, with bound function $\Gamma(\rho)$*.

In case of infinite-dimensional spaces, the symbols $\dot{+}$ and $\oplus$ indicate algebraic direct sums and topological direct sums (with closed subspaces), respectively. In finite-dimensional spaces all subspaces are closed; then we only apply the symbol $\oplus$ for the direct sums.

Topological direct sum decompositions of a Banach space are intimately connected with bounded projectors. The range of each bounded projector on a Banach space is a closed subspaces. A closed subspace of a Banach space has a topological complement if and only if it is the range of some bounded projector. A Hilbert space is more comfortable: any closed subspace M has a topological complement, and $M^\perp$ is one such complement; the orthogonal projector P onto a closed subspace is linear, idempotent and it holds that $P^* = P$ and $\|P\| = 1$.

For a densely defined operator (or even bounded operator) $K \in \mathcal{L}_c(X,Y)$ in Hilbert spaces, one can define the so-called *orthogonal generalized inverse*, denoted by $K^+ \in \mathcal{L}(Y,X)$, which is the operator version of the Moore-Penrose inverse given for matrices.

Let X and Y be Hilbert spaces, $K \in \mathcal{L}_c(X,Y)$, $\overline{\operatorname{dom} K} = Y$. Then $\ker K$ is closed. Let $\mathfrak{P} \in \mathcal{L}_b(X)$ and $\mathfrak{R} \in \mathcal{L}_b(Y)$ denote the orthoprojectors onto $(\ker K)^\perp$ and $\overline{\operatorname{im} K}$, respectively. It follows that $K = K\mathfrak{P} = \mathfrak{R}K$. One defines

$$K^+y := (K|_{(\operatorname{dom} K)\cap(\ker K)^\perp})^{-1}\mathfrak{R}y \quad \text{for all } y \in \operatorname{im} K + (\operatorname{im} K)^\perp =: \operatorname{dom} K^+.$$

This definition implies the relations

$$\begin{aligned} KK^+K &= K \text{ on } \operatorname{dom} K, \\ K^+KK^+ &= K^+ \text{ on } \operatorname{dom} K^+, \\ KK^+ &= \mathfrak{R}|_{\operatorname{dom} K^+}, \\ K^+K &= \mathfrak{P}|_{\operatorname{dom} K}, \end{aligned}$$

which show that K^+ is at the same time a particular algebraic generalized inverse.
The orthogonal generalized inverse K^+ of the densely defined closed operator K is also closed. KK^+ and K^+K are then densely defined and symmetric, but not necessarily closed. K^+K becomes closed exactly if K is bounded, and KK^+ is closed exactly if K^+ is bounded. Furthermore, K^+ is bounded exactly if $\operatorname{im} K$ is closed in Y.

Let X be a vector space and Y be a Hilbert space, $K \in \mathcal{L}(X,Y)$, $y_* \in Y$. The element $x_* \in X$ is said to be a *least-squares solution* (LSS) of the equation $Kx = y_*$ if

$$\|Kx_* - y_*\| = \inf\{\|Kx - y_*\| : x \in \operatorname{dom} K\}.$$

An LSS is also named a *quasisolution*, e.g.[71].
Let $\mathfrak{R} \in \mathcal{L}_b(Y)$ again denote the orthoprojector onto $\overline{\operatorname{im} K}$. Then, the equation $Kx = y_*$ has a LSS exactly if $\mathfrak{R}y_* \in \operatorname{im} K$, i.e., $y_* \in \operatorname{im} K + (\operatorname{im} K)^\perp$. If X is normed, then one can look for a *LSS with minimal norm.* If X is also an Hilbert space, then the LSS with minimal norm is called *pseudosolution*, e.g.[71].

There is a close relation between LSS and the orthogonal generalized inverse (cf. [57, Theorem 5.1]):

Theorem 6.3 *Let X and Y be Hilbert spaces, $K \in \mathcal{L}_c(X,Y)$ be densely defined, $y_* \in Y$. Then the following is true:*

(1) *x_* is a LSS exactly if it satisfies the equation $K^*(Kx - y_*) = 0$.*

(2) *For $y_* \in \operatorname{dom} K^+$, K^+y_* is the unique solution of minimal norm of the equation $Kx = \mathfrak{R}y_*$.*

(3) *For $y_* \in \operatorname{dom} K^+$, K^+y_* is the unique LSS of minimal norm of the equation $Kx = y_*$.*

(4) *For $y_* \in \operatorname{dom} K^+$, K^+y_* is the unique solution of minimal norm of the equation $K^*(Kx - y_*) = 0$.*

(5) *If, additionally, K is bounded, then x_* is a LSS exactly if the normal equation $K^*Kx_* = K^*y_*$ is satisfied.*

We quote [57, Theorem 7.1] and parts of [57, Theorem 7.2]:

Theorem 6.4 *Let the operator $K \in \mathcal{L}(X,Y)$ acting in Banach spaces be bounded or closed with dense domain. Then, K has a bounded inner inverse exactly if* $\ker K$ *and* $\operatorname{im} K$ *are closed and have topological complements in X and Y respectively.*

Theorem 6.5 *For the operator $K \in \mathcal{L}_b(X,Y)$ acting in Hilbert spaces the following statements are equivalent:*

(1) *K has a bounded inner inverse.*

(2) $\operatorname{im} K$ *is closed.*

(3) *K is normally solvable.*

(4) $\inf\{\|Kx - y\| : x \in X\}$ *is attained for each $y \in Y$.*

(5) *The orthogonal generalized (Moore-Penrose) inverse K^+ of K is bounded.*

(5) *For some $\gamma > 0$, it holds that $\|Kx\| \geq \|x\|$ for all $x \in M$, where $X =$* $\ker K \oplus M$.

