Utility-Indifference Hedging and Valuation via Reaction-Diffusion Systems

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This article studies the exponential utility-indifference approach to the valuation and hedging problem in incomplete markets. We consider a financial model which is driven by a system of interacting Itô and point processes. The model allows for a variety of mutual stochastic dependencies between the tradable and non-tradable factors of risk, but still permits for a constructive and fairly explicit solution. In analogy to the Black-Scholes model, the utility based price and the hedging strategy can be described by a partial differential equation. But the non-tradable factors of risk in our model demand for an interacting semi-linear system of parabolic partial differential equations. To obtain the solution for the underlying utility maximization problem, we use a verification theorem to identify the optimal martingale measure for the corresponding dual problem.

Keywords: utility-indifference, hedging, incomplete markets, point processes, utility maximization, relative entropy

1. Introduction

If the risky payoff of a contingent claim is replicable by dynamical trading, the claim can be perfectly hedged by the replicating strategy and its price is determined by no-arbitrage arguments as the replication cost. However, such claims are dynamically redundant and have apparently little reason to exists. In a less ideal model world, markets are incomplete so that contingent claims could incorporate some inevitable intrinsic risk. Hence, the issuer's valuation and his (partial) hedging strategy should take into account his attitude and preferences towards risk. We study an approach to the hedging and valuation problem which relies on exponential utility-indifference arguments. In the context of a dynamical market, this idea appeared first in Hodges & Neuberger (1989) and has been studied by many authors recently. The indifference problem and also the underlying utility optimization problem are well understood and characterized on a general but abstract level by martingale duality results, cf. Cvitanić et al. (2001), Delbaen et al. (2002), Frittelli (2000), Owen (2002), and by backward stochastic differential equations, cf. Rouge & El Karoui (2000). But it is still a challenge to obtain more constructive and explicit solutions in more specific models. For situations where the tradable asset evolves as a geometric Brownian motion and the market is incomplete because of further noise, such results have been achieved by Davis (2000, unpublished) and Henderson & Hobson (2002).

The present article provides a constructive description for the (exponential) utility-indifference price and for the corresponding hedging strategy in a model with several tradable risky assets, where incompleteness is caused by additional non predictable events which are triggered by point processes. To this end, we

develop a model for a financial market with interacting Itô and point processes. This allows for a variety of mutual stochastic dependencies between the tradable and non-tradable factors of risk affecting the claim, but nevertheless still permits for a fairly explicit solution. More specifically, the price evolutions of the marketed assets are modeled by Itô processes, and additional non-tradable factors of risk are represented by a finite state process η and driven by point processes with stochastic intensities. In such a framework, the utility based price and the hedging strategy can be described by the solution of a partial differential equation, in analogy to the Black-Scholes model. However, the non-tradable factors of risk in our model demand for an interacting system of semi-linear parabolic partial differential equations, a so-called reaction diffusion system. The solution to the underlying utility optimization problem with a potential additional liability is described by a similar reaction-diffusion system. To prove this, we use a verification theorem to identify the optimal martingale measure for the corresponding dual problem, and provide an explicit description of its density process. In the case of no additional liability, this yields in particular the density process of the minimal entropy martingale measure.
The finite state process η could be interpreted, for instance, as an abstract economic state variable, as the state of an insurance contract, or as the joint rating and default process of several companies. Depending on the interpretation, the model can be considered as a stochastic volatility model, as a model for equity linked insurance contracts, or as a hybrid model for credit risky securities.
The paper is organized as follows. Section 2 formulates the problem and summerizes some martingale duality results on the utility optimization problem. The setup for our model with interacting Itô and point processes is developed in section 3. Section 4 describes the solutions for the utility optimization problem with an additional liability and for the utility-indifference hedging and valuation problem.

2. Formulation of the Problem

In this section, we formulate the (exponential) utility-indifference valuation and hedging problem. We first introduce some notation and recall the key duality result on the underlying utility maximization problem with an additional liability.

(a) *Notation and Preliminaries*

All modeling takes place on a probability space $(\Omega, \mathcal{F}, P)$ with a finite time horizon T and a filtration $I\!\!F = (\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual conditions of right-continuity and completeness. For simplicity, let $\mathcal{F}_0$ be trivial and $\mathcal{F}_T = \mathcal{F}$. All semimartingales are taken to have right continuous paths with left limits (RCLL). For unexplained notation we refer to He et al. (1992) or Protter (1990).
Let S be an $I\!\!R^d$-valued semimartingale which is $I\!\!F$-locally bounded. We consider S as the evolution of the discounted prices for the risky assets in a financial market. The market moreover contains a risk-less asset with discounted price constant at 1. The sets of absolutely continuous and equivalent (local) martingale measures for S, and those with finite relative entropy are denoted by $I\!\!P_a$, $I\!\!P_e$, and $I\!\!P_f := \{\, Q \in I\!\!P_a \,|\, H(Q|P) < \infty\}$, respectively; with $H(Q|P)$ denoting the relative entropy of Q with respect to P. We assume that our financial model is free of arbitrage in the sense that

$$I\!\!P_e \cap I\!\!P_f \neq \emptyset\,. \tag{2.1}$$

As the set of permitted trading strategies in the financial market S, we take the space

$$\Theta_{\mathcal{M}} := \left\{\theta \in L(S) \,\middle|\, \theta\cdot S \text{ is a } (Q, I\!\!F)\text{-martingale for all } Q \in I\!\!P_f\right\} \tag{2.2}$$

with $L(S)$ denoting the set of predictable, $I\!\!R^d$-valued, and S-integrable processes. The stochastic integral $(\theta\cdot S)_t \equiv \int_0^t \theta\, dS$ represents the gains and losses from trading up to time $t \in [0,T]$. The choice of

$\Theta_{\mathcal{M}}$ is convenient from a martingale duality point of view, since $\Theta_{\mathcal{M}}$ is a linear space and the optimal strategies in the sequel are attained within $\Theta_{\mathcal{M}}$. But the choice is not as special as it might seem at first glance. In fact, the dual side of next duality result – which links (2.4) to (2.6) – is robust with respect to alternative choices for $\Theta_{\mathcal{M}}$, see Delbaen et al. (2002), Kabanov & Stricker (2002), and Schachermayer (2002, unpublished). Therefore we will base our subsequent analysis on the dual side, in order to get results that are robust in a similar sense.
Contingent claims, that is risky future payoffs, are modeled by random variables and denoted by B. We will need the integrability assumption

$$E\left[\mathrm{e}^{(\alpha+\varepsilon)B}\right] < \infty \quad \text{and} \quad E\left[\mathrm{e}^{-\varepsilon B}\right] < \infty \quad \text{for some } \varepsilon > 0 \tag{2.3}$$

which implies in particularly that B is in $L^1(Q)$ for all $Q \in I\!P_f$. The next proposition recalls central duality results from Delbaen et al. (2002) and Kabanov & Stricker (2002) on the exponential utility maximization problem

$$u(x-B) := u(x-B;\alpha) := \sup_{\theta\in\Theta_{\mathcal{M}}} E\left[-\exp\left(-\alpha\Big(x+\int_0^T \theta\, dS - B\Big)\right)\right] \tag{2.4}$$

and the corresponding dual problem

$$\sup_{Q\in I\!P_f} \left\{\alpha\, E_Q[B] - H(Q|P)\right\}. \tag{2.5}$$

Proposition 2.1. *Suppose (2.1) and (2.3). Then*
1. There exists a unique $Q^B \in I\!P_f \cap I\!P_e$ *which maximizes (2.5).*
2. The density of Q^B *has the form*

$$\frac{dQ^B}{dP} = \exp\left(-\alpha\big(c^B + \int_0^T \theta^B dS - B\big)\right)$$

with $c^B \in I\!R$ *and* $\theta^B := \theta^{B,\alpha} \in L(S)$ *such that* $\theta^B \cdot S$ *is a* Q^B*-martingale. Moreover,* θ^B *is in* $\Theta_{\mathcal{M}}$.
3. The maximal expected utility is $u(x-B;\alpha) = E\left[-\exp\left(-\alpha(x+\int_0^T \theta^B dS - B)\right)\right] = -\exp(-\alpha(x-c^B))$ *with* $\theta^B \in \Theta_{\mathcal{M}}$ *and* $c^B \in I\!R$ *from part 2. Moreover,* $u(x-B;\alpha)$ *is also given via (2.5) as*

$$u(x-B;\alpha) = -\mathrm{e}^{-\alpha x}\exp\left(\sup_{Q\in I\!P_f}\left\{\alpha E_Q[B] - H(Q|P)\right\}\right). \tag{2.6}$$

For claims B which are additionally bounded from below, the foregoing result is just a reformulation of the generalized theorem 2 of Delbaen et al. (2002) by Kabanov & Stricker (2002). Their proofs can be adapted to the slightly more general integrability assumption (2.3) on B (cf. Becherer (2001, 2003)). The measure Q^0, that corresponds to the claim $B = 0$, minimizes $H(Q|P)$ over $Q \in I\!P_a$ and is called minimal entropy martingale measure.

(*b*) *utility-indifference valuation and hedging*

We next formulate the utility-indifference approach for the pricing and hedging problem in incomplete markets, and define our major objects of interest.

Definition *If there is a unique solution* $\pi(B) = \pi(B;\alpha)$ *to the equation*

$$u(x;\alpha) = u(x+\pi-B;\alpha), \tag{2.7}$$

we call this solution the utility-indifference (selling) price *for* B.

It can be seen directly from (2.4) that equation (2.7) and, hence, $\pi(B)$ do not depend on x. The basic idea behind this valuation approach is that it yields a subjective fair premium for a risk averse issuer of the claim B. The utility-indifference price is the adjustment of the initial capital, which just compensates a potential issuer for taking the liability B in terms of maximal expected utility. That is, the investor would be indifferent between issuing the claim B for premium π and the alternative to skip the deal. In a static setting (without trading opportunities), a classical result from actuarial mathematics states that in particularly the exponential utility leads to certain desirable valuation properties. The first adaption of the static indifference approach to a dynamic setting with trading opportunities seems to be in Hodges & Neuberger (1989).

