Adaptive Nonconforming Finite Element Approximation of Eigenvalue Clusters

Dietmar Gallistl*

May 15, 2014

Abstract

This paper analyses an adaptive nonconforming finite element method for eigenvalue clusters of self-adjoint operators and proves optimal convergence rates (with respect to the concept of nonlinear approximation classes) for the approximation of the invariant subspace spanned by the eigenfunctions of the eigenvalue cluster. Applications include eigenvalues of the Laplacian and of the Stokes system.

Keywords eigenvalue problem, eigenvalue cluster, adaptive finite element method, Stokes operator, optimality

AMS subject classification 65M12, 65M60, 65N25

1 Introduction

Nonconforming finite element methods (FEMs) are of high interest in computational fluid dynamics where they provide stable low-order discretisations with favourable local mass conservation properties. Especially for eigenvalue problems, the nonconforming discretisation is even more attractive because it allows for a convenient computation of guaranteed lower eigenvalue bounds [19]. In many practical situations the eigenvalues of interest form an eigenvalue cluster where all eigenfunctions have to be discretised simultaneously in adaptive algorithms. This paper applies and generalises the technique of the recent work [33] to the nonconforming $\mathcal{P}_1$ discretisation of the Laplace and Stokes eigenvalue problems and proves optimal convergence rates of the simultaneous adaptive FEM computation for the eigenfunctions in the cluster. Optimal convergence rates for adaptive FEMs for eigenvalue problems were established in [26, 18] for simple eigenvalues and in [25] for multiple eigenvalues for conforming finite elements and in [14] for the nonconforming discretisation of the first eigenvalue of the Laplacian. The main difference to the analysis of those results is the additional difficulty that the cluster width should not enter the error estimates as an additive term. Consider a polyhedral Lipschitz domain $\Omega \subseteq \mathbb{R}^d$ for $d \geq 2$ and a simplicial triangulation $\mathcal{T}_\ell$. Let W be the invariant subspace spanned by the eigenfunctions of an eigenvalue cluster and let W_ℓ describe the linear hull of the corresponding nonconforming $\mathcal{P}_1(\mathcal{T}_\ell)$ eigenfunctions. The adaptive algorithm is driven by the explicit residual-based error estimator contributions of all discrete eigenfunctions in the cluster. The main results of this paper state that the error quantities

$$\sup_{\substack{w \in W \\ \|w\|=1}} \inf_{v_\ell \in W_\ell} \|w - v_\ell\|_{\mathrm{NC}}$$

*Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany
This work was supported by the DFG Research Center Matheon.

(in the case of the Laplace eigenproblem $-\Delta u = \lambda u$) and

$$\sup_{\substack{w\in W\\ \|w\|=1}} \inf_{v_\ell\in W_\ell} \left(|||w-v_\ell|||_{\mathrm{NC}}^2 + \|p(w)-p(v_\ell)\|^2 \right)^{1/2}$$

(in the case of the Stokes eigenproblem $-\Delta u+(Dp)^\top = \lambda u$; $\operatorname{div} u = 0$) decay as $(\operatorname{card}(\mathcal{T}_\ell)-\operatorname{card}(\mathcal{T}_0))^{-\sigma}$, provided all eigenfunctions belong to the approximation class $\mathfrak{A}_\sigma$ (resp. $\mathfrak{A}_\sigma^{\mathrm{Stokes}}$). Here, $\|\cdot\|$ denotes the L^2 norm and $|||\cdot|||_{\mathrm{NC}}$ denotes the nonconforming energy norm (i.e., the L^2 norm of the piecewise derivative). Although one can prove using the techniques of [32] or the different approach of [4] that those error quantities also control the square root of the eigenvalue error, this paper merely studies the approximation of the space W. An important methodological tool is the higher-order L^2 control for the eigenfunction approximations which is proven by means of conforming companion operators. This kind of operators were introduced in [14, 41] in the two-dimensional case and are generalised in this paper to higher space dimensions $d \geq 2$. The resulting L^2 error estimates compare the L^2 error directly with the energy error and therefore do not employ any a priori results of the eigenfunction approximation.

The proofs for optimal convergence rates of adaptive FEMs were initiated by [22, 46] and extended to nonconforming FEMs for the Poisson equation [3, 42] and the Stokes equations [2, 21, 39]. These approaches were recently unified in the axiomatic approach of [12]. The convergence of adaptive FEMs for eigenvalues was proven in [35, 36, 10]. The optimality results [26, 34, 18] concern simple eigenvalues and conforming FEMs while [14] establishes optimaliy for the nonconforming discretisation of the first Laplace eigenvalue. The first optimality analysis for an adaptive algorithm for multiple eigenvalues [25] based on conforming FEMs introduced a simultaneous bulk criterion for all discrete eigenfunctions of the multiple eigenvalue. In [33] this marking strategy was proven to lead to optimal convergence rates in the case of eigenvalue clusters. The results of this paper establish a corresponding result for the nonconforming $\mathcal{P}_1$ FEM and the first optimality result for the Stokes eigenproblem.

The remaining parts of this paper are organised as follows. Section 2 describes an abstract framework for the discretisation of eigenvalue clusters. Section 3 introduces the notation on triangulations and presents the conforming companion operators for the nonconforming $\mathcal{P}_1$ FEM in any space dimension. Section 4 is devoted to the analysis of the adaptive FEM for the eigenvalues of the Laplacian. Section 5 studies the adaptive FEM approximation of the eigenvalues of the Stokes system.

Throughout the paper standard notation on Lebesgue and Sobolev spaces is employed. The integral mean is denoted by $⨍$. The notation $a \lesssim b$ abbreviates $a \leq Cb$ for a positive generic constant C that may depend on the domain Ω and the initial triangulation $\mathcal{T}_0$ but not on the mesh-size or the eigenvalue cluster of interest. The notation $a \approx b$ stands for $a \lesssim b \lesssim a$.

2 Approximation of Eigenvalue Clusters

Let $(V, a(\cdot,\cdot))$ be a separable Hilbert space over $\mathbb{R}$ with induced norm $\|\cdot\|_a$ and let $b(\cdot,\cdot)$ be a scalar product on V with induced norm $\|\cdot\|_b$ such that the embedding $(V, \|\cdot\|_a) \hookrightarrow (V, \|\cdot\|_b)$ is compact. This paper is concerned with eigenvalue problems of the form: Find eigenpairs $(\lambda, u) \in \mathbb{R}\times V$ with $\|u\|_b = 1$ such that

$$a(u, v) = \lambda b(u, v) \quad \text{for all } v \in V. \tag{2.1}$$

It is well known from the spectral theory of selfadjoint compact operators [40, 23] that the eigenvalue problem (2.1) has countably many eigenvalues, which are real and positive with $+\infty$ as only possible accumulation point. Suppose that the eigenvalues are enumerated as

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$$

and let $(u_1, u_2, u_3, \dots)$ be some b-orthonormal system of corresponding eigenfunctions. For any $j \in \mathbb{N}$, the eigenspace corresponding to λ_j is defined as

$$E(\lambda_j) := \{u \in V \mid (\lambda_j, u) \text{ satisfies (2.1)}\} = \operatorname{span}\{u_k \mid k \in \mathbb{N} \text{ and } \lambda_k = \lambda_j\}.$$

In the present case of an eigenvalue problem of (the inverse of) a compact operator, the spaces $E(\lambda_j)$ have finite dimension. The discretisation of (2.1) is based on a family (over a countable index set I) of separable (not necessarily finite-dimensional) Hilbert spaces V_ℓ with scalar products $a_{\mathrm{NC}}(\cdot,\cdot)$ and $b_{\mathrm{NC}}(\cdot,\cdot)$ on $V + V_\ell$ with induced norms $\|\cdot\|_{a,\mathrm{NC}}$ and $\|\cdot\|_{b,\mathrm{NC}}$ such that a_{NC} and b_{NC} coincide with a and b when restricted to V

$$a_{\mathrm{NC}}|_{V\times V} = a \quad \text{and} \quad b_{\mathrm{NC}}|_{V\times V} = b.$$

The discrete eigenvalue problem seeks eigenpairs $(\lambda_\ell, u_\ell) \in \mathbb{R} \times V_\ell$ with $\|u_\ell\|_{b,\mathrm{NC}} = 1$ such that

$$a_{\mathrm{NC}}(u_\ell, v_\ell) = \lambda_\ell b_{\mathrm{NC}}(u_\ell, v_\ell) \quad \text{for all } v_\ell \in V_\ell. \tag{2.2}$$

The discrete eigenvalues can be enumerated

$$0 < \lambda_{\ell,1} \le \lambda_{\ell,2} \le \lambda_{\ell,3} \dots$$

with corresponding b_{NC}-orthonormal eigenfunctions $(u_{\ell,1}, u_{\ell,2}, u_{\ell,3} \dots)$. For a cluster of eigenvalues $\lambda_{n+1}, \dots, \lambda_{n+N}$ of length $N \in \mathbb{N}$, define the index set $J := \{n+1, \dots, n+N\}$ and the spaces

$$W := \operatorname{span}\{u_j \mid j \in J\} \quad \text{and} \quad W_\ell := \operatorname{span}\{u_{\ell,j} \mid j \in J\}.$$

The eigenspaces $E(\lambda_j)$ may differ for different $j \in J$.

Assume that the cluster is contained in a compact interval $[A, B]$ in the sense that

$$\{\lambda_j \mid j \in J\} \cup \{\lambda_{\ell,j} \mid \ell \in I, j \in J\} \subseteq [A, B].$$

This implies

$$\sup_{\ell \in I} \max_{(j,k) \in J^2} \max \left\{ \lambda_k^{-1} \lambda_{\ell,j}, \lambda_{\ell,j}^{-1} \lambda_k \right\} \le B/A. \tag{2.3}$$

Although in the applications in this paper $\dim(V_\ell)$ will be finite-dimensional, the analysis in this section admits the case $\dim(V_\ell) \in \mathbb{N} \cup \{\infty\}$. Let $J^C := \{1, \dots, \dim(V_\ell)\} \setminus J$ denote the complement of J. Assume that the cluster is separated from the remaining part of the spectrum in the sense that there exists a separation bound

$$M_J := \sup_{\ell \in I} \sup_{j \in J^C} \max_{k \in J} \frac{\lambda_k}{|\lambda_{\ell,j} - \lambda_k|} < \infty. \tag{H1}$$

Given $f \in V$, let $u \in V$ denote the unique solution to the linear problem

$$a(u, v) = b(f, v) \quad \text{for all } v \in V.$$

The quasi-Ritz projection $R_\ell u \in V_\ell$ is defined as the unique solution to

$$a_{\rm NC}(R_\ell u, v_\ell) = b_{\rm NC}(f, v_\ell) \quad \text{for all } v_\ell \in V_\ell.$$

Let P_ℓ denote the $b_{\rm NC}$-orthogonal projection onto W_ℓ and define

$$\Lambda_\ell := P_\ell \circ R_\ell. \tag{2.4}$$

For any eigenfunction $u \in W$, the function $\Lambda_\ell u \in W_\ell$ is regarded as its approximation. This approximation does not depend on the basis of W_ℓ. Notice that $\Lambda_\ell u$ is neither computable without knowledge of u nor necessarily an eigenfunction.

The following result is essentially contained in the textbook [48] and in [10] for a conforming finite element discretisation of the Laplace eigenvalue problem. The proof presented here extends the arguments of [48] to a more abstract situation.

Proposition 2.1. *Any eigenpair $(\lambda, u) \in \mathbb{R} \times W$ of (2.1) with $\|u\|_b = 1$ satisfies*

$$\|R_\ell u - \Lambda_\ell u\|_{b,\rm NC} \le M_J \|u - R_\ell u\|_{b,\rm NC} \quad \textit{and}$$
$$\|u - P_\ell u\|_{b,\rm NC} \le \|u - \Lambda_\ell u\|_{b,\rm NC} \le (1 + M_J)\|u - R_\ell u\|_{b,\rm NC}.$$

Proof. Set $v_\ell := R_\ell u - \Lambda_\ell u$ and recall $\dim(V_\ell) \in \mathbb{N} \cup \{\infty\}$. Since the eigenfunctions $(u_{\ell,j} \mid j = 1, \dots, \dim(V_\ell))$ form a $b_{\rm NC}$-orthonormal system of V_ℓ and v_ℓ is $b_{\rm NC}$-orthogonal on W_ℓ, there exist coefficients $(\alpha_j \mid j \in J^C)$ such that

$$v_\ell = \sum_{j \in J^C} \alpha_j u_{\ell,j} \quad \text{and} \quad \sum_{j \in J^C} \alpha_j^2 = \|v_\ell\|^2_{b,\rm NC}.$$

The definition of R_ℓ and the symmetry show that

$$(\lambda_{\ell,j} - \lambda) b_{\rm NC}(R_\ell u, u_{\ell,j}) = \lambda b_{\rm NC}(u - R_\ell u, u_{\ell,j}).$$

This and the orthogonality of v_ℓ and $\Lambda_\ell u$ lead to

$$\|v_\ell\|^2_{b,\rm NC} = b_{\rm NC}(R_\ell u, \sum_{j \in J^C} \alpha_j u_{\ell,j}) = b_{\rm NC}(u - R_\ell u, \sum_{j \in J^C} \alpha_j \frac{\lambda}{\lambda_{\ell,j} - \lambda} u_{\ell,j}).$$

The Cauchy inequality, the estimate (H1) and the $b_{\rm NC}$-orthogonality of the discrete eigenfunctions therefore show

$$\|v_\ell\|_{b,\rm NC} \le M_J \|u - R_\ell u\|_{b,\rm NC}.$$

The second claimed chain of inequalities follows from the projection property of P_ℓ and the triangle inequality. □

The following algebraic identity applies frequently in the analysis. It states the important property that, although $\Lambda_\ell u$ is no eigenfunction in general, $\Lambda_\ell u$ satisfies an equation that is similar to an eigenfunction property.

Lemma 2.2. *Any eigenpair $(\lambda, u) \in \mathbb{R} \times V$ of (2.1) satisfies*

$$a_{\rm NC}(\Lambda_\ell u, v_\ell) = \lambda b_{\rm NC}(P_\ell u, v_\ell) \quad \textit{for all } v_\ell \in V_\ell.$$

In other words, R_ℓ and P_ℓ commute, $P_\ell \circ R_\ell = R_\ell \circ P_\ell$.

Proof. The proof is given in [33, Lemma 2.2] and repeated here for convenient reading. The representation of $\Lambda_\ell u$ in terms of the b_{NC}-orthonormal basis $(u_{\ell,j})_{j\in J}$ reads as

$$\Lambda_\ell u = \sum_{j\in J} \alpha_j u_{\ell,j} \quad \text{with } \alpha_j = b_{\mathrm{NC}}(R_\ell u, u_{\ell,j}) \quad \text{for all } j \in J.$$

The symmetry of a_{NC} and b_{NC} proves for any $j \in J$ that

$$\alpha_j = b_{\mathrm{NC}}(R_\ell u, u_{\ell,j}) = \lambda_{\ell,j}^{-1} a_{\mathrm{NC}}(R_\ell u, u_{\ell,j}) = \lambda_{\ell,j}^{-1}\lambda b_{\mathrm{NC}}(u, u_{\ell,j}).$$

Therefore, the discrete eigenvalue problem reveals

$$\begin{aligned} a_{\mathrm{NC}}(\Lambda_\ell u, v_\ell) &= \sum_{j\in J} \alpha_j \lambda_{\ell,j} b_{\mathrm{NC}}(u_{\ell,j}, v_\ell) \\ &= \lambda \sum_{j\in J} b_{\mathrm{NC}}(b_{\mathrm{NC}}(u, u_{\ell,j}) u_{\ell,j}, v_\ell) = \lambda b_{\mathrm{NC}}(P_\ell u, v_\ell). \end{aligned}$$ □

The following result states a comparison of seminorms for the eigenfunctions. The application in the subsequent sections will be the equivalence of error estimators.

Lemma 2.3. *Suppose that*

$$\varepsilon := \max_{j\in J} \|u_j - \Lambda_\ell u_j\|_{b,\mathrm{NC}} \le \sqrt{1 + 1/(2N)} - 1 \quad \text{for all } \ell \in I. \tag{H2}$$

Then, both $(P_\ell u_j)_{j\in J}$ and $(\Lambda_\ell u_j)_{j\in J}$ form a basis of W_ℓ. For any $w_\ell \in W_\ell$ with $\|w_\ell\|_{b,\mathrm{NC}} = 1$, the coefficients of the representation $w_\ell = \sum_{j\in J} \beta_j P_\ell u_j$ and $w_\ell = \sum_{j\in J} \gamma_j \Lambda_\ell u_j$ are controlled as

$$\max\left\{ \sum_{j\in J} |\beta_j|^2, \sum_{j\in J} |\gamma_j|^2 \right\} \le 2 + 4N \qquad \text{for } N = \mathrm{card}(J). \tag{2.5}$$

For any $\ell \in I$, any seminorm ρ_ℓ on V_ℓ satisfies

$$\begin{aligned} N^{-1} \sum_{j\in J} \rho_\ell(\lambda_j P_\ell u_j)^2 &\le (B/A)^2 \sum_{j\in J} \rho_\ell(\lambda_{\ell,j} u_{\ell,j})^2 \\ &\le (B/A)^4 (2N + 4N^2) \sum_{j\in J} \rho_\ell(\lambda_j P_\ell u_j)^2 \end{aligned}$$

and

$$N^{-1} \sum_{j\in J} \rho_\ell(\Lambda_\ell u_j)^2 \le (B/A)^2 \sum_{j\in J} \rho_\ell(u_{\ell,j})^2 \le (B/A)^4 (2N + 4N^2) \sum_{j\in J} \rho_\ell(\Lambda_\ell u_j)^2.$$

Proof. The proof follows from Lemma 5.1 and Proposition 5.2 of [33]. □

3 The Nonconforming $\mathcal{P}_1$ Finite Element Space

This section introduces the necessary notation on regular simplicial triangulations and recalls some elementary facts on the nonconforming $\mathcal{P}_1$ finite element space. It furthermore generalises the companion operators from [14] to higher space dimensions.