Finally we quote the perturbation result [59, Lemma 2.2]:

Lemma 6.6 *Let X and Y be Banach spaces, $K \in \mathcal{L}_b(X,Y)$, and let $K^- \in \mathcal{L}_b(Y,X)$ be a bounded outer inverse of K. Let $B \in \mathcal{L}_b(X,Y)$ be such that*

$$\|K^-(B-K)\| < 1.$$

Then $B^- := (I + K^-(B-K))^{-1}K^-$ is a bounded outer inverse of B with $\ker B^- = \ker K^-$ *and* $\operatorname{im} B^- = \operatorname{im} K^-$. *Moreover*

$$\|B^- - K^-\| \leq \frac{K^-(B-K)K^-}{1 - \|K^-(B-K)\|}.$$

6.1.3 Nonlinear Fréchet-differentiable operators

Let X and Y be Banach spaces, $U \subseteq X$ be an open subset. We consider the map $K : \operatorname{dom} K := U \to Y$. We say that K is *Fréchet-differentiable at $u \in U$*, if there exists an operator $A \in \mathcal{L}_b(X,Y)$ such that, if we set

$$R(h) = K(u+h) - K(u) - A(h),$$

there results $R(h) = o(\|h\|$, that is $\frac{\|R(h)\|}{\|h\|} \to 0$ as $h \to 0$. Such an A will be called Fréchet-differential of K at u and denoted by $A = dK(u)$.
Let K be differentiable at all $u \in U$. Then the map $K' : U \to \mathcal{L}_b(X,Y)$, $K'(u) := dK(u)$, $u \in U$, is called the *Fréchet-derivative* of K. $K'(u)$ is also called the *linearization* of K at u.
If the derivative K' is continuous as a map from U to $\mathcal{L}_b(X,Y)$, then we will say that K *is a $\mathcal{C}^1$- operator*, see e.g.,[1, Section 1.1].

Let X and Y be Banach spaces. The $\mathcal{C}^1$-operator K acting between X and Y, with definition domain $\operatorname{dom} K$ open in X, is called a *Fredholm operator* exactly if for each $x \in \operatorname{dom} K$ the linearization $K'(x) \in \mathcal{L}_b(X,Y)$ is a Fredholm

operator, see e.g., [76, Page 317].
If $\operatorname{dom} K = X$ then $\operatorname{ind}_{fred}(K'(x))$ is independent of x, and then one sets

$$\operatorname{ind}_{fred}(K) := \operatorname{ind}_{fred}(K'(x)),$$

see [76, Section 5.15].

We quote the local inverse mapping theorem from [76, p. 259].

Theorem 6.7 *Let X, Y be Banach spaces, $r \in \mathbb{N}$, and $\mathcal{U}(x_0) \subseteq X$ be an open neighborhood of $x_0 \in X$. Let $K : \mathcal{U}(x_0) \subseteq X \to Y$ be a $\mathcal{C}^r$ map. Then K is a local $\mathcal{C}^r$-diffeomorphism at x_0 if and only if $\mathbb{K}'(x_0) : X \to Y$ is bijective.*

Often, classical *well-posedness in Hadamard's sense* of the nonlinear operator equation $Kx = p$, with K acting in Banach spaces X, Y, is unrealistic. Instead of this global requirement, one applies its local version which actually means that K is a local $\mathcal{C}^1$- diffeomorphism at the wanted solution:
Let X, Y be Banach spaces. The nonlinear equation $Kx = 0$, with a Fréchet-differentiable operator $K : \operatorname{dom} K \subseteq X \to Y$, is said to be *well-posed around* $x_* \in \operatorname{dom} K$, with $Kx_* = 0$, if the Fréchet derivative $K'(x_*) \in \mathcal{L}(X, Y)$ is a homeomorphism. Then, owing to the classical implicit function theorem in Banach spaces (e.g., [1]), for each sufficiently small $q \in Y$, in the neighborhood of x_* exists exactly one solution $x(q)$ to the equation $Kx = q$, and the inequality

$$\|x(q) - x_*\| \le \kappa \|q\|$$

is given with a constant κ.

Regular higher-index differential-algebraic operators (in their natural Banach spaces) are Fréchet-differentiable, however, $\operatorname{im} K'(x_*)$ is a nonclosed subset in Y, such that K fails to be fredholm and, to make matters worse, $K'(x)$ is no longer normally solvable. One can treat equation $Kx = 0$ as ill-posed problem applying the respective methods (e.g.,[69, 16, 71, 59, 57, 31, 21, 58]). The following implicit function theorem is a consequence of [16, Theorem 3].

Theorem 6.8 *Let X and Y be Banach spaces, Let the function $K : X \to Y$ be strongly Fréchet-differentiable at $x_* \in X$, with $K(x_*) = 0$. Let the Fréchet derivative $K'(x_*) \in \mathcal{L}_b(X, Y)$ be outer invertible, with approximate outer inverses $K_\rho^{\simeq}$ and bound function $\Gamma(\rho) = h_0 \rho^{-\gamma}$, where $\gamma < 1$. Then, whenever $z_* \in X$ satisfies $K'(x_*)z_* = 0$, and $\|z_*\| = 1$, there exists a solution $x = x_* + tz_* + o(t)$ to $K(x) \in \ker K_\rho^{\simeq}$.*

6.2 Matrix functions, varying subspaces and special function spaces

We identify matrices $M \in \mathbb{R}^{n \times m}$ and their associated operators $M \in \mathcal{L}(\mathbb{R}^m, R^n)$. By $\mathcal{C}(\Omega, \mathbb{R}^s)$ and $\mathcal{C}^k(\Omega, \mathbb{R}^s)$ we denote the linear spaces of continuous and k-times continuously differentiable functions defined on $\Omega \subseteq \mathbb{R}^n$ with values in $\mathbb{R}^s$, $k \in \mathbb{N}$ and $k = \infty$.
We apply the usual notations $L^2(\mathcal{I}, \mathbb{R}^n)$ and $H^1(\mathcal{I}, \mathbb{R}^n)$, with a compact interval $\mathcal{I}$, for the respective Lebesgue and Sobolev spaces.
By $|\cdot|$ we denote absolute values as well as norms of vectors and matrices. In

contrast, $\|\cdot\|$ is a function or operator norm. At episodes when several norms are to be distinguished, we use specific indices.

Let $s, n \geq 1$, $k \geq 0$ be integers, $\Omega \subseteq \mathbb{R}^s$ be an connected set, and let $S(z) \subseteq \mathbb{R}^n$ be a subspace for all $z \in \Omega$. We say that $S(\cdot)$ is a $\mathcal{C}^k$*-subspace* in R^n, if there exists a projector-valued function $P \in \mathcal{C}^k(\Omega, L(\mathbb{R}^s, \mathbb{R}^s))$ such that $\operatorname{im} P(z) = S(z)$ for all $z \in \Omega$. Equivalently, $S(\cdot)$ is a $\mathcal{C}^k$-subspace, if the orthoprojector function onto $S(\cdot)$ belongs to the class $\mathcal{C}^k$.
A $\mathcal{C}^k$-subspace has constant dimension on Ω. It has further local $\mathcal{C}^k$-bases. For $s = 1$, there is even a global $\mathcal{C}^k$-basis, see [50, Appendix A].

