$M \in \mathcal{M}_{loc}$ and $K \in L^2_{loc}(M)$ and $H \in L^2_{loc}(K.M)$.
Show that $HK \in L^2_{loc}(M)$ and

$$((HK).M) = (H.(K.M)).$$

- Let us show the result when $\mathcal{M}_{loc}$ is replaced by $\mathcal{H}^2_c$ and L^2_{loc} is replaced by L^2:

By Theorem 6.11 and since the associativity holds for pathwise integration we have:

$$\langle K.M, K.M \rangle \underset{\substack{\text{Theo 6.11} \\ \text{and since } K \in L^2(M)}}{=} K.\langle M, K.M \rangle \underset{\text{Theo. 6.11}}{=} K.(K.\langle M,M \rangle)$$

$$\underset{\text{Assoc. for pathwise Integ.}}{=} K^2.\langle M,M \rangle$$

Then

$$\mathbb{E}\left[\int_0^{+\infty} H_s^2 K_s^2 \, d\langle M,M\rangle_s\right] = \mathbb{E}\left[\int_0^{+\infty} H_s^2 \, d\langle K.M, K.M\rangle_s\right] \underset{H \in L^2(K.M)}{<} \infty.$$

$\Rightarrow HK \in L^2(M)$.

$\cdot\ \forall N \in \mathcal{H}_c^2,\ \langle (HK)\cdot M, N\rangle \underset{\substack{\downarrow \\ \text{Theo. 6.12}}}{=} HK \cdot \langle M, N\rangle \underset{\substack{\downarrow \\ \text{Ass. pathwise int.}}}{=} H\cdot(K\cdot\langle M,N\rangle)$

$\underset{\substack{\downarrow \\ \text{Theo. 6.12}}}{=} H\cdot\langle K\cdot M, N\rangle \underset{\substack{\downarrow \\ \text{Theo 6.12}}}{=} \langle H\cdot(K\cdot M), N\rangle$

Thus $(HK)\cdot M = H\cdot(K\cdot M)$.

– For the general cas. Note that

$HK \in L^2_{loc}(M)$ iff for some localizing sequence $(T_n)_n$,

$$\mathbb{E}\left[\int_0^{T_n} H_s^2 K_s^2 \, d\langle M,M\rangle_s\right] < \infty$$

By assumption $K \in L^2_{loc}(M)$ and $H \in L^2_{loc}(K\cdot M)$.

let S_n be a common localizing sequence for

M and $K\cdot M$:

$$\langle K\cdot M, K\cdot M\rangle^{S_n} = K\cdot\langle M, K\cdot M\rangle^{S_n} = K\cdot(K\cdot\langle M,M\rangle)^{S_n}$$

$$= K^2\cdot\langle M,M\rangle^{S_n}$$

$$\Rightarrow \mathbb{E}\left[\int_0^{T_n} H_s^2 K_s^2 \, d\langle M,M\rangle_s\right] = \mathbb{E}\left[\int_0^{+\infty} H_s^2 \, d\langle K\cdot M, K\cdot M\rangle_s^{S_n}\right] < \infty$$

$H \in L^2_{loc}(K\cdot M)$.

and

$$(H \cdot (K \cdot M))^{S_n} = H \cdot (K \cdot M)^{S_n} = H \cdot (K \cdot M^{S_n})$$

$$= HK \cdot M^{S_n} = (HK \cdot M)^{S_n}.$$

By the result proved in the case of $\mathcal{H}^2_c$ and L^2

By sending $n \to \infty$, we conclude.