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Linear Elliptic Boundary Value Problems with Non-smooth Data: Campanato Spaces of Functionals

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Dedicated to ROLF HÜNLICH *on his sixtieth birthday*

Abstract. In this paper linear elliptic boundary value problems of second order with non-smooth data (L^∞-coefficients, sets with Lipschitz boundary, regular sets, non-homogeneous mixed boundary conditions) are considered. It will be shown that such boundary value problems generate isomorphisms between certain Sobolev–Campanato spaces of functions and functionals, respectively.

1. Introduction

In this paper we consider linear elliptic operators $L : W_0^{1,2}(\Omega \cup \Gamma) \to W^{-1,2}(\Omega \cup \Gamma)$ defined as

$$\langle Lu, w\rangle := \int_\Omega (A\nabla u \cdot \nabla w + duw)\, \mathrm{d}\lambda^n, \quad w \in W_0^{1,2}(\Omega \cup \Gamma), \tag{1.1}$$

and regularity properties of solutions $u \in W_0^{1,2}(\Omega \cup \Gamma)$ to the corresponding linear elliptic boundary value problem

$$\langle Lu, w\rangle = \langle F, w\rangle, \quad w \in W_0^{1,2}(\Omega \cup \Gamma), \tag{1.2}$$

for functionals $F \in W^{-1,2}(\Omega \cup \Gamma)$. In (1.1) and (1.2) Ω is a bounded open subset of $\mathbb{R}^n$, and Γ is a relatively open subset of the boundary $\partial\Omega$ such that $\Omega \cup \Gamma$ is regular in the sense of GRÖGER [11]. Furthermore, $W_0^{1,2}(\Omega \cup \Gamma)$ and $W^{-1,2}(\Omega \cup \Gamma)$ denote the Sobolev spaces of functions $u \in W^{1,2}(\Omega)$ having trace zero on $\partial\Omega \setminus \overline{\Gamma}$ and its dual space, respectively. Hence, our variational formulation (1.2) includes natural and Dirichlet boundary conditions on the boundary parts Γ and $\partial\Omega \setminus \overline{\Gamma}$, respectively. The coefficients A and d are bounded measurable maps defined on Ω, where A is real

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symmetric $(n \times n)$-matrix valued, and d is scalar valued. Finally, we assume that there exists a real constant $\varepsilon > 0$ such that for all $\xi \in \mathbb{R}^n$ and almost all $x \in \Omega$ there hold

$$\varepsilon \le d(x) \le \frac{1}{\varepsilon} \quad \text{and} \quad \varepsilon\,|\xi|^2 \le A(x)\xi\cdot\xi \le \frac{1}{\varepsilon}\,|\xi|^2.$$

Under the above assumptions there exists a constant $\overline{p} = \overline{p}(\varepsilon, G) > 2$ such that L maps $W_0^{1,p}(\Omega\cup\Gamma)$ isomorphically onto $W^{-1,p}(\Omega\cup\Gamma)$ for all $2 \le p < \overline{p}$ (see Gröger [11]). Unfortunately, for $n \ge 3$ this result in general does not yield the Hölder continuity of the solution u to the mixed boundary value problem $Lu = F \in W^{-1,p}(\Omega\cup\Gamma)$.

In this paper we will consider appropriate function spaces for the case $n \ge 3$. Recke [15] and Griepentrog, Recke [9] have shown the existence of a parameter $n-2 < \overline{\omega} < n$ depending only on ε and G such that for all $0 \le \omega < \overline{\omega}$ and all functionals $F \in W^{-1,2,\omega}(\Omega\cup\Gamma)$ the solution $u \in W_0^{1,2}(\Omega\cup\Gamma)$ of the mixed boundary value problem $Lu = F$ belongs to the Sobolev–Campanato space

$$W_0^{1,2,\omega}(\Omega\cup\Gamma) = \bigl\{u \in W_0^{1,2}(\Omega\cup\Gamma) : \nabla u \in \mathfrak{L}^{2,\omega}(\Omega;\mathbb{R}^n)\bigr\},$$

if F belongs to the space $W^{-1,2,\omega}(\Omega\cup\Gamma)$ of all functionals $F \in W^{-1,2}(\Omega\cup\Gamma)$ with

$$\langle F, w\rangle := \int_\Omega (f\cdot\nabla w + gw)\,\mathrm{d}\lambda^n, \quad w \in W_0^{1,2}(\Omega\cup\Gamma), \tag{1.3}$$

where

$$f \in \mathfrak{L}^{2,\omega}(\Omega;\mathbb{R}^n), \quad g \in \mathfrak{L}^{2,\omega-2}(\Omega). \tag{1.4}$$

Note, that in the case $n-2 < \omega < \overline{\omega}$ for $F \in W^{-1,2,\omega}(\Omega\cup\Gamma)$ the Hölder continuity of the solution $u \in W_0^{1,2,\omega}(\Omega\cup\Gamma)$ to the boundary value problem (1.2) follows via embedding theorems. The main goal of the present paper is to overcome the following shortcoming of the above approach:

In order to prove, that an arbitrarily given functional $F \in W^{-1,2}(\Omega\cup\Gamma)$ is an element of $W^{-1,2,\omega}(\Omega\cup\Gamma)$, up to now it was necessary to repeat the whole regularity theory to get a representation of F in the form (1.3) and (1.4) via the variational formulation (1.1) and (1.2) of the elliptic problem $Lu = F$.

Generalizing the results of Rakotoson [13, 14] (for the case $\Gamma = \varnothing$) in the present paper we are able to give a more direct characterization of the space $W^{-1,2,\omega}(\Omega\cup\Gamma)$ which has the major advantage of being independent of a concrete representation (1.3) and (1.4). Nevertheless the arguments are closely related to the methods developed in Recke [15] and Griepentrog, Recke [9]. Our paper is organized as follows:

In Section 2 we collect preliminary results related to regular sets $\Omega\cup\Gamma \subset \mathbb{R}^n$ and Sobolev–Campanato spaces $W_0^{1,2,\omega}(\Omega\cup\Gamma)$.

Section 3 is devoted to the introduction of new Campanato spaces $Y^{-1,2,\omega}(\Omega\cup\Gamma)$ of functionals (see also Rakotoson [13, 14]), and among other things we prove the continuous embedding $W^{-1,2,\omega}(\Omega\cup\Gamma) \hookrightarrow Y^{-1,2,\omega}(\Omega\cup\Gamma)$ for all $0 \le \omega < n$.

In Section 4 we prove our main result (Theorem 4.12) for solutions to the variational problem (1.1) and (1.2). In fact, we will show the isomorphism property of the linear elliptic operator L between $W_0^{1,2,\omega}(\Omega\cup\Gamma)$ and $Y^{-1,2,\omega}(\Omega\cup\Gamma)$, hence, the coincidence of the spaces $W^{-1,2,\omega}(\Omega\cup\Gamma)$ and $Y^{-1,2,\omega}(\Omega\cup\Gamma)$ for all $0 \le \omega < \overline{\omega}$.

A more comprehensive treatment of the topic can be found in the doctoral thesis of the author (see Griepentrog [10]).