Lemma 6.9 *Given are two matrix functions $M \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^s))$ and $\tilde{M} \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{\tilde{s}}))$, both with constant rank on the interval $\mathcal{I}$. Additionally, let the constant-rank condition*

$$\ker \tilde{M}(t) = \ker M(t), \quad \text{for all} \quad t \in \mathcal{I}.$$

be satisfied. Then the following becomes valid:

(1) $\ker \tilde{M}(\cdot) = \ker M(\cdot)$ *is a $\mathcal{C}^1$-subspace varying in $\mathbb{R}^m$.*

(2) *The function spaces*

$$\mathcal{C}^1_M(\mathcal{I}, \mathbb{R}^m) := \{x \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : Mx \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^s)\},$$
$$H^1_M(\mathcal{I}, \mathbb{R}^m) := \{x \in L^2(\mathcal{I}, \mathbb{R}^m) : Mx \in H^1(\mathcal{I}, \mathbb{R}^s)\},$$

coincide with

$$\mathcal{C}^1_{\tilde{M}}(\mathcal{I}, \mathbb{R}^m) := \{x \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : \tilde{M}x \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^{\tilde{s}})\},$$
$$H^1_{\tilde{M}}(\mathcal{I}, \mathbb{R}^m) := \{x \in L^2(\mathcal{I}, \mathbb{R}^m) : \tilde{M}x \in H^1(\mathcal{I}, \mathbb{R}^s)\},$$

respectively.

(3) *Assume the interval $\mathcal{I}$ to be compact. Equipped with the norm*

$$\|x\|_{\mathcal{C}^1_M} := \|x\|_\infty + \|(Mx)'\|_\infty, \quad x \in \mathcal{C}^1_M(\mathcal{I}, \mathbb{R}^m),$$

the function space $\mathcal{C}^1_M(\mathcal{I}, \mathbb{R}^m)$ becomes a Banach space. The norms $\|\cdot\|_{\mathcal{C}^1_M}$ and $\|\cdot\|_{\mathcal{C}^1_{\tilde{M}}}$ are equivalent.

(4) *Assume the interval $\mathcal{I}$ to be compact. Equipped with the scalar product*

$$(x, \xi)_{H^1_M} := (x, \xi)_{L^2} + ((Mx)', (M\xi)')_{L^2}, \quad x \in H^1_M(\mathcal{I}, \mathbb{R}^m), \qquad (175)$$

the function space $H^1_M(\mathcal{I}, \mathbb{R}^m)$ becomes a Hilbert space. The associated norms $\|\cdot\|_{H^1_M}$ and $\|\cdot\|_{H^1_{\tilde{M}}}$ are equivalent.

Proof: Statement (1) is a consequence of [50, Lemma A.15].
(2) Owing to the constant-rank properties the Moore-Penrose inverses M^+ and $\tilde{M}^+$ are also continuously differentiable. It holds that $\tilde{M}^+\tilde{M} = M^+M$.
For each $x \in \mathcal{C}^1_M(\mathcal{I}, \mathbb{R}^m)$, we have $x \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$ and $\tilde{M}x = \tilde{M}\tilde{M}^+\tilde{M}x = \tilde{M}M^+Mx \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^{\tilde{s}})$, such that $x \in \mathcal{C}^1_{\tilde{M}}(\mathcal{I}, \mathbb{R}^m)$, and hence $\mathcal{C}^1_M(\mathcal{I}, \mathbb{R}^m) =$

$\mathcal{C}^1_{\tilde{M}}(\mathcal{I},\mathbb{R}^m)$.
Similarly, for each $x\in H^1_M(\mathcal{I},\mathbb{R}^m)$, we have $x\in L^2(\mathcal{I},\mathbb{R}^m)$ and $\tilde{M}x=\tilde{M}\tilde{M}^+\tilde{M}x=\tilde{M}M^+Mx\in H^1(\mathcal{I},\mathbb{R}^{\tilde{s}})$, such that $x\in H^1_{\tilde{M}}(\mathcal{I},\mathbb{R}^m)$, and hence $H^1_M(\mathcal{I},\mathbb{R}^m)=H^1_{\tilde{M}}(\mathcal{I},\mathbb{R}^m)$.
(3) The given expression defines a norm in fact. The norm equivalence results from the inequality

$$\begin{aligned}\|x\|_{\mathcal{C}^1_M} &:= \|x\|_\infty+\|(Mx)'\|_\infty=\|x\|_\infty+\|(M\tilde{M}^+\tilde{M}x)'\|_\infty\\ &=\|x\|_\infty+\|(M\tilde{M}^+)'\tilde{M}x+M\tilde{M}^+(\tilde{M}x)'\|_\infty\\ &\le\kappa(\|x\|_\infty+\|(\tilde{M}x)'\|_\infty)=\kappa\|x\|_{\mathcal{C}^1_{\tilde{M}}}.\end{aligned}$$

For proving the completeness, we consider the sequence $\{x_l\}$ being a Cauchy sequence in $\mathcal{C}^1_M$. Then, $\{x_l\}$ and $\{Mx_l\}$ are Cauchy sequences in $\mathcal{C}$, therefore there is a $x_*\in\mathcal{C}$ such that $\|x_l-x_*\|_\infty\to 0$ and $\|Mx_l-Mx_*\|_\infty\to 0$. Furthermore, $\{Mx_l\}$ is a Cauchy sequence in $\mathcal{C}^1$, and there is a $y_*\in\mathcal{C}^1$ so that $\|Mx_l-y_*\|_{\mathcal{C}^1}\to 0$. This implies $Mx_*=y_*\in\mathcal{C}^1$, thus $x_*\in\mathcal{C}^1_M$ as well as $\|x_l-x_*\|_{\mathcal{C}^1_M}\to 0$.
(4) Formula (175) defines a scalar product and induce a norm on H^1_M. Further we can proceed analogously to (3). □

Lemma 6.10 *Let $V\in\mathcal{C}(\mathcal{I},\mathcal{L}(\mathbb{R}^m))$ be a projector-valued function on the compact interval $\mathcal{I}$.*

(1) *Then, for $X=\mathcal{C}(\mathcal{I},\mathbb{R}^m)$ and $X=L^2(\mathcal{I},\mathbb{R}^m)$, the relation*

$$(\mathcal{V}x)(t):=V(t)x(t),\quad t\in\mathcal{I},\quad \textit{resp. a.e. in } \mathcal{I}, \tag{176}$$

defines a projector $\mathcal{V}\in\mathcal{L}_b(X)$ such that the topological direct sum decomposition

$$\ker\mathcal{V}\oplus\operatorname{im}\mathcal{V}=X \tag{177}$$

is valid.