3.1 Notation on Regular Triangulations

Let $\mathcal{T}_0$ be a regular simplicial triangulation of Ω in the sense of [47], i.e., $\cup\mathcal{T}_0 = \overline{\Omega}$ and any two elements of $\mathcal{T}_0$ are either disjoint or share exactly one k-dimensional face for $k \le d$ (e.g., a vertex or an edge). Throughout this paper, any regular triangulation of Ω is assumed to be admissible in the sense that it is regular and a refinement of $\mathcal{T}_0$ created by the refinement rules of [47] with proper initialisation of the refinement edges [47]. The set of all admissible refinements is denoted by $\mathbb{T}$. Given a triangulation $\mathcal{T}_\ell \in \mathbb{T}$, the piecewise constant mesh-size function $h_\ell := h_{\mathcal{T}_\ell}$ is defined by $h_\ell|_T := h_T := \operatorname{meas}(T)^{1/d}$ for any simplex $T \in \mathcal{T}_\ell$.

The set of $(d-1)$-dimensional hyper-faces (e.g., edges for $d = 2$ or faces for $d = 3$) of $\mathcal{T}_\ell$ is denoted by $\mathcal{F}_\ell$ while the interior $(d-1)$-dimensional hyper-faces are denoted by $\mathcal{F}_\ell(\Omega)$. Let every $F \in \mathcal{F}_\ell$ be equipped with a fixed normal vector ν_F. Given $F \in \mathcal{F}_\ell(\Omega)$, $F = \partial T_+ \cap \partial T_-$ shared by two simplices $(T_+, T_-) \in \mathcal{T}_\ell^2$, and a piecewise smooth function v, define the jump of v across F by

$$[v]_F := v|_{T_+} - v|_{T_-}.$$

For hyper-faces $F \subseteq \partial\Omega$ on the boundary, $[v]_F := v|_F$ denotes the trace. For a simplex T, the set of $(d-1)$-dimensional hyper-faces belonging to T is denoted by $\mathcal{F}(T)$.

The set of piecewise polynomial functions of degree $\le k$ with respect to $\mathcal{T}_\ell$ is denoted by $\mathcal{P}_k(\mathcal{T}_\ell)$. The L^2 projection onto $\mathcal{P}_k(\mathcal{T}_\ell)$ is denoted by $\Pi^k_{\mathcal{T}_\ell} \equiv \Pi^k_\ell$. The k-th order oscillations of a given function $f \in L^2(\Omega)$ is defined as

$$\operatorname{osc}_k(f, \mathcal{T}_\ell) := \|h_\ell(1 - \Pi^k_\ell)f\|_{L^2(\Omega)}.$$

The piecewise action of a differential operator is indicated by the subscript NC, i.e., the piecewise versions of D and div read as $D_{\mathrm{NC}} \equiv D_{\mathrm{NC}(\ell)}$ and $\operatorname{div}_{\mathrm{NC}} \equiv \operatorname{div}_{\mathrm{NC}(\ell)}$ e.g., $(D_{\mathrm{NC}}v)|_T = D(v|_T)$ for any $T \in \mathcal{T}_\ell$. The dependence on $\mathcal{T}_\ell$ in the notation is dropped whenever there is no risk of confusion.

3.2 Nonconforming Finite Element Space and Companion Operator

The nonconforming $\mathcal{P}_1$ finite element space, sometimes referred to as Crouzeix-Raviart finite element space [24], reads as

$$\mathfrak{CR}^1_0(\mathcal{T}_\ell) := \left\{ v_\ell \in \mathcal{P}_1(\mathcal{T}_\ell) \;\middle|\; \begin{array}{l} v_\ell \text{ is continuous in the interior hyper-faces'} \\ \text{midpoints and vanishes in the midpoints} \\ \text{of hyper-faces on the boundary} \end{array} \right\}.$$

Let, throughout this subsection, $V_\ell := V(\mathcal{T}_\ell) := \mathfrak{CR}^1_0(\mathcal{T}_\ell)$ and $V := H^1_0(\Omega)$. Given an admissible refinement $\mathcal{T}_{\ell+m} \in \mathbb{T}(\mathcal{T}_\ell)$ of $\mathcal{T}_\ell$, define the operator $\mathcal{I}^{\mathfrak{CR}}_\ell : V + V_{\ell+m} \to V_\ell$ by

$$\int_F (v - \mathcal{I}^{\mathfrak{CR}}_\ell v)\, ds = 0 \quad \text{for all } F \in \mathcal{F}_\ell \text{ and all } v \in V + V_{\ell+m}.$$

Note that $\mathcal{I}^{\mathfrak{CR}}_\ell$ is indeed well-defined for functions in $\mathfrak{CR}^1_0(\mathcal{T}_{\ell+m})$. A (piecewise) integration by parts proves the projection property $D_{\mathrm{NC}}\mathcal{I}^{\mathfrak{CR}}_\ell = \Pi^0_\ell D$, i.e.,

$$\int_T D_{\mathrm{NC}}\mathcal{I}^{\mathfrak{CR}}_\ell v\, dx = \int_T Dv\, dx \quad \text{for all } T \in \mathcal{T}_\ell \text{ and all } v \in V + V_{\ell+m}. \tag{3.1}$$

The proof of the approximation and stability property

$$\|h_T^{-1}(v - \mathcal{I}^{\mathfrak{CR}}_\ell v)\|_{L^2(T)} + \|D_{\mathrm{NC}}(v - \mathcal{I}^{\mathfrak{CR}}_\ell v)\|_{L^2(T)} \lesssim \|(1 - \Pi^0_\ell)D_{\mathrm{NC}}v\|_{L^2(T)} \tag{3.2}$$

for any $v \in V + V_{\ell+m}$ and any $T \in \mathcal{T}_\ell$ follows from the discrete Friedrichs inequality [9, Thm. 10.6.12] and a scaling argument.

The remaining parts of this subsection present conforming companion operators. The idea behind these operators is to design for a nonconforming finite element function v_ℓ some conforming companion $J_{d+1}v_\ell \in V$ with certain conservation properties. For $d = 2$ these kind of operators have been constructed by [14] and independently by [41]. The following result extends [14] to any dimension $d \geq 2$.

Proposition 3.1 (companion operator in any space dimension)**.** *Given any $v_\ell \in V_\ell$ there exists some $J_{d+1}v_\ell \in \mathcal{P}_{d+1}(\mathcal{T}_\ell) \cap V$ such that $v_\ell - J_{d+1}v_\ell$ is L^2 orthogonal onto the space $\mathcal{P}_0(\mathcal{T}_\ell)$ of piecewise constants, it enjoys the integral mean property*

$$\Pi_\ell^0(D_{\mathrm{NC}}(v_\ell - J_{d+1}v_\ell)) = 0, \tag{3.3}$$

and it satisfies the approximation and stability property

$$\begin{aligned}\|h_\ell^{-1}(v_\ell - J_{d+1}v_\ell)\|_{L^2(\Omega)} + \|D_{\mathrm{NC}}(v_\ell - J_{d+1}v_\ell)\|_{L^2(\Omega)}&\\ \lesssim \min_{v\in V}\|D_{\mathrm{NC}}(v_\ell - v)\|_{L^2(\Omega)}&.\end{aligned} \tag{3.4}$$

Proof. The design follows in three steps.

Step 1. The operator $J_1 : V_\ell \to P_1(\mathcal{T}_\ell) \cap V$ acts on any function $v_\ell \in V_\ell$ by averaging the function values at each interior vertex z, i.e.,

$$J_1v_\ell(z) = \operatorname{card}(\mathcal{T}_\ell(z))^{-1} \sum_{T\in\mathcal{T}_\ell(z)} v_\ell|_T(z) \quad \text{for all } z \in \mathcal{N}_\ell(\Omega)$$

where $\mathcal{T}_\ell(z) := \{T \in \mathcal{T}_\ell \mid z \in T\}$ is the set of simplices that contain the vertex z. This operator is also known as enriching operator in the context of fast solvers [8]. The proof of the approximation property

$$\|h_\ell^{-1}(v_\ell - J_1v_\ell)\|_{L^2(\Omega)} \lesssim \min_{v\in V}\|D_{\mathrm{NC}}(v_\ell - v)\|_{L^2(\Omega)} \tag{3.5}$$

is included in [11, Thm. 5.1] for $d = 2$. A generalisation to higher dimensions is outlined in the proof of [13, Thm. 4.9]. This and an inverse estimate [9] imply the stability property

$$\|D_{\mathrm{NC}}(v_\ell - J_1v_\ell)\|_{L^2(\Omega)} \lesssim \min_{v\in V}\|D_{\mathrm{NC}}(v_\ell - v)\|_{L^2(\Omega)}. \tag{3.6}$$

Step 2. Given any hyper-face $F = \operatorname{conv}\{z_1, \ldots, z_d\}$ with nodal $\mathcal{P}_1$ conforming basis functions $\varphi_1, \ldots, \varphi_d \in \mathcal{P}_1(\mathcal{T}_\ell) \cap V$, the quadratic edge-bubble function

$$\flat_F := \frac{(2d-1)!}{(d-1)!}\prod_{j=1}^{d}\varphi_j$$

is supported on the patch of F (that is the union of simplices which F belongs to) and satisfies $⨍_F \flat_F\, ds = 1$. For any function $v_\ell \in V_\ell$ the operator $J_d : V_\ell \to \mathcal{P}_d(\mathcal{T}_\ell) \cap V$ acts as

$$J_dv_\ell := J_1v_\ell + \sum_{F\in\mathcal{F}_\ell(\Omega)} \left(⨍_F (v_\ell - J_1v_\ell)\, ds\right) \flat_F.$$

An immediate consequence of this choice reads as

$$⨍_F J_dv_\ell\, ds = ⨍_F v_\ell\, ds \quad \text{for all } F \in \mathcal{F}_\ell.$$

An integration by parts shows the integral mean property of the gradients $\Pi_\ell^0 DJ_d = D_{\mathrm{NC}}$, i.e,

$$\int_T DJ_d v_\ell \, dx = \int_T D_{\mathrm{NC}} v_\ell \, dx \quad \text{for all } T \in \mathcal{T}_\ell.$$

Let $T \in \mathcal{T}_\ell$ with $F \in \mathcal{F}(T)$. The scaling $\|\flat_F\|_{L^2(\Omega)} \lesssim h_T^{d/2}$ and the Hölder and trace inequalities [30] show

$$\begin{aligned} h_T^{-1} \Big\| \sum_{F\in\mathcal{F}(T)} \Big(⨍_F (v_\ell - J_1 v_\ell)\, ds \Big) \flat_F \Big\|_{L^2(T)} \\ \lesssim h_T^{(d-2)/2} \sum_{F\in\mathcal{F}(T)} \Big| ⨍_F (v_\ell - J_1 v_\ell)\, ds \Big| \\ \lesssim h_T^{-1/2} \sum_{F\in\mathcal{F}(T)} \|v_\ell - J_1 v_\ell\|_{L^2(F)} \\ \lesssim h_T^{-1} \|v_\ell - J_1 v_\ell\|_{L^2(T)} + \|D_{\mathrm{NC}}(v_\ell - J_1 v_\ell)\|_{L^2(T)}. \end{aligned}$$

This, the triangle inequality and the properties (3.5)–(3.6) yield

$$\|h_\ell^{-1}(v_\ell - J_d v_\ell)\|_{L^2(\Omega)} \lesssim \min_{v\in V} \|D_{\mathrm{NC}}(v_\ell - v)\|_{L^2(\Omega)}. \tag{3.7}$$

The stability property of J_d follows with an inverse estimate [9]

$$\|D_{\mathrm{NC}}(v_\ell - J_d v_\ell)\|_{L^2(\Omega)} \lesssim \|h_\ell^{-1}(v_\ell - J_d v_\ell)\|_{L^2(\Omega)} \lesssim \min_{v\in V} \|D_{\mathrm{NC}}(v_\ell - v)\|_{L^2(\Omega)}.$$

Step 3. On any simplex $T = \operatorname{conv}\{z_1, \dots, z_{d+1}\}$ with nodal basis functions $\varphi_1, \dots, \varphi_{d+1}$, the volume bubble function is defined by

$$\flat_T := \frac{(2d+1)!}{d!} \prod_{j=1}^{d+1} \varphi_j \in H_0^1(\operatorname{int}(T)) \tag{3.8}$$

and satisfies $⨍_T \flat_T \, dx = 1$. Define

$$J_{d+1} v_\ell := J_d v_\ell + \sum_{T\in\mathcal{T}_\ell} \Big(⨍_T (v_\ell - J_d v_\ell)\, dx \Big) \flat_T.$$

The difference $v_\ell - J_{d+1} v_\ell$ is L^2-orthogonal to all piecewise constant functions. Since $\flat_T$ vanishes on all $F \in \mathcal{F}_\ell$, J_{d+1} enjoys the integral mean property $\Pi_\ell^0 DJ_{d+1} = D_{\mathrm{NC}}$. The Hölder inequality and (3.7) imply

$$\Big| ⨍_T (v_\ell - J_d v_\ell)\, dx \Big| \lesssim h_T^{-d/2} \|v_\ell - J_d v_\ell\|_{L^2(T)} \lesssim h_T^{-(d-2)/2} \min_{v\in V} \|D_{\mathrm{NC}}(v_\ell - v)\|_{L^2(\Omega)}.$$

The scaling $\|D\flat_T\|_{L^2(\Omega)} \approx h_T^{(d-2)/2}$ and the triangle inequality prove the stability property

$$\|D_{\mathrm{NC}}(v_\ell - J_{d+1} v_\ell)\|_{L^2(\Omega)} \lesssim \min_{v\in V} \|D_{\mathrm{NC}}(v_\ell - v)\|_{L^2(\Omega)}.$$

A piecewise Poincaré inequality proves the approximation property

$$\|h_\ell^{-1}(v_\ell - J_{d+1} v_\ell)\|_{L^2(\Omega)} \lesssim \min_{v\in V} \|D_{\mathrm{NC}}(v_\ell - v)\|_{L^2(\Omega)}. \qquad \square$$

4 Eigenvalues of the Laplacian

This section studies the adaptive nonconforming FEM approximation of the Laplace eigenproblem. Subsection 4.1 presents L^2 and best-approximation estimates for the linear Poisson problem. Subsection 4.2 introduces the discretisation of the eigenvalue problem. A 'theoretical' (i.e., non-computable) error estimator and its discrete reliability are analysed in Subsection 4.3. Subsections 4.4–4.5 present the practical AFEM and prove contraction and optimal convergence rates.

4.1 Nonconforming FEM for the Poisson Model Problem

This subsection revisits the nonconforming $\mathcal{P}_1$ discretisation of the linear Poisson equation. Let $V := H^1_0(\Omega)$ be equipped with the scalar products

$$a(v,w) := (Dv, Dw)_{L^2(\Omega)} \quad \text{and} \quad b(v,w) := (v,w)_{L^2(\Omega)}$$

and induced norms $|||v||| := a(v,v)^{1/2}$ and $\|v\| := b(v,v)^{1/2}$. Given $f \in L^2(\Omega)$, the weak formulation of the Poisson problem $-\Delta u = f$ under homogeneous Dirichlet boundary conditions reads as

$$a(u,v) = b(f,v) \quad \text{for all } v \in V. \tag{4.1}$$

The nonconforming finite element discretisation is based on the space $V_\ell := \mathfrak{CR}^1_0(\mathcal{T}_\ell)$ and the scalar product

$$a_{\mathrm{NC}}(v_\ell, w_\ell) := (D_{\mathrm{NC}} v_\ell, D_{\mathrm{NC}} w_\ell)_{L^2(\Omega)} \quad \text{for all } (v_\ell, w_\ell) \in V_\ell^2$$

with norm $|||\cdot|||_{\mathrm{NC}} := a_{\mathrm{NC}}(\cdot,\cdot)$ and seeks $u_\ell \equiv R_\ell u \in V_\ell$ such that

$$a_{\mathrm{NC}}(u_\ell, v_\ell) = b(f, v_\ell) \quad \text{for all } v_\ell \in V_\ell. \tag{4.2}$$

A posteriori and a priori error estimates as well as best-approximation properties for this problem are well-studied in the literature [6, 27, 37, 17]. Error estimates in the L^2 norm require a modification of the usual duality argument for conforming finite element methods. The following proposition establishes an L^2 error estimate. The main ingredient is the use of the companion operator J_{d+1}. For $d = 2$, this result was first obtained by [14] and [15]. A similar approach has independently been developed by [41] for $d = 2$. The result presented here compares the L^2 error directly with the energy error and therefore uses no a priori results of the eigenfunction approximation. This is important as the L^2 control will usually lead to higher-order terms which can be absorbed for $\|h_0\|_\infty \ll 1$.

Let $0 < s \le 1$ indicate the elliptic regularity index of the Poisson problem $-\Delta u = f$ with homogeneous Dirichlet boundary conditions in the sense that $\|u\|_{H^{1+s}(\Omega)} \le C(s)\|f\|_{L^2(\Omega)}$.