2. Preliminary results concerning Campanato spaces

Throughout the paper we will assume $n \geq 3$. The symbol $|\cdot|$ is used for the absolute value, and for the Euclidean norm in $\mathbb{R}^n$. By $\mathfrak{e}_j$ we denote the j-th unit vector in $\mathbb{R}^n$ and furthermore, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we write $\hat{x} = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x,r) := \{\xi \in \mathbb{R}^n : |\xi - x| < r\}$ and $E_1(x,r) := \{\xi \in \mathbb{R}^n : |\xi - x| < r,\ \xi_n - x_n < 0\}$ the open ball and the open halfball, respectively. In the case $x = 0$, $r = 1$ we shortly write B and E_1.

As usual, for subsets G of $\mathbb{R}^n$ we write G°, $\overline{G}$ and ∂G for the interior, the closure and the (topological) boundary of G, respectively.

By λ^n we will denote the n-dimensional Lebesgue measure on the σ-algebra of Lebesgue-measurable subsets of $\mathbb{R}^n$. Let Ω be a bounded open subset of $\mathbb{R}^n$. We write $L^\infty(\Omega)$ and $L^\infty(\Omega;\mathbb{R}^n)$, for the sets of bounded measurable maps from Ω into $\mathbb{R}$ and $\mathbb{R}^n$, respectively. Analogously, for $1 \leq p < \infty$ we write $L^p(\Omega)$ and $L^p(\Omega;\mathbb{R}^n)$ for the Lebesgue spaces of p-integrable functions from Ω into $\mathbb{R}$ and $\mathbb{R}^n$, respectively.

2.1. Campanato spaces and Sobolev–Campanato spaces

For $1 \leq p < \infty$, $0 \leq \omega < n + p$ we denote by $\mathfrak{L}^{p,\omega}(\Omega)$ the *Campanato space*, i.e. the space of all $u \in L^p(\Omega)$ such that

$$[u]^p_{\mathfrak{L}^{p,\omega}(\Omega)} := \sup_{\substack{x\in\Omega \\ r>0}} r^{-\omega} \int_{\Omega[x,r]} |u - u_{\Omega[x,r]}|^p \,\mathrm{d}\lambda^n < \infty. \tag{2.1}$$

In (2.1) we used the notation

$$\Omega[x,r] := \Omega \cap B(x,r), \quad u_{\Omega[x,r]} := \frac{1}{\lambda^n(\Omega[x,r])} \int_{\Omega[x,r]} u \,\mathrm{d}\lambda^n. \tag{2.2}$$

The space $\mathfrak{L}^{p,\omega}(\Omega)$ is a Banach space with the norm

$$\|u\|_{\mathfrak{L}^{p,\omega}(\Omega)} := \left\{ \|u\|^p_{L^p(\Omega)} + [u]^p_{\mathfrak{L}^{p,\omega}(\Omega)} \right\}^{1/p}. \tag{2.3}$$

Analogously, by $\mathfrak{L}^{p,\omega}(\Omega;\mathbb{R}^n)$ we denote the space of all $f \in L^p(\Omega,\mathbb{R}^n)$ with components in $\mathfrak{L}^{p,\omega}(\Omega)$, and the norm in $\mathfrak{L}^{p,\omega}(\Omega;\mathbb{R}^n)$ is defined similarly to (2.3). Finally, for the sake of simplicity, for $\omega \leq 0$ we will use the notation $\mathfrak{L}^{p,\omega}(\Omega) := L^p(\Omega)$.

The usual Sobolev space $W^{1,p}(\Omega)$ will be equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \left\{ \|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^n)} \right\}^{1/p}.$$

For $0 \leq \omega < n + p$ we denote by $W^{1,p,\omega}(\Omega)$ the *Sobolev–Campanato space*, i.e. the space of all $u \in W^{1,p}(\Omega)$ such that $\nabla u \in \mathfrak{L}^{p,\omega}(\Omega;\mathbb{R}^n)$. The space $W^{1,p,\omega}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{1,p,\omega}(\Omega)} := \left\{ \|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{\mathfrak{L}^{p,\omega}(\Omega;\mathbb{R}^n)} \right\}^{1/p}.$$

The following well-known (cf., e.g., TROIANIELLO [17]) property of Campanato spaces will be used repeatedly in our paper: If $r_0 > 0$ is fixed and if the supremum in (2.1) is taken over $0 < r \le r_0$, only, then the corresponding r_0-depending norm, defined analogously to (2.3), is equivalent to the original norm in $\mathfrak{L}^{p,\omega}(\Omega)$. Moreover, we will use the following theorem (cf. KUFNER, JOHN, FUČIK [12], GIAQUINTA [7] or TROIANIELLO [17]) that describes embedding properties of Campanato spaces.

Theorem 2.1. *Let* $1 \le p_1 \le p_2 < \infty$ *and* $0 \le \omega_1 < n + p_1$, $0 \le \omega_2 < n + p_2$ *such that* $(\omega_1 - n)/p_1 \le (\omega_2 - n)/p_2$. *Then we have* $\mathfrak{L}^{p_2,\omega_2}(\Omega) \hookrightarrow \mathfrak{L}^{p_1,\omega_1}(\Omega)$.

A bijective map Φ between two subsets of $\mathbb{R}^n$ such that Φ and Φ^{-1} are Lipschitz continuous is called *Lipschitz transformation*; $L > 0$ is said to be a Lipschitz constant of a Lipschitz transformation Φ if it is one for both Φ and Φ^{-1}.

In order to formulate further properties of Campanato spaces (equivalence to Morrey and Hölder spaces, multiplier, embedding and transformation properties) we have to suppose certain minimal regularity of the boundary $\partial\Omega$. Hence, let us introduce the following usual terminology (using notation (2.2)):

Definition 2.2. Let $a > 0$. An open set $\Omega \subset \mathbb{R}^n$ is said to have property (a) if for all sufficiently small $r > 0$ we have $\lambda^n(\Omega[x, r]) \ge a r^n$ for all $x \in \Omega$.

The results, summarized in the following theorem, are classical (cf. CAMPANATO [1, 2, 3, 4], GIUSTI [8]).

Theorem 2.3. *Let* $1 \le p < \infty$ *and suppose that* $\Omega \subset \mathbb{R}^n$ *has property* (a). *Then the following holds:*

(i) *Let* $0 \le \omega < n$ *and* $u \in L^p(\Omega)$. *Then* $u \in \mathfrak{L}^{p,\omega}(\Omega)$ *if and only if*

$$\|u\|^p_{L^{p,\omega}(\Omega)} := \sup_{\substack{x\in\Omega\\ r>0}} r^{-\omega} \int_{\Omega[x,r]} |u|^p \, d\lambda^n < \infty, \tag{2.4}$$

and the so called Morrey norm defined by (2.4) *is an equivalent norm on* $\mathfrak{L}^{p,\omega}(\Omega)$.

