b) Many problems with convergence types!
Have to write which convergence is considered,
otw/ we assume ~~at~~ a.s.

$\mathbb{E}\left[\int_0^t \phi_s dB_s\right] = 0$:

~~Let $(\phi_n^t) \subset \mathcal{E}$ with $\|\phi_n^t - \phi\|_{\Lambda_2(t)} \to 0$.~~

~~Then~~ ~~$\|\phi_n^t - \phi 1_{[0,t]}$~~

~~$\|I(\phi_n) - I(\phi)$~~

Let $\phi_n^t \subset \mathcal{E}$ s.t. $\|\phi_n^t - \underbrace{\phi 1_{[0,t]}}_{\in \Lambda_2}\|_{\Lambda_2} \to 0$.

Then

$$\|\underbrace{I(\phi_n^t)}_{=0\,\forall n} - \underbrace{I(\phi 1_{[0,t]})}_{=\int_0^t \phi_s dB_s}\|_{\boxed{L^2}} \to 0,$$ and therefore

! Note that we do <u>not</u> get a.s. convergence (see above).

we also get L^1-conv. and in particular

$$\mathbb{E}\left[\int_0^t \phi_s dB_s\right] = \lim_n \mathbb{E}\left[I(\phi_n^t)\right] = 0.$$

Similarly

$$\mathbb{E}\left[\left(\int_0^t \phi_s \, dB_s\right)^2\right] \overset{L^2\text{- conv.}}{=} \lim_n \mathbb{E}\left[I(\phi_n^t)^2\right]$$

$$\overset{\text{Isom.}}{=} \lim_n \|\phi_n^t\|^2_{\mathcal{A}_2} = \cancel{\lim} \, \|\phi \mathbb{1}_{[0,t]}\|^2_{\mathcal{A}_2}$$

$$= \cancel{8} \, \mathbb{E}\left[\int_0^t \phi_s^2 \, ds\right].$$

Here we used: $X_n \xrightarrow{L^2} X \Rightarrow |\mathbb{E}[X_n] - \mathbb{E}[X]| \leq \mathbb{E}[|X_n - X|] = \|X_n - X\|_{L^1} \leq \|X_n - X\|_{L^2} \to 0$, and $X_n \xrightarrow{L^2} X \Rightarrow X_n^2 \xrightarrow{L^1} X^2 \Rightarrow \mathbb{E}[X_n^2] \to \mathbb{E}[X^2]$ (or: $\mathbb{E}[X_n^2] = \|X_n\|^2_{L^2} \to \|X\|^2_{L^2} = \mathbb{E}[X^2]$)

c) Simply use polarisation:

$$\int_0^t \phi_s \, dB_s \int_0^t \psi_s \, dB_s = \frac{1}{4}\left(\left(\int_0^t \phi_s \, dB_s + \int_0^t \psi_s \, dB_s\right)^2 - \left(\int_0^t \phi_s \, dB_s - \int_0^t \psi_s \, dB_s\right)^2\right)$$

$$\overset{\substack{\text{linearity of} \\ \text{stoch int}}}{=} \frac{1}{4}\left(\left(\int_0^t (\phi_s + \psi_s) \, dB_s\right)^2 - \left(\int_0^t (\phi_s - \psi_s) \, dB_s\right)^2\right)$$

$$\Rightarrow \mathbb{E}[\cdot] \overset{a)}{=} \frac{1}{4}\left(\mathbb{E}\left[\int_0^t (\phi_s + \psi_s)^2 \, ds\right] - \mathbb{E}\left[\int_0^t (\phi_s - \psi_s)^2 \, ds\right]\right)$$

$$= \mathbb{E}\left[\cancel{\phi} \int_0^t \phi_s \psi_s \, ds\right].$$