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Exercises

Exercise 5.1 (5 Points)
Let $f : \mathbb{Q}_+ \longrightarrow \mathbb{R}$, where $\mathbb{Q}_+ := \mathbb{Q} \cap \mathbb{R}_+$, be such that for all $k \in \mathbb{N}$ and for all $a, b \in \mathbb{Q}$ with $a < b$

$$\sup_{s \in [0,k] \cap \mathbb{Q}_+} |f(s)| < \infty \text{ and } D([a,b]; [0,k] \cap \mathbb{Q}_+; f) < \infty,$$

where $D([a,b]; I; f)$ denotes the number of downcrossings of $f|_I$ across the interval $[a,b]$.

a) Show that the limit from the right $f(t_+) = \lim_{s \downarrow\downarrow t, s \in \mathbb{Q}_+} f(s)$ exits for all $t \geq 0$ and the limit from the left $f(t_-) = \lim_{s \uparrow\uparrow t, s \in \mathbb{Q}_+} f(s)$ exits for all $t > 0$.

b) Show that the function $t \longmapsto f(t_+), t \geq 0$, is càdlàg.

Exercise 5.2 (10 Points)
Let $(\tau_k)_{k \in \mathbb{N}}$ be a sequence of independent exponentially distributed random variables with parameter $\lambda > 0$. Define $N_t := \sup\{k \geq 1 : \tau_1 + \cdots + \tau_k \leq t\}$ for $t \geq 0$, where $\sup \emptyset = 0$. N is called *Poisson process with parameter* λ.

a) Show that N_t has a Poisson distribution with parameter λt.
Hint: The Gamma distribution μ with parameters $(k, \theta) \in \mathbb{N} \times (0, \infty)$ has characteristic function $\hat{\mu}(u) = (1 - \theta i u)^{-k}$ and cumulative distribution function $\mu([0,t]) = 1 - e^{-t/\theta} \sum_{l=0}^{k-1} \frac{(t/\theta)^l}{l!}$.

b) Show that the exponential distribution is memoryless, i.e. $\mathbb{P}(\tau_1 > t | \tau_1 > s) = \mathbb{P}(\tau_1 > t - s)$ for any $0 \leq s \leq t$. Conclude that N has independent and stationary increments, i.e. the increments of N are independent and $N_t - N_s$ is Poisson distributed with parameter $\lambda(t-s)$.
Hint: First show that for any $k, l \in \mathbb{N}$ and any bounded and measurable $f : \mathbb{R}^k \to \mathbb{R}$ we have

$$\mathbb{E}[f(\tau_1, \ldots, \tau_k) \mathbf{1}_{\{N_s = k\}} \mathbf{1}_{\{N_t - N_s = l\}}] = \mathbb{E}[f(\tau_1, \ldots, \tau_k) \mathbf{1}_{\{N_s = k\}}] \mathbb{P}(N_{t-s} = l).$$

c) Conclude that N is a Markov process in the filtration generated by N.

d) Show that for every $t \geq 0$, N is almost surely continuous in t.

e) Calculate $\mathbb{P}(N \text{ is continuous on } [0,t])$ for $t \geq 0$.

Exercise 5.3 (5 Points)
Let N be a Poisson process with parameter $\lambda > 0$ and $(Y_k)_{k \in \mathbb{N}}$ be a sequence of i.i.d. random variables, independent of N. Let F be the cumulative distribution function of Y_1. The process X defined by $X_t := \sum_{k=1}^{N_t} Y_k$ for $t \geq 0$, is called a (λ, F)-*compound Poisson process.*

a) Show that X has stationary and independent increments.

b) Show that if $E[|Y_1|] < \infty$, then $(X_t - t\lambda E[Y_1])_{t \geq 0}$ is a martingale in the filtration generated by X.

Due date: May 25, 2016. You may submit your solutions in groups of two.