(2) *If the additional matrix-valued function $D\in\mathcal{C}^1(\mathcal{I},\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n))$ is given, and*

$$DV(I-D^+D)=0,\quad DVD^+\in\mathcal{C}^1(\mathcal{I},\mathcal{L}(\mathbb{R}^n)), \tag{178}$$

then statement (1) is also valid for $X=\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ and $X=H^1_D(\mathcal{I},\mathbb{R}^m)$.

Proof: (1) $\mathcal{V}$ is idempotent since $V(t)^2=V(t)$ for all t. Since $V(t)$ and $I-V(t)$ are uniformly bounded on $\mathcal{I}$, the projectors $\mathcal{V},I-\mathcal{V}\in\mathcal{L}(X)$ are bounded. Then, their nullspaces are closed, hence (177) is valid.
(2) For $x\in\mathcal{C}^1_D$ it holds that $D\mathcal{V}x=D\mathcal{V}D^+Dx\in\mathcal{C}^1$ and $(D\mathcal{V}x)'=(D\mathcal{V}D^+)'Dx+D\mathcal{V}D^+(Dx)'$. We have $\mathcal{V}x\in\mathcal{C}^1_D$ in fact, and $\|\mathcal{V}x\|_{\mathcal{C}^1_D}\le\gamma\|x\|_{\mathcal{C}^1_D}$.
As bounded operators, $\mathcal{V},I-\mathcal{V}$ have closed nullspaces. Analogous arguments apply for $X=H^1_D(\mathcal{I},\mathbb{R}^m)$. □

We emphasize that $\operatorname{im}\mathcal{V}$ and $\ker\mathcal{V}$ are infinite-dimensional subspaces of the function space X, though $\operatorname{im}V(t),\ker V(t)\subseteq\mathbb{R}^m$ are finite-dimensional.
Often, if no confusion looms, we apply the same letter V also instead of $\mathcal{V}$.

6.3 Basics concerning regular DAEs

We collect basic facts on the DAE

$$f((Dx)'(t), x(t), t) = 0, \tag{179}$$

which exhibits the involved derivative by means of an extra matrix valued function D. The function $f : \mathbb{R}^n \times \mathcal{D}_f \times \mathcal{I}_f \longrightarrow \mathbb{R}^m$, $\mathcal{D}_f \times \mathcal{I}_f \subseteq \mathbb{R}^m \times \mathbb{R}$ open, is continuous and has continuous partial derivatives f_y and f_x with respect to the first two variables $y \in \mathbb{R}^n$, $x \in \mathcal{D}_f$. The partial Jacobian $f_y(y, x, t)$ is everywhere singular. The matrix function $D : \mathcal{I}_f \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ is continuously differentiable and $D(t)$ has constant rank r on the given interval $\mathcal{I}_f$. Then, $\operatorname{im} D$ is a $\mathcal{C}^1$-subspace in $\mathbb{R}^n$. We refer to [50] for proofs, motivation, and more details.

6.3.1 Regular DAEs, regularity regions

Definition 6.11 *The DAE (179) has a properly involved derivative, also called properly stated leading term, if* $\ker f_y$ *is another* $\mathcal{C}^1$*-subspace varying in* $\mathbb{R}^n$*, and the transversality condition*

$$\ker f_y(y, x, t) \oplus \operatorname{im} D(t) = \mathbb{R}^n, \quad (y, x, t) \in \mathbb{R}^n \times D_f \times \mathcal{I}_f, \tag{180}$$

is valid.

Below, we always assume the DAE (179) to have a properly stated leading term. To simplify matters we further assume the nullspace $\ker f_y(y, x, t)$ to be independent of y. Then, the transversality condition (180) pointwise induces a projector matrix $R(x, t) \in \mathcal{L}(\mathbb{R}^n)$, the so-called *border projector*, such that

$$\operatorname{im} R(x, t) = \operatorname{im} D(t),\ \ker R(x, t) = \ker f_y(y, x, t),\ (y, x, t) \in \mathbb{R}^n \times \mathcal{D}_f \times \mathcal{I}_f. \tag{181}$$

Since both subspaces $\operatorname{im} D$ and $\ker f_y$ are $\mathcal{C}^1$-subspaces, the border projector function $R : \mathcal{D}_f \times \mathcal{I}_f \to \mathcal{L}(\mathbb{R}^n)$ is continuously differentiable, see [50, Lemma A.20].

Note that, if the subspace $\ker f_y(y, x, t)$ actually depends on y. then we can modify the DAE by letting $\tilde{f}(y, x, t) := f(D(t)D(t)^+ y, x, t)$ such that $\ker \tilde{f}_y(y, x, t) = (\operatorname{im} D(t))^\perp$ solely depends on t.

Next we depict the notion of regularity regions of a DAE (179). For this aims we introduce *admissible matrix function sequences* and associated projector functions (cf. [50]). Denote

$$\begin{aligned}
A(x^1, x, t) &: = f_y(D(t)x^1 + D'(t)x, x, t) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m),\\
B(x^1, x, t) &: = f_x(D(t)x^1 + D'(t)x, x, t) \in \mathcal{L}(\mathbb{R}^m),\\
G_0(x^1, x, t) &: = A(x^1, x, t)D(t) \in \mathcal{L}(\mathbb{R}^m),\\
B_0(x^1, x, t) &: = B(x^1, x, t) \in \mathcal{L}(\mathbb{R}^m) \quad \text{for } \ x^1 \in \mathbb{R}^m, x \in \mathcal{D}_f, t \in \mathcal{I}_f.
\end{aligned}$$

The transversality condition (180) implies $\ker G_0(x^1, x, t) = \ker D(t)$. We introduce projector valued functions $Q_0, P_0, \Pi_0 \in \mathcal{C}(\mathcal{I}_f, \mathcal{L}(\mathbb{R}^m))$ such that for all $t \in \mathcal{I}_f$

$$\operatorname{im} Q_0(t) = N_0(t) := \ker D(t), \quad \Pi_0(t) := P_0(t) := I - Q_0(t). \tag{182}$$

Since D has constant rank, the orthoprojector function onto N_0 is as smooth as D. Therefore, as Q_0 we can choose the orthoprojector function onto N_0 which is even continuously differentiable. Next we determine the generalized inverse $D(x,t)^-$ of $D(t)$ pointwise for all arguments by

$$\begin{aligned} D(x,t)^- D(t) D(x,t)^- &= D(x,t)^-, \\ D(t) D(x,t)^- D(t) &= D(t), \\ D(x,t)^- D(t) &= P_0(t), \\ D(t) D(x,t)^- &= R(x,t). \end{aligned}$$

The resulting function D^- is continuous, if P_0 is continuously differentiable then so is also D^-.