Proposition 4.1 (L^2 error estimate for the linear problem)**.** *The exact solution u to* (4.1) *and the discrete solution u_ℓ to* (4.2) *satisfy*

$$\|u - u_\ell\| \lesssim \|h_0\|_\infty^s |||u - u_\ell|||_{\mathrm{NC}}.$$

Proof. Let $e := u - u_\ell$ and let $z \in V$ denote the solution of

$$a(z,v) = b(e,v) \quad \text{for all } v \in V.$$

Recall the companion operator J_{d+1} from Proposition 3.1. Since $\Pi^0_\ell(u_\ell - J_{d+1}u_\ell) = 0$, it holds that

$$\begin{aligned} \|e\|^2 &= b(J_{d+1}u_\ell - u_\ell, e) + b(e, u - J_{d+1}u_\ell) \\ &= b(J_{d+1}u_\ell - u_\ell, (1 - \Pi^0_\ell)e) + a(z, u - J_{d+1}u_\ell). \end{aligned} \tag{4.3}$$

Piecewise Poincaré inequalities and (3.4) lead to

$$b(J_{d+1}u_\ell - u_\ell, (1-\Pi_\ell^0)e) \lesssim \|h_0\|_\infty^2 \|\!|e|\!\|_{\mathrm{NC}}^2.$$

Since e is perpendicular to the conforming finite element functions in $\mathcal{P}_1(\mathcal{T}) \cap V$ and since $\Pi_\ell^0 D_{\mathrm{NC}}(u_\ell - J_{d+1}u_\ell) = 0$, the Scott-Zhang quasi-interpolation $z_C \in \mathcal{P}_1(\mathcal{T}) \cap V$ of z [45] satisfies

$$\begin{aligned} a(z, u - J_{d+1}u_\ell) &= a_{\mathrm{NC}}(e, z) + a_{\mathrm{NC}}(u_\ell - J_{d+1}u_\ell, z) \\ &= a_{\mathrm{NC}}(e, z - z_C) + a_{\mathrm{NC}}(u_\ell - J_{d+1}u_\ell, z - z_C). \end{aligned}$$

The Cauchy inequality and (3.4) imply

$$a_{\mathrm{NC}}(e, z - z_C) + a_{\mathrm{NC}}(u_\ell - J_{d+1}u_\ell, z - z_C) \lesssim \|\!|e|\!\|_{\mathrm{NC}} \|\!|z - z_C|\!\|_{\mathrm{NC}}.$$

Standard a priori error estimates [9] and the elliptic regularity imply

$$\|\!|z - z_C|\!\| \lesssim \|h_0\|_\infty^s \|z\|_{H^{1+s}(\Omega)} \lesssim \|h_0\|_\infty^s \|e\|.$$

The combination of the above estimates proves

$$\|e\| \lesssim \|h_0\|_\infty^s \|\!|e|\!\|_{\mathrm{NC}}. \qquad \square$$

The next result states a best-approximation property in any space dimension. It generalises some recent results of the medius analysis [7, 37, 17] to arbitrary space dimensions. The result is stated with a refined oscillation term $\mathrm{osc}_1(f, \mathcal{T}_\ell)$. This will be important for the analysis of eigenvalue problems.

Proposition 4.2 (best-approximation property)**.** *The solution $u \in V$ to* (4.1) *with right-hand side $f \in L^2(\Omega)$ and the discrete solution $u_\ell \in V_\ell$ to* (4.2) *satisfy*

$$\|\!|u - u_\ell|\!\|_{\mathrm{NC}} \lesssim \|(1-\Pi_\ell^0)Du\| + \mathrm{osc}_1(f, \mathcal{T}_\ell).$$

Proof. The projection property (3.1) of the nonconforming interpolation operator $\mathcal{I}_\ell^{\mathfrak{CR}}$ and the Pythagoras theorem show that

$$\|\!|u - u_\ell|\!\|_{\mathrm{NC}}^2 = \|\!|u_\ell - \mathcal{I}_\ell^{\mathfrak{CR}} u|\!\|_{\mathrm{NC}}^2 + \|\!|u - \mathcal{I}_\ell^{\mathfrak{CR}} u|\!\|_{\mathrm{NC}}^2.$$

Since $\|\!|u - \mathcal{I}_\ell^{\mathfrak{CR}} u|\!\|_{\mathrm{NC}} = \|(1-\Pi_\ell^0)Du\|$, it remains to estimate the first term on the right-hand side. Set $\varphi_\ell := u_\ell - \mathcal{I}_\ell^{\mathfrak{CR}} u$. The properties of the companion operator from Proposition 3.1 show that

$$\begin{aligned} \|\!|u_\ell - \mathcal{I}_\ell^{\mathfrak{CR}} u|\!\|_{\mathrm{NC}}^2 &= a_{\mathrm{NC}}(u_\ell - u, \varphi_\ell) \\ &= b(f, \varphi_\ell - J_{d+1}\varphi_\ell) + ((1-\Pi_\ell^0)Du, D_{\mathrm{NC}}(J_{d+1} - 1)\varphi_\ell)_{L^2(\Omega)}. \end{aligned}$$

The approximation and stability properties (3.4) show that this is bounded by

$$(\|h_\ell f\| + \|(1-\Pi_\ell^0)Du\|) \|\!|\varphi_\ell|\!\|_{\mathrm{NC}}.$$

The efficiency $\|h_\ell f\| \lesssim \|(1-\Pi_\ell^0)Du\| + \mathrm{osc}_1(f, \mathcal{T}_\ell)$ in the spirit of [49] follows from arguments similar to those of [33, Prop. 3.1]. This concludes the proof. $\square$

4.2 Discretisation of the Laplace Eigenvalue Problem

The Laplace eigenvalue problem seeks eigenpairs $(\lambda, u) \in \mathbb{R} \times V$ with $\|u\| = 1$ such that

$$a(u, v) = \lambda b(u, v) \quad \text{for all } v \in V. \tag{4.4}$$

The finite element discretisation based on a regular triangulation $\mathcal{T}_\ell$ seeks discrete eigenpairs $(\lambda_\ell, u_\ell) \in \mathbb{R} \times V_\ell$ with $\|u_\ell\| = 1$ and

$$a_{\mathrm{NC}}(u_\ell, v_\ell) = \lambda_\ell b(u_\ell, v_\ell) \quad \text{for all } v_\ell \in V_\ell. \tag{4.5}$$

Adopt the notation of Section 2 with exact and discrete eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \quad \text{and} \quad 0 < \lambda_{\ell,1} \leq \dots \leq \lambda_{\ell,\dim(V_\ell)}$$

and their corresponding b-orthonormal systems of eigenfunctions

$$(u_1, u_2, u_3, \dots) \quad \text{and} \quad (u_{\ell,1}, u_{\ell,2}, \dots, u_{\ell,\dim(V_\ell)}).$$

Recall the definitions of Section 2: The set $J = \{n+1, \dots, n+N\}$ describes the eigenvalue cluster of interest and $W := \mathrm{span}\{u_j \mid j \in J\}$ and $W_\ell := \mathrm{span}\{u_{\ell,j} \mid j \in J\}$ are the exact and discrete invariant subspaces (not necessarily eigenspaces) related to the cluster. In the present situation, the quasi-Ritz projection R_ℓ maps the solution $u \in V$ of the linear problem (4.1) to the solution $R_\ell u$ of the discrete linear problem (4.2). With the L^2 projection $P_{\mathcal{T}_\ell} := P_\ell$ onto W_ℓ let $\Lambda_{\mathcal{T}_\ell} := \Lambda_\ell := P_\ell \circ R_\ell$.

The remaining parts of this subsection prove an L^2 error estimate as well as a best-approximation result.

Proposition 4.3 (L^2 error control)**.** *Provided $\|h_0\|_\infty \ll 1$, any eigenpair $(\lambda, u) \in \mathbb{R} \times W$ with $\|u\| = 1$ satisfies for some constant C_{L^2} and the separation constant M_J from* (H1) *(Section 2) that*

$$\|u - P_\ell u\| \leq \|u - \Lambda_\ell u\| \lesssim (1 + M_J)\|u - R_\ell u\| \leq C_{L^2}(1 + M_J)\|h_0\|_\infty^s \| u - \Lambda_\ell u \|_{\mathrm{NC}}.$$

Proof. Note that $R_\ell u$ solves (4.2) with right-hand side $f := \lambda u$. The combination of Proposition 2.1 with Proposition 4.1 and Proposition 4.2 yields

$$\|u - P_\ell u\| \leq \|u - \Lambda_\ell u\| \lesssim (1 + M_J)\|h_0\|_\infty^s (\| u - \Lambda_\ell u \|_{\mathrm{NC}} + \mathrm{osc}_1(\lambda u, \mathcal{T}_\ell)).$$

Provided $\|h_0\|_\infty \ll 1$, the oscillation term can be absorbed. □

Proposition 4.4 (best-approximation property)**.** *Provided $\|h_0\|_\infty \ll 1$, any eigenpair $(\lambda, u) \in \mathbb{R} \times W$ of* (4.4) *with $\|u\| = 1$ satisfies*

$$\| u - \Lambda_\ell u \|_{\mathrm{NC}} \lesssim \|(1 - \Pi_\ell^0) D u\|.$$

Proof. The triangle inequality proves for the quasi-Ritz projection $R_\ell u$ that

$$\| u - \Lambda_\ell u \|_{\mathrm{NC}} \leq \| u - R_\ell u \|_{\mathrm{NC}} + \| R_\ell u - \Lambda_\ell u \|_{\mathrm{NC}}.$$

Set $\varphi_\ell := R_\ell u - \Lambda_\ell u$. The definition of R_ℓ and the discrete problem (cf. Lemma 2.2) prove that

$$\| R_\ell u - \Lambda_\ell u \|_{\mathrm{NC}}^2 = a_{\mathrm{NC}}(R_\ell u - \Lambda_\ell u, \varphi_\ell) = \lambda b(u - P_\ell u, \varphi_\ell).$$

Hence, the Cauchy and discrete Friedrichs inequalities [9, Thm. 10.6.12] and the L^2 control from Proposition 4.3 prove that

$$|||R_\ell u - \Lambda_\ell u|||_{\mathrm{NC}} \lesssim \lambda(1+M_J)\|h_0\|_\infty^s |||u-\Lambda_\ell u|||_{\mathrm{NC}}.$$

The combination of the foregoing estimates with Proposition 4.2 results in

$$|||u-\Lambda_\ell u|||_{\mathrm{NC}} \lesssim \|(1-\Pi_\ell^0)Du\| + \lambda(1+M_J)\|h_0\|_\infty^s |||u-\Lambda_\ell u|||_{\mathrm{NC}} + \mathrm{osc}_1(\lambda u, \mathcal{T}_\ell).$$

If $\|h_0\|_\infty \ll 1$ is sufficiently small, the higher-order terms on the right-hand side can be absorbed. □

4.3 Theoretical Error Estimator and Discrete Reliability

The analysis relies on a theoretical, non-computable error estimator that does not depend on the choice of the discrete eigenfunctions. This idea was first presented in [25]. Given an eigenpair (λ, u), the error estimator includes the elementwise residuals in terms of $P_\ell u$ and $\Lambda_\ell u$. More precisely, define, for any $T \in \mathcal{T}_\ell$,

$$\mu_\ell^2(T,\lambda,u) := h_T^2\|\lambda P_\ell u\|_{L^2(T)}^2 + \sum_{F\in\mathcal{F}(T)} h_T^{-1}\|[\Lambda_\ell u]_F\|_{L^2(F)}^2$$

and, for any subset $\mathcal{K} \subseteq \mathcal{T}_\ell$,

$$\mu_\ell^2(\mathcal{K},\lambda_j,u_j) := \sum_{T\in\mathcal{K}} \mu_\ell^2(T,\lambda_j,u_j) \quad \text{and} \quad \mu_\ell^2(\mathcal{K}) := \sum_{j\in J} \mu_\ell^2(\mathcal{K},\lambda_j,u_j).$$

The following shorthand notation for higher-order terms will be frequently used in the remaining parts of this section. For $(\ell, m) \in \mathbb{N}_0^2$ define (with the constant C_{L^2} from Proposition 4.3)

$$\boldsymbol{r}_{\ell,m} := \|h_0\|_\infty^s \lambda(1+M_J)C_{L^2}\sqrt{|||u-\Lambda_\ell u|||^2 + |||u-\Lambda_{\ell+m}u|||^2}. \tag{4.6}$$

The theoretical error estimator satisfies the following discrete reliability.

Proposition 4.5 (discrete reliability)**.** *There exists a constant $C_{\mathrm{drel}} \approx 1$ solely dependent on $\mathcal{T}_0$ with $\|h_0\|_\infty \ll 1$ such that any eigenpair $(\lambda, u) \in \mathbb{R}\times W$ of (4.4) with $\|u\| = 1$ satisfies*

$$2|||\Lambda_{\ell+m}u - \Lambda_\ell u|||^2 \le C_{\mathrm{drel}}^2\big(\mu_\ell^2(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m},\lambda,u) + \boldsymbol{r}_{\ell,m}^2\big).$$

Proof. Let $v_{\ell+m}$ denote the best-approximation (with respect to the norm $|||\cdot|||_{\mathrm{NC}}$) of $\Lambda_\ell u$ in $V_{\ell+m}$. The Pythagoras theorem reads as

$$|||(\Lambda_{\ell+m}-\Lambda_\ell)u|||_{\mathrm{NC}}^2 = |||\Lambda_{\ell+m}u - v_{\ell+m}|||_{\mathrm{NC}}^2 + \min_{w_{\ell+m}\in V_{\ell+m}} |||w_{\ell+m}-\Lambda_\ell u|||_{\mathrm{NC}}^2.$$

The second term has been estimated in [13, Thm. 3.1] by means of the jumps of $\Lambda_\ell u$. For the analysis of the first term, let $\varphi_{\ell+m} := \Lambda_{\ell+m}u - v_{\ell+m}$. The projection property (3.1) of the nonconforming interpolation and the discrete eigenvalue problems (cf. Lemma 2.2) reveal that

$$\begin{aligned} |||\Lambda_{\ell+m}u - v_{\ell+m}|||_{\mathrm{NC}}^2 &= a_{\mathrm{NC}}((\Lambda_{\ell+m}-\Lambda_\ell)u, \varphi_{\ell+m}) \\ &= \lambda b((P_{\ell+m}-P_\ell)u, \varphi_{\ell+m}) + \lambda b(P_\ell u, (1-\mathcal{I}_\ell^{\mathfrak{CR}})\varphi_{\ell+m}). \end{aligned}$$

The L^2 error estimate from Proposition 4.3 and the approximation and stability property (3.2) conclude the proof. □

The reliability of the error estimator is an immediate consequence.

Proposition 4.6 (reliability and efficiency). *Provided* $\|h_0\|_\infty \ll 1$, *any eigenpair* $(\lambda, u) \in \mathbb{R} \times W$ *of* (4.4) *with* $\|u\| = 1$ *satisfies*

$$|||u - \Lambda_\ell u|||_{\mathrm{NC}}^2 \le C_{\mathrm{drel}}^2 \mu_\ell^2(\mathcal{T}_\ell, \lambda, u). \tag{4.7}$$

For some constant $C_{\mathrm{eff}} \approx 1$, *it holds that*

$$\mu_\ell(\mathcal{T}_\ell, \lambda, u)^2 \le C_{\mathrm{eff}}^2 |||u - \Lambda_\ell u|||_{\mathrm{NC}}^2. \tag{4.8}$$

Proof. The reliability

$$2|||u - \Lambda_\ell u|||_{\mathrm{NC}}^2 \le C_{\mathrm{drel}}^2 \left(\mu_\ell^2(\mathcal{T}_\ell, \lambda, u) + \|h_0\|_\infty^{2s} \lambda^2 (1 + M_J)^2 |||u - \Lambda_\ell u|||_{\mathrm{NC}}^2\right)$$

follows from the discrete reliability on a sequence of meshes $\mathcal{T}_{\ell+m}$ with $\|h_{\ell+m}\|_\infty \to 0$ and the a priori convergence result of Proposition 4.4. Provided the initial mesh is sufficiently fine, the higher-order terms on the right-hand side can be absorbed. The efficiency

$$2\mu_\ell^2(\mathcal{T}_\ell, \lambda, u) \le C_{\mathrm{eff}}^2 \left(1 + \lambda \|h_0\|_\infty^{1+s}(1 + M_J) C_{L^2}\right)^2 |||u - \Lambda_\ell u|||_{\mathrm{NC}}^2$$

follows from the triangle inequality and the L^2 error control from Proposition 4.3 combined with the standard arguments of [49]. The assumption $\|h_0\|_\infty \ll 1$ implies

$$\mu_\ell^2(\mathcal{T}_\ell, \lambda, u) \le C_{\mathrm{eff}}^2 |||u - \Lambda_\ell u|||_{\mathrm{NC}}^2.$$ □

4.4 Adaptive Algorithm and Contraction Property

This subsection presents the adaptive algorithm and proves the contraction property.

For any simplex $T \in \mathcal{T}_\ell$, the explicit residual-based error estimator consists of the sum of the residuals of the computed discrete eigenfunctions $(u_{\ell,j})_{j\in J}$,

$$\eta_\ell^2(T) := \sum_{j\in J} \left(h_T^2 \|\lambda_{\ell,j} u_{\ell,j}\|_{L^2(T)}^2 + \sum_{F\in\mathcal{F}(T)} h_T^{-1} \|[u_{\ell,j}]_F\|_{L^2(F)}^2 \right).$$

Let, for any subset $\mathcal{K} \subseteq \mathcal{T}$,

$$\eta_\ell^2(\mathcal{K}) := \sum_{T\in\mathcal{K}} \eta_\ell^2(T).$$

For simple eigenvalues this type of error estimator was introduced by [28]. The adaptive algorithm is driven by this computable error estimator and runs the following loop.

