Humboldt-Universität zu Berlin

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Stochastic analysis
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Exercise sheet 9

Exercises

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where $(\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions, and let B be an $(\mathcal{F}_t)$-Brownian motion. We recall that $\mathcal{H}_c^2$ denotes the set of continuous martingales M such that $M_0 = 0$ and $\sup_{t\geq 0} \mathbb{E}[M_t^2] < \infty$.

Exercise 9.1 (5 Points)
Prove that $\mathcal{H}_c^2$ is a Hilbert space for the scalar product $(.,.)_{\mathcal{H}_c^2} := \mathbb{E}[\langle .,. \rangle_\infty]$.

Exercise 9.2 (3+4 Points)
This exercise is to solve without using neither Itô's formula nor Integration by parts.

a) Show that for every $t \geq 0$: $\int_0^t s dB_s = tB_t - \int_0^t B_s ds$.

   Hint: $tB_t = \sum_{k=0}^{n-1}(t_{k+1}B_{t_{k+1}} - t_k B_{t_k})$ for $t_k = kt/n$, $k = 0, \ldots, n$.

b) Show that for every $t \geq 0$: $\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds$.

   Hint: Writing B_t^3 as telescope sum might be helpful, as well as using a Taylor expansion of x^3.

Exercise 9.3 (3+3+3 Points)
Let $(B_t^x)_{t\geq 0}$ be a three-dimensional Brownian motion, started in $x \in \mathbb{R}^3 \backslash \{0\}$ (i.e. $B_t^x = x + B_t$, and $B_t = (B_t^1, B_t^2, B_t^3)^\top$, where the B^i are independent Brownian motions on $\mathbb{R}$). We define the process Y by: for $t \geq 0$,

$$Y_t := 0, \text{ if } B_t^x = 0 \text{ and } Y_t := 1/|B_t^x| \text{ otherwise.}$$

Show that:

a) The process Y is a local martingale and almost surely continuous.

b) Y is not a martingale.
   Hint: Show that $\lim_{t\to\infty} \mathbb{E}[Y_t] = 0$.

c) Y is bounded in L^2 (i.e. $\sup_{t\geq 0} \mathbb{E}[|Y_t|^2] < \infty$), and therefore in particular uniformly integrable.

Due date: June 22, 2016. You may submit your solutions in groups of two.