Remark: From proposition 2.1, one readily obtains (cf. (4.6) in Delbaen et al. (2002)) the following general formula for the utility-indifference price

$$\pi(B;\alpha) = \sup_{Q\in \mathbb{P}_f} \left\{ E_Q[B] - \frac{1}{\alpha}\Big(H(Q|P) - H(Q^0|P)\Big)\right\} . \tag{2.8}$$

Many nice properties of the utility-indifference price can be inferred directly from formula (2.8). In essence, $\pi(B)$ yields a valuation method which basically inherits all the "desirable properties" from the (static) exponential premium principle in actuarial mathematics, which is consistent with the no-arbitrage principle, and which moreover constitutes a so-called convex measure of risk; cf. Becherer (2001, 2003) for details and further references.

If one is interested in constructive and explicit results on $\pi(B)$, the general formula (2.8) however seems a bit too abstract. It will be the main task of the next sections to obtain such results under additional structural assumptions on the financial market model.

Next, we are going to define a utility-indifference price *process* for the claim B. Using part 3 of proposition 2.1, we start with the simple observation that $\pi(B) = c^B - c^0$ and that both c^B and c^0 can be interpreted as certainty equivalents. This observation can be extended: There is a unique semimartingale $(c_t^B)_{t\in[0,T]}$ such that the density process of Q^B can be written as

$$\frac{dQ^B}{dP}\bigg|_{\mathcal{F}_t} = \exp\left(-\alpha\Big(c_0^B + \int_0^t \theta^B\, dS - c_t^B\Big)\right), \quad t\in[0,T]. \tag{2.9}$$

In fact, (c_t^B) is determined by θ^B and B as

$$c_t^B = \frac{1}{\alpha}\log E_P\left[\exp\Big(\alpha(B - \int_t^T \theta^B\, dS)\Big)\Big|\mathcal{F}_t\right], \quad t\in[0,T], \tag{2.10}$$

with $c_0^B = c^B$ and $c_T^B = B$, in particular. This allows for the following interpretation. An investor who maximizes his expected utility under terminal liability B follows the optimal trading strategy θ^B. At time t, he then faces the effective liability $B - \int_t^T \theta^B\, dS$ which is the difference between B and the gains & losses from trade that he is going to realize in the time left. By (2.10), c_t^B is the current (exponential) certainty equivalent of this effective liability at time t. That means, c_t^B is the "time–t–certain" liability that is as good as the remaining effective liability in terms of expected exponential utility. Analogous considerations lead to a process (c_t^0) which corresponds to the claim $B=0$. Note that $c_0^B - c_0^0 = \pi(B)$

and $c_T^B - c_T^0 = B$. This gives rise to

Definition *Suppose the assumptions for proposition 2.1 hold. The process* $\pi_t = \pi_t(B;\alpha) := c_t^B - c_t^0$, $t \in [0,T]$, *is called the utility-indifference price process of* B.

Following the same rationale as in the valuation approach, we define the utility-indifference (partial) hedging strategy $\psi(B;\alpha)$ as the adjustment of the optimal strategy without liability that is necessary to obtain a strategy that is optimal in presence of the terminal liability B.

Definition *Under the assumptions for proposition 2.1, we define the* utility-indifference hedging strategy ψ *for* B *by*

$$\psi(B) = \psi(B;\alpha) := \theta^{B,\alpha} - \theta^{0,\alpha}\,. \tag{2.11}$$

The hedging strategy $\psi(B)$ is unique in the sense that the stochastic integral $\int \psi(B)\,dS$ is unique. To see this, note that θ^B and θ^0 are unique in the same sense by proposition 2.1. More formally, the strategies are uniquely defined in the corresponding quotient space.
The next proposition is basically a reformulation of proposition 3.4 from Grandits & Rheinländer (2002); see Becherer (2001, 2003) for details. We will use this result in the sequel as a verification theorem to identify the optimal measure Q^B for the dual problem (2.5), and thereby obtain the optimal strategy θ^B for the primal problem (2.4).

Proposition 2.2. *Assume (2.3). Suppose the density of some* $\bar{Q} \in I\!P_a$ *takes the form*

$$\frac{d\bar{Q}}{dP} = \exp\left(-\alpha\Big(\bar{c} + \int_0^T \bar{\theta}\,dS - B\Big)\right) \quad \textit{for some } \bar{c} \in I\!R \textit{ and } \bar{\theta} \in L(S),$$

where $\bar{\theta}\cdot S$ *is a* $\bar{Q}$*-martingale of bounded mean oscillation (BMO), and* $\exp\Big(\alpha\big(B + \varepsilon\int_0^T \bar{\theta}\,dS\big)\Big) \in L^1(P)$ *for some* $\varepsilon > 0$. *Then,* $\bar{Q}$ *is in* $I\!P_e \cap I\!P_f$ *and solves the dual problem from proposition 2.1, i.e.* $\bar{Q} = Q^B$, $\bar{\theta} = \theta^B$, *and* $\bar{c} = c^B$.

3. A financial model with interacting Itô and point processes

In this section, we develop a model for a financial market (incomplete, in general), where the prices of the risky assets evolve as Itô processes. An additional finite state process, driven by a point process, represents further non-tradable factors of uncertainty and risk. The model allows for a variety of mutual dependencies between tradable and non-tradable factors of risk.

(*a*) *Model setup and assumptions*

We start with $m \in I\!N$ and a domain D in $I\!R^d$; typical examples are $D = (0,\infty)^d$ and $D = I\!R^d$. Let (S,η) be the solution of the following system of stochastic differential equations (SDEs) with paths in $D \times \{1,\dots,m\}$:

$$S_0 \in D, \quad dS_t = \Gamma(t,S_t,\eta_{t-})\,dt + \Sigma(t,S_t,\eta_{t-})\,dW_t, \tag{3.1}$$

$$\eta_0 \in \{1,\dots,m\}, \quad d\eta_t = \sum_{k,j=1}^{m} (j-k) I_k(\eta_{t-})\,dN_t^{kj}, \tag{3.2}$$

where $\Gamma : [0,T] \times D \times \{1,\ldots,m\} \to I\!R^d$ and $\Sigma : [0,T] \times D \times \{1,\ldots,m\} \to I\!R^{d\times d}$ are C^1 with respect to $(t,x) \in [0,T] \times D$, $I_k := I_{\{k\}}$ is the indicator function on the set $\{k\}$, $W = (W^i)_{i=1,\ldots,d}$ is an $I\!R^d$-valued $(P, I\!F)$-Brownian motion, and $N = (N^{kj})_{k,j=1,\ldots,m}$ is a multivariate $I\!F$-adapted point process such that

$$(N^{kj}_t) \quad \text{has } (P, I\!F)\text{-intensity } \lambda^{kj}(t, S_t) \quad \text{for } k,j = 1,\ldots,m, \tag{3.3}$$

with bounded C^1 functions $\lambda^{kj} : [0,T] \times D \to [0,\infty)$. Let

$$dM^{kj} := dN^{kj} - \lambda^{kj}(t, S_t)\, dt\,, \quad k,j \in \{1,\ldots,m\}\,,$$

denote the compensated $(P, I\!F)$-point processes. Since $[M^{kj}] = N^{kj}$ and the intensities λ^{kj} are bounded, a simple time change argument via theorem II.16 in Brémaud (1981) shows that $[M^{kj}]_T$ has all exponential moments and is therefore in every $L^p(P)$, and M^{kj} is in the martingale space $\mathcal{H}^p(P)$ for any $p \in [1,\infty)$. Due to the mutual dependences between S, η and N, (3.1)–(3.3) constitutes a non-standard SDE system, and we comment below on its construction and properties.
If $D \subseteq (0,\infty)^d$, one can rewrite (3.1) as a generalized Black-Scholes model. With the notations $\frac{dS}{S} = \left(\frac{dS^i}{S^i}\right)_{i=1,\ldots,d}$, $\gamma(t,x,k) = \mathrm{diag}\left(\frac{1}{x^i}\right)_{i=1,\ldots,d} \Gamma(t,x,k)$, and $\sigma(t,x,k) = \mathrm{diag}\left(\frac{1}{x^i}\right)_{i=1,\ldots,d} \Sigma(t,x,k)$, we get

$$\frac{dS_t}{S_t} = \gamma(t, S_t, \eta_{t-})\, dt + \sigma(t, S_t, \eta_{t-})\, dW_t\,. \tag{3.4}$$

(b) Construction by a change of measure

If we look at (3.1)–(3.3), we see that the process η enters the coefficients in the SDE (3.1) for the dynamics of S. In turn, the intensities in (3.3) of the point process N driving η depend on the process S. At first sight, it seems difficult to construct models with such mutual dependences since they form a non-standard SDE system where the solution (S,η) via (3.3) also affects a part of the driving process. However, the problem can be reduced to the case where $N = (N^{kj})$ is a standard multivariate Poisson process. Then η is an autonomous process, and S is well-defined by (3.1) under certain conditions. Having a solution for this simpler case, we can then construct the desired (t,S)-dependent intensities for N by a change of measure. More precisely, we start with a filtered probability space $(\Omega, \mathcal{F}', I\!F', P')$ carrying a d-dimensional $(P', I\!F')$-Brownian motion $W = (W^i)_{i=1,\ldots,d}$ and a multivariate $I\!F'$-adapted point process $N = (N^{kj})_{k,j=1,\ldots,m}$ with constant $(P', I\!F')$-intensity 1 for any k and j. In other words,

$$N^{kj},\ k,j = 1,\ldots,m, \text{ are independent standard Poisson processes under } P'. \tag{3.5}$$

We assume that $\mathcal{F}'_0$ is trivial, $\mathcal{F}'_T = \mathcal{F}'$, and $I\!F'$ satisfies the usual conditions. Then (3.2) defines a unique autonomous process η, and given this process, there is a solution S to (3.1) under suitable assumptions on the coefficients; see the examples in §3 *c*. Setting

$$dP := \mathcal{E}\left(\sum_{k,j=1,\ldots,m} \int \left(\lambda^{kj}(t,S_t) - 1\right) \left(dN^{kj}_t - dt\right) \right)_T dP' \tag{3.6}$$

defines a probability measure $P \ll P'$ such that N has the $(P, I\!F')$-intensities (3.3); see Brémaud (1981), theorems VI.2.3 and VI.2.4. By Girsanov's theorem, W is a (local) $(P, I\!F')$-martingale whose covariance process $\langle W\rangle$ is the same under P' and P since it can be computed path-wise, and so W is also a $(P, I\!F')$-Brownian motion. Finally, let $(\Omega, \mathcal{F}, I\!F, P)$ be the standard P-completion of $(\Omega, \mathcal{F}', I\!F', P)$. Then one can show that $I\!F$ satisfies the usual conditions, W is a Brownian motion, and N is a multivariate point process with the desired intensities (3.3) under $(P, I\!F)$. Moreover, (S,η) solves (3.1)–(3.2) under $(P, I\!F)$.