(ii) *Let* $0 \le \omega < n$. *Then for all* $u \in \mathfrak{L}^{p,\omega}(\Omega)$ *and* $v \in L^\infty(\Omega)$ *the product* uv *belongs to* $\mathfrak{L}^{p,\omega}(\Omega)$, *again, and there exists a constant* $c > 0$ *such that*

$$\|uv\|_{\mathfrak{L}^{p,\omega}(\Omega)} \le c\,\|u\|_{\mathfrak{L}^{p,\omega}(\Omega)}\,\|v\|_{L^\infty(\Omega)} \quad \textit{for all } u \in \mathfrak{L}^{p,\omega}(\Omega),\ v \in L^\infty(\Omega).$$

(iii) *Let* $n < \omega < n + p$. *Then* $\mathfrak{L}^{p,\omega}(\Omega)$ *is isomorphic to the Hölder space* $C^{0,\alpha}(\overline{\Omega})$ *with* $\alpha = (\omega - n)/p$.

(iv) *Let* Ψ *be a Lipschitz transformation from an open neighborhood of* Ω *into* $\mathbb{R}^n$ *and* $0 \le \omega < n+p$. *Then there exists a constant* $c > 0$ *such that for the transformation* $\Psi_* u := u \circ \Psi : \Omega \to \mathbb{R}$ *of a function* $u : \Psi(\Omega) \to \mathbb{R}$ *we have*

$$\begin{aligned} \|\Psi_* u\|_{\mathfrak{L}^{p,\omega}(\Omega)} &\le c\,\|u\|_{\mathfrak{L}^{p,\omega}(\Psi(\Omega))} && \textit{for all } u \in \mathfrak{L}^{p,\omega}(\Psi(\Omega)),\\ \|\Psi_* u\|_{W^{1,p,\omega}(\Omega)} &\le c\,\|u\|_{W^{1,p,\omega}(\Psi(\Omega))} && \textit{for all } u \in W^{1,p,\omega}(\Psi(\Omega)). \end{aligned}$$

2.2. Campanato spaces on Lipschitz hypersurfaces

For the introduction of Campanato spaces on hypersurfaces in $\mathbb{R}^n$ we give the following definition of Lipschitz hypersurfaces in $\mathbb{R}^n$ and sets with Lipschitz boundary:

Definition 2.4. (i) A subset M of $\mathbb{R}^n$ is called *Lipschitz hypersurface* in $\mathbb{R}^n$ if for each $x_0 \in M$ there exist an open neighborhood U of x_0 and a Lipschitz transformation Φ from U onto B such that $\Phi(x_0) = 0$ and $U \cap M = \{x \in U : \Phi_n(x) = 0\}$.

(ii) A bounded subset Ω of $\mathbb{R}^n$ is called *set with Lipschitz boundary* (see Giusti [8]) if for each $x_0 \in \partial\Omega$ there exist an open neighborhood U of x_0 and a Lipschitz transformation Φ from U onto B such that $\Phi(x_0) = 0$ and $\Phi(U \cap \Omega) = E_1$.

Remark 2.5. Every set with Lipschitz boundary is an open subset of $\mathbb{R}^n$ having property (a). Moreover, the following holds: If $\Omega \subset \mathbb{R}^n$ is a bounded open set and $\Upsilon = \mathbb{R}^n \setminus \overline{\Omega}$ its exterior, then Ω is a set with Lipschitz boundary if and only if $\partial\Omega$ is a Lipschitz hypersurface in $\mathbb{R}^n$ with $\partial\Omega = \partial\Upsilon$.

Let $\Omega \subset \mathbb{R}^n$ be a set with Lipschitz boundary and M a relatively open subset of $\partial\Omega$. By $\lambda_{\partial\Omega}$ we denote the $(n-1)$-dimensional Lebesgue measure on the σ-algebra of Lebesgue-measurable subsets of $\partial\Omega$. Note, that on the σ-algebra of Lebesgue measurable subsets of $\partial\Omega$ it is equal to the (suitably normalized) $(n-1)$-dimensional Hausdorff measure (cf. Simon [16] and Evans, Gariepy [6]).

For $1 \le p < \infty$ we write $L^p(M)$ and $L^\infty(M)$ for the Lebesgue spaces of p-integrable functions and bounded measurable maps from M into $\mathbb{R}$, respectively.

For $1 \le p < \infty$, $0 \le \omega < n-1+p$ we denote by $\mathfrak{L}^{p,\omega}(M)$ the *Campanato space*, i.e. the space of all $u \in L^p(M)$ such that

$$[u]^p_{\mathfrak{L}^{p,\omega}(M)} := \sup_{\substack{x\in M\\ r>0}} r^{-\omega} \int_{M[x,r]} |u - u_{M[x,r]}|^p \, \mathrm{d}\lambda_{\partial\Omega} < \infty. \tag{2.5}$$

In (2.5) we used the notation

$$M[x,r] := M \cap B(x,r), \quad u_{M[x,r]} := \frac{1}{\lambda_{\partial\Omega}(M[x,r])} \int_{M[x,r]} u \, \mathrm{d}\lambda_{\partial\Omega}. \tag{2.6}$$

The space $\mathfrak{L}^{p,\omega}(M)$ is a Banach space with the norm

$$\|u\|_{\mathfrak{L}^{p,\omega}(M)} := \left\{ \|u\|^p_{L^p(M)} + [u]^p_{\mathfrak{L}^{p,\omega}(M)} \right\}^{1/p}. \tag{2.7}$$

For the sake of simplicity, for $\omega \le 0$ we will use the notation $\mathfrak{L}^{p,\omega}(M) := L^p(M)$.

If $r_0 > 0$ is fixed and if the supremum in (2.5) is taken over $0 < r \le r_0$, only, then the corresponding r_0-depending norm, defined analogously to (2.7), is equivalent to the original norm in $\mathfrak{L}^{p,\omega}(M)$. Moreover, we have (see Griepentrog [10])

Theorem 2.6. *Let $1 \le p_1 \le p_2 < \infty$ and $0 \le \omega_1 < n-1+p_1$, $0 \le \omega_2 < n-1+p_2$ such that $(\omega_1 - n + 1)/p_1 \le (\omega_2 - n + 1)/p_2$. Then $\mathfrak{L}^{p_2,\omega_2}(M) \hookrightarrow \mathfrak{L}^{p_1,\omega_1}(M)$.*

For the formulation of further properties of Campanato spaces on Lipschitz hypersurfaces (equivalence to Morrey and Hölder spaces, multiplier and embedding properties) we want to suppose property (a) of the boundary part M of $\partial\Omega$. Having in mind notation (2.6), we introduce the following terminology:

Definition 2.7. Let $a > 0$ and $\Omega \subset \mathbb{R}^n$ be a set with Lipschitz boundary. A relatively open subset M of $\partial\Omega$ is said to have property (a) if for all sufficiently small $r > 0$ we have $\lambda_{\partial\Omega}(M[x,r]) \geq ar^{n-1}$ for all $x \in M$.

Remark 2.8. For every set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ has property (a).

