Humboldt-Universität zu Berlin

Prof. Dr. Nicolas Perkowski
Dr. Achref Bachouch

Stochastic analysis
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Exercise sheet 12

Exercises

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where $(\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions, and let B be an $(\mathcal{F}_t)$-Brownian motion.

Exercise 12.1 (2+2 Points)
Let $a, \sigma > 0$.

a) Show that the process X, given by $X_t = x_0 e^{-at} + \sigma \int_0^t e^{-a(t-s)} dB_s$, $t \geq 0$, solves the SDE: $dX_t = -aX_t dt + \sigma dB_t$, $X_0 = x_0 \in \mathbb{R}$.

b) Find the explicit solution to the SDE: $dX_t = (-aX_t + \beta)dt + \sigma dB_t$, $X_0 = x_0 \in \mathbb{R}$, where $\beta > 0$.

Exercise 12.2 (10 Points)
Let μ, σ and $r > 0$. Consider the SDE: $dS_t = S_t(\mu dt + \sigma dB_t)$, $S_0 = 1$.

a) Use the Ansatz $S_t = f(t, B_t)$, $t \geq 0$, for some $f \in C^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ to find the explicit solution S.

b) Set $\theta = -\frac{\mu - r}{\sigma}$. Let Q the probability measure defined by $dQ = L_1 d\mathbb{P}$, with $L_1 := \exp\{\theta B_1 - \frac{\theta^2}{2}\}$. Show that $(W_t)_{t\in[0,1]}$, defined by $W_t = B_t - \theta t$, $t \in [0, 1]$, is a Q-Brownian motion.

c) Let $\widetilde{\mathbb{P}}$ the probability measure defined by $d\widetilde{\mathbb{P}} = Z_1 dQ$, with $Z_1 := \exp\{\sigma W_1 - \frac{\sigma^2}{2}\}$. Show that: $dS_t = S_t((r + \sigma^2)dt + \sigma d\widetilde{B}_t)$, $t \in [0, 1]$, where $(\widetilde{B}_t)_{t\in[0,1]}$ is $\widetilde{\mathbb{P}}$-Brownian motion.

d) Consider $(P_t)_{t\in[0,1]}$ defined by $P_t = e^{rt}, t \in [0, 1]$. Show that $\left(\frac{S_t}{P_t}\right)_{t\in[0,1]}$ is a Q-martingale.

e) Show that $\left(\frac{P_t}{S_t}\right)_{t\in[0,1]}$ is a $\widetilde{\mathbb{P}}$-martingale.

Exercise 12.3 (4 Points)
Let $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ be the unit sphere in $\mathbb{R}^d$, and let $K = \{x \in \mathbb{R}^d : |x| < 1\}$ be the open unit ball. Let $f : S^{d-1} \to \mathbb{R}$ be continuous. Assume h is a solution of the Dirichlet problem on K with boundary condition f, that is $h \in C^2(\overline{K}, \mathbb{R})$ and h satisfies

$$\begin{aligned} \Delta h(x) &= 0, \qquad x \in K, \\ h(x) &= f(x), \qquad x \in S^{d-1}. \end{aligned}$$

Here $\overline{K} = K \cup S^{d-1}$ is the closure of K and Δ is the Laplace operator. Show that if $(W_t^x)_{t\geq 0}$ is a d-dimensional Brownian motion started in x, and if $\tau_K = \inf\{t \geq 0 : W_t^x \in K^c\}$ is the exit time from K, then $h(x) = \mathbb{E}[f(W_{\tau_K}^x)]$.

Exercise 12.4 (4 Points)
Let now $(\Omega, \mathcal{F})$ be a new measurable space and let $(B_t)_{t\geq 0}$ be a continuous stochastic process with $B_0 = 0$. Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on $\mathcal{F}$ such that B is a Brownian motion under $\mathbb{P}$, and such that $\tilde{B}_t = B_t - t$, $t \geq 0$, is a Brownian motion under $\mathbb{Q}$. Show that $\mathbb{P}$ and $\mathbb{Q}$ are mutually singular.

Hint: Consider the behavior of B_t for $t \to \infty$ under $\mathbb{P}$ and under $\mathbb{Q}$.

Due date: July 13, 2016. You may submit your solutions in groups of two.