(c) *Some examples*

We provide some examples where a solution (S, η) to the system (3.1)–(3.2) exists. We can restrict ourselves to the simpler case (3.5) where N is a standard multidimensional Poisson process. A model with intensities of the form (3.3) is then easily constructed by the change of measure (3.6).

Example: Let $D = I\!R^d$, and suppose that Γ and Σ are globally Lipschitz in $(x, k) \in D \times \{1, \dots, m\}$, uniformly in $t \in [0, T]$, and that (3.5) holds. Then there exists (Protter (1990), theorem V.7) a unique solution (S, η) to the system (3.1) – (3.2).

Example: Let $D = (0, \infty)^d$ and suppose that γ and σ depend only on (t, k), but not on x. If (3.5) holds, η is given by (3.2). Having that γ and σ are continuous in $t \in [0, T]$ for any $k \in \{1, \dots, m\}$, the unique solution S to (3.4) is given by $S^i = S_0^i\, \mathcal{E}\left(\int \gamma^i(u, \eta_{u-})\, du + \left(\int \sigma(u, \eta_{u-})\, dW_u\right)^i\right)$, $i = 1, \dots, d$.

Example: Let $D = I\!R^d$ and suppose that for each $k \in \{1, \dots, m\}$, the functions $\Gamma(t, x, k)$ and $\Sigma(t, x, k)$ are globally Lipschitz-continuous in x, uniformly in $t \in [0, T]$. Under (3.5), η is defined by (3.2) and an induction argument yields the existence of a strong solution S to (3.1) as follows. We start with $\tau^0 := 0$ and define the sequence $\tau^{n+1} := \inf\{t > \tau^n \mid \Delta\eta_t \neq 0\} \wedge T$ of jump times of η. Since N is non-exploding, $P[\tau^n \geq T] \to 1$ for $n \to \infty$ and so it is enough to show by induction that (3.1) has a unique solution on $[\![0, \tau^n]\!]$ for all n. Suppose that

$$X^n \text{ is the unique solution to the SDE (3.1) on } [\![0, \tau^n]\!]. \tag{3.7}$$

By theorem V.7 in Protter (1990), the SDE

$$X_t^{n+1} = X_{t\wedge\tau^n}^n + \int_0^t I_{]\!]\tau^n,\tau^{n+1}]\!]}\Gamma(s, X_s^{n+1}, \eta_{\tau^n})\, ds + \int_0^t I_{]\!]\tau^n,\tau^{n+1}]\!]}\Sigma(s, X_s^{n+1}, \eta_{\tau^n})\, dW_s \tag{3.8}$$

has a unique solution $(X_t^{n+1})_{t\in[0,T]}$. By construction, we have $X^{n+1} = X^n$ on $[\![0, \tau^n]\!]$ and $\eta_{s-} = \eta_{\tau^n}$ for $(\omega, s) \in]\!]\tau^n, \tau^{n+1}]\!]$. Combining this with (3.7) and (3.8) yields

$$\begin{aligned} X_t^{n+1} &= X_{t\wedge\tau^n}^n + \int_0^t I_{]\!]\tau^n,\tau^{n+1}]\!]}\Gamma(s, X_s^{n+1}, \eta_{s-})\, ds + \int_0^t I_{]\!]\tau^n,\tau^{n+1}]\!]}\Sigma(s, X_s^{n+1}, \eta_{s-})\, dW_s \\ &= S_0 + \int_0^t \Gamma(s, X_s^{n+1}, \eta_{s-})\, ds + \int_0^t \Sigma(s, X_s^{n+1}, \eta_{s-})\, dW_s, \quad t \in [0, T]. \end{aligned}$$

Hence X^{n+1} is a solution to the SDE (3.1) on $[\![0, \tau^{n+1}]\!]$. To see that this solution is unique on $[\![0, \tau^{n+1}]\!]$, recall that we have uniqueness on $[\![0, \tau^n]\!]$ by hypothesis. Thus any solution to (3.1) must satisfy (3.8) on $[\![0, \tau^{n+1}]\!]$ and therefore coincides with X^{n+1} on $[\![0, \tau^{n+1}]\!]$.

4. Solution via Reaction-Diffusion Systems

In this section we will show that the solution for the dual (and thereby also the primal) utility maximization problem with additional liability is given via the unique solution to a system of interacting semi-linear partial differential equations (PDEs) of parabolic type. Such systems are often called reaction-diffusion systems. From the solutions to the utility maximization problem, we will then derive a similar PDE-type description for the utility-indifference price process and the corresponding hedging strategy.

(a) Assumptions and preparations

Supposing the Itô and point process model from §3 a, we consider claims B of the form

$$B \quad = \quad h(S_T, \eta_T) + \int_0^T f(t, S_t, \eta_t)\, dt + \sum_{t \leq T} \sum_{\substack{k,j=1 \\ k \neq j}}^{m} I_{\{\eta_{t-}=k, \eta_t=j\}} f^{kj}(t, S_t) \tag{4.1}$$

with continuous bounded functions $h : D \times \{1, \ldots, m\} \to I\!R$, $f : [0, T] \times D \times \{1, \ldots, m\} \to I\!R$ and $f^{kj} : [0, T] \times D \to I\!R$, $k, j \in \{1, \ldots, m\}$. Furthermore, $f(\cdot, \cdot, k)$ and $f^{kj}(\cdot, \cdot)$ are supposed to be of class C^1 on $[0, T] \times D$ for all $k, j \in \{1, \ldots, m\}$.

The claim B is the sum of three components with the following interpretations. The first term describes a terminal payoff that depends on the final state (S_T, η_T). The second term models payments which are made continuously at rate $f(t, S_t, \eta_t)$. The third term specifies further payments for each time when η jumps from one state to another.

Lemma 4.1. *A claim B of the form (4.1) possesses all exponential moments under P, i.e. $\exp(\beta|B|)$ is in $L^1(P)$ for all $\beta \in I\!R$. In particular, B satisfies condition (2.3).*

Proof. It suffices to prove $E[\exp(\beta N'_T)] < \infty$ for any $\beta < \infty$ with $N' := \sum_{k,j} N^{kj}$. To see this, note that the first two terms in (4.1) are bounded by hypothesis and the third term is bounded in absolute value by $\sup_{k,j} ||f^{kj}||_\infty N'_T$. By construction, (N'_t) is a point process with bounded intensity $\lambda'_t := \sum_{k,j} \lambda^{kj}(t, S_t)$. But then a simple time change argument (cf. Brémaud (1981), th.II.16) shows that N'_T is dominated by a random variable which is Poisson distributed, thus having all exponential moments. □

For the next results we will need the assumption that

$$\Sigma \text{ is invertible and } \Sigma^{-1}\Gamma \text{ is bounded on } [0, T] \times D \times \{1, \ldots, m\}. \tag{4.2}$$

This condition implies in particular that there exists an equivalent local martingale measure with finite relative entropy for S, i.e. $I\!P_e \cap I\!P_f \neq \emptyset$, since $d\widehat{Q} := \mathcal{E}(-\int \Sigma^{-1}\Gamma(t, S_t, \eta_{t-})\, dW_t)_T\, dP$ defines an element of $I\!P_e \cap I\!P_f$. But in general, $I\!P_e$ is not a singleton and the market is incomplete. The existence of Σ^{-1} and the continuity and differentiability properties of Σ imply that

$$(t, x) \mapsto \Sigma^{-1}(t, x, k) \quad \text{is continuous and of class } C^1 \text{ on } [0, T] \times D \text{ for any } k. \tag{4.3}$$

For claims B of the form (4.1), the subsequent theorem will show how the solution to the (dual) optimal investment problem is described by the solution to the following system of m partial differential equations with terminal condition at T and $k = 1, \ldots, m$ (subscripts are denoting partial derivatives). Such PDEs are often called reaction-diffusion systems.

$$\begin{aligned}
& v^B_t(t, x, k) + \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t, x, k)\, v^B_{x^i x^j}(t, x, k) \\
& - \frac{1}{2\alpha} ||\Sigma^{-1}(t, x, k)\Gamma(t, x, k)||^2 + f(t, x, k) \\
& + \sum_{\substack{j=1 \\ j \neq k}}^{m} \lambda^{kj}(t, x) \frac{1}{\alpha} \left(\mathrm{e}^{\alpha(v^B(t,x,j) - v^B(t,x,k) + f^{kj}(t,x))} - 1 \right) \qquad = \quad 0, \quad (t, x) \in [0, T) \times D, \\
& \qquad \text{and} \qquad v^B(T, x, k) \qquad = \quad h(x, k), \quad x \in D,
\end{aligned} \tag{4.4}$$

with $a = (a^{ij})_{i,j \in \{1,\ldots,d\}} := \Sigma\, \Sigma^{\mathrm{tr}}$ and $||\Sigma^{-1}\Gamma||^2 := (\Sigma^{-1}\Gamma)^{\mathrm{tr}}(\Sigma^{-1}\Gamma) = \Gamma^{\mathrm{tr}} a^{-1} \Gamma$.