Definition 6.12 *Let the DAE (179) have a properly involved derivative. $\mathcal{G} \subseteq \mathcal{D}_f \times \mathcal{I}_f$ be open connected.*
For the given level $\kappa \in \mathbb{N}$, we call the sequence $G_0, \ldots, G_\kappa$ an admissible matrix function sequence associated with the DAE (179) on the set $\mathcal{G}$, if it is built pointwise for all $(x,t) \in \mathcal{G}$ and all arising $x^j \in \mathbb{R}^m$ by the rule:
set $G_0 := AD$, $B_0 := B$, $N_0 := \ker G_0$,
for $i \geq 1$:

$$G_i := G_{i-1} + B_{i-1} Q_{i-1}, \tag{183}$$

$$N_i := \ker G_i, \quad \widehat{N_i} := (N_0 + \cdots + N_{i-1}) \cap N_i,$$

find a complement X_i such that $N_0 + \cdots + N_{i-1} = \widehat{N_i} \oplus X_i$,
choose a projector Q_i such that $\operatorname{im} Q_i = N_i$ and $X_i \subseteq \ker Q_i$,
set $P_i := I - Q_i$, $\Pi_i := \Pi_{i-1} P_i$,

$$B_i := B_{i-1} P_{i-1} - G_i D^- (D \Pi_i D^-)' D \Pi_{i-1}, \tag{184}$$

and, additionally,

(a) *the matrix function G_i has constant rank r_i on $\mathbb{R}^{mi} \times \mathcal{G}$, $i = 0, \ldots, \kappa$,*

(b) *the intersection $\widehat{N_i}$ has constant dimension $u_i := \dim \widehat{N_i}$ there,*

(c) *the product function Π_i is continuous and $D \Pi_i D^-$ is continuously differentiable on $\mathbb{R}^{mi} \times \mathcal{G}$, $i = 0, \ldots, \kappa$.*

The projector functions $Q_0, \ldots, Q_\kappa$ linked with an admissible matrix function sequence are said to be admissible themselves.
An admissible matrix function sequence G_0, …, G_κ is said to be regular admissible, if

$$\widehat{N_i} = \{0\} \quad \textit{for all} \quad i = 1, \ldots, \kappa.$$

Then, also the projector functions $Q_0, \ldots, Q_\kappa$ are called regular admissible.
The numbers $r_0 = \operatorname{rank} G_0, \ldots, r_\kappa = \operatorname{rank} G_\kappa$ and $u_1, \ldots, u_\kappa$ are named characteristic values of the DAE on $\mathcal{G}$.

To shorten the wording we often speak simply of *admissible projector functions* having in mind the admissible matrix function sequence built with these admissible projector functions. Admissible projector functions are always cross-linked with their matrix function sequence. Changing a projector function yields a new matrix function sequence.

We refer to [50] for many useful properties of the admissible matrix function sequences. A special instance of a sequence is given in Example 4.4. It always holds that

$$r_0 \leq \cdots \leq r_{\kappa-1} \leq r_\kappa.$$

The notion of *characteristic values* makes sense, since these values are independent of the special choice of admissible projector functions and invariant under regular transformations.

In case of a linear constant coefficient DAE, the construct simplifies to a sequence of matrices. In particular, the second term in the definition of B_i disappears. It is aging ([30]) that a pair $\{E, F\}$ of $m \times m$ matrices E, F is regular with Kronecker index μ exactly if an admissible sequence of matrices starting with $G_0 = AD = E$, $B_0 := F$ yields

$$r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m. \tag{185}$$

Thereby, neither the factorization nor the special choice of admissible projectors do matter. The characteristic values describe the structure of the Weierstraß-Kronecker form (9): we have $l = \sum_{j=0}^{\mu-1}(m - r_j)$ and the nilpotent part N contains altogether $s = m - r_0$ Jordan blocks, among them $r_i - r_{i-1}$ Jordan blocks of order i, $i = 1, \ldots, \mu$.

For linear DAEs with time-varying coefficients, the term $(\cdot)'$ in (184) means the derivative in time, and all matrix functions are functions in time. In general, the term $(\cdot)'$ in (184) stands for the total derivative in jet variables and then the matrix function G_i depends on the basic variables $(x, t) \in \mathcal{G}$ and, additionally, on the jet variables $x^1, \ldots, x^{i+1} \in \mathbb{R}^m$. Owing to the total derivative $(D\Pi_i D^-)'$ the new variable $x^{i+2} \in \mathbb{R}^m$ comes in at this level, see [50, Section 3.2].
Owing to the constant-rank conditions, the terms $D\Pi_i D^-$ are basically continuous. It may happen that, for making these terms continuously differentiable, the data function f must satisfy additional smoothness requirements. A precise description of those smoothness is much too involved and a all overall sufficient condition, say $f \in \mathcal{C}^m$, is much too superficial. To indicate that there might be additional smoothness demands we restrict us to the wording f *be sufficiently smooth*.

The next definition ties regularity up to the inequalities (185) and so generalizes regularity of matrix pencils for time-varying linear DAEs as well as for nonlinear DAEs. We emphasize that regularity is supported by several constant-rank conditions.

Definition 6.13 *Let the DAE (179) have a properly involved derivative. Let $\mathcal{G} \subseteq \mathcal{D}_f \times \mathcal{I}_f$ be an open, connected subset. The DAE (179) is said to be*

(1) *regular on $\mathcal{G}$ with tractability index 0, if $r_0 = m$,*

(2) *regular on $\mathcal{G}$ with tractability index μ, if an admissible matrix function sequence exists such that (185) is valid on $\mathcal{G}$.*

(3) *regular on $\mathcal{G}$, if it is, on $\mathcal{G}$, regular with any index (i.e., case (1) or (2) apply).*

The open connected subset $\mathcal{G}$ is called a regularity region or regularity domain. A point $(\bar{x},\bar{t}) \in \mathcal{D}_f \times \mathcal{I}_f$ is a regular point, if there is a regularity region $\mathcal{G} \ni (\bar{x},\bar{t})$. If $\mathcal{D} \subseteq \mathcal{D}_f$ is an open subset and $\mathcal{I} \subseteq \mathcal{I}_f$ is a compact subinterval, then the DAE (179) is said to be regular on $\mathcal{D} \times \mathcal{I}$, if there is a regularity region $\mathcal{G}$ such that $\mathcal{D} \times \mathcal{I} \subset \mathcal{G}$.