As mentioned above, we want to summarize results comparable to Theorem 2.3 but now for Campanato spaces on Lipschitz hypersurfaces (see Griepentrog [10]):

Theorem 2.9. *Let $1 \leq p < \infty$ and Ω be a set with Lipschitz boundary. If the relatively open subset M of $\partial\Omega$ has property* (a), *then the following is true:*

(i) *Let $0 \leq \omega < n-1$ and $u \in L^p(M)$. Then $u \in \mathfrak{L}^{p,\omega}(M)$ if and only if*

$$\|u\|^p_{L^{p,\omega}(M)} := \sup_{\substack{x\in M\\ r>0}} r^{-\omega} \int_{M[x,r]} |u|^p \,\mathrm{d}\lambda_{\partial\Omega} < \infty, \tag{2.8}$$

and the so called Morrey norm defined by (2.8) *is an equivalent norm in $\mathfrak{L}^{p,\omega}(M)$.*

(ii) *Let $0 \leq \omega < n-1$. Then for all $u \in \mathfrak{L}^{p,\omega}(M)$ and $v \in L^\infty(M)$ the product uv belongs to $\mathfrak{L}^{p,\omega}(M)$, again, and there exists a constant $c > 0$ such that*

$$\|uv\|_{\mathfrak{L}^{p,\omega}(M)} \leq c\,\|u\|_{\mathfrak{L}^{p,\omega}(M)}\,\|v\|_{L^\infty(M)} \quad \text{for all } u \in \mathfrak{L}^{p,\omega}(M),\ v \in L^\infty(M).$$

(iii) *Let $n-1 < \omega < n-1+p$. Then $\mathfrak{L}^{p,\omega}(M)$ is isomorphic to the Hölder space $C^{0,\alpha}(\overline{M})$ with $\alpha = (\omega - n + 1)/p$.*

2.3. Regular sets

Let us define the following sets for $x \in \mathbb{R}^n$ and $r > 0$:

$$\begin{aligned}
B_2(x,r) &:= \{\xi \in \mathbb{R}^n : |\xi - x| < r,\ \xi_n - x_n = 0\},\\
E_1(x,r) &:= \{\xi \in \mathbb{R}^n : |\xi - x| < r,\ \xi_n - x_n < 0\},\\
E_2(x,r) &:= \{\xi \in \mathbb{R}^n : |\xi - x| < r,\ \xi_n - x_n \leq 0\},\\
E_3(x,r) &:= \{\xi \in E_2(x,r) : \xi_1 - x_1 > 0 \text{ or } \xi_n - x_n < 0\}.
\end{aligned}$$

Here and later on in the case $x = 0$ and $r = 1$ we shortly write B_2, E_1, E_2 and E_3, respectively. For the treatment of mixed boundary value problems we will use the following terminology of regular sets $G \subset \mathbb{R}^n$ which is equivalent to the original concept introduced by Gröger [11]. Additionally, we collect some frequently used properties of regular sets (cf. Griepentrog, Recke [9]).

Definition 2.10. A bounded subset G of $\mathbb{R}^n$ is called *regular*, if for each $x_0 \in \partial G$ there exist an open neighborhood U of x_0 in $\mathbb{R}^n$ and a Lipschitz transformation Φ from U onto B such that $\Phi(x_0) = 0$ and $\Phi(U \cap G) \in \{E_1, E_2, E_3\}$.

Remark 2.11. Every set with Lipschitz boundary is a regular set. Vice versa, the interior of a regular set is a set with Lipschitz boundary. Moreover, the closure of a regular set is regular, too.

Lemma 2.12. *If $G \subset \mathbb{R}^n$ is a regular set and Ψ a Lipschitz transformation from an open neighborhood of $\overline{G}$ onto another open subset of $\mathbb{R}^n$, then $\Psi(G)$ is regular.*

Lemma 2.13. *For every regular subset G of $\mathbb{R}^n$ there exists an atlas of charts $(\Phi_1, U_1), \ldots, (\Phi_m, U_m)$ of the following type: There exist points $x_1, \ldots, x_m \in \overline{G}$, open neighborhoods $U_1, \ldots, U_m$ of $x_1, \ldots, x_m$ in $\mathbb{R}^n$, and Lipschitz transformations $\Phi_1, \ldots, \Phi_m$ from $U_1, \ldots, U_m$ into $\mathbb{R}^n$, respectively, such that*

$$\partial G \subset \bigcup_{j \in I} U_j, \quad \bigcup_{j \in I_0} \overline{U_j} \subset G^\circ, \quad \overline{G} \subset \bigcup_{j=1}^{m} U_j, \tag{2.9}$$

with $I_0 = \big\{j \in \{1, \ldots, m\} : x_j \in G^\circ\big\}$, $I = \big\{j \in \{1, \ldots, m\} : x_j \in \partial G\big\}$ and

$$\Phi_j(x_j) = 0, \quad \Phi_j(U_j) = B, \quad \Phi_j(U_j \cap G) \in \{B, E_1, E_2, E_3\} \tag{2.10}$$

for all $j \in \{1, \ldots, m\}$. The subfamily $\big\{(\Phi_j, U_j) : j \in I\big\}$ is an atlas of ∂G.

2.4. Sobolev–Campanato spaces on regular sets

Throughout this section we will assume, that $G \subset \mathbb{R}^n$ is a regular set, $U \subset \mathbb{R}^n$ is a relatively open subset of G and, finally, that $V \subset \mathbb{R}^n$ is a relatively open subset of U. Before considering Sobolev–Campanato spaces on regular sets we want to present embedding and trace properties of Sobolev–Campanato spaces $W^{1,2,\omega}(\Omega)$ on sets with Lipschitz boundary (see Giusti [8], Griepentrog, Recke [9]):

Theorem 2.14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and M be a relatively open subset of $\partial\Omega$. Then, for $0 \le \omega < n$ the following is true:*
(i) *$W^{1,2}(\Omega)$ is continuously embedded into $L^{2n/(n-2)}(\Omega)$.*
(ii) *$W^{1,2,\omega}(\Omega)$ is continuously embedded into $\mathfrak{L}^{2,\omega+2}(\Omega)$.*
(iii) *The trace operator γ_M maps $W^{1,2}(\Omega)$ continuously into $L^{2(n-1)/(n-2)}(M)$.*
(iv) *The trace operator γ_M maps $W^{1,2,\omega}(\Omega)$ continuously into $\mathfrak{L}^{2,\omega+1}(M)$.*

In the sequel we will work with the following notation, which is usual in the theory of mixed boundary value problems (cf., e.g., Troianiello [17], Gröger [11]). By $W_0^{1,2}(U)$ we denote the closure in $W^{1,2}(U^\circ)$ of the set

$$C_0^\infty(U) := \big\{u|_{U^\circ} : u \in C_0^\infty(\mathbb{R}^n),\ \mathrm{supp}(u) \cap (\overline{U} \setminus U) = \varnothing\big\}. \tag{2.11}$$

Furthermore, for $0 \leq \omega < n+2$ we consider closed subspaces of the Sobolev–Campanato spaces defined as

$$W_0^{1,2,\omega}(U) := W_0^{1,2}(U) \cap W^{1,2,\omega}(U^\circ)$$

and equipped with the norm of $W^{1,2,\omega}(U^\circ)$. For the sake of completeness we write down the following principles concerning extension, transformation, and restriction of Sobolev space functions (see Griepentrog, Recke [9] and Griepentrog [10]):