Notation: Let $\mathrm{grad}_x v := (v_{x^i})_{i=1\dots,d}$ denote the gradient of $v(t,x,k)$ with respect to x, and let $C_b^{1,2}([0,T)\times D\times\{1,\dots,m\},I\!R)$ denote the space of continuous bounded functions $v : [0,T]\times D\times\{1,\dots,m\}\to I\!R$ with $(t,x)\mapsto v(t,x,k)$ being of class $C^{1,2}$ on $[0,T)\times D$ for any k.

As a crucial condition, the next theorem 4.3 requires that

$$\text{there is an unique solution } v^B\in C_b^{1,2}([0,T)\times D\times\{1,\dots,m\},I\!R) \text{ to the PDE (4.4)}. \tag{4.5}$$

By proposition 4.2, an additional regularity condition on (basically) the coefficient functions of the SDE for S is already sufficient to establish (4.5). To this end, consider the SDE

$$X_t^{t,x,k}=x\in D\,,\quad dX_s^{t,x,k}=\Gamma(s,X_s^{t,x,k},k)\,dt+\Sigma(s,X_s^{t,x,k},k)\,dW_s\,,\quad s\in[t,T]\,, \tag{4.6}$$

which has a unique strong solution for any $(t,x,k)\in[0,T]\times D\times\{1,\dots,k\}$ up to a possibly finite explosion time as Γ and Σ are C^1-functions, cf. Kunita (1984), th.II.5.2.

Proposition 4.2. *Suppose that the payoff functions h,f and f^{kj} satisfy the conditions following (4.1), (4.2) holds, D satisfies (5.4), and $X^{t,x,k}$ does not explode but stays in D up to time T, i.e.*

$$P\left[X_s^{t,x,k}\in D \ \ \textit{for all } s\in[t,T]\right]=1\quad \textit{for any } (t,x,k)\in[0,T]\times D\times\{1,\dots,m\}. \tag{4.7}$$

Then, there is a unique classical solution to the PDE-system (4.4) in the sense that (4.5) holds.

Comparing the SDEs (3.1) and (4.6) for S and $X^{t,x,k}$, condition (4.7) might be seen as an additional regularity condition on S which already was supposed to stay in in the domain D. We note that conditions (4.7) and (5.4) can be readily verified in the examples from §3 *c*.

Proof. Since (4.2) allows for a change of measure argument which makes the drift of $X^{t,x,k}$ vanish, assumption (4.7) holds with $X^{t,x,k}$ from (4.6) if and only if it also holds for the unique strong solution to $dX_s^{t,x,k}=\Sigma(s,X_s^{t,x,k},k)\,dW_s$, $s\in[t,T]$, instead. Hence, proposition 5.1 yields that there is a unique solution $v^B\in C_b^{1,2}([0,T)\times D\times\{1,\dots,m\},I\!R)$ for the PDE-system (4.4). More precisely, proposition 5.1 yields the existence of a unique solution $(v^k(t,x))_{k=1,\dots,m}$ in $C_b^{1,2}([0,T)\times D,I\!R^m)$ to the PDE-system (5.3) with $b_k:=0$, $\Sigma_k:=\Sigma(\cdot,\cdot,k)$, $f^k(t,x):=f(t,x,k)-\frac{1}{2\alpha}||\Sigma^{-1}\Gamma(t,x,k)||^2$, $g^k(t,x,v):=f^k(t,x)+\sum_{j\neq k}\frac{1}{\alpha}\lambda^{kj}(t,x)\big(\exp(\alpha(v^j-v^k+f^{kj}(t,x)))-1\big)$, and $h^k(x):=h(x,k)$. Thus, $v^B(t,x,k):=v^k(t,x)$ is the unique solution in $C_b^{1,2}([0,T)\times D\times\{1,\dots,m\},I\!R)$ to the interacting PDE-system (4.4). □

(*b*) *Utility maximization with an additional liability*

Now we are in position to formulate the central theorem of this paper.

Theorem 4.3. *Suppose (4.1), (4.2) and (4.5) hold. Then, the optimal measure $Q^B\in I\!P_e\cap I\!P_f$ for the dual problem (2.5) is described by the unique solution $v^B\in C_b^{1,2}([0,T)\times D\times\{1,\dots,m\},I\!R)$ to the PDE-system (4.4) as follows:*

$$\begin{aligned}\frac{dQ^B}{dP} &= \exp\left(-\alpha\Big(v^B(0,S_0,\eta_0)+\int_0^T\theta_u^B\,dS_u-B\Big)\right)\quad \textit{with}\\ \theta_t^B &= \mathrm{grad}_x v^B(t,S_t,\eta_{t-})+\frac{1}{\alpha}\left(\Sigma\,\Sigma^{\mathrm{tr}}(t,S_t,\eta_{t-})\right)^{-1}\Gamma\,(t,S_t,\eta_{t-})\,,\quad t\in[0,T)\,,\end{aligned}$$

and $\theta_T^B=0$. Moreover, θ^B is in $\Theta_{\mathcal{M}}$.

Letting $\theta^B_T = 0$ is arbitrary since the value of θ^B_T does not affect the stochastic integral $\int \theta^B \, dS$ – provided that the latter is well-defined of $[0,T]$. However, the equation for θ^B_t in the theorem does not extend to all $t \in [0,T]$ since the gradient may not exist at time T.
The proof of this theorem will be broken into several steps, each of them being a lemma. In the statement of those lemmata, we will suppose that we are dealing with the situation and under the assumptions described in theorem 4.3, but will not mention this each time. In a first step, we show that a certain stochastic exponential defines an equivalent local martingale measure Q, which will later be identified as the optimal measure Q^B for the dual problem.

Lemma 4.4. *The stochastic exponential*

$$\begin{aligned} Z \;&:=\; \mathcal{E}\Bigg(-\int \Sigma^{-1}(u, S_u, \eta_{u-})\Gamma(u, S_u, \eta_{u-})\, dW_u \\ &\qquad + \sum_{\substack{k,j=1\\k\neq j}}^{m} \int I_k(\eta_{u-}) \left(\mathrm{e}^{\alpha(v^B(u,S_u,j) - v^B(u,S_u,k) + f^{kj}(u,S_u))} - 1 \right) dM^{kj}_u \Bigg) \end{aligned} \tag{4.8}$$

is a $\mathcal{H}^p(P)$-martingale for any $p < \infty$, and $dQ := Z_T \, dP$ is an element of $I\!P_e$.

Proof. For ease of notation, we define the short-hand notations

$$\begin{aligned} dN^{\eta j}_t &:= \sum_{k=1}^m I_k(\eta_{t-})\, dN^{kj}_t\,, & dM^{\eta j}_t &:= \sum_{k=1}^m I_k(\eta_{t-})\, dM^{kj}_t\,, \\ \lambda^{\eta j}(t,x) &:= \sum_{k=1}^m I_k(\eta_{t-})\, \lambda^{kj}(t,x)\,, & f^{\eta j}(t,x) &:= \sum_{k=1}^m I_k(\eta_{t-}) f^{kj}(t,x)\,. \end{aligned} \tag{4.9}$$

Note that $dN^{\eta j}_t = dM^{\eta j}_t + \lambda^{\eta j}(t, S_t)\, dt$. With $v := v^B$ and

$$\xi^j(t, S_t, \eta_{t-}) := I_{\{\eta_{t-}\neq j\}} \left(\mathrm{e}^{\alpha(v(t,S_t,j) - v(t,S_t,\eta_{t-}) + f^{\eta j}(t,S_t))} - 1 \right), \quad j = 1,\dots,m\,, \tag{4.10}$$

we can write the stochastic exponential Z as

$$Z = \mathcal{E}\left(-\int \Sigma^{-1}(t,S_t,\eta_{t-})\Gamma(t,S_t,\eta_{t-})\, dW_t + \sum_{j=1}^m \int \xi^j(t,S_t,\eta_{t-})\, dM^{\eta j}_t \right). \tag{4.11}$$

Since $\xi^j > -1$ holds for all j, the process Z is strictly positive. To reduce notation, we make the standing convention that functions in stochastic integrals are evaluated at (t, S_t, η_{t-}) unless specified otherwise. Then we have by Yor's formula

$$Z = \mathcal{E}\left(-\int \Sigma^{-1}\Gamma \; dW \right) \, \mathcal{E}\left(\sum_{j=1}^m \int \xi^j \, dM^{\eta j} \right). \tag{4.12}$$

Thanks to Novikov's criterion, the boundedness of $\Sigma^{-1}\Gamma$ implies that

$$\mathcal{E}\left(-\int \Sigma^{-1}\Gamma \; dW \right) \quad \text{is in the martingale space } \mathcal{H}^p(P) \text{ for any } p \in [1,\infty)\,. \tag{4.13}$$

