

The numeraire portfolio for unbounded semimartingales

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Abstract. Asset prices discounted by a tradable numeraire N should be (local) martingales under some measure Q that is equivalent to the original probability measure P . Instead of studying the set of equivalent martingale measures with respect to a prespecified numeraire, we will look for a tradable numeraire N^P such that the discounted asset prices become martingales with respect to the original measure P . N^P is called *(P-)numeraire portfolio*. Since the above martingale condition is too stringent to obtain a general existence result, we define a (generalized) numeraire portfolio by a weaker requirement. This N^P is characterized as the solution to several optimization problems.

Key Words: numeraire portfolio, change of numeraire, martingale measures, growth optimal portfolio, relative entropy

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1 Introduction

We consider a financial market with one riskless asset (savings account) and d risky assets (stocks). For simplicity assume the value of the savings account to be identically one. The (discounted) prices of the risky assets are described by a d -dimensional semimartingale $S = (S^i)_{i=1, \dots, d}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$ where T is a finite time horizon. For a predictable S -integrable process ϑ the value process of the (self-financing) trading strategy ϑ with initial capital 1 is given by $1 + \int \vartheta dS$. If this value process is strictly positive, it is called a tradable numeraire. \mathcal{N} denotes the set of all tradable numeraires.

To motivate the idea of the numeraire portfolio, let us consider the pricing of a European option modelled by an \mathcal{F}_T -measurable random variable $H \geq 0$. It is well known that any arbitrage-free valuation is essentially of the form $H \mapsto E^Q [H/N_T]$ where N is an appropriate tradable numeraire and Q a probability measure equivalent to P such that all asset prices discounted by N are (local) Q -martingales. If the state price density $\frac{dQ}{dP} \frac{1}{N_T}$ is not unique, arbitrage arguments alone typically only give an interval of arbitrage-free prices. So one has to choose Q and N in order to decide on one of the possible arbitrage-free valuations of H . There are two dual approaches for making this choice:

1. Fix $N \in \mathcal{N}$ and choose $Q \sim P$ such that (1)
 $1/N$ and S/N become (local) Q -martingales.
2. Fix $Q \sim P$ and choose $N \in \mathcal{N}$ such that (2)
 $1/N$ and S/N become (local) Q -martingales.

In the first approach one fixes a priori some appropriate tradable numeraire N , e.g., the riskless numeraire $S^0 \equiv 1$. As the set of equivalent martingale measures for the asset prices discounted by N is not a singleton, a choice of Q has to be made and this is typically done according to some optimality criterion involving the original measure P . Examples are the minimal entropy martingale measure [11], the minimal martingale measure [10, 20] or the variance optimal martingale measure [21]. Pricing problems, e.g. for the exchange option studied by Margrabe [19], often become more convenient under another numeraire than $N = S^0$ (say). Given a martingale measure Q corresponding to N , the methodology of Geman, El Karoui and Rochet [9] provides an efficient way for a change to another numeraire by a simultaneous change of the martingale measure.

In the second approach a certain probability Q is fixed a priori - a natural choice is the original probability P . Here the situation is somewhat simpler because one can show that there is at most one tradable numeraire satisfying the requirement in (2). So this numeraire is the unique ‘right’ tradable numeraire with respect to a given probability measure Q . In the situation of $Q := P$, Long [18] called this numeraire the *(P-)numeraire portfolio*. The original probability measure P becomes an equivalent martingale measure with respect to the P -numeraire portfolio if this satisfies (2). The first appearance of such an idea seems to be in Vasicek [23] (on p. 184). Suggested applications of this concept use the following two aspects: From a technical point of view, well-known results can be proved from another perspective without the usual change to some risk-neutral measure. On the empirical side the original probability measure P models the ‘true world’ probability which can be investigated by statistical methods. Long [18], for example, studied the application of measuring abnormal stock returns by discounting NYSE stock returns by empirical proxies of the numeraire portfolio.

Long [18] proved existence of a tradable numeraire N satisfying (2) in the case of finite Ω and discrete time and in the case where the asset prices S evolve according to a sufficiently regular multidimensional diffusion model. In these cases the numeraire portfolio turns out to be growth-optimal. More properties and applications in the diffusion case, where the numeraire portfolio is mean-variance efficient and therefore related to the CAPM-theory, are shown in Bajeux-Besnainou and Portait [3] and Johnson [13]. A close connection to the concept of value preservation was very recently pointed out by Korn and Schäl [15, 16]. For a survey discussion and further references see Artzner [2].

The basic goal of the current paper is to study the idea of the numeraire portfolio under minimal assumptions on the model. Section 2 contains the general setting. Section 3 states essential results on the *growth-optimal* numeraire that is known to be closely related to the P -numeraire portfolio. As examples in Section 5 show, the (local) martingale condition

(2) is too stringent to obtain a general existence result. For this reason we define the (generalized) P -numeraire portfolio in Section 4 by a weaker requirement such that our definition is a consistent extension of the original numeraire portfolio concept introduced by Long [18]. From general duality results from Kramkov and Schachermayer [17] on utility maximization, the existence of the P -numeraire portfolio N^P can be shown under very mild conditions which are of ‘no-arbitrage’ type. N^P is characterized as the solution to several optimization problems. In the case of continuous asset prices $1/N^P$ coincides with the density process of the minimal martingale measure.

