Humboldt-Universität zu Berlin
Prof. Dr. Nicolas Perkowski
Dr. Achref Bachouch

Stochastic analysis
Summer semester 2016
Exercise sheet 13

Exercises

Exercise 13.1 (Points)
Let $(B_t)_{t\geq 0}$ be a Brownian motion. Our aim is to construct the *local time* $(L_t)_{t\geq 0}$ which measures how much time B spends in 0.

a) Consider $g \in C^1(\mathbb{R},\mathbb{R})$ such that g' is continuously differentiable on $\mathbb{R}\backslash\{x_1,\dots,x_k\}$ for some $x_1,\dots,x_k \in \mathbb{R}$. Further assume that $|g''(x)| \leq M$ for $x \in \mathbb{R}\backslash\{x_1,\dots,x_k\}$. Show that Itô's formula is still valid for g, i.e. that

$$g(B_t) = g(B_0) + \int_0^t g'(B_s)dB_s + \frac{1}{2}\int_0^t g''(B_s)1_{\mathbb{R}\backslash\{x_1,\dots,x_k\}}(B_s)ds.$$

Now define for $\epsilon > 0$

$$g_\epsilon(x) = \begin{cases} \frac{1}{2}(\epsilon + \frac{x^2}{\epsilon}), & x \in (-\epsilon,\epsilon) \\ |x|, & x \in (-\epsilon,\epsilon)^c. \end{cases}$$

Show that:

b) $g_\epsilon(B_t) = g_\epsilon(B_0) + \int_0^t g'_\epsilon(B_s)dB_s + \frac{1}{2\epsilon}\lambda(\{s \in [0,t] : B_s \in (-\epsilon,\epsilon)\})$, where λ is Lebesgue measure;

c) $\lim_{\epsilon\to 0}\int_0^t g'_\epsilon(B_s)1_{(-\epsilon,\epsilon)}(B_s)dB_s = 0$ in $L^2(\mathbb{P})$;

d) $|B_t| = |B_0| + \int_0^t \operatorname{sgn}(B_s)dB_s + L_t$, where $\operatorname{sgn}(x) = 1$ if $x \geq 0$ and $\operatorname{sgn}(x) = -1$ if $x < 0$, and

$$L_t = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\lambda(\{s \in [0,t] : B_s \in (-\epsilon,\epsilon)\}) \text{ in } L^2(\mathbb{P}).$$

Exercise 13.2 (Points)
The aim of this exercise is to construct a *pathwise* solution to the one-dimensional SDE

$$X_t = x + \int_0^t \left(b(X_s) + \frac{1}{2}\sigma(X_s)\sigma'(X_s)\right)ds + \int_0^t \sigma(X_s)dB_s = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)\circ dB_s, \quad (1)$$

where $\int_0^t \sigma(X_s)\circ dB_s = \int_0^t \sigma(X_s)dB_s + \frac{1}{2}\langle\sigma(X),B\rangle_t$ denotes the Stratonovich integral of Exercise 10.1 and where $\sigma \in C_b^2(\mathbb{R},\mathbb{R})$, i.e. $\sigma \in C^2$ and σ as well as its derivatives up to order 2 are bounded, and $b \in C(\mathbb{R},\mathbb{R})$ is Lipschitz continuous. The following method is called *Doss-Sussmann transformation.*

Let $u \in C^2(\mathbb{R}^2,\mathbb{R})$ be the solution to the ODE

$$\partial_x u(x,y) = \sigma(u(x,y)), \quad u(0,y) = y.$$

Let B be a Brownian motion, let $x \in \mathbb{R}$, and write $D_t(\omega)$ for the solution to the ODE

$$\frac{dD_t(\omega)}{dt} = b(u(B_t(\omega),D_t(\omega)))\exp\left(-\int_0^{B_t(\omega)}\sigma'(u(z,D_t(\omega)))dz\right), \qquad D_0(\omega) = x.$$

Prove that the pathwise defined $X_t(\omega) := u(B_t(\omega),D_t(\omega))$ is the unique solution to (**??**).

Hint: Use your knowledge about u to derive an ODE for $\partial_y u(x,y)$ which can be solved explicitly.

Exercise 13.3 (Points)

Let B be a Brownian motion on $\mathbb{R}$ and let $b, \sigma \in C(\mathbb{R}, \mathbb{R})$ be bounded and Lipschitz continuous and such that $\inf_{x\in\mathbb{R}} |\sigma(x)| > 0$. Consider the solution X to

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s,$$

and find a function $s : \mathbb{R} \to \mathbb{R}$ such that $s(X)$ is a local martingale. We call s a *scale function* of X.

Bonus: For solutions to multidimensional SDEs it is usually not possible to find a scale function explicitly. Intuitively, why is that?

Due date: July 20, 2016. You may submit your solutions in groups of two.