The process ξ^j is bounded since v and the f^{kj}'s are so. Hence, there is some constant $c < \infty$ such that

$$\begin{aligned} \mathcal{E}\left(\sum_{j=1}^m \int \xi^j \, dM^{\eta j} \right)_t \;&=\; \prod_{j=1}^m \mathcal{E}\left(\int \xi^j \, dM^{\eta j} \right)_t \\ &=\; \prod_{j=1}^m \left(\mathrm{e}^{-\int_0^t \xi^j_s \lambda^{\eta j}(s,S_s)\, ds} \prod_{0<s\le t} (1 + \xi^j_{s-}\, \Delta N^{\eta j}_s) \right) \\ &\le\; \mathrm{e}^{mTc\|\lambda\|} (1+c)^{\sum_{k,j=1}^m N^{kj}_T}\,, \qquad t \in [0,T]\,, \end{aligned} \tag{4.14}$$

with $||\lambda|| := \sup_{k,j\in\{1,\dots,m\}} ||\lambda^{kj}||_\infty < \infty$. Therefore, the exponential on the left hand side in (4.14) is bounded (uniformly in t) by the random variable on the right hand side. We are going to show that this bound is in $L^p(P)$ for any $p \in [1,\infty)$ and thereby obtain that

$$\mathcal{E}\left(\sum_{j=1}^{m}\int \xi^j \, dM^{\eta j}\right) \quad \text{is in the martingale space } \mathcal{H}^p(P) \text{ for any } p \in [1,\infty)\,. \tag{4.15}$$

For the integrability in question, it suffices to verify that $E[\exp(\beta N'_T)] < \infty$ holds for any $\beta < \infty$ with $N' := \sum_{k,j} N^{kj}$. This has been done in the proof of lemma 4.1. By (4.12), (4.13), (4.15) and Hölder's inequality we obtain that Z is in the martingale space $\mathcal{H}^p(P)$ for any $p < \infty$.
So, $dQ := Z_T \, dP$ defines a probability measure which is equivalent to P as $Z > 0$. Girsanov's theorem and Lévy's characterization yield that

$$dS_t = \Sigma(t, S_t, \eta_{t-})\, d\widetilde{W}_t \tag{4.16}$$

for a Q-Brownian motion $\widetilde{W} := W + \int \Sigma^{-1}\Gamma \, dt$, so that Q is indeed in $\mathbb{P}_e$. □

The next step introduces our candidate $\bar\theta$ for the optimal strategy θ^B, and establishes some nice integrability conditions. Those shall later enable us to apply the verification result of proposition 2.2.

Lemma 4.5. *Define* $(\bar\theta_t)_{t\in[0,T]}$ *by* $\bar\theta_t := \mathrm{grad}_x v^B(t, S_t, \eta_{t-}) + \frac{1}{\alpha}\left(\Sigma\Sigma^{\mathrm{tr}}(t, S_t, \eta_{t-})\right)^{-1}\Gamma\ (t, S_t, \eta_{t-})$ *for* $t \in [0,T)$, *and* $\bar\theta_T := 0$. *Let* Q *be the measure from lemma 4.4. Then*

$$\bar\theta \text{ is } S\text{-integrable and } \quad \left(\int_0^t \bar\theta \, dS\right)_{t\in[0,T]} \quad \text{is a } Q\text{-BMO-martingale,} \tag{4.17}$$

$$\text{and there exists } \varepsilon > 0 \text{ such that} \quad \exp\left(\alpha\left(B + \varepsilon\int_0^T \bar\theta_t \, dS_t\right)\right) \in L^1(P)\,. \tag{4.18}$$

Proof. Let us first show (4.17). From the definition of $\bar\theta$, we have that $\bar\theta$ is left-continuous with right limits on $[0,T)$, and thus S-integrable in the sense of local Q-martingales up to any $t \in [0,T)$. To obtain (4.17), it suffices to prove that

$$\left(\int_0^t \bar\theta \, dS\right)_{t\in[0,T)} \quad \text{is a } Q\text{-BMO-martingale} \tag{4.19}$$

and thereby in the martingale space $\mathcal{H}^2(Q)$ on $[0,T)$. Since $d[S]$ is absolutely continuous, it follows that $\left(\int \theta dS\right)_{t\in[0,T]}$ is well-defined in $\mathcal{H}^2(Q)$, and even in $\mathrm{BMO}(Q)$ by the form of the BMO-norm (cf. He et al. (1992), ch. X.1). Denoting $v := v^B$, the definition of $\bar\theta$ together with (4.16) implies that $\int_0^t \bar\theta \, dS = \int_0^t \mathrm{grad}_x v \, dS + \frac{1}{\alpha}\int_0^t \Sigma^{-1}\Gamma \, d\widetilde{W}$ for $t \in [0,T)$. To establish (4.19) it suffices to show that $\int \mathrm{grad}_x v \, dS$ is a Q-BMO-martingale since $\Sigma^{-1}\Gamma$ is bounded by hypothesis. By (4.13),

$$d\widehat{Q} := \mathcal{E}\left(-\int \Sigma^{-1}\Gamma \, dW\right)_T dP$$

defines a probability measure which is in $\mathbb{P}_e$. Again by Girsanov's theorem, the processes M^{kj} are also $\widehat{Q}$-martingales and the multivariate point process (N^{kj}) has the same bounded intensities $\lambda^{kj}(t, S_t)$ under $\widehat{Q}$ as under P. Recalling (4.10), it follows from (4.12) and the definition of $\widehat{Q}$ that the density process of Q with respect to $\widehat{Q}$ is given by the stochastic exponential

$$\mathcal{E}\left(\sum_{j=1}^{m}\int \xi_t^j \, dM_t^{\eta j}\right) = \mathcal{E}\left(\sum_{\substack{k,j=1\\k\neq j}}^{m}\int I_k(\eta_{t-})\left(\mathrm{e}^{\alpha(v(t,S_t,j)-v(t,S_t,k)+f^{kj}(t,S_t))} - 1\right) dM_t^{kj}\right).$$

By theorem VI.3 in Brémaud (1981) this implies that N^{kj} has $(Q, \mathbb{F})$-intensity

$$\tilde{\lambda}_t^{kj} := \lambda^{kj}(t, S_t)\left(1 + I_k(\eta_{t-})I_{\{k\neq j\}}\left(\mathrm{e}^{\alpha(v(t,S_t,j)-v(t,S_t,k)+f^{kj}(t,S_t))} - 1\right)\right). \tag{4.20}$$

Let $\widetilde{M}^{kj} := N^{kj} - \int \tilde{\lambda}_t^{kj} dt$ denote the compensated $(Q, \mathbb{F})$-point processes, and note that each $\tilde{\lambda}^{kj}$ is bounded since the functions v, λ^{kj} and f^{kj} are so. Hence, each $\widetilde{M}^{kj}$, $k, j \in \{1, \ldots, m\}$, is in $\mathcal{M}^2(Q, \mathbb{F})$. By Itô's formula, we obtain – recalling our convention that functions in integrands are evaluated at (t, S_t, η_{t-}) unless specified otherwise –

$$\mathrm{grad}_x v\, dS_t = dv(t, S_t, \eta_t) - \Delta v(t, S_t, \eta_t) - \left(v_t + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}\, v_{x^i x^j}\right) dt\,, \quad t \in [0, T)\,,$$

with $\Delta v(t, S_t, \eta_t) := v(t, S_t, \eta_t) - v(t-, S_{t-}, \eta_{t-}) = v(t, S_t, \eta_t) - v(t, S_t, \eta_{t-})$. Substituting

$$\Delta v(t, S_t, \eta_t) = \sum_{\substack{k,j=1\\k\neq j}}^{m} I_k(\eta_{t-})(v(t, S_t, j) - v(t, S_t, k))dN_t^{kj}$$

and $dN_t^{kj} = d\widetilde{M}_t^{kj} + \tilde{\lambda}_t^{kj} dt$, and using the PDE for v leads to

$$\mathrm{grad}_x v\, dS_t = dv(t, S_t, \eta_t) + L_t\, dt + \sum_{k,j=1}^{m} H_t^{kj}\, d\widetilde{M}^{kj}\,, \quad t \in [0, T)\,, \tag{4.21}$$

with L and H^{kj}, $k, j = 1, \ldots, m$, given by

$$\begin{aligned} L_t &:= f(t, S_t, \eta_{t-}) - \frac{1}{2\alpha}||\Sigma^{-1}(t, S_t, \eta_{t-})\Gamma(t, S_t, \eta_{t-})||^2 \\ &\quad + \sum_{\substack{j=1\\j\neq\eta_{t-}}}^{m} \frac{\lambda^{\eta j}(t,S_t)}{\alpha}\left(\mathrm{e}^{\alpha(v(t,S_t,j)-v(t,S_t,\eta_{t-})+f^{\eta j}(t,S_t))} - 1\right) - \tilde{\lambda}_t^{\eta j}\left(v(t, S_t, j) - v(t, S_t, \eta_{t-})\right), \\ H_t^{kj} &:= -I_k(\eta_{t-})I_{\{k\neq j\}}(v(t, S_t, j) - v(t, S_t, k))\,. \end{aligned}$$

Note that L and H^{kj} are bounded. Hence, the stochastic integral with respect to $\widetilde{M}$ on the right hand side of eq. (4.21) is a square integrable Q-martingale. Using also the boundedness of $\tilde{\lambda}^{kj}$, we conclude that there is some constant $c < \infty$ such that

$$\begin{aligned} &E_Q\left[\left[\int H^{kj} d\widetilde{M}^{kj}\right]_T - \left[\int H^{kj} d\widetilde{M}^{kj}\right]_t \Big| \mathcal{F}_t\right] \\ &= E_Q\left[\int_t^T (H^{kj})^2 d[\widetilde{M}^{kj}]\Big|\mathcal{F}_t\right] = E_Q\left[\int_t^T (H_u^{kj})^2 \tilde{\lambda}_u^{kj}\, du\Big|\mathcal{F}_t\right] \leq (T-t)c \end{aligned} \tag{4.22}$$

for all k, j and $t \in [0, T]$. By the form of the BMO-norm (cf. He et al. (1992), ch.X.1), we conclude that the $d\widetilde{M}$-integral in (4.21) is even a Q-BMO-martingale. Hence, (4.21) shows that $(\int \mathrm{grad}_x v\, dS)_{t\in[0,T)}$ equals the sum of a bounded process (the first two terms on the right) and a Q-BMO-martingale (the third term on the right). This implies that $(\int \mathrm{grad}_x v\, dS)_{t\in[0,T)}$ is a square integrable Q-martingale and is moreover even a Q-BMO-martingale. The latter assertion uses that BMO-martingales constitute a linear space which contains all bounded martingales. This establishes (4.19) and thereby (4.17).