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ be a filtered probability space with a filtration satisfying the usual conditions, where the time horizon $T > 0$ is finite. For simplicity, we assume $\mathcal{F}_T = \mathcal{F}$. All stochastic processes will be indexed by $t \in [0, T]$. All semimartingales considered in the sequel are assumed to have right-continuous paths with left limits. For an \mathbb{R}^d -valued semimartingale S and an S -integrable predictable \mathbb{R}^d -valued process ϑ we denote by $\vartheta \cdot S$ the semimartingale defined by $(\vartheta \cdot S)_t = \int_0^t \vartheta_u dS_u$. Let $L(S)$ denote the space of all \mathbb{R}^d -valued predictable processes that are integrable with respect to S . For unexplained terminology from stochastic analysis we refer to Dellacherie/Meyer [8] and Jacod [12].

We consider a model of a financial market consisting of $d + 1$ assets: one savings account and d stocks with prices $S = (S^i)_{i=1, \dots, d}$. S is assumed to be a semimartingale. For simplicity we assume the value of the savings account to be constant at one so that the asset prices are expressed in units of the savings account.

A (self-financing) portfolio strategy is defined by a pair $(V_0, \vartheta) \in \mathbb{R} \times L(S)$, where V_0 specifies the initial capital and ϑ^i the number of shares of each stock i held in the dynamically varying portfolio. The corresponding value process is given by $V = V_0 + \vartheta \cdot S$. We denote by

$$\mathcal{N} := \{N \mid N = 1 + \vartheta \cdot S, \vartheta \in L(S), N > 0\} \quad (3)$$

the family of all strictly positive value processes starting at one. These processes are called *tradable numeraires*.

Definition 2.1 *A probability measure $Q \sim P$ is called an equivalent local martingale measure for \mathcal{N} if all $N \in \mathcal{N}$ are local Q -martingales. The set of all such measures is denoted by $\mathcal{M}^e(\mathcal{N})$.*

Remark 2.2 By Corollaire 3.5 from [1] every equivalent local martingale measure for S (in the usual sense) is an element of $\mathcal{M}^e(\mathcal{N})$, i.e. $\mathcal{M}^e(S) \subseteq \mathcal{M}^e(\mathcal{N})$. Moreover, a simple argument shows that $Q \in \mathcal{M}^e(\mathcal{N})$ if and only if every stochastic integral $\vartheta \cdot S$ that is locally bounded from below is a local Q -martingale. This implies by taking $\vartheta := 1$ that $\mathcal{M}^e(\mathcal{N})$ coincides with the set $\mathcal{M}^e(S)$ of equivalent local martingale measures for S if S is locally

bounded from below. If S fails to be locally bounded from below the situation may be more complicated: If S is a local Q -martingale, then Q is an equivalent local martingale measure for \mathcal{N} (see [1]), but the converse is not true in general (see [7]).

The next proposition shows that $\mathcal{M}^e(\mathcal{N})$ is not empty if and only if the financial market S satisfies the condition of *No Free Lunch with Vanishing Risk* (NFLVR), i.e.,

$$\bar{C} \cap L_+^\infty(\Omega, \mathcal{F}, P) = \{0\},$$

where $L_+^\infty(\Omega, \mathcal{F}, P)$ is the set of nonnegative P -a.s. bounded random variables and \bar{C} denotes the L^∞ -closure of

$$C := \left\{ f \in L^\infty(\Omega, \mathcal{F}, P) \mid \begin{array}{l} \exists \vartheta \in L(S) \text{ s.t. } \vartheta \cdot S \text{ is } P\text{-a.s. uniformly} \\ \text{bounded from below and } f \leq (\vartheta \cdot S)_T \end{array} \right\}. \quad (4)$$

This ‘no-arbitrage’ type concept was introduced by Delbaen and Schachermayer; see [5, 7] for details and an economic interpretation.

Proposition 2.3 *S satisfies (NFLVR) if and only if $\mathcal{M}^e(\mathcal{N}) \neq \emptyset$.*

Proof: \Leftarrow : Let $Q \in \mathcal{M}^e(\mathcal{N})$. Let $f^n \in C$ such that $f^n \xrightarrow{L^\infty} f$ where $f \in L_+^\infty$. By the definition of C there exist $\vartheta^n \in L(S)$ such that $f^n \leq (\vartheta^n \cdot S)_T$ and $\vartheta^n \cdot S$ is uniformly bounded from below. Let $Q \in \mathcal{M}^e(\mathcal{N})$. Then $\vartheta^n \cdot S$ is a local Q -martingale by Remark 2.2 and bounded from below. So we obtain $E^Q[f^n] \leq E^Q[(\vartheta^n \cdot S)_T] \leq 0$. Hence $E^Q[f] \leq 0$ and it follows that $f = 0$.

\Rightarrow : By Theorem 1.1 in [7] S satisfies (NFLVR) if and only if there is a probability measure $Q \sim P$ such that S is a Q -sigma-martingale. The latter means that there exist an \mathbb{R}^d -valued (local) Q -martingale \tilde{M} and a strictly positive \tilde{M} -integrable predictable process φ such that $S = S_0 + \varphi \cdot \tilde{M}$. For any $N = 1 + \vartheta \cdot S$ in \mathcal{N} we have $N = 1 + \vartheta \cdot (\varphi \cdot \tilde{M}) = 1 + (\vartheta \varphi) \cdot \tilde{M}$. Applying Corollaire 3.5 from [1] we obtain $Q \in \mathcal{M}^e(\mathcal{N})$. \square

Motivated by Proposition 2.3 we impose throughout the remainder of this paper the

$$\text{STANDING ASSUMPTION: } \mathcal{M}^e(\mathcal{N}) \neq \emptyset. \quad (6)$$

If $\mathcal{M}^e(\mathcal{N}) = \{Q\}$ is a singleton, we have a martingale representation result even though S need not be a local Q -martingale:

Proposition 2.4 *Assume $\mathcal{M}^e(\mathcal{N}) = \{Q\}$. Then for every local Q -martingale M there is $\vartheta \in L(S)$ such that $M = M_0 + \vartheta \cdot S$.*