Example 6.14 (Regularity regions) *We write the DAE*

$$\begin{aligned} x_1'(t) + x_1(t) &= 0, \\ x_2(t)x_2'(t) - x_3(t) &= 0, \\ x_1(t)^2 + x_2(t)^2 - 1 - \gamma(t) &= 0, \end{aligned}$$

in the form (179), with $n = 2,\ m = k = 3$,

$$f(y,x,t) = \begin{bmatrix} y_1 + x_1 \\ x_2 y_2 - x_3 \\ x_1^2 + x_2^2 - \gamma(t) - 1 \end{bmatrix}, \quad f_y(y,x,t) = \begin{bmatrix} 1 & 0 \\ 0 & x_2 \\ 0 & 0 \end{bmatrix},$$
$$D(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

for $y \in \mathbb{R}^2$, $x \in \mathcal{D}_f = \mathbb{R}^3$, $t \in \mathcal{I}_f = \mathbb{R}$.
The derivative is properly involved on the open subsets $\mathbb{R}^2 \times \mathcal{G}_+$ and $\mathbb{R}^2 \times \mathcal{G}_-$, $\mathcal{G}_+ := \{x \in \mathbb{R}^3 : x_2 > 0\} \times \mathcal{I}_f$, $\mathcal{G}_- := \{x \in \mathbb{R}^3 : x_2 < 0\} \times \mathcal{I}_f$. We have there

$$G_0 = AD = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x_2^1 & -1 \\ 2x_1 & 2x_2 & 0. \end{bmatrix}.$$

Letting

$$Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \textit{yields} \quad G_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2x_2 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

G_1 is singular but has constant rank. Since $N_0 \cap N_1 = \{0\}$ we find a projector function Q_1 such that $N_0 \subseteq \ker Q_1$. We choose

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{x_2} & 0 \end{bmatrix}, P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{x_2} & 1 \end{bmatrix}, \Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, D\Pi_1 D^- = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and obtain $B_1 = B_0 P_0 Q_1$, and then

$$G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2x_2 + x_2^1 & -1 \\ 0 & 2x_2 & 0 \end{bmatrix}.$$

The matrix $G_2 = G_2(x^1,x,t)$ is nonsingular for all arguments (x^1,x,t) with $x_2 \neq 0$. The admissible matrix function sequence terminates at this level. The open connected subsets $\mathcal{G}_+$ and $\mathcal{G}_-$ are regularity regions, here both with characteristics $r_0 = 2$, $r_1 = 2$, $r_2 = 3$, and tractability index $\mu = 2$. □

For regular DAEs, all intersections $\widehat{N_i}$ are trivial ones, thus $u_i = 0,\ i \geq 1$. Namely, because of the inclusions

$$\widehat{N_i} \subseteq N_i \cap N_{i+1} \subseteq N_{i+1} \cap N_{i+2} \subseteq \cdots \subseteq N_{\mu-1} \cap N_\mu,$$

for reaching a nonsingular G_μ, which means $N_\mu = \{0\}$, it is necessary to have $\widehat{N_i} = \{0\}$, $i \geq 1$. This is a useful condition for checking regularity in practice.

Observe that each open connected subset of a regularity region is again a regularity region. A regularity region consist of regular points having uniform characteristics. The union of regularity regions is, if it is connected, a regularity region, too. Further, the nonempty intersection of two regularity regions is also a regularity region. Only regularity regions with uniform characteristics may yield nonempty intersections. *Maximal regularity regions* are then bordered by so-called critical points. Solutions may cross the borders of maximal regularity regions and undergo there bifurcations et cetera, see examples in [50, 56, 49]. No doubt, much further research is needed to elucidate these phenomena.

6.3.2 The structure of linear DAEs

The general DAE (179) captures linear DAEs

$$A(t)(Dx)'(t) + B(t)x(t) - q(t) = 0 \tag{186}$$

as $f(y, x, t) := A(t)y + B(t)x - q(t),\ t \in \mathcal{I}_f$. Now, admissible matrix function sequences depend only on time t; and hence, we speak on *regularity intervals* instead of regions. A regularity interval is open by definition. We say that the linear DAE with properly leading term is *regular on the compact interval* $[t_a, t_e]$, if there is an accommodating regularity interval, or equivalently, if all points of $[t_a, t_e]$ are regular.
If the linear DAE is regular on the interval $\mathcal{I}$, then it is also regular on each subinterval of $\mathcal{I}$ with the same characteristics. This sounds as a triviality; however, there is a continuing profound debate about some related questions, cf. Subsection 4.4.

If the linear DAE (186) is regular on the interval $\mathcal{I}$, then (see [50, Section 2.4]) it can be decoupled by admissible projector functions into an *inherent regular ODE* (IERODE)

$$u' - (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1}G_\mu^{-1}B_\mu D^- u = D\Pi_{\mu-1}G_\mu^{-1}q \tag{187}$$

and a triangular subsystem of several equations including differentiations

$$\begin{bmatrix} 0 & \mathcal{N}_{01} & \cdots & \mathcal{N}_{0,\mu-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & \mathcal{N}_{\mu-2,\mu-1} \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ (Dv_1)' \\ \vdots \\ (Dv_{\mu-1})' \end{bmatrix}$$

$$+ \begin{bmatrix} I & \mathcal{M}_{01} & \cdots & \mathcal{M}_{0,\mu-1} \\ & I & \ddots & \vdots \\ & & \ddots & \mathcal{M}_{\mu-2,\mu-1} \\ & & & I \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{\mu-1} \end{bmatrix} + \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_{\mu-1} \end{bmatrix} D^- u = \begin{bmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \\ \vdots \\ \mathcal{L}_{\mu-1} \end{bmatrix} q.$$