Lemma 2.15. *The zero extension map R_U on $W_0^{1,2}(V)$ defined as*

$$R_U u := \begin{cases} u & \lambda^n\text{-almost everywhere on } V^\circ, \\ 0 & \lambda^n\text{-almost everywhere on } U^\circ \setminus V^\circ, \end{cases} \qquad u \in W_0^{1,2}(V),$$

is a bounded linear operator from $W_0^{1,2}(V)$ into $W_0^{1,2}(U)$. Moreover, we have

$$\|R_U u\|_{W_0^{1,2}(U)} = \|u\|_{W_0^{1,2}(V)} \quad \text{for all } u \in W_0^{1,2}(V).$$

Lemma 2.16. *If Ψ is a Lipschitz transformation of an open neighborhood of $\overline{G}$ onto another open subset of $\mathbb{R}^n$, then u belongs to $W_0^{1,2}(\Psi(U))$ if and only if $\Psi_* u$ is an element of $W_0^{1,2}(U)$, and*

$$\Psi_* R_{\Psi(U)} u = R_U \Psi_* u \quad \text{for all } u \in W_0^{1,2}(\Psi(U)).$$

Let $x \in B$, $r > 0$, and $k \in \{1,2\}$. Furthermore, let $P : B \to E_2$ be the projection defined as $Px := (\hat{x}, -|x_n|) \in E_2$ for $x = (\hat{x}, x_n) \in B$. Finally, using the notation $D(x,r) := B(x,r) \cup B(Px,r)$, for $u : B \cap D(x,r) \to \mathbb{R}$ we define the odd part $T_1(x,r)u : E_1[Px,r] \to \mathbb{R}$ and the even part $T_2(x,r)u : E_1[Px,r] \to \mathbb{R}$ of $2u$ by

$$(T_k(x,r)u)(y) := u(y) + (-1)^k u(\hat{y}, -y_n), \quad y \in E_1[Px,r].$$

respectively. Then, we have $2u = T_1(x,r)u + T_2(x,r)u$ and

Lemma 2.17. *For $x \in B$, $r > 0$, and $k \in \{1,2\}$ the operator $T_k(x,r)$ maps $W_0^{1,2}(B \cap D(x,r))$ continuously into $W_0^{1,2}(E_k[Px,r])$, and*

$$T_k R_B u = R_{E_k} T_k(x,r) u \quad \text{for all } u \in W_0^{1,2}(B \cap D(x,r)).$$

Let $k \in \{1,2\}$. For $u : E_1 \to \mathbb{R}$ we define the antireflection $R_1 u : B \to \mathbb{R}$ and the reflection $R_2 u : B \to \mathbb{R}$ onto the unit ball B by

$$(R_k u)(x) := \begin{cases} u(x) & \text{for } x \in E_1, \\ (-1)^k u(\hat{x}, -x_n) & \text{for } x \in B \setminus E_1, \end{cases}$$

respectively. Then, we have the following statement

Lemma 2.18. *For $0 \le \omega < n$ and $k \in \{1,2\}$ the operator R_k maps $W_0^{1,2,\omega}(E_k)$ continuously into $W_0^{1,2,\omega}(B)$. Moreover, R_2 maps $W^{1,2,\omega}(E_1)$ continuously into $W^{1,2,\omega}(B)$, and*

$$\|R_1u\|^2_{W^{1,2,\omega}(B)} \le 2\,\|u\|^2_{W^{1,2,\omega}(E_1)} \le 2\,\|R_1u\|^2_{W^{1,2,\omega}(B)} \quad \text{for all } u \in W_0^{1,2,\omega}(E_1),$$
$$\|R_2u\|^2_{W^{1,2,\omega}(B)} \le 2\,\|u\|^2_{W^{1,2,\omega}(E_1)} \le 2\,\|R_2u\|^2_{W^{1,2,\omega}(B)} \quad \text{for all } u \in W^{1,2,\omega}(E_1).$$

In the sequel we also need the generalization of the above reflection operations to vector and matrix valued functions. For $f : E_1 \to \mathbb{R}^n$ we define the antireflection $R_1 f : B \to \mathbb{R}^n$ and the reflection $R_2 f : B \to \mathbb{R}^n$ by

$$(R_1f)_j := R_1f_j \quad \text{for } j \in \{1,\dots,n-1\}, \quad \text{and} \quad (R_1f)_n := R_2f_n,$$
$$(R_2f)_j := R_2f_j \quad \text{for } j \in \{1,\dots,n-1\}, \quad \text{and} \quad (R_2f)_n := R_1f_n.$$

Let $0 < \varepsilon \le 1$ be a real constant. By $\mathfrak{S}(n)$ and $\mathfrak{S}(\varepsilon, n)$ we denote the spaces of all real symmetric $(n\times n)$-matrices and all real positive definite $(n\times n)$-matrices having the spectrum in the interval $[\,\varepsilon, 1/\varepsilon\,]$, respectively. For $A : E_1 \to \mathfrak{S}(n)$ we define the reflection $R_2A : B \to \mathfrak{S}(n)$ by

$$(R_2A)\mathfrak{e}_i := R_2(A\mathfrak{e}_i) \quad \text{for } i \in \{1,\dots,n-1\}, \quad \text{and} \quad (R_2A)\mathfrak{e}_n := R_1(A\mathfrak{e}_n).$$

Notice, that for $A : E_1 \to \mathfrak{S}(\varepsilon, n)$ we have $R_2A : B \to \mathfrak{S}(\varepsilon, n)$.

3. Campanato spaces of functionals

Throughout this section we assume, that $G \subset \mathbb{R}^n$ is a regular set, $U \subset \mathbb{R}^n$ is a relatively open subset of G and, finally, that $V \subset \mathbb{R}^n$ is a relatively open subset of U.

3.1. Definition

Let $W^{-1,2}(U)$ be the dual space to $W_0^{1,2}(U)$ and $\langle\ ,\ \rangle_U$ the dual pairing between these spaces. We define the norm of an element $F \in W^{-1,2}(U)$ by

$$\|F\|_{W^{-1,2}(U)} := \sup\left\{|\langle F, w\rangle_U| : w \in W_0^{1,2}(U),\ \|w\|_{W_0^{1,2}(U)} \le 1\right\}.$$

To localize a functional $F \in W^{-1,2}(U)$ we do the following: We define the mapping $F \mapsto F|_V$ from $W^{-1,2}(U)$ into $W^{-1,2}(V)$ as the adjoint operator to the extension map $R_U : W_0^{1,2}(V) \to W_0^{1,2}(U)$, that means,

$$\langle F|_V, w\rangle_V := \langle F, R_U w\rangle_U, \quad w \in W_0^{1,2}(V).$$

Obviously, the property of the extension operator R_U (see Lemma 2.15) yields

$$\|F|_V\|_{W^{-1,2}(V)} \le \|F\|_{W^{-1,2}(U)} \quad \text{for all } F \in W^{-1,2}(U).$$

Moreover, we have the following norm identity

$$\|F\|_{W^{-1,2}(U)} = \sup_{\substack{x\in U^\circ \\ r>0}} \left\|F|_{U[x,r]}\right\|_{W^{-1,2}(U[x,r])} \quad \text{for all } F \in W^{-1,2}(U).$$

Now, we construct Campanato spaces of functionals as subspaces of $W^{-1,2}(U)$ by the following modification of the $W^{-1,2}(U)$-norm (cf. Rakotoson [13, 14]).