Proof: By Proposition 2.3 S satisfies the condition (NFLVR) and this implies ([7], Theorem 1.1) $S = S_0 + \varphi \cdot \tilde{M}$, where \tilde{M} is an \mathbb{R}^d -valued (local) martingale with respect to some probability measure equivalent to P and φ is a strictly positive \tilde{M} -integrable predictable

process. Without loss of generality we can assume $\widetilde{M}_0 = 0$. Corollaire 3.5 from [1] implies that every equivalent local martingale measure for \widetilde{M} is an element of $\mathcal{M}^e(\mathcal{N})$. Hence Q is the unique equivalent local martingale measure for \widetilde{M} . Therefore \widetilde{M} has the strong predictable representation property ([12] Corollaire 11.4). By an additional localization argument ([12] Proposition 2.38) this implies that every local Q -martingale M has a representation $M = M_0 + H \cdot \widetilde{M}$, where H is integrable with respect to \widetilde{M} in the sense of local martingales. But $1/\varphi$ is S -integrable since φ is \widetilde{M} -integrable and $\varphi > 0$. Hence $H \cdot \widetilde{M} = H \cdot (1/\varphi \cdot S) = (H/\varphi) \cdot S$ and $\vartheta := H/\varphi$ is an element of $L(S)$. \square

The following lemma summarizes results about the structure of \mathcal{N} that are needed in the sequel.

Lemma 2.5 1. \mathcal{N} is convex.

2. For $N^1, N^2 \in \mathcal{N}$ and for stopping times σ, τ with $0 \leq \sigma \leq \tau \leq T, A \in \mathcal{F}_\sigma$, the process

$$N := I_{\llbracket 0, \sigma \rrbracket} N^1 + I_{\llbracket \sigma, \tau \rrbracket} \left(I_A \frac{N_\sigma^1}{N_\sigma^2} N^2 + I_{A^c} N^1 \right) + I_{\llbracket \tau, T \rrbracket} \left(I_A \frac{N_\sigma^1}{N_\sigma^2} \frac{N_\tau^2}{N_\tau^1} + I_{A^c} \right) N^1$$

is an element of \mathcal{N} .

3. Every $N \in \mathcal{N}$ can be written as a stochastic exponential, and the set

$$\mathcal{L} := \left\{ L \mid L \text{ semimartingale, } L_0 = 0, \mathcal{E}(L) \in \mathcal{N} \right\}$$

contains 0 and is predictably convex, i.e., for any predictable $[0, 1]$ -valued process H and $L^1, L^2 \in \mathcal{L}$, the process $L := H \cdot L^1 + (1 - H) \cdot L^2$ is an element of \mathcal{L} .

Proof: 1. This is obvious.

3.: Every $N \in \mathcal{N}$ is a local Q -martingale for $Q \in \mathcal{M}^e(\mathcal{N})$ and strictly positive, hence $N_- > 0$. Taking $L := (1/N_-) \cdot N$ we get $N = \mathcal{E}(L) \equiv 1 + N_- \cdot L$. $0 \in \mathcal{L}$ follows from $1 \in \mathcal{N}$. In order to show predictable convexity of \mathcal{L} let $N^1, N^2 \in \mathcal{N}$ and $N^i = 1 + \vartheta^i \cdot S = \mathcal{E}(L^i)$ for $i = 1, 2$. Define $L := H \cdot L^1 + (1 - H) \cdot L^2$ and $N := \mathcal{E}(L)$. From the equality $\vartheta^i dS = dN^i = N_-^i dL^i$ for $i = 1, 2$ it follows

$$dN = N_- (H dL^1 + (1 - H) dL^2) = \left(\frac{N_- H}{N_-^1} \vartheta^1 + \frac{N_- (1 - H)}{N_-^2} \vartheta^2 \right) dS. \quad (7)$$

From $N^i > 0$ follows $\Delta L^i > -1$ for $i = 1, 2$, hence $N > 0$ and by (7) we obtain $N \in \mathcal{N}$. This implies $L \in \mathcal{L}$.

2.: Let $N^i = \mathcal{E}(L^i)$ for $i = 1, 2$, $H := I_{\llbracket 0, \sigma \rrbracket} + I_{\llbracket \sigma, \tau \rrbracket} I_{A^c} + I_{\llbracket \tau, T \rrbracket}$ and $L := H \cdot L^1 + (1 - H) \cdot L^2$. We show that $N = \mathcal{E}(L)$: Since $dL = I_{\llbracket 0, \sigma \rrbracket} dL^1 + I_{\llbracket \sigma, \tau \rrbracket} (I_{A^c} dL^1 + I_A dL^2) + I_{\llbracket \tau, T \rrbracket} dL^1$, we obtain $dN_t = dN_t^1 = N_{t-}^1 dL_t^1 = N_{t-} dL_t$ for $(\omega, t) \in \llbracket 0, \sigma \rrbracket$. For $(\omega, t) \in \llbracket \sigma, \tau \rrbracket$ we have

$$\begin{aligned} dN_t &= I_A \frac{N_\sigma^1}{N_\sigma^2} N_{t-}^2 dL_t^2 + I_{A^c} N_{t-}^1 dL_t^1 \\ &= N_{t-} d(I_A \cdot L^2)_t + N_{t-} d(I_{A^c} \cdot L^1)_t = N_{t-} dL_t, \end{aligned}$$

and finally for $(\omega, t) \in \llbracket \tau, T \rrbracket$ we observe

$$dN_t = \left(I_A \frac{N_\sigma^1 N_\tau^2}{N_\sigma^2 N_\tau^1} + I_{A^c} \right) N_{t-}^1 dL_t^1 = N_{t-} dL_t.$$

So N solves $N = 1 + N_- \cdot L$. Hence $N = \mathcal{E}(L)$ and by part 3 this proves the claim. \square

3 The growth-optimal numeraire

It is well known that there is a close relationship between the numeraire portfolio and the growth-optimal numeraire. In fact, both coincide in the cases studied in [18]. We shall see in the next section that this remains true in general.