Algorithm 4.7 (nonconforming AFEM for the Laplace eigenproblem).
Input: Initial triangulation $\mathcal{T}_0$, bulk parameter $0 < \theta \le 1$.
for $\ell = 0, 1, 2, \dots$
Solve. Compute discrete eigenpairs $(\lambda_{\ell,j}, u_{\ell,j})_{j\in J}$ of (4.5) with respect to $\mathcal{T}_\ell$.
Estimate. Compute local contributions of the error estimator $\left(\eta_\ell^2(T)\right)_{T\in\mathcal{T}_\ell}$.
Mark. Choose a minimal subset $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta\eta_\ell^2(\mathcal{T}_\ell) \le \eta_\ell^2(\mathcal{M}_\ell)$.
Refine. Generate $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ with the refinement rules of [47].
end for
Output: Triangulations $(\mathcal{T}_\ell)_\ell$ and discrete solutions $\left((\lambda_{\ell,j}, u_{\ell,j})_{j\in J}\right)_\ell$.

The first important observation is that, by Lemma 2.3, the non-computable error estimator $\mu_\ell(\mathcal{M}_\ell)$ satisfies the bulk criterion

$$\tilde{\theta}\mu_\ell(\mathcal{T}_\ell) \le \mu_\ell(\mathcal{M}_\ell)$$

for the modified bulk parameter

$$\tilde{\theta} := \left((B/A)^4(2N^2 + 4N^3)\right)^{-1} \theta < 1. \tag{4.9}$$

The following proposition states the error estimator reduction property.

Proposition 4.8 (error estimator reduction for μ_ℓ). *Provided the assumptions* (H1) *and* (H2) *(see Lemma 2.3) hold, there exist constants* $0 < \rho_1 < 1$ *and* $0 < K < \infty$ *such that* $\mathcal{T}_\ell$ *and its one-level refinement* $\mathcal{T}_{\ell+1}$ *generated by Algorithm 4.7 and any eigenfunction* $u \in W$ *with* $\|u\| = 1$ *and eigenvalue* λ *satisfy (with* $\boldsymbol{r}_{\ell,1}$ *from* (4.6)*) that*

$$\mu_{\ell+1}^2(\mathcal{T}_{\ell+1}, \lambda, u) \le \rho_1 \mu_\ell^2(\mathcal{T}_\ell, \lambda, u) + K\left(\|\!|\Lambda_{\ell+1}u - \Lambda_\ell u|\!\|_{\mathrm{NC}}^2 + \|h_0\|_\infty^2 \boldsymbol{r}_{\ell,1}^2\right).$$

Proof. The standard techniques of [22, 46] and the bulk criterion (4.9) lead to a constant $\tilde{K}$ such that

$$\begin{aligned}&\mu_{\ell+1}^2(\mathcal{T}_{\ell+1}, \lambda, u)\\ &\quad\le \rho_1 \mu_\ell^2(\mathcal{T}_\ell, \lambda, u) + \tilde{K}\left(\|\!|\Lambda_{\ell+1}u - \Lambda_\ell u|\!\|_{\mathrm{NC}}^2 + \|h_{\ell+1}\lambda(P_{\ell+1} - P_\ell)u\|^2\right).\end{aligned}$$

The triangle inequality for the term $\|h_{\ell+1}\lambda(P_{\ell+1} - P_\ell)u\|$ and the L^2 error control from Proposition 4.3 prove the result. □

The next technical result is needed for the reduction of the volume contribution of the error estimator. Inequalities of this type were previously utilised in [42] for $d = 2$ for the linear Poisson problem and in [13] for boundary value problems for $d \ge 2$.

Lemma 4.9 (control of the volume contribution). *Provided* $\|h_0\|_\infty \ll 1$*, any triangulation* $\mathcal{T}_\ell \in \mathbb{T}$ *and any admissible refinement* $\mathcal{T}_{\ell+m}$ *of* $\mathcal{T}_\ell$ *satisfy for any* $0 < \delta < \infty$ *and any eigenpair* $(\lambda, u) \in \mathbb{R} \times W$ *of* (4.4) *with* $\|u\| = 1$ *that*

$$\begin{aligned}&\|h_{\ell+m}\lambda P_{\ell+m}u\|_{L^2(\Omega)}^2 + (1 + \delta^{-1})(1 - 2^{-2/d})\|h_\ell \lambda P_\ell u\|_{L^2(\cup(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m}))}^2\\ &\qquad\qquad \le 2(1 + \delta)\|h_0\|_\infty^2 \boldsymbol{r}_{\ell,m}^2 + (1 + \delta^{-1})\|h_\ell \lambda P_\ell u\|_{L^2(\Omega)}^2.\end{aligned}$$

Proof. The triangle and Young inequalities prove for any $0 < \delta < \infty$ that

$$\begin{aligned}&\|h_{\ell+m}\lambda P_{\ell+m}u\|_{L^2(\Omega)}^2\\ &\quad \le (1 + \delta)\|h_{\ell+m}\lambda(P_{\ell+m}u - P_\ell u)\|_{L^2(\Omega)}^2 + (1 + \delta^{-1})\|h_{\ell+m}\lambda P_\ell u\|_{L^2(\Omega)}^2.\end{aligned}$$

The relation $h_{\ell+m}^d \le h_\ell^d/2$ on $\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m}$ proves

$$\|h_\ell \lambda P_\ell u\|_{L^2(\cup(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m}))}^2 \le (1 - 2^{-2/d})^{-1}(\|h_\ell \lambda P_\ell u\|_{L^2(\Omega)}^2 - \|h_{\ell+m}\lambda P_\ell u\|_{L^2(\Omega)}^2).$$

The preceding two displayed formulas together with Proposition 4.3 prove the result. □

In the case of nonconforming discretisations of eigenvalue problems, the Galerkin orthogonality is violated at two points. First, the nonlinearity leads to a perturbation of the right-hand side. Furthermore, the nonconforming finite element functions are not admissible test functions in the continuous problem and, thus, additional techniques enter the analysis. The notion of "quasi-orthogonality" traces back to [20].

Proposition 4.10 (quasi-orthogonality). *Under the hypothesis $\|h_0\|_\infty \ll 1$ there exists a constant $C_{\rm qo}$ such that any eigenpair $(\lambda, u) \in \mathbb{R} \times W$ of (4.4) with $\|u\| = 1$, any $\mathcal{T}_\ell \in \mathbb{T}$, and any admissible refinement $\mathcal{T}_{\ell+m}$ of $\mathcal{T}_\ell$ satisfy*

$$|2a_{\rm NC}(u - \Lambda_{\ell+m}u, \Lambda_{\ell+m}u - \Lambda_\ell u)| \\ \leq C_{\rm qo}\big(\|h_\ell \lambda P_\ell u\|_{L^2(\cup\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})} + \boldsymbol{r}_{\ell,m}\big)\|\!|u - \Lambda_{\ell+m}u|\!\|_{\rm NC}.$$

Proof. Some algebraic manipulations with the projection property (3.1) of the nonconforming interpolation and the discrete eigenvalue problems (cf. Lemma 2.2) reveal

$$\begin{aligned} &a_{\rm NC}((1-\Lambda_{\ell+m})u, (\Lambda_{\ell+m} - \Lambda_\ell)u) \\ &= a_{\rm NC}(\Lambda_{\ell+m}u, \mathcal{I}^{\mathfrak{CR}}_{\ell+m}(1-\Lambda_{\ell+m})u) - a_{\rm NC}(\Lambda_\ell u, \mathcal{I}^{\mathfrak{CR}}_{\ell}(1-\Lambda_{\ell+m})u) \\ &= \lambda b(P_{\ell+m}u, \mathcal{I}^{\mathfrak{CR}}_{\ell+m}(1-\Lambda_{\ell+m})u) - \lambda b(P_\ell u, \mathcal{I}^{\mathfrak{CR}}_{\ell}(1-\Lambda_{\ell+m})u) \\ &= \lambda b(P_\ell u, (\mathcal{I}^{\mathfrak{CR}}_{\ell+m} - \mathcal{I}^{\mathfrak{CR}}_{\ell})(1-\Lambda_{\ell+m})u) + \lambda b((P_{\ell+m} - P_\ell)u, \mathcal{I}^{\mathfrak{CR}}_{\ell+m}(1-\Lambda_{\ell+m})u). \end{aligned}$$

Since $\mathcal{I}^{\mathfrak{CR}}_{\ell+m}v|_T = \mathcal{I}^{\mathfrak{CR}}_{\ell}v|_T$ for all $T \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}$, the first term of the right-hand side can be controlled with (3.2) as

$$\lambda b(P_\ell u, (\mathcal{I}^{\mathfrak{CR}}_{\ell+m} - \mathcal{I}^{\mathfrak{CR}}_{\ell})(1-\Lambda_{\ell+m})u) \\ \lesssim \|h_\ell \lambda P_\ell u\|_{L^2(\cup\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})}\|D_{\rm NC}(1-\Lambda_{\ell+m})u\|_{L^2(\cup\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})}.$$

For the second term, the discrete Friedrichs inequality [9, Thm. 10.6.12] and the stability of $\mathcal{I}^{\mathfrak{CR}}_{\ell}$ reveal

$$\lambda b((P_{\ell+m} - P_\ell)u, \mathcal{I}^{\mathfrak{CR}}_{\ell+m}(1-\Lambda_{\ell+m})u) \lesssim \lambda\|(P_{\ell+m} - P_\ell)u\|\|\!|u - \Lambda_{\ell+m}u|\!\|_{\rm NC}.$$

The triangle inequality and Proposition 4.3 control the term $\lambda\|(P_{\ell+m} - P_\ell)u\|$ by $\boldsymbol{r}_{\ell,m}$ from (4.6). This concludes the proof. □

The following contraction property implies the convergence of the adaptive algorithm.

Proposition 4.11 (contraction property). *Under the condition $\|h_0\|_\infty \ll 1$, there exist $0 < \rho_2 < 1$ and $0 < \beta, \gamma < \infty$ such that, for any eigenpair $(\lambda, u) \in \mathbb{R} \times W$ with $\|u\| = 1$, the term $\xi_\ell^2 := \mu_\ell^2(\mathcal{T}_\ell, \lambda, u) + \beta\|\!|u - \Lambda_\ell u|\!\|^2_{\rm NC} + \gamma\|h_\ell P_\ell u\|^2$ satisfies*

$$\xi_{\ell+1}^2 \leq \rho_2 \xi_\ell^2 \quad \text{for all } \ell \in \mathbb{N}_0.$$

Proof. Throughout the proof, the following shorthand notation applies

$$\begin{aligned} \boldsymbol{e}_\ell &:= \|\!|u - \Lambda_\ell u|\!\|_{\rm NC}, & \boldsymbol{e}_{\ell+1} &:= \|\!|u - \Lambda_{\ell+1} u|\!\|_{\rm NC}, \\ \mu_\ell^2 &:= \mu_\ell^2(\mathcal{T}_\ell, \lambda, u), & \mu_{\ell+1}^2 &:= \mu_{\ell+1}^2(\mathcal{T}_{\ell+1}, \lambda, u). \end{aligned}$$

The error estimator reduction from Proposition 4.8 and elementary algebraic manipulations plus the quasi-orthogonality (Proposition 4.10) lead to

$$\begin{aligned} &\mu_{\ell+1}^2 + K\boldsymbol{e}_{\ell+1}^2 \\ &\quad\leq \rho_1\mu_\ell^2 + K\Big(\boldsymbol{e}_\ell^2 + 2a(u - \Lambda_{\ell+1}u, (\Lambda_\ell - \Lambda_{\ell+1})u) + \|h_0\|_\infty^2 \boldsymbol{r}_{\ell,1}^2\Big) \\ &\quad\leq \rho_1\mu_\ell^2 + K\Big(\boldsymbol{e}_\ell^2 + C_{\rm qo}\big(\|h_\ell\lambda P_\ell u\|_{L^2(\cup\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1})} + \boldsymbol{r}_{\ell,1}\big)\boldsymbol{e}_{\ell+1} + \|h_0\|_\infty^2 \boldsymbol{r}_{\ell,1}^2\Big). \end{aligned}$$

This and the Young inequality for any $0 < \varepsilon < 1$ lead to

$$\begin{aligned}&\mu_{\ell+1}^2 + K(1 - C_{\mathrm{qo}}\varepsilon/2)\boldsymbol{e}_{\ell+1}^2\\&\quad\le \rho_1\mu_\ell^2 + K\Big(\boldsymbol{e}_\ell^2 + C_{\mathrm{qo}}/\varepsilon\big(\|h_\ell\lambda P_\ell u\|^2_{L^2(\cup\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})} + \boldsymbol{r}_{\ell,1}^2\big) + \|h_0\|_\infty^2\boldsymbol{r}_{\ell,1}^2\Big).\end{aligned}$$

The reliability (4.7) proves for any $0 < \zeta < \infty$ that this is bounded by

$$\begin{aligned}&(\rho_1 + K\zeta C_{\mathrm{drel}}^2)\mu_\ell^2\\&\quad + K\Big((1-\zeta)\boldsymbol{e}_\ell^2 + C_{\mathrm{qo}}/\varepsilon(\|h_\ell\lambda P_\ell u\|^2_{L^2(\cup\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})} + \boldsymbol{r}_{\ell,1}^2) + \|h_0\|_\infty^2\boldsymbol{r}_{\ell,1}^2\Big).\end{aligned}$$

Lemma 4.9 states for any $0 < \delta < \infty$ and $c_d := (1 - 2^{-2/d})$ that

$$\|h_\ell\lambda P_\ell u\|^2_{L^2(\cup(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1}))} \le \frac{2\delta\|h_0\|_\infty^2\boldsymbol{r}_{\ell,1}^2}{c_d} + \frac{\|h_\ell\lambda P_\ell u\|^2}{c_d} - \frac{\|h_{\ell+1}\lambda P_{\ell+1}u\|^2}{(1+\delta^{-1})c_d}.$$

Altogether,

$$\begin{aligned}&\mu_{\ell+1}^2 + K\left((1 - C_{\mathrm{qo}}\varepsilon/2)\boldsymbol{e}_{\ell+1}^2 + \frac{C_{\mathrm{qo}}\|h_{\ell+1}\lambda P_{\ell+1}u\|^2}{\varepsilon(1+\delta^{-1})c_d}\right)\\&\quad\le (\rho_1 + K\zeta C_{\mathrm{drel}}^2)\mu_\ell^2 + K\Bigg((1-\zeta)\boldsymbol{e}_\ell^2\\&\qquad\qquad + \Big(\varepsilon^{-1}C_{\mathrm{qo}}(1 + 2\delta\|h_0\|_\infty^2/c_d) + \|h_0\|_\infty^2\Big)\boldsymbol{r}_{\ell,1}^2 + \frac{C_{\mathrm{qo}}\|h_\ell\lambda P_\ell u\|^2}{\varepsilon c_d}\Bigg).\end{aligned}$$

Define

$$t(h_0,\varepsilon,\delta) := C_{\mathrm{drel}}^2\|h_0\|_\infty^{2s}\lambda^2(1+M_J)^2C_{L^2}^2K\big(\varepsilon^{-1}C_{\mathrm{qo}}(1 + \frac{2\delta\|h_0\|_\infty^2}{c_d}) + \|h_0\|_\infty^2\big).$$

Recall the definition (4.6) of $\boldsymbol{r}_{\ell,1}$. The reliability (4.7) implies

$$K\Big(\varepsilon^{-1}C_{\mathrm{qo}}(1 + 2\delta\|h_0\|_\infty^2/c_d) + \|h_0\|_\infty^2\Big)\boldsymbol{r}_{\ell,1}^2 \le t(h_0,\varepsilon,\delta)(\mu_\ell^2 + \mu_{\ell+1}^2).$$

This and the fact that $\|h_\ell\lambda P_\ell u\|^2 \le \mu_\ell^2$ together with the foregoing estimates prove

$$\begin{aligned}&(1 - t(h_0,\varepsilon,\delta))\mu_{\ell+1}^2 + K\left((1 - C_{\mathrm{qo}}\varepsilon/2)\boldsymbol{e}_{\ell+1}^2 + \frac{C_{\mathrm{qo}}\|h_{\ell+1}\lambda P_{\ell+1}u\|^2}{\varepsilon(1+\delta^{-1})c_d}\right)\\&\quad\le (\rho_1 + K\zeta C_{\mathrm{drel}}^2 + t(h_0,\varepsilon,\delta) + K\varepsilon)\mu_\ell^2\\&\qquad\qquad + K\Big((1-\zeta)\boldsymbol{e}_\ell^2 + \Big(\frac{C_{\mathrm{qo}}}{\varepsilon c_d} - \varepsilon\Big)\|h_\ell\lambda P_\ell u\|^2\Big).\end{aligned}$$

Hence, for

$$\beta := \frac{K(1 - C_{\mathrm{qo}}\varepsilon/2)}{1 - t(h_0,\varepsilon,\delta)},\qquad \gamma := \frac{KC_{\mathrm{qo}}}{\varepsilon(1+\delta^{-1})c_d(1 - t(h_0,\varepsilon,\delta))},$$

and

$$\begin{aligned}\rho_2 := \max\Bigg\{&\frac{\rho_1 + K\zeta C_{\mathrm{drel}}^2 + t(h_0,\varepsilon,\delta) + K\varepsilon}{1 - t(h_0,\varepsilon,\delta)},\\&\frac{1-\zeta}{1 - C_{\mathrm{qo}}\varepsilon/2},\quad (1+\delta^{-1})(C_{\mathrm{qo}} - \varepsilon^2 c_d)/C_{\mathrm{qo}}\Bigg\},\end{aligned}$$

it follows that

$$\mu_{\ell+1} + \beta\boldsymbol{e}_{\ell+1}^2 + \gamma\|h_{\ell+1}\lambda P_{\ell+1}u\|^2 \le \rho_2(\mu_\ell + \beta\boldsymbol{e}_\ell^2 + \gamma\|h_\ell\lambda P_\ell u\|^2).$$

Choose $\delta := C_{\mathrm{qo}}/(\varepsilon^2 c_d)$ and $\varepsilon < 2\zeta C_{\mathrm{qo}}^{-1}$. The choice of sufficiently small ζ, ε and $\|h_0\|_\infty$ yields $\rho_2 < 1$. □

4.5 Optimal Convergence Rates

Let, for any $m \in \mathbb{N}$, the set of triangulations in $\mathbb{T}$ whose cardinality differs from that of $\mathcal{T}_0$ by m or less be denoted by

$$\mathbb{T}(m) := \{\mathcal{T} \in \mathbb{T} \mid \operatorname{card}(\mathcal{T}) - \operatorname{card}(\mathcal{T}_0) \le m\}.$$

Define the seminorm

$$|u|_{\mathcal{A}_\sigma} := \sup_{m\in\mathbb{N}} m^\sigma \inf_{\mathcal{T}\in\mathbb{T}(m)} \|(1-\Pi^0_{\mathcal{T}})Du\|$$

and the approximation class

$$\mathfrak{A}_\sigma := \left\{ v \in V \;\middle|\; |v|_{\mathcal{A}_\sigma} < \infty \right\}.$$

Define the following alternative set, also referred to as approximation class

$$\mathfrak{A}_\sigma^{\mathrm{NC},\Delta} := \left\{ u \in V \;\middle|\; |u|_{\mathfrak{A}_\sigma^{\mathrm{NC},\Delta}} < \infty \right\}$$

for

$$|u|_{\mathfrak{A}_\sigma^{\mathrm{NC},\Delta}} := \sup_{m\in\mathbb{N}} m^\sigma \inf_{\mathcal{T}\in\mathbb{T}(m)} \| u - \Lambda_{\mathcal{T}} u \|_{\mathrm{NC}}$$

for the eigenfunction approximation $\Lambda_{\mathcal{T}} u$ with respect to a triangulation $\mathcal{T}$. Proposition 4.4 proves that these two approximation classes are equivalent in the sense that any eigenfunction $u \in W$ belongs to $\mathfrak{A}_\sigma$ if and only if it belongs to $\mathfrak{A}_\sigma^{\mathrm{NC},\Delta}$. The following theorem states optimality of Algorithm 4.7. The proof follows in the remaining parts of this section.