Next, we prove (4.18). By definition of $\bar\theta$, we have $\bar\theta\, dS = \mathrm{grad}_x v\, dS + \frac{1}{\alpha}\Gamma^{\mathrm{tr}}(\Sigma\Sigma^{\mathrm{tr}})^{-1}\, dS$. Replacing the first addend by (4.21) and substituting $d\widetilde{M}^{kj} = dN^{kj} - \tilde\lambda^{kj}\, dt = dM^{kj} - (\tilde\lambda^{kj} - \lambda^{kj})\, dt$ then leads to

$$\begin{aligned}\bar\theta_t\, dS_t = dv(t,S_t,\eta_t) + \Bigg(&\frac{1}{2\alpha}||\Sigma^{-1}\Gamma||^2 + f + \sum_{j=1}^m I_{\{\eta_{t-}\neq j\}}\lambda^{\eta j}(t,S_t)\Bigg\{\\ &\frac{1}{\alpha}\left(\mathrm{e}^{\alpha(v(t,S_t,j)-v(t,S_t,\eta_{t-})+f^{\eta j}(t,S_t))}-1\right) - \Big(v(t,S_t,j)-v(t,S_t,\eta_{t-})\Big)\Bigg\}\Bigg)\, dt \qquad (4.23)\\ &+\frac{1}{\alpha}\left(\Sigma^{-1}\Gamma\right)^{\mathrm{tr}} dW_t - \sum_{j=1}^m \left(v(t,S_t,j)-v(t,S_t,\eta_{t-})\right) dM_t^{\eta j}\,, \qquad t\in[0,T]\,.\end{aligned}$$

More precisely, we obtain the previous equality at first only for $t\in[0,T)$ but it extends to all of $[0,T]$ since both sides describe continuous processes on $[0,T]$. The dt-integrand and the process v on the right hand side of (4.23) are bounded, and the stochastic integrals with respect to W and M are P-BMO-martingales. The latter assertion follows by similar arguments as in (4.22) since the integrands again are bounded. Hence, $(\int_0^t \bar\theta\, dS)_{t\in[0,T]}$ equals a bounded process plus a P-BMO-martingale, and by the John-Nirenberg inequality there is some $\bar\varepsilon > 0$ such that

$$\exp\left(\alpha\,\bar\varepsilon\int_0^T \bar\theta\, dS\right)\in L^1(P)\,.$$

In combination with lemma 4.1 and Hölder's inequality, this establishes (4.18). ☐

In a next step, we will show how our candidate Z for the density process of Q^B can be represented as an (ordinary) exponential by means of the solution v^B to the PDE-system (4.4).

Lemma 4.6. *The density process Z from lemma 4.4 can be written as*

$$\begin{aligned}Z_t \;=\; \exp\Bigg(&-\alpha\Big\{v^B(0,S_0,\eta_0) + \int_0^t \bar\theta_u\, dS_u - v^B(t,S_t,\eta_t)\\ &-\int_0^t f(u,S_u,\eta_u)du - \sum_{u\le t}\sum_{\substack{k,j=1\\k\neq j}}^m I_{\{\eta_{u-}=k,\eta_u=j\}}f^{kj}(u,S_u)\Big\}\Bigg) \qquad (4.24)\end{aligned}$$

for $t\in[0,T]$ and with $\bar\theta$ given by lemma 4.5.

Proof. We have to show $Z=\widetilde Z$ for $\widetilde Z$ denoting the exponential at the right-hand side of (4.24). In the sequel, we use that $\int f(u,S_u,\eta_u)du$ is indistinguable from $\int f(u,S_u,\eta_{u-})du$. To simplify notation, again let $v := v^B$ and recall our convention that all functions in integrands are evaluated at (t,S_t,η_{t-}) unless specified otherwise. Then Itô's formula gives

$$\begin{aligned}\frac{d\widetilde Z_t}{\widetilde Z_{t-}} \;=\;& \alpha\left(v_t + \frac{1}{2}\sum_{i,j=1}^d a^{ij}v_{x^ix^j} + (\mathrm{grad}_x v - \bar\theta)^{\mathrm{tr}}\,\Gamma + \frac{\alpha}{2}||\Sigma^{\mathrm{tr}}(\mathrm{grad}_x v - \bar\theta)||^2 + f\right)dt\\ &+\alpha(\mathrm{grad}_x v - \bar\theta)^{\mathrm{tr}}\Sigma\, dW_t + \frac{\Delta\widetilde Z_t}{\widetilde Z_{t-}}\,,\quad t\in[0,T)\,,\end{aligned}$$

and substituting the definition of $\bar\theta$ leads to

$$\begin{aligned}\frac{d\widetilde Z_t}{\widetilde Z_{t-}} \;=\;& \alpha\left(v_t + \frac{1}{2}\sum_{i,j=1}^d a^{ij}v_{x^ix^j} - \frac{1}{2\alpha}||\Sigma^{-1}\Gamma||^2 + f\right)dt \qquad (4.25)\\ &-\left(\Sigma^{-1}\Gamma\right)^{\mathrm{tr}} dW_t + \frac{\Delta\widetilde Z_t}{\widetilde Z_{t-}}\,,\quad t\in[0,T)\,.\end{aligned}$$

Recalling the short-hand notations $N^{\eta j}$, $M^{\eta j}$, $\lambda^{\eta j}$ and $f^{\eta j}$ from (4.9), we have

$$
\begin{aligned}
\frac{\Delta \widetilde{Z}_t}{\widetilde{Z}_{t-}} &= \exp\left(\alpha\Big(v(t,S_t,\eta_t) - v(t,S_t,\eta_{t-}) + \sum_{\substack{k,j=1\\k\neq j}}^{m} f^{kj}(t,S_t) I_{\{\eta_{t-}=k,\eta_t=j\}}\Big)\right) - 1 \\
&= \sum_{j=1}^{m} I_{\{\eta_{t-}\neq j\}} \left(\mathrm{e}^{\alpha(v(t,S_t,j)-v(t,S_t,\eta_{t-})+f^{\eta j}(t,S_t))} - 1\right) \Delta N_t^{\eta j}\,, \quad t\in[0,T)\,,
\end{aligned}
$$

and $\Delta N_t^{\eta j} = dN_t^{\eta j} = dM_t^{\eta j} + \lambda^{\eta j}(t,S_t)\,dt$. Substituting these terms into (4.25) leads to

$$
\begin{aligned}
\frac{d\widetilde{Z}_t}{\widetilde{Z}_{t-}} \quad = \quad & \alpha\left(v_t + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij} v_{x^i x^j} - \frac{1}{2\alpha}||\Sigma^{-1}\Gamma||^2 + f \right. \\
& \left. + \frac{1}{\alpha}\sum_{j=1}^{m} I_{\{\eta_{t-}\neq j\}} \lambda^{\eta j}(t,S_t) \left(\mathrm{e}^{\alpha(v(t,S_t,j)-v(t,S_t,\eta_{t-})+f^{\eta j}(t,S_t))} - 1\right)\right) dt \\
& - \quad \left(\Sigma^{-1}\Gamma\right)^{\mathrm{tr}} \; dW_t \; + \; \sum_{j=1}^{m} I_{\{\eta_{t-}\neq j\}} \left(\mathrm{e}^{\alpha(v(t,S_t,j)-v(t,S_t,\eta_{t-})+f^{\eta j}(t,S_t))} - 1\right) dM_t^{\eta j}
\end{aligned}
$$

for $t\in[0,T)$, and by using the PDE for v this simplifies to

$$
\frac{d\widetilde{Z}_t}{\widetilde{Z}_{t-}} = -(\Sigma^{-1}\Gamma)^{\mathrm{tr}}\, dW_t + \sum_{j=1}^{m} \xi^j\, dM_t^{\eta j}\,, \quad t\in[0,T)\,, \tag{4.26}
$$

with ξ^j given by (4.10). Now, the definition (4.11) of Z as a stochastic exponential and the SDE (4.26) for $\widetilde{Z}$ imply that $Z_t = \widetilde{Z}_t$ for $t\in[0,T)$. This equality extends to all of $[0,T]$ since both Z and $\widetilde{Z}$ are a.s. continuous at T. The latter follows from $N_T = N_{T-}$ and $M_T = M_{T-}$ (P-a.s.). □

Using the previous lemmata, we now have everything in place to finalize the

Proof. (Theorem 4.3) We apply the verification result of proposition 2.2. The density Z_T of the measure Q from lemma 4.4 can be written by lemmata 4.6 and 4.5 just in the form, which is required for the verification result. Therefore, $Q = Q^B$ is the solution for the dual problem, and $\bar\theta = \theta^B$ is the optimal strategy. This proves theorem 4.3. □

In the previous lemmata, we moreover have derived two explicit descriptions for the $I\!F$-density process of the optimal measure Q^B for the dual problem from proposition 2.1. We state this as a separate result.

Corollary 4.7. *Suppose the assumptions for theorem 4.3 hold. Then, the density process with respect to P of the optimal measure Q^B to (2.5) is given by the process Z from lemmata 4.4 and 4.6, and $\bar\theta = \theta^B$. In particular, the density process of Q^B can be represented as a stochastic exponential (4.8) and also as an ordinary exponential (4.24) by means of the solution v^B to the PDE system (4.4).*

Comparing the exponential form (4.24) of the density process of Q^B with the general representation (2.9) in §2 gives an interpretation for the term

$$
c_t^B = v^B(t,S_t,\eta_t) + \int_0^t f(u,S_u,\eta_u)du + \sum_{u\le t}\sum_{\substack{k,j\\k\neq j}} I_{\{\eta_{u-}=k,\eta_u=j\}} f^{kj}(u,S_u) \tag{4.27}
$$

which appears in the exponent of the density process. Following eq. (2.10) plus subsequent remarks, the quantity (4.27) represents the current (exponential) certainty equivalent of the remaining effective liability at time t.