Definition 3.1 *We call a tradable numeraire growth-optimal and denote it by N^{\log} if it solves the maximization problem*

$$u := \sup_{N \in \mathcal{N}} E[\log N_T]. \quad (8)$$

Justification for the name is given by Proposition 3.3 which summarizes the essential properties of N^{\log} .

Remark 3.2 The supremum in (8) is a priori only taken over all $N \in \mathcal{N}$ for which $E[\log N_T]$ is defined. If $u < \infty$ then $\log^+ N_T \in L^1(P)$ for every $N \in \mathcal{N}$. To see this, observe that for $\varepsilon \in (0, 1)$ $N^\varepsilon := \varepsilon + (1 - \varepsilon)N \geq \varepsilon$ is bounded away from zero, $N^\varepsilon \in \mathcal{N}$ by Lemma 2.5, and the concavity of \log implies $E[\log^+ N_T^\varepsilon] \geq (1 - \varepsilon)E[\log^+ N_T]$. Hence $E[\log N_T]$ is well-defined with values in $\mathbb{R} \cup \{-\infty\}$ if u is finite. If $0 \leq \sigma \leq \tau \leq T$ then part 2 of Lemma 2.5 (with $N^1 = 1$, $N^2 = N$, $A = \Omega$) gives an element $N' \in \mathcal{N}$ such that $N'_T = N_\tau / N_\sigma$. The same argument as above shows that $E[\log N_\tau / N_\sigma]$ is also well-defined with values in $\mathbb{R} \cup \{-\infty\}$ if u is finite. This will be used later.

Proposition 3.3 *Assume $u < \infty$.*

1. *The growth-optimal numeraire N^{\log} is unique.*
2. *N^{\log} maximizes simultaneously all intertemporal conditional expected growth rates: For any stopping times σ, τ with $0 \leq \sigma \leq \tau \leq T$ we have*

$$E \left[\log \frac{N_\tau^{\log}}{N_\sigma^{\log}} \middle| \mathcal{F}_\sigma \right] \geq E \left[\log \frac{N_\tau}{N_\sigma} \middle| \mathcal{F}_\sigma \right] \quad \forall N \in \mathcal{N}. \quad (9)$$

Proof: 2.: Using Remark 3.2 we have for $0 \leq \sigma \leq \tau \leq T$

$$E \left[\log N_T^{\log} \right] = E \left[\log N_\sigma^{\log} \right] + E \left[\log \frac{N_\tau^{\log}}{N_\sigma^{\log}} \right] + E \left[\log \frac{N_T^{\log}}{N_\tau^{\log}} \right].$$

The finiteness of the left-hand side now implies $\log(N_\tau^{\log}/N_\sigma^{\log}) \in L^1(P)$. Define $N \in \mathcal{N}$ according to part 2 of Lemma 2.5 with $N^1 = N^{\log}$ and arbitrary $N^2 \in \mathcal{N}$, $A \in \mathcal{F}_\sigma$. Since $E[\log N_T]$ is well-defined by Remark 3.2, the log-optimality of N^{\log} implies

$$\begin{aligned} 0 \leq E \left[\log N_T^{\log} - \log N_T \right] &= E \left[-\log \left(I_A \frac{N_\sigma^{\log}}{N_\sigma^2} \frac{N_\tau^2}{N_\tau^{\log}} + I_{A^c} \right) \right] \\ &= E \left[I_A \left(\log \frac{N_\tau^{\log}}{N_\sigma^{\log}} - \log \frac{N_\tau^2}{N_\sigma^2} \right) \right]. \end{aligned}$$

Using the P -integrability of $\log(N_\tau^{\log}/N_\sigma^{\log})$ we obtain part 2.

1.: The second part implies in particular the equality $E[\log N_t^{\log}] = \sup_{N \in \mathcal{N}} E[\log N_t]$ for all $t \in [0, T]$. Hence the uniqueness of N_t^{\log} for every $t \in [0, T]$ is a consequence of the convexity of \mathcal{N} and the strict concavity of the logarithm. \square

4 The numeraire portfolio

Definition 4.1 Let $Q \sim P$. A tradable numeraire in \mathcal{N} is called the Q -numeraire portfolio¹ and denoted by N^Q if N/N^Q is a Q -supermartingale for all $N \in \mathcal{N}$.

There can be at most one Q -numeraire portfolio. To see this, assume N^1 and N^2 are both Q -numeraire portfolios. Then N^1/N^2 and also the reciprocal N^2/N^1 are Q -supermartingales. Using Jensen's inequality we conclude that N^1 and N^2 are equal.

Remark 4.2 The definition is a consistent extension of the numeraire portfolio concept (2) introduced by Long [18]. In fact, a tradable numeraire N satisfying condition (2) (i.e. S/N and $1/N$ are (local) Q -martingales for some $Q \sim P$) defines (locally) a martingale measure $d\tilde{Q} := 1/N_{T(n)} dQ$ for S . By Corollaire 3.5 in [1] this implies that any tradable numeraire N' is a local \tilde{Q} -martingale. Thus N'/N is a local Q -martingale for all $N' \in \mathcal{N}$ and hence N is the unique Q -numeraire portfolio.

We shall confine ourselves in the sequel to the P -numeraire portfolio N^P . This is without loss of generality since we could choose Q instead of P as the original measure of our probability space. The following proposition shows that the condition defining the P -numeraire portfolio is equivalent to growth-optimality. Hence as in the cases studied in the literature the numeraire portfolio is equal to the growth optimal portfolio.