Theorem 4.12 (optimal convergence rates)**.** *Provided the bulk parameter $\theta \ll 1$ and the initial mesh-size $\|h_0\|_\infty \ll 1$ are sufficiently small, Algorithm 4.7 computes sequences of triangulations $(\mathcal{T}_\ell)_\ell$ and discrete eigenpairs $((\lambda_{\ell,j}, u_{\ell,j})_{j\in J})_\ell$ with optimal rate of convergence in the sense that, for some constant C_{opt},*

$$\sup_{\ell\in\mathbb{N}} \big(\operatorname{card}(\mathcal{T}_\ell) - \operatorname{card}(\mathcal{T}_0)\big)^{2\sigma} \sum_{j\in J} \| u_j - \Lambda_\ell u_j \|^2_{\mathrm{NC}} \le C_{\mathrm{opt}} \sum_{j\in J} |u_j|^2_{\mathfrak{A}_\sigma^{\mathrm{NC},\Delta}}.$$

Proposition 4.4 implies the following immediate consequence.

Corollary 4.13. *Provided the bulk parameter $\theta \ll 1$ and the initial mesh-size $\|h_0\|_\infty \ll 1$ are sufficiently small, Algorithm 4.7 computes triangulations $(\mathcal{T}_\ell)_\ell$ and discrete eigenpairs $((\lambda_{\ell,j}, u_{\ell,j})_{j\in J})_\ell$ with optimal rate of convergence in the sense that*

$$\sup_{\ell\in\mathbb{N}} (\operatorname{card}(\mathcal{T}_\ell) - \operatorname{card}(\mathcal{T}_0))^\sigma \sup_{\substack{w\in W\\ \|w\|=1}} \inf_{v_\ell\in W_\ell} \| w - v_\ell \|_{\mathrm{NC}} \lesssim \left(\sum_{j\in J} |u_j|^2_{\mathfrak{A}_\sigma} \right)^{1/2}. \qquad \square$$

The remaining part of this subsection is devoted to the proof of Theorem 4.12 which follows the methodology of [46, 22] as in [33]. The optimality proof of this section is concerned with the simultaneous error of all eigenfunction approximations. Consider

$$\Xi_\ell^2 := \mu_\ell^2(\mathcal{T}_\ell) + \beta \sum_{j\in J} \| u_j - \Lambda_\ell u_j \|^2_{\mathrm{NC}} + \gamma \sum_{j\in J} \| h_\ell \lambda_j P_\ell u_j \|^2 \quad \text{for all } \ell \in \mathbb{N}_0$$

for the parameters β and γ from Proposition 4.11. The proof excludes the pathological case $\Xi_0 = 0$. Choose $0 < \tau \le \sum_{j\in J} |u_j|^2_{\mathfrak{A}_\sigma^{\mathrm{NC},\Delta}} / \Xi_0^2$, and set $\varepsilon(\ell) := \sqrt{\tau}\, \Xi_\ell$. Let $N(\ell) \in \mathbb{N}$ be minimal with the property

$$\sum_{j\in J} |u_j|^2_{\mathfrak{A}_\sigma^{\mathrm{NC},\Delta}} \le \varepsilon(\ell)^2\, N(\ell)^{2\sigma}.$$

Let for a fixed $\ell \in \mathbb{N}$, $\widetilde{\mathcal{T}}_\ell \in \mathbb{T}$ denote the optimal triangulation of cardinality

$$\operatorname{card}(\widetilde{\mathcal{T}}_\ell) \le \operatorname{card}(\mathcal{T}_0) + N(\ell)$$

in the sense that the projection $\widetilde{\Lambda} := \Lambda_{\widetilde{\mathcal{T}}_\ell}$ with respect to $\widetilde{\mathcal{T}}_\ell$ satisfies

$$\sum_{j\in J} \|u_j - \widetilde{\Lambda} u_j\|_{\mathrm{NC}}^2 \le N(\ell)^{-2\sigma} \sum_{j\in J} |u_j|^2_{\mathfrak{A}_\sigma^{\mathrm{NC},\Delta}} \le \varepsilon(\ell)^2 \tag{4.10}$$

and define $\widehat{\mathcal{T}}_\ell := \mathcal{T}_\ell \otimes \widetilde{\mathcal{T}}_\ell$ as the overlay [22], that is the smallest common refinement of $\mathcal{T}_\ell$ and $\widetilde{\mathcal{T}}_\ell$. The arguments of [22, 33] lead to

$$\operatorname{card}(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell) \le N(\ell) \le 2 \Big(\sum_{j\in J} |u_j|^2_{\mathfrak{A}_\sigma^{\mathrm{NC},\Delta}} \Big)^{1/(2\sigma)} \varepsilon(\ell)^{-1/\sigma}. \tag{4.11}$$

Let $\widehat{\Lambda} := \Lambda_{\widehat{\mathcal{T}}_\ell}$ denote the projection with respect to $\widehat{\mathcal{T}}_\ell$.

Lemma 4.14. *Provided $\|h_0\|_\infty \ll 1$, it holds that*

$$\sum_{j\in J} \|u_j - \widehat{\Lambda} u_j\|_{\mathrm{NC}}^2 \lesssim \varepsilon(\ell)^2.$$

Proof. Recall that by definition of the overlay [22] the triangulations $\widehat{\mathcal{T}}_\ell$ and $\widetilde{\mathcal{T}}_\ell$ are nested. Hence, the best-approximation result of Proposition 4.4 and (4.10) prove

$$\sum_{j\in J} \|u_j - \widehat{\Lambda} u_j\|_{\mathrm{NC}}^2 \lesssim \sum_{j\in J} \|u_j - \widetilde{\Lambda} u_j\|_{\mathrm{NC}}^2 \le \varepsilon(\ell)^2. \qquad \square$$

Lemma 4.15 (key argument). *Provided $\|h_0\|_\infty \ll 1$, there exists $C_2 \approx 1$ such that*

$$\mu_\ell^2(\mathcal{T}_\ell) \le C_2 \mu_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell).$$

Proof. The triangle inequality and the Young inequality imply for any $j \in J$, that

$$\|u_j - \Lambda_\ell u_j\|_{\mathrm{NC}}^2 \le 2\|u_j - \widehat{\Lambda} u_j\|_{\mathrm{NC}}^2 + 2\|\widehat{\Lambda} u_j - \Lambda_\ell u_j\|_{\mathrm{NC}}^2.$$

Hence, the discrete reliability from Proposition 4.5 leads to

$$\begin{aligned}\|u_j - \Lambda_\ell u_j\|_{\mathrm{NC}}^2 \le & (2 + C_{\mathrm{drel}}^2 \lambda_j^2 \|h_0\|_\infty^{2s} (1+M_J)^2 C_{L^2}^2) \|u_j - \widehat{\Lambda} u_j\|_{\mathrm{NC}}^2 \\ & + C_{\mathrm{drel}}^2 \lambda_j^2 \|h_0\|_\infty^{2s} (1+M_J)^2 C_{L^2}^2 \|u_j - \Lambda_\ell u_j\|_{\mathrm{NC}}^2 \\ & + C_{\mathrm{drel}}^2 \mu_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell, \lambda_j, u_j).\end{aligned}$$

The term with $\|u_j - \Lambda_\ell u_j\|_{\mathrm{NC}}^2$ can be absorbed for sufficiently small $\|h_0\|_\infty \ll 1$. Therefore, Lemma 4.14 implies for constants $C_3 \approx 1 \approx C_4$ and $\|h_0\|_\infty \ll 1$ that

$$\sum_{j\in J} \|u_j - \Lambda_\ell u_j\|_{\mathrm{NC}}^2 \le C_3 \varepsilon(\ell)^2 + C_4 \mu_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell).$$

Let C_{eq} denote the constant of $C_3 \Xi_\ell^2 \le C_{\mathrm{eq}} \mu_\ell^2(\mathcal{T}_\ell)$ (which exists by reliability). The efficiency (4.8), the definition of $\varepsilon(\ell)$ and the preceding estimates prove

$$\begin{aligned} C_{\mathrm{eff}}^{-2} \mu_\ell^2(\mathcal{T}_\ell) &\le C_3 \varepsilon(\ell)^2 + C_4 \mu_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell) \\ &\le \tau C_{\mathrm{eq}} \mu_\ell^2(\mathcal{T}_\ell) + C_4 \mu_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell). \end{aligned}$$

For a sufficiently small choice of τ, the constant $C_2 := (C_{\mathrm{eff}}^{-2} - \tau C_{\mathrm{eq}})^{-1} C_4$ is positive. $\square$

The finish of the optimality proof follows the arguments of [22, 46]. The proof is identical to that of [33, Lemma 7.3] and therefore omitted.

Lemma 4.16 (finish of the optimality proof)**.** *The choice*

$$0 < \theta \leq 1 \big/ \big(C_2(B/A)^4(2N^2 + 4N^3)\big)$$

implies the existence of a constant $C(\sigma)$ such that

$$\big(\operatorname{card}(\mathcal{T}_\ell) - \operatorname{card}(\mathcal{T}_0)\big)^\sigma \left(\sum_{j\in J} \|u_j - \Lambda_\ell u_j\|_{\mathrm{NC}}^2\right)^{1/2} \leq C(\sigma)\Big(\sum_{j\in J} |u_j|^2_{\mathfrak{A}_\sigma^{\mathrm{NC},\Delta}}\Big)^{1/2}. \qquad \square$$

5 Eigenvalues of the Stokes System

This section studies the adaptive nonconforming FEM approximation of the Stokes eigenproblem. Subsection 5.1 presents new L^2 and best-approximation estimates for the linear Stokes equations. Subsection 5.2 introduces the discretisation of the eigenvalue problem. A theoretical error estimator and its discrete reliability are analysed in Subsection 5.3. Subsections 5.4–5.5 present the practical AFEM and prove contraction and optimal convergence rates. Whenever there is no significant modification compared to the case of the eigenvalues of the Laplacian, the arguments are merely sketched.

5.1 Nonconforming Discretisation of the Stokes Equations

One important advantage of the nonconforming $\mathcal{P}_1$ finite element method is that it provides a stable low-order discretisation of the Stokes equations [24]. The strong form of the linear Stokes equations for a given force f seeks the velocity field u and the pressure p such that

$$-\Delta u + (Dp)^\top = f \text{ and } \operatorname{div} u = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.$$

Conforming finite elements satisfying the constraint $\operatorname{div} u = 0$ pointwise a.e. are rather complicated, see [44, 38]. The nonconforming $\mathcal{P}_1$ finite element satisfies favourable local mass-conservation property for the piecewise divergence.

Let $V := [H^1_0(\Omega)]^d$ and $M := L^2_0(\Omega) := \{q \in L^2(\Omega) \mid \int_\Omega q\,dx = 0\}$ and define the bilinear form

$$a(v,w) := (Dv, Dw)_{L^2(\Omega)} \quad \text{for all } (v,w) \in V^2$$

with induced norm $\|\cdot\|$. Furthermore define

$$b(v,q) := -(\operatorname{div} v, q)_{L^2(\Omega)} \quad \text{for all } (v,q) \in V \times M$$

and set $c(\cdot,\cdot) := (\cdot,\cdot)_{L^2(\Omega)}$ with $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$.

Given $f \in [L^2(\Omega)]^d$, the linear Stokes problem seeks $(u,p) \in V \times M$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= c(f,v) && \text{for all } v \in V,\\ b(u,q) &= 0 && \text{for all } q \in M. \end{aligned} \tag{5.1}$$

This mixed system can be reformulated as an elliptic problem. Let $Z := \{v \in V \mid \operatorname{div} v = 0\}$ denote the space of divergence-free vector fields. Problem (5.1) is equivalent to

$$a(u,v) = c(f,v) \quad \text{for all } v \in Z \tag{5.2}$$

and the pressure variable p plays the role of a Lagrange multiplier. The equivalence with (5.1) follows from the Ladyzhenskaya lemma [9, 1] which states that the divergence operator $\operatorname{div}: V \to M$ has a continuous right-inverse. Note that (5.1) carries more information than (5.2) in the sense that the pressure variable p extracts information from $f \in [L^2(\Omega)]^d$ even if f is zero as an element of the dual space $Z^\star$.

The nonconforming $\mathcal{P}_1$ finite element discretisation of the linear Stokes equations is based on the nonconforming finite element space $V_\ell := \left[\mathfrak{CR}^1_0(\mathcal{T}_\ell)\right]^d$ and $M_\ell := \mathcal{P}_0(\mathcal{T}_\ell) \cap L^2_0(\Omega)$ and the bilinear forms

$$a_{\mathrm{NC}}(v_\ell, w_\ell) := (D_{\mathrm{NC}} v_\ell, D_{\mathrm{NC}} w_\ell)_{L^2(\Omega)} \quad \text{for all } (v_\ell, w_\ell) \in V_\ell^2$$

with induced norm $|||\cdot|||_{\mathrm{NC}}$ and

$$b_{\mathrm{NC}}(v_\ell, q_\ell) := -(\operatorname{div}_{\mathrm{NC}} v_\ell, q_\ell)_{L^2(\Omega)} \quad \text{for all } (v_\ell, q_\ell) \in V_\ell \times M_\ell.$$

The nonconforming FEM seeks $(u_\ell, p_\ell) \in V_\ell \times M_\ell$ such that

$$\begin{aligned} a_{\mathrm{NC}}(u_\ell, v_\ell) + b_{\mathrm{NC}}(v_\ell, p_\ell) &= c(f, v_\ell) && \text{for all } v_\ell \in V_\ell, \\ b_{\mathrm{NC}}(u_\ell, q_\ell) &= 0 && \text{for all } q_\ell \in M_\ell. \end{aligned} \tag{5.3}$$

The well-posedness follows from the discrete inf-sup condition [5]

$$0 < \beta \le \inf_{q_\ell \in M_\ell \setminus \{0\}} \sup_{v_\ell \in V_\ell \setminus \{0\}} \frac{b_{\mathrm{NC}}(v_\ell, q_\ell)}{|||v_\ell|||_{\mathrm{NC}} \, \|q_\ell\|}. \tag{5.4}$$

Obviously, the discrete solution u_ℓ of (5.3) is piecewise divergence-free, $\operatorname{div}_{\mathrm{NC}} u_\ell = 0$. The equivalent formulation based on the space $Z_\ell := \{v_\ell \in V_\ell \mid \operatorname{div}_{\mathrm{NC}} v_\ell = 0\}$ reads as

$$a_{\mathrm{NC}}(u_\ell, v_\ell) = b(f, v_\ell) \quad \text{for all } v_\ell \in Z_\ell. \tag{5.5}$$

Note that the nonconforming interpolation operator $\mathcal{I}^{\mathfrak{CR}}_\ell$ maps the space Z onto Z_ℓ. This follows from the projection property (3.1). It is well-established in the literature [29] and follows from the discrete inf-sup condition (5.4) of the system (5.3) that the error in the pressure variable can be controlled as

$$\|p - p_\ell\| \lesssim \|h_\ell f\| + |||u - u_\ell|||_{\mathrm{NC}}. \tag{5.6}$$

The main difference to the analysis of the Laplace operator is that the pressure variable enters the analysis even if one considers the elliptic formulations (5.2) and (5.5). One reason is that the companion operator J_{d+1} from Proposition 3.1 does not map the space Z_ℓ on Z only. Also the efficiency error estimate of the volume term $\|h_\ell f\|$ leads to a pressure term on the right-hand side.

The following best-approximation result has been proved by [16] with techniques from the medius analysis [37] for the case $d = 2$,

$$\|p - p_\ell\| + |||u - u_\ell|||_{\mathrm{NC}} \lesssim \|(1 - \Pi^0_\ell) p\| + \|(1 - \Pi^0_\ell) Du\| + \operatorname{osc}_0(f, \mathcal{T}_\ell).$$

The following result gives a generalisation to $d \ge 2$ space dimensions with a refined oscillation term.