The subspace $\operatorname{im} D\Pi_{\mu-1}$ is an invariant subspace for the IERODE (187). This structural decoupling is associated with the decomposition

$$x = D^- u + v_0 + v_1 + \cdots + v_{\mu-1}.$$

The coefficients are continuous and explicitly given in terms of an admissible matrix function sequence as

$$\begin{aligned}
\mathcal{N}_{01} &:= -Q_0 Q_1 D^- \\
\mathcal{N}_{0j} &:= -Q_0 P_1 \cdots P_{j-1} Q_j D^-, && j = 2, \dots, \mu-1, \\
\mathcal{N}_{i,i+1} &:= -\Pi_{i-1} Q_i Q_{i+1} D^-, \\
\mathcal{N}_{ij} &:= -\Pi_{i-1} Q_i P_{i+1} \cdots P_{j-1} Q_j D^-, && j = i+2, \dots, \mu-1,\ i = 1, \dots, \mu-2, \\
\mathcal{M}_{0j} &:= Q_0 P_1 \cdots P_{\mu-1} \mathcal{M}_j D\Pi_{j-1} Q_j, && j = 1, \dots, \mu-1, \\
\mathcal{M}_{ij} &:= \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} \mathcal{M}_j D\Pi_{j-1} Q_j, && j = i+1, \dots, \mu-1,\ i = 1, \dots, \mu-2, \\
\mathcal{L}_0 &:= Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1}, \\
\mathcal{L}_i &:= \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} G_\mu^{-1}, && i = 1, \dots, \mu-2, \\
\mathcal{L}_{\mu-1} &:= \Pi_{\mu-2} Q_{\mu-1} G_\mu^{-1}, \\
\mathcal{H}_0 &:= Q_0 P_1 \cdots P_{\mu-1} \mathcal{K} \Pi_{\mu-1}, \\
\mathcal{H}_i &:= \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} \mathcal{K} \Pi_{\mu-1}, && i = 1, \dots, \mu-2, \\
\mathcal{H}_{\mu-1} &:= \Pi_{\mu-2} Q_{\mu-1} \mathcal{K} \Pi_{\mu-1},
\end{aligned}$$

with

$$\mathcal{K} := (I - \Pi_{\mu-1}) G_\mu^{-1} B_{\mu-1} \Pi_{\mu-1} + \sum_{l=1}^{\mu-1} (I - \Pi_{l-1})(P_l - Q_l)(D\Pi_l D^-)' D\Pi_{\mu-1},$$

$$\mathcal{M}_j := \sum_{k=0}^{j-1} (I - \Pi_k)\{P_k D^-(D\Pi_k D^-)' - Q_{k+1} D^-(D\Pi_{k+1} D^-)'\} D\Pi_{j-1} Q_l D^-, \quad l = 1, \dots, \mu-1.$$

The IERODE is always uncoupled of the second subsystem, but the latter is tied to the IERODE (187) if among the coefficients $\mathcal{H}_0, \dots, \mathcal{H}_{\mu-1}$ is at least one who does not vanish. One speaks about a *fine decoupling*, if $\mathcal{H}_1 = \cdots = \mathcal{H}_{\mu-1} = 0$, and about a *complete decoupling*, if $\mathcal{H}_0 = 0$, additionally. A complete decoupling is given, exactly if the coefficient $\mathcal{K}$ vanishes identically.
If the DAE (186) is regular and the original data are sufficiently smooth, then the DAE (186) is called *fine*. For fine DAEs, fine and complete decouplings always exist and can be constructed, see [50, Subsection 2.4.3]. Example 4.4 shows an instance of completely decoupling projectors.

It is noteworthy that, if $Q_0, \dots, Q_{\mu-1}$ generate a complete decoupling for a constant coefficient DAE $Ex'(t) + Fx(t) = 0$, then $\Pi_{\mu-1}$ is the spectral projector of the matrix pencil $\{E, F\}$. This way, the projector function $\Pi_{\mu-1}$ associated with a complete decoupling of a fine time-varying DAE represents the generalization of the spectral projector.

6.3.3 Linearizations

Given is now a reference function $x_* \in \mathcal{C}^1_D(\mathcal{I}_*, \mathbb{R}^m)$ on an individual interval $\mathcal{I}_* \subseteq \mathcal{I}_f$, whose values belong to $\mathcal{D}_f$. For each such reference function we may consider the linearization of the (179) along x_*, that is, the linearized DAE

$$A_*(t)(Dx)'(t) + B_*(t)x(t) = q(t), \quad t \in \mathcal{I}_*, \tag{188}$$

with coefficients

$$A_*(t) := f_y((Dx_*)'(t), x_*(t), t), \quad B_*(t) := f_x((Dx_*)'(t), x_*(t), t), \quad t \in \mathcal{I}_*.$$

The linear DAE (188) inherits from the nonlinear DAE (179) the properly stated leading term.

We denote by $\mathcal{C}^m_{ref}(\mathcal{G})$ the set of all $\mathcal{C}^m$ functions x_*, defined on individual intervals $\mathcal{I}_{x_*}$, and with graph in $\mathcal{G}$, that is, $(x_*(t), t) \in \mathcal{G}$ for $t \in \mathcal{I}_{x_*}$. Clearly, then we have also $x_* \in \mathcal{C}^1_D(\mathcal{I}_{x_*}, \mathbb{R}^m)$. By the smoothness of the reference functions x_* and the function f we ensure that also the coefficients A_* and B_* are sufficiently smooth for regularity.

Next we adapt the necessary and sufficient regularity condition from [50, Theorem 3.33] to our somewhat simpler situation.

Theorem 6.15 *Let the DAE (179) have a properly involved derivative and let f be sufficiently smooth. Let $\mathcal{G} \subseteq \mathcal{D}_f \times \mathcal{I}_f$ be an open connected set. Then the following statements are valid:*

(1) *The DAE (179) is regular on $\mathcal{G}$ if the linearized DAE (188) along each arbitrary reference function $x_* \in \mathcal{C}^m_{ref}(\mathcal{G})$ is regular, and vice versa.*

(2) *If the DAE (179) is regular on $\mathcal{G}$ with tractability index μ and characteristic values $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$, then all linearized DAEs (188) along reference functions $x_* \in \mathcal{C}^m_{ref}(\mathcal{G})$ are regular with uniform index μ and characteristics $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$.*

(3) *If all linearized DAEs (188) along reference functions $x_* \in \mathcal{C}^m_{ref}(\mathcal{G})$ are regular, then they have uniform index and characteristics, and the nonlinear DAE (179) is also regular on $\mathcal{G}$, with the same index and characteristics.*

Corollary 6.16 *Let the DAE (179) have a properly involved derivative and let f be sufficiently smooth. Let $\mathcal{D} \subseteq \mathcal{D}_f$ be an open connected set and $\mathcal{I} \subset \mathcal{I}_f$ be a compact interval. Then the following statements are valid:*

(1) *The DAE (179) is regular on $\mathcal{D} \times \mathcal{I}$ if the linearized DAE (188) along each arbitrary reference function $x_* \in \mathcal{C}^m(\mathcal{I}, \mathbb{R}^m)$ with values in $\mathcal{D}$ is regular, and vice versa.*