Proposition 4.3 Assume $u < \infty$. A numeraire is growth-optimal if and only if it is equal to the P -numeraire portfolio, i.e., $N^P = N^{\log}$.

¹This is a slight abuse of terminology: we do not distinguish here between a portfolio and the corresponding value process in order to keep the usual term 'numeraire portfolio'.

Note that this result shows in particular that the existence of the P -numeraire portfolio implies the existence of the growth-optimal portfolio and vice versa.

Proof: Assume N^P exists. From $\log a \leq a - 1$ for $a > 0$ we get that $\log y \geq \log x + 1 - x/y$ for all $x, y > 0$. Hence $\log N_T^P \geq \log N_T + 1 - N_T/N_T^P$ for all $N \in \mathcal{N}$. Taking $N = 1$ and using the property defining N^P shows $(\log N_T^P)^- \in L^1(P)$. Hence $E[\log N_T^P]$ is well-defined with values in $\mathbb{R} \cup \{\infty\}$. So we get $E[\log N_T^P] \geq E[\log N_T]$ for any $N \in \mathcal{N}$. This proves the growth-optimality of the numeraire portfolio. To prove the converse assume now that N^{\log} exists. We proceed as Kramkov and Schachermayer [17] (proof of (3.17) in Lemma 3.9) in order to obtain (10): Let $N \in \mathcal{N}, \varepsilon \in (0, 1)$ and define $N^\varepsilon := (1 - \varepsilon)N^{\log} + \varepsilon N \in \mathcal{N}$. Since $u < \infty$ we get

$$\begin{aligned} 0 &\leq E\left[\log N_T^{\log} - \log N_T^\varepsilon\right] = E\left[\int_{N_T^\varepsilon}^{N_T^{\log}} \frac{1}{z} dz\right] \\ &\leq E\left[\frac{N_T^{\log} - N_T^\varepsilon}{N_T^\varepsilon}\right] = E\left[\frac{\varepsilon(N_T^{\log} - N_T)}{(1 - \varepsilon)N_T^{\log} + \varepsilon N_T}\right]. \end{aligned}$$

Thus $E[N_T/N_T^\varepsilon] \leq E[N_T^{\log}/N_T^\varepsilon]$, and for $\varepsilon \downarrow 0$ we get by Fatou's lemma

$$E\left[\frac{N_T}{N_T^{\log}}\right] \leq 1 \quad \text{for all } N \in \mathcal{N}. \quad (10)$$

For $N' \in \mathcal{N}$ and $0 \leq s \leq t \leq T$ define $A := \left\{ \left(N_s^{\log}/N_s' \right) E\left[N_t'/N_t^{\log} \mid \mathcal{F}_s \right] > 1 \right\}$. By part 2 of Lemma 2.5 (take $N^2 = N', N^1 = N^{\log}$), (10) implies

$$1 \geq E\left[I_A \frac{N_s^{\log} N_t'}{N_s' N_t^{\log}} + I_{A^c} \right] = E\left[I_A \frac{N_s^{\log}}{N_s'} E\left[\frac{N_t'}{N_t^{\log}} \mid \mathcal{F}_s \right] + I_{A^c} \right].$$

By the definition of A we obtain $P[A] = 0$ and conclude that N^{\log} is the P -numeraire portfolio. \square

Definition 4.4 We denote by

$$\mathcal{SM} := \left\{ Z \mid Z \geq 0, Z_0 = 1, ZN \text{ is a } P\text{-supermartingale for all } N \in \mathcal{N} \right\} \quad (11)$$

the family of all P -supermartingale densities for \mathcal{N} .

\mathcal{SM} is not empty because for any $Q \in \mathcal{M}^e(\mathcal{N})$ the density process of Q with respect to P is in \mathcal{SM} . Note that each $Z \in \mathcal{SM}$ is actually a P -supermartingale because $1 \in \mathcal{N}$. This implies $\log^+ Z_T \in L^1(P)$ and hence $E[\log 1/Z_T]$ is always well-defined and takes values in $\mathbb{R} \cup \{+\infty\}$. In the spirit of Kramkov and Schachermayer [17] we can now set up the dual problem to (8):

$$h := \inf_{Z \in \mathcal{SM}} E\left[\log \frac{1}{Z_T}\right] \quad (12)$$

If $Z_T = dQ/dP$ is the density of a probability measure $Q \sim P$, the right-hand side is the relative entropy $H_{\mathcal{F}_T}(P|Q) := E^Q[(1/Z_T) \log(1/Z_T)]$ of P with respect to Q on (Ω, \mathcal{F}_T) .

The full duality picture is described by the following theorem, which is an application of a general duality theorem on utility maximization due to Kramkov and Schachermayer [17]; see also Karatzas et al. [14] for previous convex duality results for such problems.

Theorem 4.5 *Assume (6) (NFLVR) and $u < \infty$ (or equivalently $h < \infty$). Then*

1. *There exists a unique solution $N^{\log} \in \mathcal{N}$ to the log-optimization problem (8) which is also optimal with respect to any previous time horizon, i.e.*

$$\exists! N^{\log} \in \mathcal{N} \quad \text{such that} \quad E \left[\log N_\tau^{\log} \right] = \sup_{N \in \mathcal{N}} E[\log N_\tau]$$

for all stopping times $\tau \leq T$.

2. *There exists a unique solution to the ‘reverse entropy’-minimization problem (12) which also minimizes the ‘reverse entropy’ with respect to any previous time horizon, i.e.*

$$\exists! \widehat{Z} \in \mathcal{SM} \quad \text{such that} \quad E \left[\log \frac{1}{\widehat{Z}_\tau} \right] = \inf_{Z \in \mathcal{SM}} E \left[\log \frac{1}{Z_\tau} \right]$$

for all stopping times $\tau \leq T$.

