Functional monotone class theorem

Theorem. *Let Ω be a set and H be a vector space of bounded functions from Ω to $\mathbb{R}$ such that*

1. the constant function 1 is an element of H,
2. if $(f_n)_{n\in\mathbb{N}} \subseteq H$, $f_n \geq 0$, such that $f_n \nearrow f$ for a bounded function $f : \Omega \to \mathbb{R}$, then $f \in H$.

If $C \subseteq H$ and C is closed under multiplication (i.e. $f, g \in C \Rightarrow f \bullet g \in C$), then $\sigma(C) \subseteq H$.

Proof. See e.g. planetmath or Williams (1991). $\square$

Exercise 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with random variables X, Y. Let further $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra such that $Y \perp \mathcal{G}$ and $X \in \mathcal{G}$. Then

$$\mathbb{E}\left[\left. f\left(X, Y\right)\right| \mathcal{G}\right] = g\left(X\right)$$

for $g(x) = \int f(x,y)\, dP_Y(y)$, where f is any bounded measurable function.

Exercise 2. Let $X = (X_t, t \geq 0)$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $X_0 = 0$ and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration. Then it is equivalent:

(i) X has independent increments.

(ii) $X_t - X_s \perp \mathcal{F}_s$ for all $t \geq s$.

Exercise 3. Consider the path space E^T with canonical σ-algebra $\mathcal{E}^{\otimes T} = \sigma(\pi_t : t \in T)$, where $\pi_t : E^T \to E, \pi_t(f) = f(t)$ are the coordinate projections. Let further $\pi_S : E^T \to E^S, \pi_S(f) = (f(t))_{t\in S}$ be the projection to E^S, $S \subseteq T$. Show that $X : E^T \to \mathbb{R}$ is $\mathcal{E}^{\otimes T}$-measurable if and only if there exists a $\mathcal{E}^{\otimes S}$-measurable function $g : E^S \to \mathbb{R}$ for some countable $S \subseteq T$ such that $X = g \circ \pi_S$.

References

Williams, D. (1991). *Probability with Martingales*. Cambridge mathematical textbooks. Cambridge University Press.