Definition 3.1. Let $0 \le \omega < n$ be a real constant. A functional F from $W^{-1,2}(U)$ should belong to the *Campanato space* $Y^{-1,2,\omega}(U)$, if and only if the supremum

$$\|F\|^2_{Y^{-1,2,\omega}(U)} := \sup_{\substack{x\in U^\circ\\ r>0}} r^{-\omega} \left\|F|_{U[x,r]}\right\|^2_{W^{-1,2}(U[x,r])} \tag{3.1}$$

has a finite value. In that case we define the norm of $F \in Y^{-1,2,\omega}(U)$ by (3.1).

Remark 3.2. If $r_0 > 0$ is a given radius and if we take the supremum in the definition (3.1) for $0 < r \le r_0$, only, then the corresponding r_0-dependent norm, defined analogously to (3.1), is an equivalent norm on $Y^{-1,2,\omega}(U)$.

Remark 3.3. Note that $r^{-\omega} \le r_0^{\sigma-\omega} r^{-\sigma}$ if $0 \le \omega \le \sigma < n$, $r_0 > 0$, $0 < r \le r_0$. This yields the continuous embedding $Y^{-1,2,\sigma}(U) \hookrightarrow Y^{-1,2,\omega}(U)$.

Remark 3.4. The spaces $Y^{-1,2,\omega}(U)$ are Banach spaces for $0 \le \omega < n$: To prove the completeness of the normed linear space $Y^{-1,2,\omega}(U)$ let $\{F_\alpha\}_{\alpha\in\mathbb{N}}$ be a Cauchy sequence in $Y^{-1,2,\omega}(U)$. Because of the embedding of $Y^{-1,2,\omega}(U)$ in $W^{-1,2}(U)$ the sequence $\{F_\alpha\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in $W^{-1,2}(U)$. Hence, it converges in $W^{-1,2}(U)$ to a functional $F \in W^{-1,2}(U)$. If we fix $\delta > 0$, we can choose $\alpha_0(\delta) \in \mathbb{N}$ such that

$$\|F_{\alpha+\beta} - F_\alpha\|_{Y^{-1,2,\omega}(U)} \le \delta \quad \text{for all } \alpha, \beta \in \mathbb{N} \text{ with } \alpha \ge \alpha_0(\delta).$$

For all $x \in U^\circ$ and $r > 0$ we get

$$r^{-\omega} \left\|(F - F_\alpha)|_{U[x,r]}\right\|^2_{W^{-1,2}(U[x,r])} \le 2r^{-\omega} \left\|(F - F_{\alpha+\beta})|_{U[x,r]}\right\|^2_{W^{-1,2}(U[x,r])} + 2\delta^2.$$

Letting $\beta \to \infty$ and taking the supremum for all $x \in U^\circ$ and $r > 0$ we arrive at the sought-for result:

$$\|F - F_\alpha\|^2_{Y^{-1,2,\omega}(U)} \le 2\delta^2 \quad \text{for all } \alpha \in \mathbb{N} \text{ with } \alpha \ge \alpha_0(\delta).$$

3.2. Invariance principles

We are going to consider several bounded linear operations on the above defined Campanato spaces of functionals.

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ and $0 \le \omega < n$. Now, for $F \in W^{-1,2}(U)$ we define by

$$\langle \chi F, w\rangle_U := \langle F, w\chi\rangle_U, \quad w \in W_0^{1,2}(U),$$

a functional $\chi F \in W^{-1,2}(U)$. There exists a real constant $c = c(\chi) > 0$ such that

$$\|\chi F\|_{W^{-1,2}(U)} \le c\, \|F\|_{W^{-1,2}(U)} \quad \text{for all } F \in W^{-1,2}(U).$$

Lemma 3.5. *Let $\chi \in C_0^\infty(\mathbb{R}^n)$ and $0 \le \omega < n$. Then, $F \mapsto \chi F$ is a bounded linear map from $Y^{-1,2,\omega}(U)$ into $Y^{-1,2,\omega}(U)$.*

Proof. If $F \in Y^{-1,2,\omega}(U)$, then by definition we get for all $x \in U^\circ$, $r > 0$

$$\left\|(\chi F)|_{U[x,r]}\right\|_{W^{-1,2}(U[x,r])} = \left\|\chi F|_{U[x,r]}\right\|_{W^{-1,2}(U[x,r])} \leq c\left\|F|_{U[x,r]}\right\|_{W^{-1,2}(U[x,r])},$$

where $c = c(\chi) > 0$ is a real constant. This proves the desired result. □

Lemma 3.6. *If $0 \leq \omega < n$, then $F \mapsto F|_V$ defines a bounded linear map from $Y^{-1,2,\omega}(U)$ into $Y^{-1,2,\omega}(V)$.*

Proof. Let $F \in Y^{-1,2,\omega}(U)$. Then, for all $x \in V^\circ$, $r > 0$ by definition we have

$$\left\|F|_{V[x,r]}\right\|_{W^{-1,2}(V[x,r])} \leq \left\|F|_{U[x,r]}\right\|_{W^{-1,2}(U[x,r])},$$

which proves the desired result. □

Another useful tool for our regularity considerations is the extension principle for functionals by reflection and antireflection, respectively. Let $x \in B$ and $r > 0$. Having in mind the continuity of the operators

$$T_k(x,r) : W_0^{1,2}(B \cap D(x,r)) \to W_0^{1,2}(E_k[Px,r]),$$

and especially the continuity of $T_k : W_0^{1,2}(B) \to W_0^{1,2}(E_k)$ for $k \in \{1,2\}$ (Lemma 2.17) we construct the mapping $F_k \mapsto R_k F_k$ from $W^{-1,2}(E_k)$ into $W^{-1,2}(B)$ as the adjoint operator of $T_k : W_0^{1,2}(B) \to W_0^{1,2}(E_k)$, that means,

$$\langle R_k F_k, w\rangle_B := \langle F_k, T_k w\rangle_{E_k}, \quad w \in W_0^{1,2}(B).$$

Because of the properties of the operators T_k (see Lemma 2.18) it follows

$$\|R_k F_k\|_{W^{-1,2}(B)} \leq \sqrt{2}\,\|F_k\|_{W^{-1,2}(E_k)} \quad \text{for all } F_k \in W^{-1,2}(E_k).$$

Lemma 3.7. *Let $k \in \{1,2\}$ and $0 \leq \omega < n$. Then, $F_k \mapsto R_k F_k$ is a bounded linear map from $Y^{-1,2,\omega}(E_k)$ into $Y^{-1,2,\omega}(B)$.*

