Humboldt-Universität zu Berlin

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Stochastic processes II: Continuous time
Summer semester 2016
Exercise sheet 4

Exercises

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a measure space, let B be a Brownian motion and write $\mathbb{F}^B := (\mathcal{F}_t^B)_{t\geq 0}$ for its canonical filtration.

Exercise 4.1 (4 Points)
Define for $a \geq 0$ the stopping time $T_a := \inf\{t \geq 0 : B_t = a\}$. Show that $(T_a)_{a\geq 0}$ has stationary and independent increments, that is for $0 \leq a \leq b$ the random variable $T_b - T_a$ is independent of $(T_c)_{0\leq c\leq a}$ and has the same distribution as T_{b-a}.

Exercise 4.2 (5 Points)
Set $Zeros = \{t \geq 0 : B_t = 0\}$. Show that the complement of $Zeros$ consists of countably many open intervals.

Exercise 4.3 (4+2 Points)
For all $t \geq 0$, set $S_t := \sup_{s\in[0,t]} B_s$. Show that:

a) For all $a \geq 0$, $b \leq a$ and $t \geq 0$

$$\mathbb{P}(S_t \geq a, B_t \leq b) = \mathbb{P}(B_t \geq 2a - b).$$

Hint: Apply the strong Markov property to $T_a := \inf\{t \geq 0 : B_t = a\}$.

b) For all $t \geq 0$, S_t has the same distribution as $|B_t|$.

Exercise 4.4 (5 Points)
Let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be a filtration smaller than $\mathbb{F}^B$ (that is $\mathcal{G}_t \subseteq \mathcal{F}_t^B$ for all $t \geq 0$). For all $t \geq 0$, set $\hat{B}_t := \mathbb{E}[B_t|\mathcal{G}_t]$. Show that if $\hat{B}$ is a pre-Brownian motion, then for all $t \geq 0$, $\hat{B}_t = B_t$ almost surely.

Hint: Use the tower property of the conditional expectation.

Due date: May 18, 2016. You may submit your solutions in groups of two.