Exercise 13.1 :

a) Choose a sequence $(g_n)_n$ of class C^2 such that

① $g_n = g$ on $\mathbb{R} \setminus \bigcup_{l=1}^{k} (x_l - \delta, x_l + \delta)$ for some fixed δ which does not depend on g_n.

② $g_n(x) \longrightarrow g(x)$, $\forall x \in \mathbb{R}$

$g_n'(x) \longrightarrow g'(x)$, $\forall x \in \mathbb{R}$

③ $|g_n''| \leq M$, $\forall n$

④ $g_n''(x) \longrightarrow g''(x)$, $\forall x \in \mathbb{R} \setminus \{x_1, \dots, x_k\}$

⑤ $\sup_n \sup_{x \in \bigcup_{l=1}^{k} (x_l - \delta, x_l + \delta)} |g_n(x)| < \infty$, and similarly for g_n'.

Then $g_n(B_t) = g_n(B_0) + \int_0^t g_n'(B_s)\, dB_s + \frac{1}{2} \int_0^t g_n''(B_s)\, ds.$

Since g is bounded on $\bigcup_{l=1}^{k} (x_l - \delta, x_l + \delta)$, we have by dominated convergence (using ⑤) that

$$\mathbb{E}\left[|g_n(B_t) - g(B_t)|^2 \right] = \mathbb{E}\left[\left| \mathbb{1}_{B_t \in \bigcup_{l=1}^{k} (x_l - \delta, x_l + \delta)} (g_n(B_t) - g(B_t)) \right|^2 \right]$$

$$\xrightarrow[n \to \infty]{} 0 .$$

Therefore, it suffices to show that

$$g_n(B_0) + \int_0^t g_n'(B_s)\, dB_s + \frac{1}{2} \int_0^t g_n''(B_s)\, ds \xrightarrow{L^2} g_t(B_0) + \int_0^t g'(B_s)\, dB_s + \frac{1}{2} \int_0^t g''(B_s) \mathbb{1}_{\{\dots\}}\, ds$$

By the triangle inequality, it suffices to show convergence for the single terms.

- $g_n(0) \longrightarrow g(0)$ by ②
- $\mathbb{E}\left[\left|\int_0^t (g_n'(B_s) - g'(B_s))\,dB_s\right|^2\right] = \mathbb{E}\left[\int_0^t |g_n'(B_s) - g'(B_s)|^2 ds\right]$

$$= \mathbb{E}\left[\int_0^t \mathbb{1}\left\{B_s \in \bigcup_{l=1}^k (x_l - \delta, x_l + \delta)\right\} (g_n'(B_s) - g'(B_s))^2 ds\right]$$

$\longrightarrow 0$ by bounded convergence, using ⑤ and that g' is continuous and thus bounded on $\bigcup_{l=1}^k (x_l - \delta, x_l + \delta)$.

- $\mathbb{E}\left[\frac{1}{4}\left(\int_0^t \left(g_n''(B_s) - g''(B_s)\, \mathbb{1}\left\{B_s \notin \bigcup_{l=1}^k \{x_l\}\right\}\right) ds\right)^2\right]$

$$\leq \frac{1}{4} t\, \mathbb{E}\left[\int_0^t g_n''(B_s) - g''(B_s)\, \mathbb{1}\left\{B_s \notin \bigcup_{l=1}^k \{x_l\}\right\})^2 ds\right]$$

Jensen to $\frac{1}{t}\lambda_{[0,t]}$

$$= \frac{t}{4}\, \mathbb{E}\left[\int_0^t (g_n''(B_s) - g''(B_s))^2\, \mathbb{1}\left\{B_s \in \bigcup_{l=1}^k (x_l - \delta, x_l + \delta)\right\} ds\right]$$

$\lambda(\{x_1, \dots, x_k\}) = 0$, define $g(x_i) = 0$

$\longrightarrow 0$, by dominated convergence (using that $|g''| \leq M$ and ③).