3. *The solutions to the problems (8) and (12) are related by $1/N^{\log} = \widehat{Z}$.*

4. *For any stopping time $\tau \leq T$ have*

$$\inf_{Z \in \mathcal{SM}} E \left[\log \frac{1}{Z_\tau} \right] = \inf_{Q \in \mathcal{M}^e(\mathcal{N})} E \left[\log \frac{1}{\left(\frac{dQ}{dP} \Big|_{\mathcal{F}_\tau} \right)} \right] = \inf_{Q \in \mathcal{M}^e(\mathcal{N})} H_{\mathcal{F}_\tau}(P|Q).$$

Proof: Applying Theorem 2.2 of [17] for the case of a logarithmic utility function, we obtain part 4 and the existence of the unique solutions N^{\log} and \widehat{Z} to the optimization problems (8) and (12). Using Proposition 3.3 we obtain part 1. Note that in the log-utility case, we have the duality in part 3 not just for the terminal values N_T^{\log} and \widehat{Z}_T as in [17] but for the processes on the entire time interval. To see this, observe that Proposition 4.3 implies $1/N^{\log} \in \mathcal{SM}$. Since we know from [17] that $1/N_T^{\log} = \widehat{Z}_T$, the uniqueness of \widehat{Z} implies the assertion of part 3. Using part 1 this implies part 2. \square

Remark 4.6 Note that the assumptions of Theorem 4.5 are of no-arbitrage character: 1) there exists an equivalent local martingale measure for \mathcal{N} such that 2) the relative entropy of P with respect to this martingale measure is finite. The first assumption is equivalent to (NFLVR) by Proposition 2.3 and the second assumption is equivalent to the assumption that the optimal expected growth up to the finite time horizon T is not infinite.

As immediate consequences we obtain the following results about the numeraire portfolio:

Corollary 4.7 (*existence*) Under the assumptions of Theorem 4.5, the P -numeraire portfolio N^P exists.

Corollary 4.8 (*equivalent characterizations*) Suppose the assumptions of Theorem 4.5 are satisfied. Then $\widehat{N} \in \mathcal{N}$ is the P -numeraire portfolio if and only if it solves one (hence every) of the optimization problems

$$E[\log \widehat{N}_T] = \sup_{N \in \mathcal{N}} E[\log N_T] \quad (13)$$

$$E[\log \widehat{N}_T] = \inf_{Z \in \mathcal{S}\mathcal{M}} E \left[\log \frac{1}{Z_T} \right] \quad (14)$$

$$E[\log \widehat{N}_T] = \inf_{Q \in \mathcal{M}^e(\mathcal{N})} H(P|Q) \quad (15)$$

This allows us to state a necessary and sufficient condition for the reciprocal of the numeraire portfolio to define an equivalent local martingale measure:

Corollary 4.9 (*equivalent local martingale measure condition*) Suppose the assumptions of Theorem 4.5 are satisfied. Then $1/N_T^P$ is the density of an equivalent local martingale measure $\widehat{Q} \in \mathcal{M}^e(\mathcal{N})$ with respect to P if and only if $\inf_{Q \in \mathcal{M}^e(\mathcal{N})} H(P|Q)$ is attained by \widehat{Q} . In this case $1/N^P$ is the density process of \widehat{Q} with respect to P .

For continuous price processes S , a result of Schweizer [22] shows that the minimal martingale measure (in the sense of Föllmer and Schweizer, see e.g. [10, 20]), if it exists, solves the above reverse entropy minimization problem. Hence we can identify in this case the discount factor given by the P -numeraire portfolio as the minimal martingale density:

Corollary 4.10 (*relation to the minimal martingale measure*) Assume S is continuous, the minimal martingale measure \widehat{P} exists and $h < \infty$. Then $1/N_t^P = \frac{d\widehat{P}}{dP} \Big|_{\mathcal{F}_t}$ for all $t \in [0, T]$.

Finally we provide a sufficient condition for the numeraire portfolio to match the requirement in approach (2), i.e., for $1/N^P$ to be a 'strict martingale density' (see Schweizer [20]). The proof of this result consists of a simple stopping argument that uses Remark 2.2.

Corollary 4.11 (*strict martingale density condition*) Assume S is locally bounded, N^P exists and $1/N^P$ is a local P -martingale. Then S/N^P is a local P -martingale.

Note that the latter condition is necessary since $1/N^P$ might fail to be a local P -martingale (see Example 6 and the subsequent remark in Section 5).

Possible generalizations:

1. Since growth-optimality and the property that defines the numeraire portfolio are local properties, the theory could be generalized to the infinite horizon setting $[0, \infty)$ by

localization; the assumptions and results then have to be formulated with respect to every T^n from an increasing sequence $(T^n)_{n \in \mathbb{N}}$ of bounded stopping times.

2. For simplicity the present paper assumes the value of the savings account to be constant. This assumption can be considerably relaxed. The basic idea is the following: If the savings account is modelled by $S^0 > 0$, one can show (in analogy to Proposition 1 in El Karoui et al. [9]) that every (suitably defined) tradable numeraire discounted by the numeraire S^0 is also a tradable numeraire (with the same trading strategy in terms of numbers of assets) with respect to the discounted market. This brings us back to the present setting. Since the discount factor $1/S^0$ cancels out in the condition that defines the numeraire portfolio, we obtain that the numeraire portfolio in the original market and the numeraire portfolio in the discounted market are equal up to the discount factor and use the same strategy. Hence the numeraire portfolio property is invariant under a change of numeraire.

