Humboldt-Universität zu Berlin

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Stochastic processes II: Continuous time
Summer semester 2016
Exercise sheet 2

Exercises

Exercise 2.1 (3+4+3 Points)
Let B be a Brownian motion.

a) Define $U_t := B_t - tB_1$ for $t \in [0,1]$. Prove that U is Gaussian, derive its mean and covariance.

b) Define $X_t := (1+t)U_{\frac{t}{1+t}}$, $t \geq 0$. Show that $(X_t)_{t\geq 0}$ is a Brownian motion. (This gives an alternative construction of Brownian motion on $[0,\infty)$).

c) Let $(W_t)_{t\geq 0}$ be a Brownian motion on $[0,\infty)$. Find functions $g, h : [0,1) \to [0,\infty)$, such that $(g(t)W_{h(t)})_{t\in[0,1)}$ has the same distribution as $(U_t)_{t\in[0,1)}$.

Exercise 2.2 (4+3 Points)
Let $(\mathcal{X}_{n,k})_{n\geq -1, 0\leq k<2^n}$ be the Haar functions and $(\phi_{n,k})_{n\geq -1, 0\leq k<2^n}$ their primitives. Define for any function $f : [0,1] \to \mathbb{R}$ and any $n \geq 0$, $0 \leq k < 2^n$

$$\langle \mathcal{X}_{n,k}, df\rangle := 2^{\frac{n}{2}}\left[2f\left(\frac{2k+1}{2^{n+1}}\right) - f\left(\frac{k}{2^n}\right) - f\left(\frac{k+1}{2^n}\right)\right]$$

and $\langle \mathcal{X}_{-1,0}, df\rangle := f(1) - f(0)$. Define $f_m(t) := \sum_{-1\leq n\leq m}\sum_{0\leq k<2^n}\langle \mathcal{X}_{n,k}, df\rangle\phi_{n,k}(t), t \in [0,1]$.

a) Assume $f(0) = 0$. Show that f_m is piecewise linear and $f_m(k2^{-m-1}) = f(k2^{-m-1})$ for every $0 \leq k \leq 2^{m+1}$.

b) Conclude that if f is a continuous function with $f(0) = 0$, then f_m converges uniformly to f, i.e

$$\lim_{m\to\infty}\sup_{t\in[0,1]}|f_m(t) - f(t)| = 0.$$

Exercise 2.3 (2+3 Points)
Let B be a Brownian motion.

a) Show that if $\alpha > 1/2$, then with probability 1 there exists no $t \geq 0$ with

$$\limsup_{s\to t}\frac{|B_s - B_t|}{|s-t|^\alpha} < \infty.$$

Hint: Adapt the proof of Theorem 2.16 of the lecture.

b) Show that if $t > 0$ is fixed, then almost surely

$$\limsup_{s\to t}\frac{|B_s - B_t|}{|s-t|^{\frac{1}{2}}} = \infty.$$

Conclude that almost surely there exists no interval $[S,T]$ with $0 \leq S < T$ on which the Brownian motion is $\frac{1}{2}$-Hölder continuous.
Hint: Use the properties of the increments of B as Brownian motion.

Due date: May 04, 2016. You may submit your solutions in groups of two.