Some research projects

by Andreas Juhl

The following research projects are concerned with problems on the cutting edge of conformal differential geometry, spectral theory on asymptotically hyperbolic (Einstein) manifolds and geometric aspects of the AdS/CFT duality. This is a rather extended area, and we formulate here only those projects for which we expect substantial progress in the near future.

(1) **Recursive structure of Q-curvatures and GJMS-operators**

In [16], it was proved that (at least) for conformally flat metrics h, and special values of the family parameter λ, the residue families $D_{2N}^{res}(h;\lambda)$ factorize into lower order residue families and GJMS operators. These factorization identities are curved analogs of multiplicity free branching rules in representation theory. They imply recursive formulae for Q-curvatures and GJMS-operators.

A central open problem is to establish these factorization identities in full generality (i.e., the full system of identities for all metrics).

In addition, it is important to resolve the resulting recursive structures. In fact, for Q-curvatures, the results in [10] revealed an intriguing combinatorial picture which deserves a further study.

The methods of [10] generated compact formulas for Q_6, Q_8, Q_{10} and similar conjectural formulae for high order Q-curvatures. Such formulae open the possibility of applications in geometric analysis. Establishing full proofs of all proposed formulae requires substantial additional efforts.

In [14] we discovered an alternative universal recursive description of Q-curvatures. This description is part of a recursive descriptions of the GJMS-operators itself. One of its features is that low order conformally covariant powers of the Laplacian are combined to high order powers. This very effective method enabled to make explicit the conformally covariant third and fourth powers of the Laplacian. It was surprising to see how Graham's extended obstruction tensors [12] appear in that connection. The methods of [14] are expected to apply to other situations as well.

The full development of the picture outlined in [14] is a challenging task. These investigations enter totally unknown territory and it seems natural to expect many new and surprising effects. Even the discovery of recursive structures in this area was quite unexpected.

(2) **Holographic formulae for Q-curvatures. Universality**

The holographic formula [13] relates the critical Q-curvature Q_n to the holographic anomaly v_n of the renormalized volume of Poincaré-Einstein spaces. The structure of the divergence terms in that formula remain to be studied more closely. Similar holographic descriptions of subcritical Q-curvatures have been proposed in [16] and remain to be proved and analyzed in full generality. The relation between the holographic anomaly v_n and the Euler-form is not well-understood in higher dimensions and for general metrics. Progress on these questions are relevant also in connection with the recent fundamental structural results of Alexakis [4], [5], [3]. The interactions of the holographic formulas with the recursive formulae are still mysterious and remain to be explored.

A central feature of the recursive formulae for Q-curvatures and GJMS-operators is their independence of the dimension of the underlying space (universality). A full and natural proof remains to be found.

(3) **Conformally covariant families and Cartan geometry**

The tractor calculus construction $D_N^T(X, M; g; \lambda)$ in [16] can be considered as kind of an induction by families of homomorphisms of semi-holonomic Verma modules. To any of the relevant families of homomorphisms of Verma modules there correspond several lifts into the semi-holonomic category. The tractor construction in [16] corresponds to the choice of *one* such lift. All other lifts should also induce conformally covariant families. The relevant induction mechanisms remain to be established and evaluated. Here the relations between tractor calculus and Cartan geometry [18] play a significant role.

(4) **Holographic duality for general metrics**

The holographic duality in [16] is a compressed formulation of intriguing relations between Q-curvatures, holographic data and the geometry of conformally invariant connections on X and M. In [16] this duality was established for conformally flat metrics. The version of the duality for general metrics is widely open. This problem is closely related to the problem of extending the theory of factorization identities to tractor families. Progress here would contribute to an understanding of the structure of the extrinsic Q-curvatures introduced in [16]. These are curvature quantities of embeddings which behave much like Branson's Q-curvatures.

For $(X, M) = (S^{n+1}, S^n)$, the holographic duality is suggested by the description of divisors of Selberg zeta functions as in [15] in terms of automorphic distributions. The connection to Selberg zeta functions, in turn, suggests a series of other dualities. The following two projects deal with two special cases.

(5) **Conformally covariant families for submanifolds of higher codimension, AdS/CFT-duality**

Here we consider analogs for submanifolds of arbitrary codimension of the conformally covariant families for hypersurfaces. The aim of this project is to understand the relations between conformal invariants of submanifolds and holographic data which are induced by associated submanifolds of Poincaré-Einstein spaces. A prominent special case concerns closed loops and associated minimal surfaces in a hyperbolic space. Such situation deserve significant interest in connection with the AdS/CFT duality [1], [2], [6], and raise a wealth of interesting geometric problems.

(6) **Q-curvature operators on differential forms**

The relation to Selberg zeta functions suggests the existence of equivariant families of differential operators $\Omega^*(S^{n+1}) \to \Omega^*(S^n)$ on differential forms. We expect that an extension of the methods in [16] will sheds new light on the theory of Q-curvature operators [8]. In particular, we expect holographic and recursive formulas for Q-curvature operators. Here the results of [7] should be relevant.

(7) **The geometric meaning of the full Q-polynomial**

In [16], we introduced the notion of Q-polynomials of Riemannian manifolds. For a Riemannian manifold (M, h) of even dimension n, its critical Q-polynomial $Q_n^{res}(h; \lambda)$ is a polynomial of degree $\frac{n}{2}$. The constant coefficient of the Q-polynomial vanishes, and its linear coefficient is the Q-curvature $Q_n(h)$ (this fact is equivalent to the holographic formula for Q_n [13]).

In [16], we also found a relation between the integral of the quadratic coefficient and the renormalized volume. The values of the Q-polynomials play a crucial role in the recursive theory. But the *geometric meanings* of the higher coefficients of the Q-polynomial are relatively less understood and deserve a detailed investigation. Analogous questions should be studied for the corresponding Q-polynomial in the framework of the tractor families $D_n^T(X, M; g; \lambda)$.

(8) **Obstruction theory**

For general background metrics g and even dimension n, the tractor families $D^T_{2N}(X, M; g; \lambda)$ of even order $2N > n$ have a finite set of remarkable simple poles. If the residues of these poles (and the contributions of the extrinsic geometry) vanish, then the corresponding values of the families yield conformally covariant powers of the Laplacian on the submanifold. These are constructions in the regime where conformally covariant powers of the Laplacian do not exist [11]. The construction yield conformally covariant powers for certain conformal classes. The consequences of this construction are not understood and remain to be analyzed.

(9) **Geometric properties of extrinsic Q-curvature**

In [16], the tractor families led to the notion of extrinsic Q-curvature Q^e_n of an immersion $M^n \hookrightarrow X^{n+1}$. If Q^e_n vanishes, then the quantity $\dot{D}^T_n(X, M, g; 0)(1)$ only depends on the intrinsic geometry of (M, g), and is a Q-like curvature quantity, i.e., under conformal changes of the metric its transformation is governed by a *linear* differential operator. These results lead to a series of new problems. In particular, one should establish geometric characterizations of the immersions with vanishing extrinsic Q-curvature. Of course, explicit formulas for Q^e_n are highly desirable. Is it possible to describe or to characterize those metrics for which the Q-like curvature coincides with the usual Q-curvature? The extrinsic Q-curvature immediately gives rise to Yamabe-problems for immersions. In the case $M^3 \hookrightarrow X^4$, the tractor Q-curvature coincides with the T-curvature of Chang-Qing [9] and corresponding Yamabe-problems have been studied recently [17].

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