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Stochastic processes II: Continuous time
Summer semester 2016
Exercise sheet 3

Exercises

Let $(\Omega, \mathcal{F})$ be a measurable space. $\mathbb{F}$ denotes the filtration $(\mathcal{F}_t)_{t\geq 0}$ and $\mathbb{F}^+$ denotes the smallest right continuous filtration containing $\mathbb{F}$. Let T be a stopping time.

Exercise 3.1 (6 Points)
Define

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \text{ for every } t \geq 0\}.$$

a) Show that $\mathcal{F}_T$ is a σ-algebra.

b) Show that if $T(\omega) \equiv t$ identically for $t \in [0, \infty]$ fixed, then $\mathcal{F}_T = \mathcal{F}_t$.

c) Show $T + t$ is a stopping time whenever $t \in [0, \infty]$.

d) Show that S is a $\mathbb{F}^+$-stopping time if and only if $\{S < t\} \in \mathcal{F}_t$, for all $t > 0$.

Exercise 3.2 (8 Points)

a) Show that

$$\mathcal{F}_T^+ = \{A \in \mathcal{F} : A \cap \{T < t\} \in \mathcal{F}_t \text{ for every } t > 0\}.$$

b) Show that T is $\mathcal{F}_T$ measurable.

c) Show that $\mathcal{F}_{T+} = \mathcal{F}_T^+$.

d) Show that if S is a stopping time with $S(\omega) \leq T(\omega)$ for every $\omega \in \Omega$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.

Exercise 3.3 (6 Points)

a) Show that if S is a stopping time, then $S \wedge T$, $S \vee T$ are stopping times and $\mathcal{F}_{S\wedge T} = F_S \cap F_T$.

   Hint: Use **3.2.d)** and that $A \cap \{S \wedge T \leq t\} = (A \cap \{S \leq t\}) \cup (A \cap \{T \leq t\})$ for every $t \geq 0$.

b) Show that if $(T_n)_{n\in\mathbb{N}}$ is a sequence of stopping times, then $\sup_{n\in\mathbb{N}} T_n$ is a stopping time and $\inf_{n\in\mathbb{N}} T_n$ is a $\mathbb{F}^+$ stopping time.

c) Bonus question: Show that if S is a $\mathbb{F}^+$-stopping time, then there exists a sequence of stopping times $(S_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} S_n(w) = S(w)$ for all $w \in \Omega$, such that every S_n only takes finitely many values, we have $S_{n+1}(\omega) \leq S_n(\omega)$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$, and $S_n(\omega) > S(\omega)$ for all $\omega \in \Omega$ with $S(\omega) < \infty$ and all $n \in \mathbb{N}$.

   Hint: Take for example $S_n = (k+1)2^{-n}$ on the set $\{S \in [k2^{-n}, (k+1)2^{-n})\}, k = 0, \ldots, n2^n - 1$, and $S_n = \infty$ on the set $\{S_n \geq n\}$.

Due date: May 11, 2016. You may submit your solutions in groups of two.