Humboldt-Universität zu Berlin

Prof. Dr. Nicolas Perkowski
Dr. Achref Bachouch

Stochastic processes II: Continuous time
Summer semester 2016
Exercise sheet 6

Exercises

Let $(B_t)_{t\geq 0}$ be a Brownian motion. For $x \in \mathbb{R}$, define $T_x := \inf\{t \geq 0 : B_t = x\}$, where $\inf \emptyset := \infty$.

Exercise 6.1 (2+2+3 Points)
For all $a \in \mathbb{R}\backslash\{0\}$, show that:

a) $\mathbb{P}(T_a < \infty) = 1$; b) $\mathbb{E}[T_a] = \infty$; c) $\mathbb{E}[\exp(-\lambda T_a)] = \exp(-|a|\sqrt{2\lambda})$, $\lambda \geq 0$.

Hint: For question b) recall from the lecture that $(B_t^2 - t)_{t\geq 0}$ is a martingale and if $a > 0$ first compute $\mathbb{E}[T_a \wedge T_{-n}]$ for $n \in \mathbb{N}$. For question c) consider the martingale $(e^{\alpha B_t - \alpha^2 t/2})_{t\geq 0}$ for suitable $\alpha \in \mathbb{R}$.

Exercise 6.2 (3+3 Points)
The process X defined by

$$X_t = \mu t + \sigma B_t, \qquad t \geq 0,$$

is called Brownian motion with drift $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$.

a) For $-a < 0 < b$ calculate the probability of X hitting b before hitting $-a$.

   Hint: For $\mu \neq 0$ choose α so that $(\exp(\alpha X_t))_{t\geq 0}$ is a martingale.

b) Show that if $\mu < 0$, then $Y := \sup_{t\geq 0} X_t$ is almost surely finite and exponentially distributed with parameter $\lambda = -2\mu/\sigma^2$.

Exercise 6.3 (3 Points)
Set $I := -\inf_{s\leq T_1} B_s$. Compute the cumulative distribution function of I.

Exercise 6.4 (2+2 Points)
Let $a > 0$ and M be a continuous positive martingale such that

$$M_0 = a \text{ and } \lim_{t\to\infty} M_t = 0.$$

a) Show that for $y > a$, $\mathbb{E}[M_{T_y \wedge t}] = a$, for all $t \geq 0$.

b) Prove that the random variable $\sup_{t\geq 0} M_t$ has the same distribution as the random variable $\frac{a}{U}$, where U is a uniformly distributed random variable on $(0, 1)$.

   Hint: Note that $\mathbb{P}(T_y < \infty) = \mathbb{P}(\sup_{t\geq 0} M_t \geq y)$ whenever $y > a$.

Due date: June 1st, 2016. You may submit your solutions in groups of two.