Humboldt-Universität zu Berlin

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Stochastic analysis
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Exercise sheet 7

Exercises

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where $(\mathcal{F}_t)_{t\geq 0}$ is a filtration satisying the usual conditions. Recall that $\mathcal{M}_{\text{loc}}$ (respectively $\mathcal{M}$) denotes the set of all continuous local martingales (respectively the set of all continuous uniformly integrable martingales) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Exercise 7.1 (3+3+3+3 Points)
Let $M \in \mathcal{M}_{\text{loc}}$.

a) Show that if M is positive, then M is a supermartingale.

b) Show that M is a uniformly integrable martingale if and only if the set

$$\{M_\tau : \tau \text{ is a finite stopping time}\}$$

is uniformly integrable.

Let $\langle M \rangle$ be the quadratic variation of M. Now assume in addition that $M_0 = 0$.

c) Show that M is a martingale and $\mathbb{E}[M_t^2] < \infty$ for every $t \geq 0$ if and only if $\mathbb{E}[\langle M \rangle_t] < \infty$ for every $t \geq 0$.

d) Show that $M \in \mathcal{M}$ and M is L^2-bounded (i.e. $\sup_{t\geq 0} \mathbb{E}[M_t^2] < \infty$) if and only if $\mathbb{E}[\langle M \rangle_\infty] < \infty$.

Exercise 7.2 (4+4 Points)
Let $M \in \mathcal{M}$, with $\mathbb{E}[M_\infty^2] < \infty$, let $(T_k)_{k\in\mathbb{N}}$ be an increasing sequence of stopping times with $\lim_{k\to\infty} T_k = \infty$, and let $C > 0$ be such that for every k the random variable H_{T_k} is $\mathcal{F}_{T_k}$-measurable and satisfies $|H_{T_k}| \leqslant C$. Then for $H_t = \sum_{k=0}^\infty H_{T_k} \mathbb{1}_{[T_k, T_{k+1})}(t),\ t \geqslant 0$, we define the stochastic integral

$$(H \cdot M)_t := \sum_{k=0}^\infty H_{T_k}(M_{T_{k+1}\wedge t} - M_{T_k \wedge t}),\ t \geq 0.$$

Show that:

a) The process $(H \cdot M)$ is a martingale in $\mathcal{M}$.

b) $\mathbb{E}[\sup_{t\geqslant 0} |(H \cdot M)_t|^2] \leq 4C^2(\mathbb{E}[M_\infty^2] - \mathbb{E}[M_0^2])$.

Due date: June 8, 2016. You may submit your solutions in groups of two.