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b) $g_\varepsilon|(-\varepsilon,\varepsilon) \in C^2$ and $g_\varepsilon|[-\varepsilon,\varepsilon]^c \in C^2$

To apply a), it suffices to show that g_ε is continuously differentiable on $\{-\varepsilon,\varepsilon\}$, and that $|g_\varepsilon''(x)| \leq M$ for $x \in \mathbb{R}\setminus\{-\varepsilon,\varepsilon\}$.

$$g_\varepsilon''(x) = \begin{cases} \frac{1}{\varepsilon}, & |x| < \varepsilon \\ 0, & |x| > \varepsilon \end{cases}$$

and $g_\varepsilon'(x) = \begin{cases} \frac{x}{\varepsilon}, & |x| < \varepsilon \\ -1, & x < -\varepsilon \\ 1, & x > \varepsilon \end{cases}$ is continuous on $\{-\varepsilon,\varepsilon\}$ (actually on $\mathbb{R}$).

Thus by a),

$$g_\varepsilon(B_t) = g_\varepsilon(0) + \int_0^t g_\varepsilon'(B_s)\,dB_s + \frac{1}{2}\int_0^t g_\varepsilon''(B_s)\,\mathbb{1}_{\mathbb{R}\setminus\{-\varepsilon,\varepsilon\}}(B_s)\,ds$$

$$= g_\varepsilon(0) + \int_0^t g_\varepsilon'(B_s)\,dB_s + \frac{1}{2\varepsilon}\underbrace{\int_0^t \mathbb{1}_{(-\varepsilon,\varepsilon)}(B_s)\,ds}_{=\ \lambda(s\in[0,t]\,:\,|B_s|<\varepsilon)}$$

c) $\mathbb{E}\left[\left(\int_0^t g_\varepsilon'(B_s)\mathbb{1}_{(-\varepsilon,\varepsilon)}(B_s)\,dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t \left(g_\varepsilon'(B_s)\mathbb{1}_{(-\varepsilon,\varepsilon)}(B_s)\right)^2 ds\right]$

$$= \mathbb{E}\left[\int_0^t \underbrace{\frac{B_s^2}{\varepsilon^2}\mathbb{1}_{(-\varepsilon,\varepsilon)}(B_s)}_{\leq \mathbb{1}_{(-\varepsilon,\varepsilon)}(B_s)}\,ds\right] \leq \int_0^t \underbrace{\mathbb{P}(|B_s|<\varepsilon)}_{\leq 1}\,ds$$

by dominated convergence: $\lim_{\varepsilon\to 0}\int_0^t \mathbb{P}(|B_s|<\varepsilon)\,ds = 0.$

j) $\frac{1}{2\varepsilon}\lambda\left(s\in[0,t] : B_s\in(-\varepsilon,\varepsilon)\right)$

$$= g_\varepsilon(B_t) - g_\varepsilon(B_0) - \int_0^t \frac{B_s}{\varepsilon}\mathbb{1}_{(-\varepsilon,\varepsilon)^c}\, dB_s - \int_0^t \mathrm{sgn}(B_s)\mathbb{1}_{(-\varepsilon,\varepsilon)^c}(B_s)\, dB_s$$

• $g_\varepsilon(0) = \frac{1}{2}\varepsilon \longrightarrow 0$ in $L^2(\mathbb{P})$; if $B_0 \neq 0$, then $\forall \varepsilon$ small $g_\varepsilon(B_0) = |B_0|$.

• $\int_0^t \frac{B_s}{\varepsilon}\mathbb{1}_{(-\varepsilon,\varepsilon)^c}\, dB_s \longrightarrow 0$ in $L^2(\mathbb{P})$ by C°).

• $g_\varepsilon(B_t) \longrightarrow |B_t|$ in $L^2(\mathbb{P})$:

$$\mathbb{E}\left[\left(g_\varepsilon(B_t) - |B_t|\right)^2\right] = \mathbb{E}\left[\mathbb{1}_{(-\varepsilon,\varepsilon)}(B_t)\, \underbrace{\left(\frac{1}{2}\left(\varepsilon + \frac{B_t^2}{\varepsilon}\right) - |B_t|\right)^2}_{\leq \left(\frac{1}{2}(1+\varepsilon)+\varepsilon\right)^2}\right]$$

$$\leq C\, \mathbb{P}(|B_t| < \varepsilon) \xrightarrow[\varepsilon \to 0]{} 0.$$