Proof. Let $k \in \{1,2\}$ be an index and F_k an element of $Y^{-1,2,\omega}(E_k)$. Then, we get for all $x \in B$, $r > 0$ and $w \in W_0^{1,2}(B[x,r])$ the relation

$$\begin{aligned}\left|\langle (R_k F_k)|_{B[x,r]}, w\rangle_{B[x,r]}\right| &= \left|\langle R_k F_k, R_B w\rangle_B\right| = \left|\langle F_k, T_k R_B w\rangle_{E_k}\right| \\ &= \left|\langle F_k, R_{E_k} T_k(x,r) R_{B\cap D(x,r)} w\rangle_{E_k}\right| \\ &= \left|\langle F_k|_{E_k[Px,r]}, T_k(x,r) R_{B\cap D(x,r)} w\rangle_{E_k[Px,r]}\right|\end{aligned}$$

by the properties of $T_k(x,r)$, T_k (see Lemma 2.17) and the extension operators. Hence,

$$\left\|(R_k F_k)|_{B[x,r]}\right\|_{W^{-1,2}(B[x,r])} \leq \sqrt{2}\,\|F_k\|_{W^{-1,2}(E_k[Px,r])},$$

which proves the desired result. □

Next we will see, how the invariance of Sobolev spaces with respect to Lipschitz transformations carries over to our new scale of Campanato spaces of functionals.

Let Ψ be a Lipschitz transformation from an open neighborhood of $\overline{G}$ onto another open subset of $\mathbb{R}^n$. Then, $\Psi(G) \subset \mathbb{R}^n$ is a regular set, too (see Lemma 2.12). Now, we are able to define the mapping $F \mapsto \Psi^* F$ from $W^{-1,2}(U)$ into $W^{-1,2}(\Psi(U))$ as the adjoint operator of $\Psi_* : W_0^{1,2}(\Psi(U)) \to W_0^{1,2}(U)$, that means,

$$\langle \Psi^* F, w\rangle_{\Psi(U)} := \langle F, \Psi_* w\rangle_U, \quad w \in W_0^{1,2}(\Psi(U)).$$

By the transformation invariance for Sobolev spaces (Theorem 2.3 and Lemma 2.16) there exists a positive constant $c = c(\Psi) > 0$ such that

$$\|\Psi^* F\|_{W^{-1,2}(\Psi(U))} \le c\,\|F\|_{W^{-1,2}(U)} \quad \text{for all } F \in W^{-1,2}(U).$$

Lemma 3.8. *Let Ψ be a Lipschitz transformation from an open neighborhood of $\overline{G}$ onto another open subset of $\mathbb{R}^n$ and $0 \le \omega < n$. Then, $F \mapsto \Psi^* F$ defines a bounded linear map from $Y^{-1,2,\omega}(U)$ into $Y^{-1,2,\omega}(\Psi(U))$.*

Proof. Let $L \ge 1$ be a Lipschitz constant for the transformation Ψ and $V = \Psi(U)$. We choose $r_0 > 0$ such that for all $y \in V^\circ$, $0 < r \le r_0$ we have the inclusion

$$\Psi^{-1}(B(y, r)) \subset B(x, Lr) \quad \text{for } x = \Psi^{-1}(y).$$

For all $y \in V^\circ$, $0 < r \le r_0$ and $w \in W_0^{1,2}(V[y, r])$ we get the relation

$$\begin{aligned} \big|\langle (\Psi^* F)|_{V[y,r]}, w\rangle_{V[y,r]}\big| &= \big|\langle \Psi^* F, R_V w\rangle_V\big| = \big|\langle F, \Psi_* R_V w\rangle_U\big| \\ &= \big|\langle F, R_U \Psi_* w\rangle_U\big| = \big|\langle F|_{U[x,Lr]}, R_{U[x,Lr]} \Psi_* w\rangle_{U[x,Lr]}\big|. \end{aligned}$$

Here we have used the properties of the extension operators with respect to the transformation Ψ (Theorem 2.3 and Lemma 2.16) and the above inclusion, respectively. Hence, there exists a constant $c = c(\Psi) > 0$ such that

$$\big\|(\Psi^* F)|_{V[y,r]}\big\|_{W^{-1,2}(V[y,r])} \le c\,\big\|F|_{U[x,Lr]}\big\|_{W^{-1,2}(U[x,Lr])},$$

which proves the result. □

3.3. Examples

Next, we consider examples of functionals from $Y^{-1,2,\omega}(G)$, which are interesting for a broad class of applications.

Theorem 3.9. *Let M be a relatively open subset of ∂G having property* (a). *Then, for all $0 \le \omega < n$ the map*

$$(f, g, h) \mapsto F(f, g, h),$$

defined by

$$\langle F(f, g, h), w\rangle_G := \int_G (f \cdot \nabla w + g w)\,\mathrm{d}\lambda^n + \int_M h \gamma_M(w)\,\mathrm{d}\lambda_{\partial G}, \quad w \in W_0^{1,2}(G),$$

is a bounded linear operator from

$$\mathfrak{L}^{2,\omega}(G^\circ;\mathbb{R}^n) \times \mathfrak{L}^{2n/(n+2),\,\omega n/(n+2)}(G^\circ) \times \mathfrak{L}^{2(n-1)/n,\,\omega(n-1)/n}(M) \quad \textit{into} \quad Y^{-1,2,\omega}(G).$$

Proof. Let $\{(\Phi_1, U_1), \dots, (\Phi_m, U_m)\}$ be an atlas of G satisfying (2.9) and (2.10). Furthermore, let $L \geq 1$ be a common Lipschitz constant for all transformations. Then, there exists a radius $r_0 > 0$ such that for all $x \in G^\circ$ the open ball $B(x, r_0)$ is included in one of the neighborhoods $U_1, \dots, U_m$. We consider the decomposition of the set $J = \{1, \dots, m\}$ into the index sets

$$I_0 = \big\{j \in J : U_j \subset\subset G^\circ\big\} \quad \text{and} \quad I = \big\{j \in J : U_j \cap \partial G \neq \varnothing\big\}.$$

(i) Obviously, we get $F(f, 0, 0) \in W^{-1,2}(G)$ by the estimate

$$|\langle F(f,0,0), w\rangle_G| \leq \|f\|_{L^2(G^\circ;\mathbb{R}^n)} \,\|\nabla w\|_{L^2(G^\circ;\mathbb{R}^n)} \quad \text{for all } w \in W_0^{1,2}(G).$$

Moreover, for all $x \in G^\circ$, $r > 0$ and $w \in W_0^{1,2}(G[x,r])$ the following holds

$$\big|\langle F(f,0,0)|_{G[x,r]}, w\rangle_{G[x,r]}\big| \leq \|f\|_{L^2(G^\circ[x,r];\mathbb{R}^n)} \,\|\nabla w\|_{L^2(G^\circ[x,r];\mathbb{R}^n)}.$$

Hence, we get

$$\big\|F(f,0,0)|_{G[x,r]}\big\|_{W^{-1,2}(G[x,r])} \leq \|f\|_{L^2(G^\circ[x,r];\mathbb{R}^n)}. \tag{3.2}$$