Proposition 5.1 (best-approximation result). *Let $f \in [L^2(\Omega)]^d$. Then, the solution $(u,p) \in V \times M$ of* (5.1) *and the discrete solution $(u_\ell, p_\ell) \in V_\ell \times M_\ell$ of* (5.3) *satisfy*

$$|||u - u_\ell|||_{\mathrm{NC}} + \|p - p_\ell\| \lesssim \|(1 - \Pi^0_\ell) Du\| + \|(1 - \Pi^0_\ell) p\| + \operatorname{osc}_1(f, \mathcal{T}_\ell).$$

Proof. The projection property (3.1) of the nonconforming interpolation operator $\mathcal{I}_\ell^{\mathfrak{CR}}$ and the Pythagoras theorem show that

$$|||u - u_\ell|||_{\mathrm{NC}}^2 = |||u_\ell - \mathcal{I}_\ell^{\mathfrak{CR}} u|||_{\mathrm{NC}}^2 + |||u - \mathcal{I}_\ell^{\mathfrak{CR}} u|||_{\mathrm{NC}}^2.$$

Since $|||u - \mathcal{I}_\ell^{\mathfrak{CR}} u|||_{\mathrm{NC}} = \|(1-\Pi_\ell^0)Du\|$, it remains to estimate the first term on the right-hand side. Set $\varphi_\ell := u_\ell - \mathcal{I}_\ell^{\mathfrak{CR}} u$. The properties of the companion operator from Proposition 3.1 and $\mathrm{div}_{\mathrm{NC}}\, u_\ell = 0 = \mathrm{div}_{\mathrm{NC}}\, \mathcal{I}_\ell^{\mathfrak{CR}} u$ show that

$$\begin{aligned} |||u_\ell - \mathcal{I}_\ell^{\mathfrak{CR}} u|||_{\mathrm{NC}}^2 &= a_{\mathrm{NC}}(u_\ell - u, \varphi_\ell) \\ &= c(f, \varphi_\ell - J_{d+1}\varphi_\ell) - b_{\mathrm{NC}}(\varphi_\ell - J_{d+1}\varphi_\ell, (1-\Pi_\ell^0)p) \\ &\qquad + ((1-\Pi_\ell^0)Du, D_{\mathrm{NC}}(J_{d+1} - 1)\varphi_\ell)_{L^2(\Omega)}. \end{aligned}$$

The approximation and stability properties (3.4) show that this is bounded by

$$(\|h_\ell f\| + \|(1-\Pi_\ell^0)p\| + |||u_\ell - \mathcal{I}_\ell^{\mathfrak{CR}} u|||_{\mathrm{NC}})|||\varphi_\ell|||_{\mathrm{NC}}.$$

The efficiency $\|h_\ell f\| \lesssim \|(1-\Pi_\ell^0)Du\| + \|(1-\Pi_\ell^0)p\| + \mathrm{osc}_1(f, \mathcal{T}_\ell)$ in the sense of [49] follows from arguments similar to those of [33, Prop. 3.1]. This and (5.6) conclude the proof. □

Remark 5.2. *One may ask whether possibly an estimate of the type*

$$|||u - u_\ell|||_{\mathrm{NC}} \lesssim \|(1-\Pi_\ell^0)Du\| + \textit{oscillations}$$

may be valid. To see that the esimate is indeed untrue consider the case of a simply-connected domain Ω *for* $d = 2$ *and the constant right-hand side* $f = (1,1)$*. Clearly,* f *is an irrotational vector field which implies that there is a function* $\psi \in H^1(\Omega)$ *such that* $f = D\psi$*. The integration by parts therefore shows that*

$$c(f, v) = 0 \quad \textit{for all } v \in Z.$$

Hence, $u = 0$ *and the right-hand side of the estimate equals zero, while the left-hand side equals* $|||u_\ell|||_{\mathrm{NC}}$*. The latter, however, is not zero because* f *does not represent the zero functional in the dual space* $Z_\ell^\star$*, although it is zero in* $Z^\star$*. This is due to the fact that the integration by parts with functions* $v_\ell \in Z_\ell$ *leads to additional jump terms.*

The next result is an L^2 error estimate for arbitrary regularity of the solution. Let $0 < s \le 1$ indicate the elliptic regularity of the problem (5.1) in the sense that [31, 43]

$$\|u\|_{H^{1+s}(\Omega)} + \|p\|_{H^s(\Omega)} \le C(s)\|f\|_{L^2(\Omega)}. \tag{5.7}$$

Proposition 5.3 (L^2 error control for the linear Stokes problem)**.** *The exact solution* $(u,p) \in V \times M$ *of the linear problem* (5.1) *and its nonconforming finite element approximation* $(u_\ell, p_\ell) \in V_\ell \times M_\ell$ *from* (5.3) *satisfy*

$$\|u - u_\ell\| \lesssim \|h_\ell\|_\infty^s(|||u - u_\ell|||_{\mathrm{NC}} + \|p - p_\ell\| + \mathrm{osc}_{1,1}(f, \mathcal{T}_\ell)).$$

Proof. Let $(z,q) \in V \times M$ denote the solution of problem (5.1) with right-hand side $e := u - u_\ell$ and set $v := u - J_{d+1}u_\ell$ for the companion operator J_{d+1} from Proposition 3.1. Since $\Pi_\ell^0(u_\ell - J_{d+1}u_\ell) = 0$, it holds that

$$\begin{aligned} \|e\|^2 &= c(J_{d+1}u_\ell - u_\ell, e) + c(e, v) \\ &= (J_{d+1}u_\ell - u_\ell, (1-\Pi_\ell^0)e)_{L^2(\Omega)} + a(z,v) + b(v,q). \end{aligned} \tag{5.8}$$

Piecewise Poincaré inequalities and (3.4) lead to

$$(J_{d+1}u_\ell - u_\ell, (1-\Pi_\ell^0)e)_{L^2(\Omega)} \lesssim \|h_0\|_\infty^2 \|e\|_{\mathrm{NC}}^2.$$

The definition of v and $\operatorname{div} u = 0 = \operatorname{div}_{\mathrm{NC}} u_\ell$ prove

$$a(z,v) + b(v,q) = a_{\mathrm{NC}}(e,z) + a_{\mathrm{NC}}((1-J_{d+1})u_\ell, z) + b_{\mathrm{NC}}(u_\ell - J_{d+1}u_\ell, q). \tag{5.9}$$

The projection property (3.1) of $\mathcal{I}_\ell^{\mathfrak{CR}}$ and the continuous and discrete problems (5.1) and (5.3) followed by the approximation and stability properties (3.2) of $\mathcal{I}_\ell^{\mathfrak{CR}}$ show for the first term on the right-hand side of (5.9) that

$$a_{\mathrm{NC}}(e,z) = a(u,z) - a_{\mathrm{NC}}(u_\ell, \mathcal{I}_\ell^{\mathfrak{CR}} z) = (f, z - \mathcal{I}_\ell^{\mathfrak{CR}} z)_{L^2(\Omega)} \lesssim \|h_\ell f\| \|(1-\Pi_\ell^0)Dz)\|.$$

Recall that $\operatorname{div}_{\mathrm{NC}} \mathcal{I}_\ell^{\mathfrak{CR}} z = \operatorname{div} z = 0$. The projection property (3.3) and the stability (3.4) of J_{d+1} show for the second term on the right-hand side of (5.9) that

$$\begin{aligned} a_{\mathrm{NC}}((1-J_{d+1})u_\ell, z) &= (D_{\mathrm{NC}}(1-J_{d+1})u_\ell, (1-\Pi_\ell^0)Dz)_{L^2(\Omega)} \\ &\le \|u - u_\ell\|_{\mathrm{NC}} \|(1-\Pi_\ell^0)Dz\|. \end{aligned}$$

Since $\Pi_\ell^0 \operatorname{div}(u_\ell - J_{d+1}u_\ell) = 0$, the third contribution of (5.9) satisfies

$$\begin{aligned} b_{\mathrm{NC}}((u_\ell - J_{d+1}u_\ell), q) &= b_{\mathrm{NC}}(u_\ell - J_{d+1}u_\ell, (1-\Pi_\ell^0)q) \\ &\le \|u_\ell - J_{d+1}u_\ell\|_{\mathrm{NC}} \|(1-\Pi_\ell^0)q\|. \end{aligned}$$

The best-approximation property (3.4) of J_{d+1} proves that $\|u_\ell - J_{d+1}u_\ell\|_{\mathrm{NC}} \lesssim \|e\|_{\mathrm{NC}}$. Altogether,

$$\begin{aligned} \|e\|^2 \lesssim \|h_0\|_\infty^2 \|e\|_{\mathrm{NC}}^2 &+ \|h_\ell f\| \|(1-\Pi_\ell^0)Dz)\| \\ &+ \|e\|_{\mathrm{NC}} \big(\|(1-\Pi_\ell^0)q\| + \|(1-\Pi_\ell^0)Dz\| \big). \end{aligned}$$

Standard a priori estimates [9] and the elliptic regularity (5.7) imply

$$\|(1-\Pi_\ell^0)Dz)\| + \|(1-\Pi_\ell^0)q\| \lesssim \|h_0\|_\infty^s \|e\|.$$

The combination of the above estimates proves

$$\|e\| \lesssim \|h_0\|_\infty^s (\|e\|_{\mathrm{NC}} + \|h_\ell f\|).$$

An efficiency estimate similar to that of [33, Prop. 3.1] proves

$$\|h_\ell f\| \lesssim \|(1-\Pi_\ell^0)Du\| + \|(1-\Pi_\ell^0)p\| + \operatorname{osc}_{1,1}(f, \mathcal{T}_\ell).$$

This concludes the proof. □

Remark 5.4. *The right-hand side in Proposition 5.3 is also an upper bound for $p - p_\ell$ in the H^{-1} norm. Although the proof is not difficult, it is not given here because the H^{-1} error control is not required in the analysis of this paper.*

5.2 Discretisation of the Stokes Eigenvalue Problem

The Stokes eigenvalue problem seeks $(\lambda, u, p) \in \mathbb{R} \times V \times M$ with $\|u\| = 1$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= \lambda\, c(u,v) \qquad \text{for all } v \in V, \\ b(u,q) &= 0 \qquad\qquad\quad \text{for all } q \in M. \end{aligned} \tag{5.10}$$

Although (λ, u, p) is rather a triple than a pair it is referred to as eigenpair and identified with the pair $(\lambda, (u, p))$. As in the foregoing section, an equivalent formulation reads as

$$a(u,v) = \lambda\, c(u,v) \quad \text{for all } v \in Z. \tag{5.11}$$

The nonconforming FEM seeks $(u_\ell, p_\ell) \in V_\ell \times M_\ell$ with $\|u_\ell\| = 1$ such that

$$\begin{aligned} a_{\mathrm{NC}}(u_\ell, v_\ell) + b_{\mathrm{NC}}(v_\ell, p_\ell) &= \lambda_\ell\, c(u_\ell, v_\ell) \qquad \text{for all } v_\ell \in V_\ell, \\ b_{\mathrm{NC}}(u_\ell, q_\ell) &= 0 \qquad\qquad\qquad \text{for all } q_\ell \in M_\ell. \end{aligned} \tag{5.12}$$

An equivalent formulation reads as

$$a_{\mathrm{NC}}(u_\ell, v_\ell) = \lambda_\ell\, c(u_\ell, v_\ell) \quad \text{for all } v_\ell \in Z_\ell. \tag{5.13}$$

The elliptic formulation on the spaces Z and Z_ℓ shows that this problem fits in the framework of Section 2 (where b from Section 2 is replaced by c) with exact and discrete eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \dots \quad \text{and} \quad 0 < \lambda_{\ell,1} \le \dots \le \lambda_{\ell,\dim(Z_\ell)}$$

and their corresponding c-orthonormal systems of eigenfunctions

$$(u_1, u_2, u_3, \dots) \in Z^{\mathbb{N}} \quad \text{and} \quad (u_{\ell,1}, u_{\ell,2}, \dots, u_{\ell,\dim(Z_\ell)}) \in Z_\ell^{\dim(Z_\ell)}.$$

The corresponding pressures are denoted by $p_1, p_2, \dots$ and $p_{\ell,1}, \dots, p_{\ell,\dim(Z_\ell)}$, respectively. Recall the definitions of Section 2: The set $J = \{n+1, \dots, n+N\}$ describes the eigenvalue cluster of interest and $W := \operatorname{span}\{u_j \mid j \in J\} \subseteq Z$ and $W_\ell := \operatorname{span}\{u_{\ell,j} \mid j \in J\} \subseteq Z_\ell$ are the exact and discrete invariant subspaces (not necessarily eigenspaces) related to the cluster. In the present situation, the quasi-Ritz projection R_ℓ maps the solution $u \in Z$ of the linear problem (5.2) to the solution $R_\ell u \in Z_\ell$ of the discrete linear problem (5.5) with discrete pressure $p(R_\ell u) \in M_\ell$ from (5.3). The L^2 projection onto W_ℓ is denoted by $P_{\mathcal{T}_\ell} := P_\ell$. Furthermore $\Lambda_{\mathcal{T}_\ell} := \Lambda_\ell := P_\ell \circ R_\ell$. In view of Lemma 2.2, the discrete pressure $p(\Lambda_\ell u) \in M_\ell$ corresponding to $\Lambda_\ell u$ is defined via

$$a_{\mathrm{NC}}(\Lambda_\ell u, v_\ell) + b_{\mathrm{NC}}(v_\ell, p(\Lambda_\ell u)) = \lambda c(P_\ell u, v_\ell) \quad \text{for all } v_\ell \in V_\ell. \tag{5.14}$$

It is not difficult to see that $p(\Lambda_\ell u)$ is well-defined: Lemma 2.2 shows that $\Lambda_\ell u$ solves the discrete source problem (5.5) with right-hand side $f = P_\ell u$. Hence, $p(\Lambda_\ell u)$ is the discrete pressure (or Lagrange multiplier) of (5.3).

The following result gives an L^2 error estimate for the eigenfunctions.

Proposition 5.5 (L^2 error estimate)**.** *Provided $\|h_0\|_\infty \ll 1$, there exists a constant C_{L^2} such that any eigenpair $(\lambda, u, p) \in \mathbb{R} \times W \times M$ of (5.10) with $\|u\| = 1$ satisfies*

$$\|u - P_\ell u\| \le \|u - \Lambda_\ell u\| \le C_{L^2}(1 + M_J)\|h_0\|_\infty^s (\|(1 - \Pi_\ell^0) D u\| + \|(1 - \Pi_\ell^0) p\|).$$

Proof. Proposition 2.1 and the L^2 error estimate from Proposition 5.3 result in the following inequality for the solution $(R_\ell u, p(R_\ell u))$ of (5.3) to the right-hand side $f := \lambda u$,

$$\begin{aligned}\|u - P_\ell u\| &\le \|u - \Lambda_\ell u\| \\ &\lesssim (1 + M_J)\|h_\ell\|_\infty^s \big(\|\!|u - R_\ell u|\!\|_{\mathrm{NC}} + \|p - p(R_\ell u)\| + \operatorname{osc}_{1,1}(\lambda u, \mathcal{T}_\ell)\big).\end{aligned}$$

The best-approximation result for the linear Stokes problem (Proposition 5.1) therefore yields

$$\|u - \Lambda_\ell u\| \lesssim (1 + M_J)\|h_\ell\|_\infty^s \big(\|(1 - \Pi_\ell^0)Du\| + \|(1 - \Pi_\ell^0)p\| + \operatorname{osc}_1(\lambda u, \mathcal{T}_\ell)\big).$$

If the initial mesh-size is sufficiently small, the discrete Friedrichs inequality [9, Thm. 10.6.12] allows to absorb the oscillation terms on the right-hand side. □

The L^2 error control and the best-approximation of the quasi-Ritz projection from Proposition 5.1 result in the following best-approximation property for the eigenfunction approximation.

Proposition 5.6 (best-approximation property)**.** *Provided the initial mesh-size is sufficiently fine $\|h_0\|_\infty \ll 1$, any eigenpair $(\lambda, u, p) \in \mathbb{R} \times W \times M$ of (5.12) with $\|u\| = 1$ satisfies*

$$\|\!|u - \Lambda_\ell u|\!\|_{\mathrm{NC}} + \|p - p(\Lambda_\ell u)\| \lesssim \|(1 - \Pi_\ell^0)Du\| + \|(1 - \Pi_\ell^0)p\|.$$

Proof. The L^2 control of Proposition 5.5 and the best-approximation result for the linear case of Proposition 5.1 enable the arguments from the proof of Proposition 4.4. The details are omitted for brevity. □

5.3 Theoretical Error Estimator and Discrete Reliability

The analysis relies on a theoretical, non-computable error estimator that does not depend on the choice of the discrete eigenfunctions. Given an eigenpair (λ, u), the theoretical error estimator includes the elementwise residuals in terms of $P_\ell u$ and $\Lambda_\ell u$. More precisely, define, for any $T \in \mathcal{T}_\ell$,

$$\mu_\ell^2(T, \lambda, u) := h_T^2 \|\lambda P_\ell u\|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}(T)} h_T^{-1} \|[\Lambda_\ell u]_F\|_{L^2(F)}^2$$

and, for any subset $\mathcal{K} \subseteq \mathcal{T}_\ell$,

$$\mu_\ell^2(\mathcal{K}, \lambda_j, u_j) := \sum_{T \in \mathcal{K}} \mu_\ell^2(T, \lambda_j, u_j) \quad \text{and} \quad \mu_\ell^2(\mathcal{K}) := \sum_{j \in J} \mu_\ell^2(\mathcal{K}, \lambda_j, u_j).$$

The following shorthand notation for higher-order terms will be frequently used in the remaining parts of this section. For $(\ell, m) \in \mathbb{N}_0^2$ define

$$\begin{aligned}\boldsymbol{r}_{\ell,m} := \|h_0\|_\infty^s (1 + M_J) C_{L^2} \Big(&\|p - p(\Lambda_\ell u)\|^2 + \|p - p(\Lambda_{\ell+m} u)\|^2 \\ &+ \|\!|u - u_\ell|\!\|_{\mathrm{NC}}^2 + \|\!|u - u_{\ell+m}|\!\|_{\mathrm{NC}}^2 \Big)^{1/2}.\end{aligned} \tag{5.15}$$

The following result states the discrete reliability for the theoretical error estimator. The discrete reliability for the linear Stokes problem was first established by [39, 21]. The proof presented here is valid for the eigenvalue problem and any space dimension.