The previous results in particular provide explicit formulae for the density of the minimal entropy martingale measure Q^0 in the special case $B = 0$. We refer to Grandits & Rheinländer (2002) and Miyahara (1996) for related results in different stochastic models. Moreover, we obtained the solution Q^B to the dual side of the utility maximization problem with an additional terminal liability B. By the duality relation from proposition 2.1, the description of the dual solution immediately implies a PDE-type description for the solution of the (primal) optimal investment problem with additional liability B.

Corollary 4.8. *Suppose the assumptions of theorem 4.3 hold. Then*

$$E\left[-\exp\left(-\alpha(x+\int_0^T \theta^B dS - B)\right)\right] = \sup_{\theta\in\Theta_{\mathcal{M}}} E\left[-\exp\left(-\alpha(x+\int_0^T \theta dS - B)\right)\right], \quad x \in I\!R,$$

and the optimal strategy $\theta^B \in \Theta_{\mathcal{M}}$ is given by theorem 4.3 and essentially described via v^B.

In the following, we will see that the solution to our utility indifference pricing and hedging problem is also given by the solution to a system of interacting semi-linear PDEs.

Let us briefly relate our results to the existing literature. PDE-type solutions for expected utility maximization in incomplete markets under an additional terminal liability have been studied by several authors. Let us single out the works by Davis (2000, unpublished), Henderson & Hobson (2002), Henderson (2002) and Musiela & Zariphopoulou (2002, unpublished), and refer to section 4.3 in Delbaen et al. (2002) for a survey with further references. Throughout, these authors consider classical incomplete models, i.e., models with a Brownian filtration where the asset prices S are given by a diffusion process, and incompleteness arises from the fact that there are more Brownian motions (sources of risk) than tradable risky assets. This modeling differentiates their work from ours. So far, more explicit results on the utility maximization problem with an additional liability in (incomplete) diffusion models have been achieved basically for the case where the price of a single tradable asset evolves as a geometric Brownian motion. With a view towards applications in life insurance, a recent article by Young & Zariphopoulou (2002) considers a model with a geometric Brownian motion for the risky asset price, and an independent point process with deterministic intensity whose first jump represents the death of the policy holder. This setting fits into our Itô and point process model. In comparison, their results rely on formal Hamilton Jacobi Bellman arguments from dynamical programming, while the present article uses martingale duality methods as verification results. Another contribution of the present article is the modeling of the tradable assets and further risky events by interacting Itô and point processes. To our knowledge, this model is new and has not been considered in the present context so far. The interacting structure allows for several mutual dependencies between tradable and non-tradable factors of risk, but still permits for fairly explicit and interpretable solutions. For the utility optimization problem and similarly for the utility-indifference problem, the solutions are described by the unique classical solution to a system of PDEs which is linear in all derivatives. All coefficients appearing in the PDE system are explicitly given in terms of parameters of the financial model. Conditions are given for "sufficient regularity", which ensure existence and uniqueness for the PDE and seem adequate for typical parameterizations in finance.

(*c*) *utility-indifference hedging and pricing*

By using the results from the last section, this section describes the solution for the indifference price process and the corresponding hedging strategy in terms of reaction diffusion systems.
Provided the assumptions for theorem 4.3 do hold for the claim B and also for the zero claim 0, the density process of the minimal entropy martingale measure $Q^0 \in I\!P_f \cap I\!P_e$ is described by the unique

solution $v^0 \in C_b^{1,2}([0,T) \times D \times \{1,\dots,m\}, I\!R)$ to the reaction-diffusion system

$$\begin{aligned} v_t^0(t,x,k) &+ \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x,k)\, v^0_{x^i x^j}(t,x,k) \\ &- \frac{1}{2\alpha}||\Sigma^{-1}(t,x,k)\Gamma(t,x,k)||^2 \\ &+ \sum_{\substack{j=1\\k\neq j}}^{m} \lambda^{kj}(t,x)\frac{1}{\alpha}\left(\mathrm{e}^{\alpha(v^0(t,x,j)-v^0(t,x,k))}-1\right) = 0\,, \quad (t,x)\in[0,T)\times D\,, \\ &\text{and} \quad v^0(T,x,k) = 0\,, \quad x\in D\,, \end{aligned} \tag{4.28}$$

with $k = 1,\dots,m$. With v^0 solving (4.28) and v^B solving (4.4), the difference $v^\pi = v^B - v^0$ solves the PDE-system

$$\begin{aligned} v_t^\pi(t,x,k) &+ \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x,k)\, v^\pi_{x^i x^j}(t,x,k) \\ &+ f(t,x,k) \\ &+ \sum_{\substack{j=1\\k\neq j}}^{m} \Lambda^{kj}(t,x)\frac{1}{\alpha}\left(\mathrm{e}^{\alpha(v^\pi(t,x,j)-v^\pi(t,x,k)+f^{kj}(t,x))}-1\right) = 0, \quad (t,x)\in[0,T)\times D\,, \\ &\text{and} \quad v^\pi(T,x,k) = h(x,k)\,, \quad x\in D\,, \end{aligned} \tag{4.29}$$

with

$$\Lambda^{kj}(t,x) := \lambda^{kj}(t,x)\mathrm{e}^{\alpha(v^0(t,x,j)-v^0(t,x,k))}\,, \quad (t,x)\in[0,T]\times D\,. \tag{4.30}$$

To see this, note that the PDEs which describe v^B and v^0 are linear in all terms but one. Calculating the difference of these non-linear terms, using the equality

$$\begin{aligned} &\mathrm{e}^{\alpha(v^B(t,x,j)-v^B(t,x,k)+f^{kj}(t,x))} - \mathrm{e}^{\alpha(v^0(t,x,j)-v^0(t,x,k))} \\ &\quad = \mathrm{e}^{\alpha(v^0(t,x,j)-v^0(t,x,k))}\left(\mathrm{e}^{\alpha(v^\pi(t,x,j)-v^\pi(t,x,k)+f^{kj}(t,x))}-1\right) \end{aligned}$$

and the definition (4.30) then leads to (4.29).

Remark: (Interpretation of the functions Λ^{kj})
Let us provide some interpretation for the functions Λ^{kj}, $k \neq j$, which appear in the PDE-system (4.29) for v^π. By eq. (4.20) in the proof of theorem 4.3 – applied for $B = 0$ – and the definition (3.2) of η, we have that $I_k(\eta_{t-})\Lambda^{kj}(t,S_t)$ with $k \neq j$ is the $(Q^0, I\!F)$-intensity of the point process $\sum_{s\le t} I_k(\eta_{s-})I_j(\eta_s)$, $t\in[0,T]$, which counts the jumps of η from k to j. In this sense, the functions Λ^{kj}, $k\neq j$, describe the intensities of the process η to jump from one state to another with respect to the minimal entropy martingale measure Q^0. In comparison, the corresponding functions with respect to the objective measure P are given by λ^{kj}, $k \neq j$.

We will need that

$$\text{there is an unique solution } v^0 \in C_b^{1,2}([0,T)\times D\times\{1,\dots,m\}\,, I\!R) \text{ to the PDE (4.28).} \tag{4.31}$$

Again, sufficient conditions for (4.31) are given by proposition 4.2. Note that the assumptions of proposition 4.2 automatically hold for the zero claim 0 when they do hold for the claim B.

Theorem 4.9. *Suppose that (4.31) holds in addition to the assumptions for theorem 4.3. Then there is a unique function $v^\pi \in C_b^{1,2}([0,T) \times D \times \{1,\dots,m\}, I\!R)$ which solves the PDE-system (4.29), and the utility-indifference price process $(\pi_t)_{t\in[0,T]}$ and the hedging strategy $\psi \in \Theta_{\mathcal{M}}$ for B are given by*

$$\begin{aligned}\pi_t(B) &= v^\pi(t,S_t,\eta_t) + \int_0^t f(u,S_u,\eta_u)\,du + \sum_{u\le t}\sum_{\substack{k,j=1\\k\neq j}}^{m} I_{\{\eta_{u-}=k,\eta_u=j\}} f^{kj}(u,S_u) \quad \textit{and}\\ \psi_t(B) &= \mathrm{grad}_x v(t,S_t,\eta_{t-})\,, \quad t\in[0,T)\,,\end{aligned}$$

and $\psi_T(B)=0$. In particular, $\pi(B;\alpha)=\pi_0(B)=v^\pi(0,S_0,\eta_0)$ is the utility-indifference price.

Again (cf. the remark following theorem 4.3) it is rather arbitrary to set $\psi_T(B)=0$ since the value of $\psi(B)$ at T does not affect the integral $\int \psi(B)\,dS$, i.e. the equivalence class of $\psi(B)$.