We conclude this section with some comments on a question that motivated the idea of the numeraire portfolio in the introduction. To which extent does the quantity $\hat{p} := E[H/N_T^P]$ provide a reasonable valuation of an European option with maturity T , modelled by an \mathcal{F}_T -measurable random variable H ? It turns out that \hat{p} corresponds to the “fair price” of Davis [4] for the case of a logarithmic utility function. Clearly, \hat{p} provides an arbitrage-free valuation if $1/N_T^P$ is the density of an equivalent local martingale measure. But the examples in Section 5 show that the latter is not the case in general. In particular, Example 6 illustrates that \hat{p} can fail to be within the interval $[\inf_{Q \in \mathcal{M}^e} E^Q[H], \sup_{Q \in \mathcal{M}^e} E^Q[H]]$ even for $H = 1$. However, the discussion by Kramkov and Schachermayer on p. 911-912 of [17] reveals that \hat{p} always can be interpreted as an arbitrage-free valuation with respect to a new numeraire, namely N^P .

5 Examples

To illustrate the theory, we now provide some examples. Example 1 shows the situation of a complete market. Examples 2 and 3 are those cases for which Long [18] introduced and studied the idea of the numeraire portfolio. In these three cases the P -numeraire portfolio is the reciprocal of the density of an equivalent (local) martingale measure for S . So the asset prices discounted by the P -numeraire portfolio are (local) martingales with respect to the original measure P . But other examples show that the general semimartingale case can lead to less well behaved situations. In particular, the numeraire portfolio may give only a local martingale or supermartingale discount factor even with respect to the constant savings account.

Example 1 (‘complete’ market): Assume there is a unique equivalent local martingale measure Q for \mathcal{N} (respectively S). Then the P -numeraire portfolio N^P exists and $1/N^P$ is the density process of Q with respect to P .

Proof: Let Q denote the unique equivalent martingale measure for \mathcal{N} (or S , respectively)

and Z the density process of P with respect to Q . Note that the uniqueness of Q implies that \mathcal{F}_0 is trivial. By Proposition 2.4 (or the strong predictable representation property of S , respectively) there exists $\vartheta \in L(S)$ such that $Z = 1 + \vartheta \cdot S$. Hence $Z \in \mathcal{N}$ and $1/Z$ is the density process of an equivalent local martingale measure for \mathcal{N} (respectively S). This implies $Z = N^P$. \square

Example 2 (finite discrete case): Assume (6) (no arbitrage) and $\text{card}(\Omega) < \infty$. Then the P -numeraire portfolio N^P exists and is log-optimal, and $1/N^P, S/N^P$ are both P -martingales.

Proof: We give a direct proof to illustrate how the martingale property follows from the first order condition of the log-optimization problem. For $Q \in \mathcal{M}^e(\mathcal{N})$ and $\vartheta \in L(S)$, the process $V := 1 + \vartheta \cdot S$ is a Q -martingale, hence $E^Q[V_T] = 1$. Denote by \mathcal{V} the set of all those processes which are in addition nonnegative. \mathcal{V} is bounded (a.s.) and closed (identifying $L^0(\Omega, \mathcal{F}, P)$ with some \mathbb{R}^m , we can view \mathcal{V} as the intersection of an affine subspace with the positive quadrant). Hence $\sup_{V \in \mathcal{V}} E[\log V_T]$ is attained by a growth-optimal $N^{\log} \in \mathcal{V}$. The optimality implies that N^{\log} is strictly positive and hence an element of \mathcal{N} . Since $N^{\log} =: 1 + \vartheta^{\log} \cdot S$ is bounded away from zero, the first order condition yields for $\vartheta \in L(S)$, $A \in \mathcal{F}_s$ and $0 \leq s < t \leq T$

$$0 = \frac{\partial}{\partial \varepsilon} E \left[\log \left(1 + \int_0^T \left(\vartheta_u^{\log} + \varepsilon I_A I_{[s,t]}(u) \vartheta_u \right) dS_u \right) \right] \Big|_{\varepsilon=0} = E \left[I_A \frac{1}{N_T^{\log}} \int_s^t \vartheta_u dS_u \right].$$

Hence $E \left[(1/N_T^{\log}) \int_s^t \vartheta_u dS_u \mid \mathcal{F}_s \right] = 0$ for all $\vartheta \in L(S)$. By taking $\vartheta = \vartheta^{\log}$ and $t = T$ this implies in particular that $1/N^{\log}$ and hence $S/N^{\log} = (S_0 + (S - S_0))/(N^{\log})$ are P -martingales. \square

Example 3 (multidimensional diffusion model): Assume that asset prices evolve according to the standard multidimensional Brownian diffusion model. Then under suitable regularity conditions the P -numeraire portfolio N^P exists and is growth-optimal, and $1/N^P, S/N^P$ are (local) P -martingales.

To be more precise, let the (discounted) prices of the risky assets be given by the stochastic differential equations

$$\frac{dS_t^i}{S_t^i} = (\mu_t^i - r_t)dt + \sum_{j=1}^n \sigma_t^{ij} dW_t^j, \quad S_0^i > 0 \quad \text{for } i = 1, \dots, d, \quad (16)$$

where $W = (W^j)_{j=1, \dots, n}$ is an n -dimensional Brownian motion and $(\mathcal{F}_t)_{0 \leq t \leq T}$ the P -augmentation of the filtration generated by W . Let the coefficients $r, \mu = (\mu^j)_{j=1, \dots, n}$ and $\sigma = (\sigma^{ij})_{i=1, \dots, d; j=1, \dots, n}$ be predictable. Assume S is well-defined, $n \geq d$ and σ_t has full rank a.s. for every t .

