Humboldt-Universität zu Berlin

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Stochastic analysis
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Exercise sheet 10

Exercises

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where $(\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions, and let B be an $(\mathcal{F}_t)$-Brownian motion.

Exercise 10.1 (5 Points)
Let $((B^n_t)_{t\geq 0})_{n\in\mathbb{N}}$ be a sequence of processes such that $t \mapsto B^n_t \in C^1(\mathbb{R}_+, \mathbb{R})$ for all n, and such that $\lim_{n\to\infty} \sup_{t\leq T} |B^n_t - B_t| = 0$ for all $T \geq 0$. Let $f \in C(\mathbb{R}, \mathbb{R})$. Show that for all $n \in \mathbb{N}$,

$$\int_0^t f(B^n_s) dB^n_s = F(B^n_t) - F(B^n_0), \qquad t \geq 0,$$

where $F(x) = \int_0^x f(y) dy$. Conclude that there exists a continuous process

$$\int_0^t f(B_s) \circ dB_s := \lim_{n\to\infty} \int_0^t f(B^n_s) dB^n_s, \qquad t \geq 0,$$

which does not depend on the approximating sequence $(B^n)_{n\in\mathbb{N}}$. $\int_0^\cdot f(B_s) \circ dB_s$ is called *Stratonovich integral* of $f(B)$ with respect to B. Show that if $f \in C^2(\mathbb{R}, \mathbb{R})$, then

$$\int_0^t f(B_s) \circ dB_s = \int_0^t f(B_s) dB_s + \frac{1}{2} \int_0^t f'(B_s) ds = \int_0^t f(B_s) dB_s + \frac{1}{2}\langle f(B), B\rangle_t, \qquad t \geq 0.$$

Exercise 10.2 (5 Points)
Let $M \in \mathcal{M}_{\text{loc}}$ and let $K \in L^2_{\text{loc}}(M)$ and $H \in L^2_{\text{loc}}(K \cdot M)$. Show that $HK \in L^2_{\text{loc}}(M)$ and

$$((HK) \cdot M) = (H \cdot (K \cdot M)).$$

Exercise 10.3 (3+4+3 Points)
We define for $n \in \mathbb{N}_0$ the function $h_n(x) := e^{x^2/2}(-1)^n \partial_x^n e^{-x^2/2}$, $x \in \mathbb{R}$.

a) Show that for $n \geq 1$ the recursion formula

$$h_{n+1}(x) = x h_n - n h_{n-1}(x), \qquad x \in \mathbb{R},$$

holds. Deduce that h_n is a polynomial of degree n for all $n \geq 0$ and that $\partial_x h_n = n h_{n-1}$ for all $n \geq 1$. h_n is called the n-th *Hermite polynomial.*

Hint: You may use without proof the generalized Leibniz rule $\partial_x^n (fg) = \sum_{k=0}^n \binom{n}{k} \partial_x^k f \partial_x^{n-k} g$.

b) Define now $H_n(x, a) := a^{n/2} h_n(x/\sqrt{a})$ for $a > 0$ and $H_n(x, 0) := x^n$. You may use without proof that $H_n \in C^2(\mathbb{R}, \mathbb{R}_+)$. Show that $(H_n(B_t, t))_{t\geq 0}$ is a (local) martingale for $n \geq 1$ and that

$$H_n(B_t, t) = \int_0^t n H_{n-1}(B_s, s) dB_s, \qquad t \geq 0.$$

c) Calculate $H_n(B_t, t)$ for $n = 1, 2, 3$.

Due date: June 29, 2016. You may submit your solutions in groups of two.