Proof. Under the given assumptions, theorem 4.3 applies to both claims B and 0. By the foregoing considerations, we know that $v^B - v^0$ solves (4.29). To see that this solution is unique, let $v^\pi \in C_b^{1,2}([0,T)\times D\times\{1,\dots,m\}, I\!R)$ denote any solution to the PDE-system (4.29). It is straightforward to verify that v^0+v^π satisfies the same PDE-system (4.4) as v^B. Hence, $v^\pi = v^B - v^0$ by the uniqueness of v^B. This implies

$$\mathrm{grad}_x v^\pi(t,x,k) = \mathrm{grad}_x v^B(t,x,k) - \mathrm{grad}_x v^0(t,x,k)\,, \quad (t,x,k)\in[0,T)\times D\times\{1,\dots,m\}\,,$$

and $v^\pi(T,S_T,\eta_T) = v^B(T,S_T,\eta_T) - 0 = h(S_T,\eta_T)$. Substituting the formulae for θ^B and θ^0 from theorem 4.3 in the definition of $\psi(B) := \theta^B - \theta^0$ yields

$$\psi_t(B) = \theta_t^B - \theta_t^0 = \mathrm{grad}_x v^B(t,S_t,\eta_{t-}) - \mathrm{grad}_x v^0(t,S_t,\eta_{t-}) = \mathrm{grad}_x v^\pi(t,S_t,\eta_{t-})\,, \quad t\in[0,T)\,,$$

and ψ is in $\Theta_{\mathcal{M}}$ since θ^B and θ^0 are in this linear space. Finally, the definition $\pi_t := c_t^B - c_t^0$ leads to the claimed form of the utility-indifference price process. To see this, recall the representation of eq. (4.27) for c_t^B and c_t^0, respectively, and recall that $v^\pi = v^B - v^0$. □

The theorem shows that the solution to our indifference pricing and hedging problem can be determined from the unique solution to the PDE-system (4.29). Note that this system involves v^0 by eq. (4.30). So one needs first the unique solution v^0 to the system (4.28) – which involves only the coefficients of the underlying financial model – in order to determine (π_t) from (4.29). In some situations this dependence on v^0 disappears and

$$\Lambda^{kj}(t,x) = \lambda^{kj}(t,x)\,, \quad (t,x)\in[0,T]\times D\,,$$

holds for all $k\neq j$. Then, eq. (4.29) alone determines the solution to our problem. Suppose for instance that the coefficients Σ and Γ in the SDE (3.1) that drives S depend only on (t,x) but do not vary in k, i.e. $\Gamma(t,x,k)=\Gamma(t,x,j)$ and $\Sigma(t,x,j)=\Sigma(t,x,k)$ for any $k,j=1,\dots,m$ and $(t,x)\in[0,T]\times D$. This means that the non-tradable stochastic risk factor (η_t) does not affect the dynamics of the asset prices S but only the asset prices can affect the intensities of η. The liability B, however, might depend on the common evolution of η and S. In this situation the solution v^0 to the PDE (4.28) does not vary in k and therefore

$$\mathrm{e}^{\alpha(v^0(t,x,j)-v^0(t,x,k))} = 1 \quad \text{for} \quad (t,x)\in[0,T]\times D,\ k,j\in\{1,\dots,m\}\,.$$

Hence, Λ^{kj} equals λ^{kj} for all k,j and the PDE-system (4.29) for $v^\pi(t,x,k)$ involves only coefficients which are directly given by the underlying financial model.

5. Appendix: An Existence and Uniqueness Result

This appendix states an existence and uniqueness result which ensures the existence of a unique classical solution to a semi-linear system of parabolic partial differential equations with interaction of exponential form. For our Itô and point process model, such reaction-diffusion systems describe the solution for the utility-indifference valuation and hedging problem, and also the solution for the utility maximization problem with an additional liability.

For the subsequent Feyman-Kac type result, we need a probabilistic framework as follows. Let $T \in (0,\infty)$ be a fixed time horizon, D a domain in $I\!R^d$, i.e. an open connected subset, and $m \in I\!N$. For $(t,x,k) \in [0,T] \times D \times \{1,\dots,m\}$, consider the following stochastic differential equation in $I\!R^d$

$$X_t^{t,x,k} = x \in D\,, \quad dX_s^{t,x,k} = b_k(s, X_s^{t,x,k})\,ds + \sum_{j=1}^{r} \Sigma_{k,j}(s, X_s^{t,x,k})\,dW_s^j\,, \quad s \in [t,T]\,, \tag{5.1}$$

for continuously differentiable functions $b_k : [0,T] \times D \to I\!R^d$ and $\Sigma_{k,j} : [0,T] \times D \to I\!R^d$, $j = 1,\dots,r$, with an r-dimensional Brownian motion $W = (W^j)_{j=1,\dots,r}$. We write b_k and each $\Sigma_{k,j}$ as a $d \times 1$ column vector and define the matrix-valued function $\Sigma_k : [0,T] \times D \to I\!R^{d\times r}$ by $\Sigma_k^{ij} := (\Sigma_{k,j})^i$. Being of class C^1, the functions b_k and Σ_k are in particular locally Lipschitz continuous in x, uniformly in t. So (5.1) has a unique (strong) solution up to a possibly finite random explosion time, cf. Protter (1990), th.V.38. We suppose additionally that for all k, t and x the solution $X^{t,x,k}$ does not explode but stays within D up to time T, i.e.

$$P\left[X_s^{t,x,k} \in D \ \text{ for all } s \in [t,T]\right] = 1\,. \tag{5.2}$$

To formulate our partial differential equation, let us define operators $\mathcal{L}^k$, $k = 1,\dots,m$, on sufficiently smooth functions $f : [0,T] \times D \to I\!R$ by

$$(\mathcal{L}^k f)(t,x) = \sum_{i=1}^{d} b_k^i(t,x) \frac{\partial f}{\partial x^i}(t,x) + \frac{1}{2} \sum_{i,j=1}^{d} a_k^{ij}(t,x) \frac{\partial^2 f}{\partial x^i \partial x^j}(t,x)$$

with $a_k(t,x) = (a_k^{ij}(t,x))_{i,j\in\{1,\dots,d\}} := \Sigma_k(t,x)\Sigma_k^{\mathrm{tr}}(t,x)$. For suitable functions $h : D \to I\!R^m$ and $g : [0,T] \times D \times I\!R^m \to I\!R^m$, we consider the system

$$\begin{aligned} \frac{\partial}{\partial t} v^k(t,x) + \mathcal{L}^k v^k(t,x) + g^k(t,x,v(t,x)) &= 0 \quad ,(t,x) \in [0,T) \times D, \\ \text{with} \quad v^k(T,x) &= h^k(x) \quad ,x \in D, \end{aligned} \tag{5.3}$$

of $k = 1,\dots,m$ interacting semi-linear partial differential equations with boundary conditions at terminal time T. These m partial differential equations are interacting via the g-term which may depend on all m components of $v(t,x) = (v^k(t,x))_{k=1,\dots,m}$ and is specified below.

We need the following assumptions on the coefficient functions of the PDE (5.3):

$$\text{There exists a sequence } (D_n)_{n\in I\!N} \text{ of bounded domains with closure } \bar{D_n} \subseteq D \quad \text{such that } \cup_{n=1}^{\infty} D_n = D, \text{ each } D_n \text{ has a } C^2\text{-boundary}, \tag{5.4}$$

$$\det a_k(t,x) \neq 0 \text{ for all } (t,x) \in [0,T] \times D \text{ and } k = 1,\dots,m, \text{ and} \tag{5.5}$$

$$h : D \to I\!R^m \text{ is bounded and continuous}\,. \tag{5.6}$$

Note that the functions b_k and $a_k = \Sigma_k\Sigma_k^{\mathrm{tr}}$ are uniformly Lipschitz-continuous on $[0,T] \times \bar{D_n}$ for any n, since they are of class C^1 on $[0,T] \times D$.

Finally, the function $g : [0,T] \times D \times I\!R^m \to I\!R^m$ is taken to be of the form

$$g^k(t,x,v) := f^k(t,x) + \sum_{j=1}^{m} I_{\{j\neq k\}} \lambda^{kj}(t,x) \frac{1}{\alpha} \left(\mathrm{e}^{\alpha(v^j - v^k + f^{kj}(t,x))} - 1 \right) , \quad k = 1, \ldots, m, \tag{5.7}$$

with $\alpha \in (0,\infty)$, and functions f^k, $f^{kj} \in C_b^1([0,T] \times D, I\!R)$ and $\lambda^{kj} \in C_b^1([0,T] \times D, [0,\infty)\,)$ which are bounded and of class C^1 with respect to both arguments on $[0,T] \times D$.
Let $C_b^{1,2}([0,T) \times D, I\!R^m)$ denote the space of continuous bounded functions $v : [0,T] \times D \to I\!R^m$ which are of class $C^{1,2}$ with respect to $(t,x) \in [0,T) \times D$. Note that the $C^{1,2}$-condition is only imposed on $[0,T) \times D$ while continuity is required on $[0,T] \times D$.

Proposition 5.1. *Suppose that the conditions (5.2) and (5.4)–(5.6) hold, and g is of the form (5.7). Then there is a unique classical solution $v \in C_b^{1,2}([0,T) \times D, I\!R^m)$ to the system (5.3) of interacting partial differential equations, and v satisfies the Feynman-Kac representation*

$$v^k(t,x) \quad = \quad E\left[h^k(X_T^{t,x,k}) + \int_t^T g^k(s, X_s^{t,x,k}, v(s, X_s^{t,x,k}))\, ds \right]$$

for $k = 1, \ldots, m$ and $(t,x) \in [0,T] \times D$.

For details on the proof we refer to Becherer (2001), theorem 7.3.2; and note that a more general result is contained in Becherer and Schweizer (2002, unpublished). A brief sketch of the main line of proof is as follows. First, one considers interacting parabolic PDE-systems where the interaction term satisfies a Lipschitz condition. Here, a fixed point argument in combination with a suitable Feynman-Kac result ensures existence and uniqueness of a classical solution. Those results can then be extended to cover the case with interaction of exponential form, by using certain monotonicity properties of the exponential interaction term in combination with truncation and stopping arguments.

Remark: Note that only local conditions are supposed for the coefficients functions b and a in order to get a classical solution on a domain D, which might be unbounded. In particular, b and a are not supposed to be bounded or of linear growth, which would exclude parameterizations for drift and volatility which are common in financial models; cf. Heath & Schweizer (2001) for examples.

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