Humboldt-Universität zu Berlin

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Stochastic analysis
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Exercise sheet 8

Exercises

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where $(\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions, and let B be an $(\mathcal{F}_t)$-Brownian motion. We recall that $\mathcal{H}_c^2$ denotes the set of continuous martingales M such that $M_0 = 0$ and $\sup_{t\geq 0} \mathbb{E}[M_t^2] < \infty$ and $\Lambda^2 := L^2(\mathbb{R}_+ \times \Omega, \mathit{Prog}, \lambda \otimes \mathbb{P})$ is the set of progressively measurable real valued processes $(\phi_t)_{t\geq 0}$ such that $\|\phi\|_{\Lambda^2} := \Big(\mathbb{E}\Big[\int_0^\infty \phi_s^2 ds\Big]\Big)^{1/2} < \infty$.

Exercise 8.1 (12 Points)
We denote by $\mathcal{E}$ the set of real valued processes ϕ of the form $\phi_t := \sum_{i=0}^{n-1} u_i \mathbb{1}_{(t_i,t_{i+1}]}(t)$, $t \geq 0$, where $n \in \mathbb{N}$, u_i is a bounded and $\mathcal{F}_{t_i}$-measurable real random varibale and $0 \leq t_0 < t_1 < \ldots < t_n$. For $\phi \in \mathcal{E}$ and $0 < t \leq \infty$, set

$$\int_0^t \phi_s dB_s := \sum_{i=0}^{n-1} u_i (B_{t_{i+1}\wedge t} - B_{t_i \wedge t}).$$

a) Prove that if $\phi \in \mathcal{E}$, then the following holds:

i) $\langle \int_0^\cdot \phi_s dB_s, \int_0^\cdot \phi_s dB_s \rangle_t = \int_0^t \phi_s^2 ds, t \geq 0.$

ii) $\mathbb{E}\Big[\int_0^\infty \phi_s dB_s\Big] = 0$ and $\mathbb{E}\Big[(\int_0^\infty \phi_s dB_s)^2\Big] = \mathbb{E}\Big[\int_0^\infty \phi_s^2 ds\Big]$.

Hint: One can show at first that if $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = \infty$ and X is an adapted integrable process such that for $t_i \leq s < t \leq t_{i+1}$, $\mathbb{E}[X_t - X_s | \mathcal{F}_s] = 0$, then X is a martingale.

Now, we recall from the lecture that the map $\phi \mapsto I(\phi) := \int_0^\infty \phi_s dB_s$ is a linear isometry on $\mathcal{E}$, then it is uniquely extended to a linear isometry on Λ^2 (denoted again I). We recall also that for $t > 0$, $\Lambda^2(t) := L^2([0,t] \times \Omega, \mathit{Prog}, \lambda|_{[0,t]} \otimes \mathbb{P})$, and for $\phi \in \Lambda^2(t)$, $\int_0^t \phi_s dB_s := I(\mathbb{1}_{[0,t]}\phi)$.

b) Prove that for $t > 0$, if $\phi \in \Lambda^2(t)$, then

$$\mathbb{E}\Big[\int_0^t \phi_s dB_s\Big] = 0 \text{ and } \mathbb{E}\Big[\Big(\int_0^t \phi_s dB_s\Big)^2\Big] = \mathbb{E}\Big[\int_0^t \phi_s^2 ds\Big].$$

c) Prove that for $t > 0$, if $\phi, \psi \in \Lambda^2(t)$, then $\mathbb{E}\Big[\int_0^t \phi_s dB_s \int_0^t \psi_s dB_s\Big] = \mathbb{E}\Big[\int_0^t \phi_s \psi_s ds\Big]$.

Exercise 8.2 (5 Points)
Prove that for all $\phi \in \Lambda^2$, there exists $M \in \mathcal{H}_c^2$ such that for all t, $M_t = \int_0^t \phi_s dB_s$ almost surely and $M_t^2 - \int_0^t \phi_s^2 ds$ is a martingale.
From now on, for $\phi \in \Lambda^2$, $\Big(\int_0^t \phi_s dB_s\Big)_t$ will denote the continuous martingale M such that for all t, $M_t = \int_0^t \phi_s dB_s$ almost surely. M is called the stochastic integral of ϕ.

Exercise 8.3 (5 Points)
Let $t \geq 0$. For ϕ and $\psi \in \Lambda^2$, set $M_t := \int_0^t \phi_s dB_s$ and $N_t := \int_0^t \psi_s dB_s$. Show that:

a) $\langle M, N \rangle_t = \int_0^t \phi_s \psi_s ds$ and $M_t N_t - \langle M, N \rangle_t$ is a martingale.

b) If T is a stopping time, then $\mathbb{E}\Big[M_t^T N_t^T\Big] = \mathbb{E}\Big[\int_0^{t\wedge T} \phi_s \psi_s ds\Big]$.

Due date: June 15, 2016. You may submit your solutions in groups of two.