Exercise 3.2:

Define $\begin{cases} t_k = \frac{k}{n}\cdot t, \; k = 0, \dots, n & \rightsquigarrow 0 = t_0 < t_1 < \dots < t_n = t \\ \Delta B_{t_k} = B_{t_{k+1}} - B_{t_k}. \\ \Delta t_k = t_{k+1} - t_k \end{cases}$

a) $\Delta(t_k B_{t_k}) := t_{k+1} B_{t_{k+1}} - t_k B_{t_k}$

$$= t_k \Delta B_{t_k} + B_{t_{k+1}}(t_{k+1} - t_k)$$

$$=: t_k \Delta B_{t_k} + B_{t_{k+1}} \Delta t_k .$$

$$t.B_t = \sum_{k=0}^{n-1} \Delta(t_k B_{t_k}) = \underbrace{\sum_{k=0}^{n-1} t_k \Delta B_{t_k}}_{I_1} + \underbrace{\sum_{k=0}^{n-1} B_{t_{k+1}} \Delta t_k}_{I_2}$$

$$I_1 = \mathbb{I}\Big(\underbrace{\sum_{k=0}^{n-1} t_k \cdot \mathbb{1}_{(t_k, t_{k+1}]}}_{\substack{\downarrow \; n \to \infty \text{ in } L^2[0,t] \\ \mathrm{id}}} \Big)$$

$$\sum_{k=0}^{n-1} \int_0^t \mathbb{1}_{(t_k, t_{k+1}]}(s)(t_k - s)^2 \, ds$$

$$= \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (t_k - s)^2 \, ds = \sum_{k=0}^{n-1} \frac{1}{3} \cdot (t_{k+1} - t_k)^3$$

$$= \frac{1}{3} n \cdot \left(\frac{t}{n}\right)^3 \xrightarrow[n \to \infty]{} 0 .$$

It remains to show that I_2, which is a Riemann integral converges also in $L^2(\Omega, \mathbb{P})$:

$$\mathbb{E}\left[\left|\int_0^t s\,dB_s - \sum_{k=0}^{n-1} B_{t_{k+1}} \Delta t_k\right|^2\right]$$

$$\leq C\, \mathbb{E}\left[\sum_{k=0}^{n-1} \int_0^{t_k} \mathbb{1}_{[t_k, t_{k+1}]}(s) . (B_{t_{k+1}} - B_s)^2\, ds\right.$$

$$= C \sum_{k=0}^{n-1} \int_0^{t_k} \mathbb{1}_{[t_k, t_{k+1}]}(s) \underbrace{\mathbb{E}\left[(B_{t_{k+1}} - B_{t_k})^2\right]}_{t_{k+1} - s} ds$$

$$= C . \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - s)\, ds$$

$$= \frac{C}{2} . n . \frac{t}{n^2} \xrightarrow[n \to \infty]{} 0 .$$

b) $x^3 = x_0^3 + 3x_0^2(x - x_0) + 6x_0\frac{(x - x_0)^2}{2} + \frac{6}{6}(x - x_0)^3$ (Taylor series for x^3)

Apply it to $B_{t_{k+1}}^3 - B_{t_k}^3$:

$$B_{t_{k+1}}^3 - B_{t_k}^3 = 3B_{t_k}^2 \Delta B_{t_k} + 3B_{t_k}(\Delta B_{t_k})^2 + (\Delta B_{t_k})^3 ,$$

then :

$$\frac{1}{3} B_t^3 = \underbrace{\sum_{k=0}^{n-1} B_{t_k}^2 \Delta B_{t_k}}_{I_3} + \underbrace{\sum_{k=0}^{n-1} B_{t_k} \Delta B_{t_k}^2}_{I_4} + \underbrace{\frac{1}{3}\sum_{k=0}^{n-1} (\Delta B_{t_k})^3}_{I_5}$$

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. $I_3 \xrightarrow[n\to\infty]{} \int_0^t B_s^2 \, dB_s$ in $L^2(\mathbb{P})$

since $\sum_{k=0}^{n-1} B_{t_k}^2 \cdot \mathbb{1}_{(t_k, t_{k+1}]}(s) \longrightarrow B_s^2$ in $L^2(\mathbb{P} \otimes \lambda[0,t])$

. I_5 : $\underbrace{\mathbb{E}\left[(I_5)^2\right] = \frac{1}{3} \sum_{k=0}^{n-1} \mathbb{E}\left[(\Delta B_{t_k})^6\right]}_{\perp\!\!\!\perp \text{ increments}} = \underbrace{\frac{1}{3} C \cdot \left(\frac{t}{n}\right)^3 \cdot n}_{\xrightarrow[n\to\infty]{} 0}$

. I_4 : We show that $I_5 \xrightarrow[n\to\infty]{} \int_0^t B_s \, ds$ in $L^2(\Omega, \mathbb{P})$:

from a, we know that : $\sum_{k=0}^{n-1} B_{t_k} \cdot \Delta t_k \xrightarrow[n\to\infty]{} \int_0^t B_s \, ds$ in $L^2(\Omega, \mathbb{P})$

(since it is a Riemann integral, we can take B_{t_k} or $B_{t_{k+1}}$).

$$\mathbb{E}\left[\left(\sum_{k=0}^{n-1} B_{t_k} (\Delta B_{t_k})^2 - \sum_{k=0}^{n-1} B_{t_k} \Delta t_k\right)^2\right]$$

$$= \sum_{k=0}^{n-1} \mathbb{E}\left[B_{t_k}\left((\Delta B_{t_k})^2 - \Delta t_k\right)^2\right] + 2 \sum_{k=0}^{n-1} \sum_{k'=0}^{k-1} \mathbb{E}\left[B_{t_k} B_{t_{k'}} \left((\Delta B_{t_k})^2 - \Delta t_k\right) \cdot \left((\Delta B_{t_{k'}})^2 - \Delta t_{k'}\right)\right]$$

$$\overset{\perp\!\!\!\perp \text{ increments}}{=} \sum_{k=0}^{n-1} \mathbb{E}\left[B_{t_k}^2\right] \underbrace{\mathbb{E}\left[\left((\Delta B_{t_k})^2 - \Delta t_k\right)^2\right]}_{= (\Delta t_k)^2 \cdot \mathbb{E}\left[G_i^2 - 1\right]} + 2 \sum_{k=0}^{n-1} \sum_{k'=0}^{k-1} \mathbb{E}\left[B_{t_k} B_{t_{k'}} \left((\Delta B_{t_{k'}})^2 - \Delta t_{k'}\right)\right] \times \underbrace{\mathbb{E}\left[(\Delta B_{t_k})^2 - \Delta t_k\right]}_{0}$$

where $G_i \sim \mathcal{N}(0,1)$

$$= \sum_{k=0}^{n-1} t_k \cdot (\Delta E_k)^2 \cdot C + 0$$

$$= \sum_{k=0}^{n-1} \frac{kt}{n} \cdot \left(\frac{t}{n}\right)^2 = \frac{t^3}{n^3} \cdot \frac{n(n-1)}{2} \xrightarrow[n \to \infty]{} 0 .$$

$\therefore$