In order to eliminate the drift terms by a Girsanov transformation, define (with tr denoting transposition and $\mathbf{1} = (1, \dots, 1)^{tr} \in \mathbb{R}^d$) $\widehat{\lambda}_t := \sigma_t^{tr} (\sigma_t \sigma_t^{tr})^{-1} (\mu_t - r_t \mathbf{1})$ and (assuming $\int_0^T |\widehat{\lambda}_t|^2 dt < \infty$ a.s.) $\widehat{Z} := \mathcal{E} \left(-\widehat{\lambda} \cdot W \right)$. If the local P -martingale \widehat{Z} is a true

P -martingale, it defines an equivalent local martingale measure $d\widehat{P} = \widehat{Z}_T dP$ which is the minimal martingale measure. By Corollary 4.10 we have $1/N^{\log} = \widehat{Z}$ so that S/N^{\log} is a local P -martingale. Under additional conditions (e.g. boundedness of the coefficients in (16)), we can obtain that S/N^{\log} is even a true P -martingale.

Example 4 ($1/N^P$ is a local martingale but fails to be a martingale): We recall just the results of a well-known counterexample of mathematical finance (see Delbaen and Schachermayer [6], Theorem 2.1, take $S := 1/X$): The incomplete market contains just one continuous stock price process S , modelled by the reciprocal of the stochastic exponential of a suitably stopped Brownian motion. Then $1/N^P = 1/N^{\log} = \widehat{Z}$ is a local P -martingale but not a martingale. So it is not the density process of any martingale measure, although an equivalent local martingale measure for S (and hence for \mathcal{N}) exists.

Example 5 (discrete time model with multiplicative i.i.d. returns): Let $S_k^i = s_0^i \prod_{j=1}^k Y_j^i$ for $k = 0, \dots, T$, $i = 1, \dots, d$ where $s_0^i > 0$ and Y_1, \dots, Y_T are i.i.d. \mathbb{R}_+^d -valued random variables. Let \mathbb{F} be the filtration generated by S . Assume (6) (no arbitrage) and $\sup_{\lambda \in \mathcal{B}} E[\log(1 + \lambda^{tr}(Y_1 - \mathbb{1}))] < \infty$ where $\mathcal{B} := \left\{ \lambda \in \mathbb{R}^d \mid E[\log^-(1 + \lambda^{tr}(Y_1 - \mathbb{1}))] < \infty \right\}$. Then

1. The P -numeraire portfolio N^P exists, N^P is growth-optimal and a corresponding optimal strategy always invests a constant fraction $\hat{\lambda}^i$ of current wealth in asset S^i for $i = 1, \dots, d$, i.e., there is $\hat{\lambda} \in \mathcal{B}$ such that for $k = 0, \dots, T$

$$N_k^P = \prod_{j=1}^k (1 + \hat{\lambda}^{tr}(Y_j - \mathbb{1})).$$

2. If the growth-optimal strategy invests a strictly positive fraction of wealth in each asset, i.e., $1 - \sum_{i=1}^d \hat{\lambda}^i > 0$ and $\hat{\lambda}^i > 0$ for $i = 1, \dots, d$, then $1/N^P$ and S/N^P are P -martingales.

We note that $\sup_{\lambda \in \mathcal{B}} E[\log(1 + \lambda^{tr}(Y_1 - \mathbb{1}))]$ is always finite if Ω is finite and if there is no arbitrage (see Example 2). We leave the proof which is completely standard to the reader.

Example 6 ($1/N^P$ (or S/N^P) is a supermartingale but fails to be a local martingale): We take the setting of Example 5 with one stock ($d = 1$) and one period ($T = 1$). Let $S_0^1 = 1$ and $Y := Y_1$ be lognormally distributed, i.e., $\log Y \sim \mathcal{N}(\mu, \sigma^2)$ with $\sigma > 0$. By the first result of Example 5 we have $N_1^P = 1 + \hat{\lambda}(Y - 1)$ for some $\hat{\lambda} \in \mathcal{B} = [0, 1]$. Let $N, N' \in \mathcal{N}$. Since $N = 1 + \vartheta \cdot S$ is a tradable numeraire if and only if $\vartheta \in [0, 1]$, we obtain that all coefficients in $E[N_1/N'_1] = (1 - \vartheta)E[1/N'_1] + \vartheta E[Y/N'_1]$ are nonnegative. It follows that N' is the P -numeraire portfolio if and only if $E[1/N'_1] \leq 1$ and $E[Y/N'_1] \leq 1$. Simple calculation yields

$$E[Y] = \exp\left(\mu + \frac{\sigma^2}{2}\right) \quad \text{and} \quad E\left[\frac{1}{Y}\right] = \exp\left(-\mu + \frac{\sigma^2}{2}\right). \quad (17)$$

We conclude that $\hat{\lambda} = 1$ if $\mu \geq \sigma^2/2$, $\hat{\lambda} = 0$ if $\mu \leq -\sigma^2/2$ and $\hat{\lambda} \in]0, 1[$ if $|\mu| < \sigma^2/2$. Hence $1/N^P$ (respectively S/N^P) is a P -supermartingale but not a martingale if $\mu > \sigma^2/2$

(respectively $\mu < -\sigma^2/2$). If $|\mu| \leq \sigma^2/2$ then $1/N^P$ and S/N^P are both P -martingales. This follows from (17) if $|\mu| = \sigma^2/2$ and from the results of Example 5 otherwise. With somewhat more effort, Example 5.1' in Kramkov and Schachermayer [17] shows that the same effect can occur even on a countable probability space with bounded asset prices.

6 Conclusion

The existing theory about the numeraire portfolio can be extended to its natural maximal degree of generality if the (local) martingale property that originally defines the numeraire portfolio is weakened to a supermartingale requirement.

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