Proposition 5.7 (discrete reliability). *There exists a constant $C_{\mathrm{drel}} \approx 1$ such that, for any eigenpair $(\lambda, u, p) \in \mathbb{R} \times W \times M$ of (5.10) with $\|u\| = 1$, any admissible refinement $\mathcal{T}_{\ell+m}$ of $\mathcal{T}_\ell \in \mathbb{T}$ and the respective discrete eigenfunction approximations $\Lambda_\ell u \in V_\ell$ and $\Lambda_{\ell+m} u \in V_{\ell+m}$ satisfy*

$$\begin{aligned} &\|p(\Lambda_{\ell+m}u) - p(\Lambda_\ell u)\|^2 \\ &\quad \lesssim \|\!|(\Lambda_{\ell+m} - \Lambda_\ell)u\|\!|^2_{\mathrm{NC}} + \|h_\ell \lambda P_\ell u\|^2_{L^2(\cup \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m})} + \boldsymbol{r}^2_{\ell,m}, \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} &2\big(\|\!|(\Lambda_{\ell+m} - \Lambda_\ell)u\|\!|^2_{\mathrm{NC}} + \|p(\Lambda_{\ell+m}u) - p(\Lambda_\ell u)\|^2\big) \\ &\qquad \le C^2_{\mathrm{drel}} \left(\mu^2_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m}) + \boldsymbol{r}^2_{\ell,m}\right). \end{aligned} \tag{5.17}$$

Proof. The discrete inf-sup condition (5.4) shows that there exists some $\varphi_{\ell+m} \in V_{\ell+m}$ with $\|\!|\varphi_{\ell+m}\|\!|_{\mathrm{NC}} = 1$ such that

$$\|p(\Lambda_{\ell+m}u) - p(\Lambda_\ell u)\| \lesssim b_{\mathrm{NC}}(\varphi_{\ell+m}, p(\Lambda_{\ell+m}u) - p(\Lambda_\ell u)).$$

The discrete eigenvalue problems on the levels $\ell + m$ and ℓ (recall Lemma 2.2 and (5.14)), some algebra and the integral mean property (3.1) of the nonconforming interpolation operator $\mathcal{I}^{\mathfrak{CR}}_\ell$ show that

$$\begin{aligned} &b(\varphi_{\ell+m}, p(\Lambda_{\ell+m}u) - p(\Lambda_\ell u)) \\ &\quad = c(\lambda(P_{\ell+m} - P_\ell)u, \varphi_{\ell+m}) + c(\lambda P_\ell u, (1 - \mathcal{I}^{\mathfrak{CR}}_\ell)\varphi_{\ell+m}) \\ &\qquad - a_{\mathrm{NC}}((\Lambda_{\ell+m} - \Lambda_\ell)u, \varphi_{\ell+m}). \end{aligned}$$

Proposition 5.5 and the discrete Friedrichs inequality [9, Thm. 10.6.12] control the first term on the right-hand side as

$$c(\lambda(P_{\ell+m} - P_\ell)u, \varphi_{\ell+m}) \lesssim \boldsymbol{r}_{\ell,m}.$$

This, the approximation and stability properties (3.2) and the discrete Friedrichs inequality [9, Thm. 10.6.12] for $\varphi_{\ell+m}$ prove (5.16).

Let $v_{\ell+m}$ denote the best-approximation with respect to the norm $\|\!|\cdot\|\!|_{\mathrm{NC}}$ of $\Lambda_\ell u$ in $V_{\ell+m}$. The Pythagoras theorem

$$\|\!|(\Lambda_{\ell+m} - \Lambda_\ell)u\|\!|^2_{\mathrm{NC}} = \|\!|\Lambda_{\ell+m}u - v_{\ell+m}\|\!|^2_{\mathrm{NC}} + \|\!|v_{\ell+m} - \Lambda_\ell u\|\!|^2_{\mathrm{NC}} \tag{5.18}$$

prove together with (5.16) that

$$\begin{aligned} &\|p(\Lambda_{\ell+m}u) - p(\Lambda_\ell u)\|^2 + \|\!|(\Lambda_{\ell+m} - \Lambda_\ell)u\|\!|^2_{\mathrm{NC}} \\ &\qquad \lesssim \|\!|\Lambda_{\ell+m}u - v_{\ell+m}\|\!|^2_{\mathrm{NC}} + \|\!|v_{\ell+m} - \Lambda_\ell u\|\!|^2_{\mathrm{NC}} \\ &\qquad\quad + \|h_\ell \lambda P_\ell u\|_{\cup(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m})} + \boldsymbol{r}^2_{\ell,m}. \end{aligned} \tag{5.19}$$

Set $\phi_{\ell+m} := \Lambda_{\ell+m}u - v_{\ell+m}$. Elementary algebra and the projection property (3.1) show

$$\begin{aligned} \|\!|\Lambda_{\ell+m}u - v_{\ell+m}\|\!|^2_{\mathrm{NC}} &= a_{\mathrm{NC}}(\Lambda_{\ell+m}u - v_{\ell+m}, \phi_{\ell+m}) \\ &= a_{\mathrm{NC}}(\Lambda_{\ell+m}u, \phi_{\ell+m}) - a_{\mathrm{NC}}(\Lambda_\ell u, \mathcal{I}^{\mathfrak{CR}}_\ell \phi_{\ell+m}). \end{aligned}$$

The discrete eigenvalue problem (5.12) and the identity (5.14) show that this equals

$$\begin{aligned} &a_{\mathrm{NC}}(\Lambda_{\ell+m}u, \phi_{\ell+m}) - a_{\mathrm{NC}}(\Lambda_\ell u, \mathcal{I}^{\mathfrak{CR}}_\ell \phi_{\ell+m}) \\ &\quad = c(\lambda P_{\ell+m}u, \phi_{\ell+m}) - c(\lambda P_\ell u, \mathcal{I}^{\mathfrak{CR}}_\ell \phi_{\ell+m}) \\ &\qquad - b_{\mathrm{NC}}(\phi_{\ell+m}, p(\Lambda_{\ell+m}u)) + b_{\mathrm{NC}}(\mathcal{I}^{\mathfrak{CR}}_\ell \phi_{\ell+m}, p(\Lambda_\ell u)). \end{aligned}$$

Since the velocity approximations $\Lambda_\ell u \in W_\ell$ and $\Lambda_{\ell+m} u \in W_{\ell+m}$ are piecewise divergence-free, the projection property of $\mathcal{I}_\ell^{\mathfrak{CR}}$ shows that

$$\begin{aligned} b_{\mathrm{NC}}(\phi_{\ell+m}, p(\Lambda_{\ell+m})) - b_{\mathrm{NC}}(\mathcal{I}_\ell^{\mathfrak{CR}}\phi_{\ell+m}, p(\Lambda_\ell u)) \\ = b_{\mathrm{NC}}(v_{\ell+m} - \Lambda_\ell u, p(\Lambda_\ell u) - p(\Lambda_{\ell+m} u)). \end{aligned}$$

The combination of the foregoing three displayed formulae yields

$$\begin{aligned} &|||\Lambda_{\ell+m} u - v_{\ell+m}|||_{\mathrm{NC}}^2 \\ &\quad = \lambda c(P_{\ell+m} u - P_\ell u, \phi_{\ell+m}) + \lambda c(P_\ell u, \phi_{\ell+m} - \mathcal{I}_\ell^{\mathfrak{CR}}\phi_{\ell+m}) \\ &\qquad + b_{\mathrm{NC}}(v_{\ell+m} - \Lambda_\ell u, p(\Lambda_{\ell+m} u) - p(\Lambda_\ell u)). \end{aligned} \tag{5.20}$$

Proposition 5.5 and the discrete Friedrichs inequality [9, Thm. 10.6.12] control the first contribution as

$$\lambda c(P_{\ell+m} - P_\ell u, \phi_{\ell+m}) \lesssim \boldsymbol{r}_{\ell,m} |||\phi_{\ell+m}|||_{\mathrm{NC}}.$$

The approximation and stability properties (3.2) of $\mathcal{I}_\ell^{\mathfrak{CR}}$ and the fact that $\mathcal{I}_\ell^{\mathfrak{CR}}\phi_{\ell+m}|_T = \phi_{\ell+m}|_T$ for all $T \in \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m}$ prove for the second term of (5.20) that

$$c(\lambda P_\ell u, \phi_{\ell+m} - \mathcal{I}_\ell^{\mathfrak{CR}}\phi_{\ell+m}) \lesssim \|h_\ell \lambda P_\ell u\|_{L^2(\cup\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})} |||\phi_{\ell+m}|||_{\mathrm{NC}}.$$

Therefore, the combination of (5.19)–(5.20) and the Young inequality prove for some constant $C \approx 1$ that

$$\begin{aligned} &\|p(\Lambda_{\ell+m} u) - p(\Lambda_\ell u)\|^2 + |||\Lambda_{\ell+m} u - \Lambda_\ell u|||_{\mathrm{NC}}^2 \\ &\qquad \le C\Big(\|h_\ell \lambda P_\ell u\|_{L^2(\cup\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})}^2 + \boldsymbol{r}_{\ell,m}^2 + |||v_{\ell+m} - \Lambda_\ell u|||_{\mathrm{NC}}^2\Big) \\ &\qquad\qquad + \frac{1}{2}|||\phi_{\ell+m}|||_{\mathrm{NC}}^2 + \frac{1}{2}\|p(\Lambda_{\ell+m} u) - p(\Lambda_\ell u)\|^2. \end{aligned}$$

The Pythagoras theorem implies the stability $|||\phi_{\ell+m}|||_{\mathrm{NC}} \le |||(\Lambda_{\ell+m} - \Lambda_\ell)u|||_{\mathrm{NC}}$. Hence, the terms on the right-hand side with the prefactor 1/2 can be absorbed. The estimate

$$|||v_{\ell+m} - \Lambda_\ell u|||_{\mathrm{NC}}^2 \lesssim \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m}} \sum_{F\in\mathcal{F}(T)} h_T^{-1} \|[\Lambda_\ell u]_F\|_{L^2(F)}^2$$

is proven in [13, Thm. 3.1] and bounds the second contribution on the right-hand side of (5.18). □

As in Subsection 4.3, the following reliability and efficiency are an immediate consequence of the discrete reliability.

Corollary 5.8 (reliability and efficiency)**.** *Provided* $\|h_0\|_\infty \ll 1$, *any eigenpair* $(\lambda, u, p) \in \mathbb{R} \times W \times M$ *of* (5.10) *with* $\|u\| = 1$ *satisfies*

$$|||u - \Lambda_\ell u|||_{\mathrm{NC}}^2 + \|p - p(\Lambda_\ell u)\|^2 \le C_{\mathrm{drel}}^2 \mu_\ell^2(\mathcal{T}_\ell, \lambda, u) \tag{5.21}$$

and, for some constant $C_{\mathrm{eff}} \approx 1$,

$$\mu_\ell(\mathcal{T}_\ell, \lambda, u)^2 \le C_{\mathrm{eff}}^2(|||u - \Lambda_\ell u|||_{\mathrm{NC}}^2 + \|p - p(\Lambda_\ell u)\|^2). \tag{5.22}$$

Proof. Let $(\mathcal{T}_{\ell+m} \mid m \in \mathbb{N})$ be a sequence of nested refinements of $\mathcal{T}_\ell$ with $\|h_{\ell+m}\|_\infty \to 0$ as $m \to \infty$. The a priori convergence results (for instance Proposition 5.6) and the discrete reliability prove the reliability. The efficiency follows from the standard techniques of [49]. Higher-order terms are absorbed for $\|h_0\|_\infty \ll 1$. □

5.4 Adaptive Algorithm and Contraction Property

This section presents the adaptive algorithm and the contraction property.

For any simplex $T \in \mathcal{T}_\ell$, the explicit residual-based error estimator consists of the sum of the residuals of the computed discrete eigenfunctions $(u_{\ell,j})_{j\in J}$,

$$\eta_\ell^2(T) := \sum_{j\in J}\left(h_T^2 \|\lambda_{\ell,j} u_{\ell,j}\|_{L^2(T)}^2 + \sum_{F\in\mathcal{F}(T)} h_T^{-1} \|[u_{\ell,j}]_F\|_{L^2(F)}^2 \right).$$

Let, for any subset $\mathcal{K} \subseteq \mathcal{T}$,

$$\eta_\ell^2(\mathcal{K}) := \sum_{T\in\mathcal{K}} \eta_\ell^2(T).$$

For the linear Stokes problem this type of error estimator without pressure contribution was introduced by [29].

The adaptive algorithm is driven by this computable error estimator and runs the following loop.

Algorithm 5.9 (AFEM for the Stokes eigenvalue problem)**.**
Input: Initial triangulation $\mathcal{T}_0$, bulk parameter $0 < \theta \leq 1$.
for $\ell = 0, 1, 2, \dots$
Solve. Compute discrete eigenpairs $(\lambda_{\ell,j}, u_{\ell,j}, p_{\ell,j})_{j\in J}$ of (5.13) with respect to $\mathcal{T}_\ell$.
Estimate. Compute local contributions of the error estimator $\big(\eta_\ell^2(T)\big)_{T\in\mathcal{T}_\ell}$.
Mark. Choose a minimal subset $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta\eta_\ell^2(\mathcal{T}_\ell) \leq \eta_\ell^2(\mathcal{M}_\ell)$.
Refine. Generate $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ with the refinement rules of [47].
end for
Output: Triangulations $(\mathcal{T}_\ell)_\ell$ and discrete solutions $\big((\lambda_{\ell,j}, u_{\ell,j}, p_{\ell,j})_{j\in J}\big)_\ell$.

The proof of the contraction property follows in a similar way as for the eigenvalues of the Laplacian. The error estimator reduction is identical to that of Proposition 4.8.

Proposition 5.10 (quasi-orthogonality)**.** *Under the hypothesis* $\|h_0\|_\infty \ll 1$ *there exists a constant* C_{qo} *such that any eigenpair* $(\lambda, u, p) \in \mathbb{R}\times W \times M$ *of* (5.10) *with* $\|u\| = 1$*, any* $\mathcal{T}_\ell \in \mathbb{T}$ *and any admissible refinement* $\mathcal{T}_{\ell+m}$ *of* $\mathcal{T}_\ell$ *satisfy*

$$\begin{aligned}&|2a_{\mathrm{NC}}(u - \Lambda_{\ell+m}u, \Lambda_{\ell+m}u - \Lambda_\ell u)|\\ &\qquad \leq C_{\mathrm{qo}}(\|h_\ell \lambda P_\ell u\|_{L^2(\cup\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})} + \boldsymbol{r}_{\ell+m})\|u - \Lambda_{\ell+m}u\|_{\mathrm{NC}}.\end{aligned}$$

Proof. The nonconforming interpolation operator $\mathcal{I}_\ell^{\mathfrak{CR}}$ maps functions from Z as well as functions from $Z_{\ell+m}$ to the space Z_ℓ, i.e., it preserves the (piecewise) divergence-free property. Hence, the proof of Proposition 4.10 applies almost verbatim. The details are omitted. □

Note that the quasi-orthogonality is stated for the velocity approximations only. A quasi-orthogonality of the pressure as in [39] is not needed in this analysis.

Proposition 5.11 (contraction property)**.** *Under the condition* $\|h_0\|_\infty \ll 1$*, there exist* $0 < \rho_2 < 1$ *and* $0 < \beta, \gamma < \infty$ *such that, for any eigenpair* $(\lambda, u, P) \in \mathbb{R}\times W\times M$ *of* (5.10) *with* $\|u\| = 1$*, the term* $\xi_\ell^2 := \mu_\ell^2(\mathcal{T}_\ell, \lambda, u) + \beta\|u - \Lambda_\ell u\|^2 + \gamma\|h_\ell P_\ell u\|^2$ *satisfies*

$$\xi_{\ell+1}^2 \leq \rho_2 \xi_\ell^2 \quad \text{for all } \ell \in \mathbb{N}_0.$$

Proof. The proof essentially follows the steps from Proposition 4.11. The pressure variable only arises in higher-order terms that are controlled by the error estimator. The details are omitted for brevity. □

5.5 Optimal Convergence Rates

This subsection establishes optimal convergence rates of Algorithm 5.9. For the linear Stokes problem, the optimal convergence of AFEMs has been proven in [2, 39, 21].

