Humboldt-Universität zu Berlin

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Stochastic analysis
Summer semester 2016
Exercise sheet 11

Exercises

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where $(\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions, and let B be an $(\mathcal{F}_t)$-Brownian motion.

Exercise 11.1 (2 Points)
Let M be an adapted and integrable càdlàg process such that $\mathbb{E}[M_T] = \mathbb{E}[M_0]$ for every bounded stopping time T. Show that M is a martingale.

Exercise 11.2 (5 Points)
Let $D \in \mathcal{M}_{\text{loc}}$ be such that $D_t > 0$ for all $t \geq 0$. Show that there exists a unique $L \in \mathcal{M}_{\text{loc}}$ such that $D_t = \exp\{L_t - \frac{1}{2}\langle L, L\rangle_t\}$, for all $t \geq 0$, and that L satisfies: for all $t \geq 0$, $L_t = \log D_0 + \int_0^t D_s^{-1} dD_s$.

Exercise 11.3 (5 Points)
Let $B^i, i = 1, 2, 3$ three Brownian motions, such that B^1 and B^2 are independent. Show that it is impossible to have $\sigma(B_s^3 : 0 \leq s \leq t) = \sigma(B_s^1, B_s^2 : 0 \leq s \leq t)$.

Exercise 11.4 (5 Points)
Let $M_t := \int_0^t \mathbb{1}_{B_s > 0} dB_s$, for $t \geq 0$.

a) Justify that M is martingale.

b) Find an increasing process A such that $(M_t^2 - A_t)_{t\geq 0}$ is a martingale.

c) We admit that there exists an increasing process C such that $A(C(t)) = t$, for all $t \geq 0$. Prove that $(M_{C_t})_{t\geq 0}$ is a $(\mathcal{G}_t)$-martingale where $(\mathcal{G}_t)_{t\geq 0}$ is a filtration to specify.

Exercise 11.5 (2+2+2 Points)
Let X be an Itô process, that is $X = X_0 + \int_0^\cdot a_s ds + \int_0^\cdot b_s dB_s$, where X_0 is $\mathcal{F}_0$-measurable and a and b are progressively measurable processes such that $\int_0^t |b_s|^2 ds < \infty$ and $\int_0^t |a_s| ds < \infty$ for all $t \geq 0$. We assume that X has the following (Itô) representations

$$\begin{aligned} X &= X_0^{(1)} + \int_0^\cdot a_s^{(1)} ds + \int_0^\cdot b_s^{(1)} dB_s \\ &= X_0^{(2)} + \int_0^\cdot a_s^{(2)} ds + \int_0^\cdot b_s^{(2)} dB_s. \end{aligned}$$

a) Show that almost surely $X_0^{(1)} = X_0^{(2)}$ and $\mathbb{P} \otimes \lambda$-almost everywhere $a^{(1)} = a^{(2)}$ and $b^{(1)} = b^{(2)}$, where λ denotes the Lebesgue measure.

b) Show that X is deterministic if and only if $b^{(1)} \equiv 0$ and $a^{(1)}$ and $X_0^{(1)}$ are deterministic.

c) Show that if $a^{(1)} \equiv 0$, $X_0^{(1)} = 0$ and $\langle X, X\rangle$ is deterministic, then X is a Gaussian process and a local martingale.

Due date: July 06, 2016. You may submit your solutions in groups of two.