Humboldt-Universität zu Berlin

Prof. Dr. Nicolas Perkowski
Dr. Achref Bachouch

Stochastic processes II: Continuous time
Summer semester 2016
Exercise sheet 1

Exercises

Exercise 1.1

a) Let $m \in \mathbb{R}^d$ and $C \in \mathbb{R}^{d\times d}$ a symmetric and positive semi-definite matrix, and let $X \sim \mathcal{N}(m, C)$. Prove that the coordinates $(X_1, \ldots, X_d)$ are independent if and only if C is a diagonal matrix.

b) Give an example of two Gaussian random variables X and Y that are uncorrelated, but not independent. (Of course in that case the couple (X, Y) cannot be Gaussian).

Exercise 1.2

Let B be a pre-Brownian motion. For a step function f on $\mathbb{R}_+$ i.e. $f := \sum_{i=1}^n \lambda_i \mathbb{1}_{[t_{i-1},t_i]}$, where $n \in \mathbb{N}$, $\lambda_i \in \mathbb{R}$ and $0 = t_0 < t_1 < \ldots < t_n$, set

$$\xi(f) := \sum_{i=1}^{n} \lambda_i (B_{t_i} - B_{t_{i-1}}).$$

a) Prove that if f and g are step functions on $\mathbb{R}_+$, then $E[\xi(f)\xi(g)] = \int_{\mathbb{R}_+} f(t)g(t)dt$.
Conclude that ξ is an isometry on the set of step functions on $\mathbb{R}_+$ equipped with the norm of $L^2(\mathbb{R}_+, \mathbb{R})$.

b) Prove that ξ can be uniquely extended to an isometry on $L^2(\mathbb{R}_+, \mathbb{R})$.
Hint: Use the density of continuous compactly supported functions in $L^2(\mathbb{R}_+, \mathbb{R})$.

c) Show that ξ is a white noise on $\mathbb{R}_+$, i.e. a Gaussian measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ with Lebesgue measure as density.

Exercise 1.3

Prove that $(a) \Leftrightarrow (b) \Leftrightarrow (c)$, where

(a) B is a pre-Brownian motion.

(b) (i) $B_0 = 0$ a.s.
(ii) $\forall t > s \geq 0$, the random variable $B_t - B_s \sim \mathcal{N}(0, t-s)$ and is independent of the variables $(B_r)_{r\leq s}$.

(c) (i) B has independent increments,
(ii) B is a centered Gaussian process and for all $t \geq 0$, $E[B_t^2] = t$.

Exercise 1.4

Let $(B_t)_{t\geq 0}$ be a pre-Brownian motion. Show that

a) $(-B_t)_{t\geq 0}$ is a pre-Brownian motion.

b) for any $a \geq 0$, $(B_{t+a} - B_a)_{t\geq 0}$ is a pre-Brownian motion independent of the variables $(B_r)_{r\leq a}$.

c) for any $c \neq 0$, $(cB_{t/c^2})_{t\geq 0}$ is a pre-Brownian motion.

Due date: Wednesday, **April** 27^{th}. You may submit your solutions in groups of two.