Define the seminorm

$$|(u,p)|_{\mathfrak{A}_\sigma^{\text{Stokes}}} := \sup_{m\in\mathbb{N}} m^\sigma \inf_{\mathcal{T}\in\mathbb{T}(m)} (\|(1-\Pi_\mathcal{T}^0)Du\| + \|(1-\Pi_\mathcal{T}^0)p\|)$$

and the approximation class

$$\mathfrak{A}_\sigma^{\text{Stokes}} := \left\{(v,q)\in V\times M \;\middle|\; |(v,q)|_{\mathfrak{A}_\sigma^{\text{Stokes}}} < \infty\right\}.$$

The set $\mathfrak{A}_\sigma^{\text{Stokes}}$ does not depend on the finite element method and instead concerns the approximability of the derivative and the pressure variable by piecewise constant functions. The following alternative set, also referred to as approximation class, is used for proving optimal convergence rates

$$\mathfrak{A}_\sigma^{\text{NC,Stokes}} := \left\{(u,p)\in V\times M \;\middle|\; |u|_{\mathfrak{A}_\sigma^{\text{NC,Stokes}}} < \infty\right\}$$

for

$$|(u,p)|_{\mathfrak{A}_\sigma^{\text{NC,Stokes}}} := \sup_{m\in\mathbb{N}} m^\sigma \inf_{\mathcal{T}\in\mathbb{T}(m)} (\|u-\Lambda_\mathcal{T} u\|_{\text{NC}} + \|p-p(\Lambda_\mathcal{T} u)\|).$$

Proposition 5.6 establishes the equivalence of those two approximation classes in the sense that any eigenfunction $(u,p)\in W\times M$ satisfies $(u,p)\in\mathfrak{A}_\sigma^{\text{Stokes}}$ if and only if $(u,p)\in\mathfrak{A}_\sigma^{\text{NC,Stokes}}$. The following theorem states optimality of Algorithm 5.9. The proof will be outlined in the remaining parts of this section.

Theorem 5.12 (optimal convergence rates)**.** *Provided the bulk parameter $\theta \ll 1$ and the initial mesh-size $\|h_0\|_\infty \ll 1$ are sufficiently small, Algorithm 5.9 computes sequences of triangulations $(\mathcal{T}_\ell)_\ell$ and discrete eigenpairs $((\lambda_{\ell,j},u_{\ell,j},p_{\ell,j})_{j\in J})_\ell$ with optimal rate of convergence in the sense that, for some constant C_{opt}, it holds that*

$$\sup_{\ell\in\mathbb{N}} \big(\operatorname{card}(\mathcal{T}_\ell)-\operatorname{card}(\mathcal{T}_0)\big)^{2\sigma} \sum_{j\in J} \big(\|u_j-\Lambda_\ell u_j\|_{\text{NC}}^2 + \|p_j - p(\Lambda_\ell u_j)\|^2\big) \le C_{\text{opt}} \sum_{j\in J} |(u_j,p_j)|^2_{\mathfrak{A}_\sigma^{\text{NC,Stokes}}}.$$

Let for any $w\in W$ with the representation $w=\sum_{j\in J}\alpha_j u_j$ the corresponding pressure be defined as $p(w) := \sum_{j\in J}\alpha_j p_j$. For any $v_\ell\in W_\ell$ with representation $v_\ell = \sum_{j\in J}\beta_j\Lambda_\ell u_j$ define $p(v_\ell) := \sum_{j\in J}\beta_j p(\Lambda_\ell u_j)$. Proposition 5.6 implies the following immediate consequence.

Corollary 5.13. *Provided the bulk parameter $\theta\ll 1$ and the initial mesh-size $\|h_0\|_\infty \ll 1$ are sufficiently small, Algorithm 5.9 computes triangulations $(\mathcal{T}_\ell)_\ell$ and discrete eigenpairs $((\lambda_{\ell,j},u_{\ell,j},p_{\ell,j})_{j\in J})_\ell$ with optimal rate of convergence in the sense that*

$$\sup_{\ell\in\mathbb{N}}(\operatorname{card}(\mathcal{T}_\ell)-\operatorname{card}(\mathcal{T}_0))^\sigma \sup_{\substack{w\in W\\ \|w\|=1}} \inf_{v_\ell\in W_\ell} \big(\|w-v_\ell\|_{\text{NC}}^2 + \|p(w)-p(v_\ell)\|^2\big)^{1/2} \lesssim \left(\sum_{j\in J}|(u_j,p_j)|^2_{\mathfrak{A}_\sigma^{\text{Stokes}}}\right)^{1/2}. \qquad \square$$

The proof of optimal convergence rates is almost identical to that presented in Subsection 4.5. The only difference is that the pressure term appears in certain estimates. The modifications are sketched in the remaining part of this subsection.

Consider

$$\Xi_\ell^2 := \mu_\ell^2(\mathcal{T}_\ell) + \beta \sum_{j\in J} |||u_j - \Lambda_\ell u_j|||_{\mathrm{NC}}^2 + \gamma \sum_{j\in J} \|h_\ell \lambda_j P_\ell u_j\|^2 \quad \text{for all } \ell \in \mathbb{N}_0$$

for the parameters β and γ from Proposition 5.11. Choose

$$0 < \tau \leq \sum_{j\in J} |(u_j, p_j)|^2_{\mathfrak{A}_\sigma^{\mathrm{NC,Stokes}}} / \Xi_0^2$$

and set $\varepsilon(\ell) := \sqrt{\tau}\, \Xi_\ell$. Let $N(\ell) \in \mathbb{N}$ be minimal with the property

$$\sum_{j\in J} |(u_j, p_j)|^2_{\mathfrak{A}_\sigma^{\mathrm{NC,Stokes}}} \leq \varepsilon(\ell)^2\, N(\ell)^{2\sigma}.$$

Let $\widetilde{\mathcal{T}}_\ell \in \mathbb{T}$ denote the optimal triangulation of cardinality

$$\mathrm{card}(\widetilde{\mathcal{T}}_\ell) \leq \mathrm{card}(\mathcal{T}_0) + N(\ell)$$

in the sense that the projection $\widetilde{\Lambda} := \Lambda_{\widetilde{\mathcal{T}}_\ell}$ with respect to $\widetilde{\mathcal{T}}_\ell$ satisfies

$$\sum_{j\in J} \left(|||u_j - \widetilde{\Lambda} u_j|||^2 + \|p_j - p(\widetilde{\Lambda} u_j)\|^2 \right) \leq N(\ell)^{-2\sigma} \sum_{j\in J} |u_j|^2_{\mathfrak{A}_\sigma^{\mathrm{NC,Stokes}}} \leq \varepsilon(\ell)^2 \tag{5.23}$$

and define $\widehat{\mathcal{T}}_\ell := \mathcal{T}_\ell \otimes \widetilde{\mathcal{T}}_\ell$ as the overlay. The arguments of [22, 33] lead to

$$\mathrm{card}(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell) \leq N(\ell) \leq 2 \left(\sum_{j\in J} |u_j|^2_{\mathfrak{A}_\sigma^{\mathrm{NC,Stokes}}} \right)^{1/(2\sigma)} \varepsilon(\ell)^{-1/\sigma}.$$

Let $\widehat{\Lambda} := \Lambda_{\widehat{\mathcal{T}}_\ell}$ denote the projection with respect to $\widehat{\mathcal{T}}$.

Lemma 5.14. *Provided $\|h_0\|_\infty \ll 1$, it holds that*

$$\sum_{j\in J} \left(|||u_j - \widehat{\Lambda} u_j|||_{\mathrm{NC}}^2 + \|p_j - p(\widehat{\Lambda} u_j)\|^2 \right) \lesssim \varepsilon(\ell)^2.$$

Proof. As in the proof of Lemma 4.14, recall that by definition of the overlay [22] the triangulations $\widehat{\mathcal{T}}_\ell$ and $\widetilde{\mathcal{T}}_\ell$ are nested. Hence, the best-approximation result of Proposition 5.6 and (5.23) prove

$$\begin{aligned}\sum_{j\in J} &\left(|||u_j - \widehat{\Lambda} u_j|||_{\mathrm{NC}}^2 + \|p_j - p(\widehat{\Lambda} u_j)\|^2 \right) \\ &\lesssim \sum_{j\in J} \left(|||u_j - \widetilde{\Lambda} u_j|||_{\mathrm{NC}}^2 + \|p_j - p(\widetilde{\Lambda} u_j)\|^2 \right) \leq \varepsilon(\ell)^2. \qquad \square\end{aligned}$$

Lemma 5.15 (key argument)**.** *Provided $\|h_0\|_\infty \ll 1$, there exists $C_2 \approx 1$ such that*

$$\mu_\ell^2(\mathcal{T}_\ell) \leq C_2 \mu_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell).$$

Proof. The discrete reliability from Proposition 5.7, the efficiency from Corollary 5.8 and the arguments of Lemma 4.15 lead to the desired estimate. The details are omitted for brevity. □

The finish of the optimality proof is identical to that of [33, Lemma 7.3] and therefore omitted.

Lemma 5.16 (finish of the optimality proof)**.** *The choice*

$$0 < \theta \leq 1 \big/ \big(C_2 (B/A)^4 (2N^2 + 4N^3)\big)$$

implies the existence of some constant $C(\sigma)$ *such that*

$$\sup_{\ell \in N} \big(\operatorname{card}(\mathcal{T}_\ell) - \operatorname{card}(\mathcal{T}_0)\big)^\sigma \left(\sum_{j \in J} \big(|\!|\!| u_j - \Lambda_\ell u_j |\!|\!|_{\mathrm{NC}}^2 + \| p_j - p(\Lambda_\ell u_j) \|^2 \big) \right)^{1/2} \leq C(\sigma) \left(\sum_{j \in J} |(u_j, p_j)|^2_{\mathfrak{A}_\sigma^{\mathrm{NC,Stokes}}} \right)^{1/2}. \qquad \square$$

References

[1] Gabriel Acosta, Ricardo G. Durán, and María A. Muschietti. Solutions of the divergence operator on John domains. *Adv. Math.*, 206(2):373–401, 2006.

[2] Roland Becker and Shipeng Mao. Quasi-optimality of adaptive nonconforming finite element methods for the Stokes equations. *SIAM J. Numer. Anal.*, 49(3):970–991, 2011.

[3] Roland Becker, Shipeng Mao, and Zhongci Shi. A convergent nonconforming adaptive finite element method with quasi-optimal complexity. *SIAM J. Numer. Anal.*, 47(6):4639–4659, 2010.

[4] D. Boffi, R. G. Duran, F. Gardini, and L. Gastaldi. A posteriori error analysis for nonconforming approximation of multiple eigenvalues. *arXiv e-Prints*, 1404.5560, 2014.

[5] Daniele Boffi, Franco Brezzi, and Michel Fortin. *Mixed Finite Element Methods and Applications*, volume 44 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2013.

[6] Dietrich Braess. *Finite Elements. Theory, Fast Solvers, and Applications in Elasticity Theory*. Cambridge University Press, Cambridge, third edition, 2007.

[7] Dietrich Braess. An a posteriori error estimate and a comparison theorem for the nonconforming P_1 element. *Calcolo*, 46(2):149–155, 2009.

[8] Susanne C. Brenner. Two-level additive Schwarz preconditioners for nonconforming finite element methods. *Math. Comp.*, 65(215):897–921, 1996.

[9] Susanne C. Brenner and L. Ridgway Scott. *The Mathematical Theory of Finite Element Methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[10] Carsten Carstensen and Joscha Gedicke. An oscillation-free adaptive FEM for symmetric eigenvalue problems. *Numer. Math.*, 118(3):401–427, 2011.

[11] C. Carstensen, M. Eigel, R.H.W. Hoppe, and C. Löbhard. A review of unified a posteriori finite element error control. *Numer. Math. Theory Methods Appl.*, 5(4):509–558, 2012.

[12] C. Carstensen, M. Feischl, M. Page, and D. Praetorius. Axioms of adaptivity. *Comput. Math. Appl.*, 67(6):1195–1253, 2014.

[13] C. Carstensen, D. Gallistl, and M. Schedensack. Discrete reliability for Crouzeix-Raviart FEMs. *SIAM J. Numer. Anal.*, 51(5):2935–2955, 2013.

[14] C. Carstensen, D. Gallistl, and M. Schedensack. Adaptive nonconforming Crouzeix-Raviart FEM for eigenvalue problems. *Math. Comp.*, 2014. In print.

[15] C. Carstensen and E.-J. Park. Convergence and optimality of adaptive least squares finite element methods. 2013. In preparation.

[16] C. Carstensen, D. Peterseim, K. Köhler, and M. Schedensack. Comparison results for the Stokes equations. *Appl. Numer. Math.*, 1401.5915v1, 2014. In press, `http://dx.doi.org/10.1016/j.apnum.2013.12.005`.

[17] C. Carstensen, D. Peterseim, and M. Schedensack. Comparison results of finite element methods for the Poisson model problem. *SIAM J. Numer. Anal.*, 50(6):2803–2823, 2012.

[18] Carsten Carstensen and Joscha Gedicke. An adaptive finite element eigenvalue solver of quasi-optimal computational complexity. *SIAM J. Numer. Anal.*, 50(3):1029–1057, 2012.

[19] Carsten Carstensen and Joscha Gedicke. Guaranteed lower bounds for eigenvalues. *Math. Comp.*, 2014. In print.

[20] Carsten Carstensen and Ronald H. W. Hoppe. Convergence analysis of an adaptive nonconforming finite element method. *Numer. Math.*, 103(2):251–266, 2006.

[21] Carsten Carstensen, Daniel Peterseim, and Hella Rabus. Optimal adaptive nonconforming FEM for the Stokes problem. *Numer. Math.*, 123(2):291–308, 2013.

[22] J.M. Cascon, Ch. Kreuzer, R. H. Nochetto, and K. G. Siebert. Quasi-optimal convergence rate for an adaptive finite element method. *SIAM J. Numer. Anal.*, 46(5):2524–2550, 2008.

[23] Françoise Chatelin. *Spectral Approximation of Linear Operators.* Computer Science and Applied Mathematics. Academic Press Inc., New York, 1983.

[24] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge*, 7(R-3):33–75, 1973.

[25] Xiaoying Dai, Lianhua He, and Aihui Zhou. Convergence rate and quasi-optimal complexity of adaptive finite element computations for multiple eigenvalues. *arXiv e-Prints*, 1210.1846v2, 2013.

[26] Xiaoying Dai, Jinchao Xu, and Aihui Zhou. Convergence and optimal complexity of adaptive finite element eigenvalue computations. *Numer. Math.*, 110(3):313–355, 2008.

[27] E. Dari, R. Duran, C. Padra, and V. Vampa. A posteriori error estimators for nonconforming finite element methods. *RAIRO Modél. Math. Anal. Numér.*, 30(4):385–400, 1996.

[28] E. A. Dari, R. G. Durán, and C. Padra. A posteriori error estimates for non-conforming approximation of eigenvalue problems. *Appl. Numer. Math.*, 62(5):580–591, 2012.

[29] Enzo Dari, Ricardo Durán, and Claudio Padra. Error estimators for nonconforming finite element approximations of the Stokes problem. *Math. Comp.*, 64(211):1017–1033, 1995.

[30] Daniele Antonio Di Pietro and Alexandre Ern. *Mathematical Aspects of Discontinuous Galerkin Methods*, volume 69 of *Mathématiques & Applications (Berlin).* Springer, Heidelberg, 2012.

[31] E. B. Fabes, C. E. Kenig, and G. C. Verchota. The Dirichlet problem for the Stokes system on Lipschitz domains. *Duke Math. J.*, 57(3):769–793, 1988.

[32] Dietmar Gallistl. Morley finite element method for the eigenvalues of the biharmonic operator. 2014. In preparation.

[33] Dietmar Gallistl. An optimal adaptive FEM for eigenvalue clusters. 2014. Matheon Preprint #1059, DFG Research Center Matheon,
`http://opus4.kobv.de/opus4-matheon/files/1287/EVclusterAFEM.pdf`.

[34] Eduardo M. Garau and Pedro Morin. Convergence and quasi-optimality of adaptive FEM for Steklov eigenvalue problems. *IMA J. Numer. Anal.*, 31(3):914–946, 2011.

[35] Eduardo M. Garau, Pedro Morin, and Carlos Zuppa. Convergence of adaptive finite element methods for eigenvalue problems. *Math. Models Methods Appl. Sci.*, 19(5):721–747, 2009.

[36] S. Giani and I. G. Graham. A convergent adaptive method for elliptic eigenvalue problems. *SIAM J. Numer. Anal.*, 47(2):1067–1091, 2009.

[37] Thirupathi Gudi. A new error analysis for discontinuous finite element methods for linear elliptic problems. *Math. Comp.*, 79(272):2169–2189, 2010.

[38] Johnny Guzmán and Michael Neilan. Conforming and divergence-free Stokes elements on general triangular meshes. *Math. Comp.*, 83(285):15–36, 2014.

[39] Jun Hu and Jinchao Xu. Convergence and optimality of the adaptive nonconforming linear element method for the Stokes problem. *J. Sci. Comput.*, 55(1):125–148, 2013. Preliminary version published 2007 as technical report `http://www.math.pku.edu.cn:8000/var/preprint/7297.pdf`.

[40] Tosio Kato. *Perturbation Theory for Linear Operators*, volume 132 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, 1966.

[41] ShiPeng Mao and ZhongCi Shi. On the error bounds of nonconforming finite elements. *Sci. China Math.*, 53(11):2917–2926, 2010.

[42] H. Rabus. A natural adaptive nonconforming FEM of quasi-optimal complexity. *Comput. Methods Appl. Math.*, 10(3):315–325, 2010.

[43] Giuseppe Savaré. Regularity results for elliptic equations in Lipschitz domains. *J. Funct. Anal.*, 152(1):176–201, 1998.

[44] L. R. Scott and M. Vogelius. Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials. *RAIRO Modél. Math. Anal. Numér.*, 19(1):111–143, 1985.

[45] L. Ridgway Scott and Shangyou Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.

[46] Rob Stevenson. Optimality of a standard adaptive finite element method. *Found. Comput. Math.*, 7(2):245–269, 2007.

[47] Rob Stevenson. The completion of locally refined simplicial partitions created by bisection. *Math. Comp.*, 77(261):227–241, 2008.

[48] Gilbert Strang and George J. Fix. *An Analysis of the Finite Element Method*. Prentice-Hall Series in Automatic Computation. Prentice-Hall Inc., Englewood Cliffs, N. J., 1973.

[49] Rüdiger Verfürth. *A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*. Advances in numerical mathematics. John Wiley & Sons., Chichester, 1996.