(2) *If the DAE (179) is regular on $\mathcal{D} \times \mathcal{I}$ with tractability index μ and characteristic values $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$, then all linearized DAEs (188) along reference functions $x_* \in \mathcal{C}^m(\mathcal{I}, \mathbb{R}^m)$ with values in $\mathcal{D}$ are regular with uniform index μ and characteristics $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$.*

(3) *If all linearized DAEs (188) along reference functions $x_* \in \mathcal{C}^m(\mathcal{I}, \mathbb{R}^m)$ with values in $\mathcal{D}$ are regular, then they have uniform index and characteristics, and the nonlinear DAE (179) is also regular on $\mathcal{D} \times \mathcal{I}$, with the same index and characteristics.*

Proof: Statement (1) is a consequence of the Statements (2) and (3).
Statement (2) follows from the construction of the admissible matrix function sequences. Namely, for each $x_* \in \mathcal{C}^m(\mathcal{I}, \mathbb{R}^m)$, with values in $\mathcal{D}$, we have

$$\begin{aligned} G_0(x_*'(t), x_*(t), t) &=: G_{*\,0}(t), \\ B_{i-1}(x_*^{(i+1)}(t), \cdots, x_*'(t), x_*(t), t) &=: B_{*\,i-1}(t), \\ G_i(x_*^{(i+1)}(t), \cdots, x_*'(t), x_*(t), t) &=: G_{*\,i}(t),\ t \in \mathcal{I}, \quad i = 1, \dots, \mu, \end{aligned}$$

which represents an admissible matrix function sequence for the linearized along x_* DAE.
Statement (3) proves along the lines of [50, Theorem 3.33] by means of so-called widely orthogonal projector functions. The prove given in [50] also works, if one supposes solely compact individual intervals $\mathcal{I}_{x_*}$.
By Lemma 6.17 below, each reference function given on an individual compact interval can be extended to belong to $x_* \in \mathcal{C}^m(\mathcal{I}, \mathbb{R}^m)$, with values in $\mathcal{D}$. □

Lemma 6.17 *Let $\mathcal{D} \subseteq \mathbb{R}^m$ be an open set and $\mathcal{I} \subset \mathbb{R}$ be a compact interval. Let $\mathcal{I}_* \subset \mathcal{I}$ be a compact subinterval and $s \in \mathbb{N}$.*
Then, for each function $x_* \in \mathcal{C}^s(\mathcal{I}_*, \mathbb{R}^m)$, with values in $\mathcal{D}$, there is an extension $f \in \mathcal{C}^s(\mathcal{I}, \mathbb{R}^m)$, with values in $\mathcal{D}$.

Proof: It suffices to verify the statement for the case $\mathcal{I} = [t_a, t_e]$, $\mathcal{I}_* = [t_a, t_0]$, $t_0 < t_e$. We put

$$\begin{aligned} f(t) &:= x_*(t), \quad t \in [t_a, t_0], \\ f(t) &:= x_0 + \mathrm{e}^{-\alpha(t-t_0)} p(t), \quad p(t) := (t-t_0)p_1 + \cdots + \frac{1}{s!}(t-t_0)^s p_s, \quad t \in (t_0, t_e]. \end{aligned}$$

Letting $x_0 := x_*(t_0)$ we have a continuous function f. We derive for $t > t_0$ and $j = 1, \dots, s$:

$$f^{(j)}(t) = \mathrm{e}^{-\alpha(t-t_0)} p^{(j)}(t) - \sum_{i=1}^{j-1} \alpha^{j-i} \binom{j}{i} f^{(i)}(t) - \alpha^j \binom{j}{0} (f(t) - x_0).$$

Fixing successively the coefficients p_j, for $j = 1, \dots, s$, by

$$p_j = x_*^{(j)}(t_0) + \sum_{i=1}^{j-1} \alpha^{j-i} \binom{j}{i} x_*^{(i)}(t_0),$$

we ensure that $f \in \mathcal{C}^s(\mathcal{I}, \mathbb{R}^m)$. It remains to show that the values of f remain in $\mathcal{D}$. We compute for $t > t_0$ and positive, sufficiently large *alpha*:

$$\begin{aligned} |f(t) - x_0| &= |\mathrm{e}^{-\alpha(t-t_0)} p(t)| \\ &= |(t-t_0)\mathrm{e}^{-\alpha(t-t_0)} p_1 + \cdots + \frac{1}{s!}(t-t_0)^s \mathrm{e}^{-\alpha(t-t_0)} p_s| \\ &= \frac{1}{\alpha}\ |\alpha(t-t_0)\mathrm{e}^{-\alpha(t-t_0)} p_1 + \cdots + \frac{1}{s!}\alpha^s (t-t_0)^s \mathrm{e}^{-\alpha(t-t_0)} \frac{1}{\alpha^{s-1}} p_s| \le c\frac{1}{\alpha}. \end{aligned}$$

The last inequality holds true, since the expressions $\alpha^j (t-t_0)^j \mathrm{e}^{-\alpha(t-t_0)}$ and $|\frac{1}{\alpha^{j-1}} p_j|$ are bounded.
Together with $x_0 \in \mathcal{D}$ there is a ball $B(x_0, \varepsilon) \subset \mathcal{D}$. Choosing a sufficient large α we arrive at $c\frac{1}{\alpha} < \varepsilon$ and we are done. □

6.4 List of symbols and abbreviations

$\mathcal{L}(X, Y)$	set of linear operators from X to Y
$\mathcal{L}_b(X, Y)$	set of bounded linear operators from X to Y
$\mathcal{L}_c(X, Y)$	set of closed linear operators from X to Y
X^*	dual space
K^*	dual and adjoint operator
K^-	outer, inner and generalized inverses
K^+	orthogonal generalized (Moore-Penrose) inverse
$\operatorname{dom} K$	definition domain of the map K
$\ker K$	nullspace (kernel) of the operator K
$\operatorname{im} K$	image (range) of the operator K
$\operatorname{ind}\{E, F\}$	Kronecker index of the matrix pair $\{E, F\}$
$\operatorname{ind}_{fred}(K)$	Fredholm index of the operator K
$\langle\cdot,\cdot\rangle$	scalar product in $\mathbb{R}^m$, dual pairing
$(\cdot,\cdot)$	scalar product in function spaces
$\lvert\cdot\rvert$	vector and matrix norms
$\lVert\cdot\rVert$	norms on function spaces, operator norms
$\dot{+}$	algebraic direct sum
$\oplus$	topological direct sum
DAE	differential-algebraic equation
ODE	ordinary differential equation
IVP	initial value problem
BVP	boundary value problem
LSS	least squares solution

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