• $\int_0^t \mathrm{sgn}(B_s)\mathbb{1}_{(-\varepsilon,\varepsilon)^c}(B_s)\, dB_s \longrightarrow \int_0^t \mathrm{sgn}(B_s)\, dB_s$ in $L^2(\mathbb{P})$:

$$\mathbb{E}\left[\left(\int_0^t \mathrm{sgn}(B_s)\left(\mathbb{1}_{(-\varepsilon,\varepsilon)^c} - 1\right)(B_s)\, dB_s\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^t \mathbb{1}_{(-\varepsilon,\varepsilon)}(B_s)\, ds\right] \longrightarrow 0.$$

Exercise 13.2:

$$X_t(\omega) := u(B_t(\omega), D_t(\omega))$$

We know that $dD_t = b(u(B_t, D_t)) \exp\left\{-\int_0^{B_t} \sigma'(u(z, D_t))\,dz\right\} dt$

Then, applying Itô to $X_t = u(B_t, D_t)$:

$$dX_t = \partial_x u(B_t, D_t)\,dB_t + \partial_y u(B_t, D_t)\,dD_t$$

$$+ \frac{1}{2}\partial_{xx} u(B_t, D_t)\,dt$$

$$= \sigma(u(B_t, D_t))\,dB_t + \partial_y u(B_t, D_t)\,dD_t + \frac{1}{2}\partial_{xx} u(B_t, D_t)\,dt$$

(I)

On the other hand,

$$\partial_x u(x,y) = \sigma(u(x,y))$$

$$\Rightarrow \underbrace{\partial_y \partial_x u(x,y)}_{\partial_x \partial_y u(x,y)} = \partial_y \sigma(u(x,y)) = \sigma'(u(x,y))\,\partial_y u(x,y)$$

Setting $v(x,y) = \partial_y u(x,y)$, we get

$$\begin{cases} \partial_x v(x,y) = \sigma'(u(x,y))\,v(x,y) \\ v(0,y) = 1 \end{cases}$$

$$\Rightarrow v(x,y) = \exp\left\{\int_0^x \sigma'(u(z,y))\,dz\right\}$$

$$\Rightarrow \partial_y u(B_t, D_t)\, dD_t = v(B_t, D_t)\, dD_t$$

$$= \exp\left\{\int_0^{B_t} \sigma'(u(z, D_t))\, dz\right\} dD_t$$

$$= b(u(B_t, D_t))\, dt \quad , \quad (\mathrm{II})$$

. We also note that

$$\partial_x \partial_x u(x,y) = \partial_x \sigma(u(x,y)) = \sigma'(u(x,y))\partial_x u(x,y)$$

$$= \sigma'(u(x,y))\, \sigma(u(x,y))$$

$$(\mathrm{III}).$$

Using (II) and (III) in (I), we get

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt + \frac{1}{2}\sigma'(X_t)\sigma(X_t)\, dt$$

$$= b(X_t)\, dt + \sigma(X_t) \circ dB_t \quad : \quad (\mathrm{IV})$$

Moreover, $X_0 := u(B_0, D_0) = u(0, x) = x$. (V)

Thus, we deduce that $X_t := u(B_t, D_t)$ is the unique solution of the SDE (IV) & (V) by the strong existence and uniqueness theorem, Theorem 7.6.

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Exercice 13.3 :

We need $s : \mathbb{R} \to \mathbb{R}$ s.t. $s(X)$ is a local martingale.

↳ By Itô, $s(X_t) = s(X_0) + \int_0^t \mathcal{L}s(X_s)\, ds + M_t^s$,

where $\begin{cases} (M_t^s)_t \text{ is a local martingale} \\ \text{and} \\ \mathcal{L}s(X_s) = b(X_s)s'(X_s) + \frac{1}{2}\sigma^2(X_s)s''(X_s) \end{cases}$

Thus, it suffices to consider s solution of

the linear ODE $\mathcal{L}s = 0$

i.e. $b(x)\, s'(x) + \frac{1}{2}\sigma^2(x)\, s''(x) = 0$

that is $s''(x) + \frac{2b(x)}{\sigma^2(x)} s'(x) = 0$, when $\sigma^2(x) \neq 0$

. In dimension 1, we can find an explicit solution to this linear ODE. For example (since $\inf_{x \in \mathbb{R}} |\sigma(x)| > 0$)

a solution is $s(x) = \int_0^x \exp\left\{ \int_0^y -\frac{2b(z)}{\sigma^2(z)} dz \right\} dy$.

. In dimension > 1, it becomes a PDE, then finding an explicit solution is usually not possible.