(ii) Because of $W_0^{1,2}(G) \hookrightarrow L^{2n/(n-2)}(G^\circ)$ it follows $F(0, g, 0) \in W^{-1,2}(G)$ by

$$|\langle F(0,g,0), w\rangle_G| \leq \|g\|_{L^{2n/(n+2)}(G^\circ)} \,\|w\|_{L^{2n/(n-2)}(G^\circ)} \quad \text{for all } w \in W_0^{1,2}(G).$$

Moreover, for all $x \in G^\circ$, $0 < r \leq r_0$ and $w \in W_0^{1,2}(G[x,r])$ we have the relation

$$\big|\langle F(0,g,0)|_{G[x,r]}, w\rangle_{G[x,r]}\big| \leq \|g\|_{L^{2n/(n+2)}(G^\circ[x,r])} \,\|w\|_{L^{2n/(n-2)}(G^\circ[x,r])}.$$

Case $B(x, r_0) \subset U_j$ *for a certain index* $j \in I_0$: Then, for all $0 < r \leq r_0$ we have $B(x, r) \subset G^\circ$ and for all $w \in W_0^{1,2}(G[x,r])$ it follows

$$\|w\|_{L^{2n/(n-2)}(G^\circ[x,r])} \leq c_1 \,\|\nabla w\|_{L^2(G^\circ[x,r];\mathbb{R}^n)},$$

where $c_1 > 0$ is a positive constant depending only on n.

Case $B(x, r_0) \subset U_j$ *for a certain index* $j \in I$: Introducing the notation

$$z = \Phi_j(x) \in E_1 \quad \text{and} \quad V(r) = \Phi_j^{-1}(B(z, Lr)),$$

we get for all $0 < r \leq r_0/L^2$ the inclusions

$$\Phi_j(G[x,r]) \subset E_2[z, Lr] \quad \text{and} \quad G[x,r] \subset G \cap V(r).$$

Hence, for all $0 < r \leq r_0/L^2$ and $w \in W_0^{1,2}(G[x,r])$ the following holds

$$w_j = (R_{G\cap V(r)} w) \circ \Phi_j^{-1} \in W_0^{1,2}(E_2[z, Lr])$$

with the estimate

$$\begin{aligned}\|w\|_{L^{2n/(n-2)}(G^\circ[x,r])} &\le c_2\,\|w_j\|_{L^{2n/(n-2)}(E_1[z,Lr])}\\ &\le c_3\,\|\nabla w_j\|_{L^2(E_1[z,Lr];\mathbb{R}^n)} \le c_4\,\|\nabla w\|_{L^2(G^\circ[x,r];\mathbb{R}^n)},\end{aligned}$$

where $c_2, c_3, c_4 > 0$ depend only on n and L. Summing up we get the existence of a constant $c_5 = c_5(n, L) > 0$, such that for all $x \in G^\circ$, $0 < r \le r_0/L^2$ and $w \in W_0^{1,2}(G[x,r])$ we have

$$\left|\langle F(0,g,0)|_{G[x,r]}, w\rangle_{G[x,r]}\right| \le c_5\,\|g\|_{L^{2n/(n+2)}(G^\circ[x,r])}\,\|\nabla w\|_{L^2(G^\circ[x,r];\mathbb{R}^n)},$$

hence,

$$\left\|F(0,g,0)|_{G[x,r]}\right\|_{W^{-1,2}(G[x,r])} \le c_5\,\|g\|_{L^{2n/(n+2)}(G^\circ[x,r])}. \tag{3.3}$$

(iii) Due to the regularity of the set $G \subset \mathbb{R}^n$ and Remark 2.8 both M and ∂G have property (a). Because of the equivalence between Morrey and Campanato norm for parameters $0 \le \sigma < n-1$ (see Theorem 2.9) we can extend $h \in \mathfrak{L}^{2(n-1)/n,\,\omega(n-1)/n}(M)$ by zero to a function which belongs to $\mathfrak{L}^{2(n-1)/n,\,\omega(n-1)/n}(\partial G)$. Hence, it suffices to consider only the case $M = \partial G$. The continuity of the trace operator γ_M from $W_0^{1,2}(G)$ into $L^{2(n-1)/(n-2)}(M)$ yields $F(0,0,h) \in W^{-1,2}(G)$:

$$|\langle F(0,0,h), w\rangle_G| \le \|h\|_{L^{2(n-1)/n}(M)}\,\|\gamma_M(w)\|_{L^{2(n-1)/(n-2)}(M)} \quad \text{for all } w \in W_0^{1,2}(G).$$

Moreover, for all $x \in G^\circ$, $0 < r \le r_0$ and $w \in W_0^{1,2}(G[x,r])$ we have the relation

$$\left|\langle F(0,0,h)|_{G[x,r]}, w\rangle_{G[x,r]}\right| \le \|h\|_{L^{2(n-1)/n}(M[x,r])}\|\gamma_M(w)\|_{L^{2(n-1)/(n-2)}(M[x,r])}.$$

To prove further estimates it is sufficient to consider $x \in G^\circ$ and $0 < r \le r_0$ such that $M[x,r]$ is nonempty. For such points $x \in G^\circ$ there exists an index $j \in I$ with the property $B(x, r_0) \subset U_j$. Using again the notation

$$z = \Phi_j(x) \in E_1 \quad\text{and}\quad V(r) = \Phi_j^{-1}(B(z, Lr)),$$

we get for all $0 < r \le r_0/L^2$ with $M[x,r] \ne \varnothing$ the inclusions

$$\Phi_j(G[x,r]) \subset E_2[z, Lr] \quad\text{and}\quad G[x,r] \subset G \cap V(r).$$

For all $0 < r \le r_0/L^2$ with $M[x,r] \ne \varnothing$ and all $w \in W_0^{1,2}(G[x,r])$ we have

$$w_j = (R_{G\cap V(r)} w) \circ \Phi_j^{-1} \in W_0^{1,2}(E_2[z, Lr])$$

and the relation

$$\begin{aligned}\|\gamma_M(w)\|_{L^{2(n-1)/(n-2)}(M[x,r])} &\le c_6\left\|\gamma_{B_2[z,Lr]}(w_j)\right\|_{L^{2(n-1)/(n-2)}(B_2[z,Lr])}\\ &\le c_7\,\|\nabla w_j\|_{L^2(E_1[z,Lr];\mathbb{R}^n)} \le c_8\,\|\nabla w\|_{L^2(G^\circ[x,r])},\end{aligned}$$

where $c_6, c_7, c_8 > 0$ depend only on n and L. Hence, we have proved the estimate

$$\left|\langle F(0,0,h)|_{G[x,r]}, w\rangle_{G[x,r]}\right| \le c_8\,\|h\|_{L^{2(n-1)/n}(M[x,r])}\,\|\nabla w\|_{L^2(G^\circ[x,r];\mathbb{R}^n)}$$

for all $x \in G^\circ$, $0 < r \le r_0/L^2$ and $w \in W_0^{1,2}(G[x,r])$, in other words,

$$\big\|F(0,0,h)|_{G[x,r]}\big\|_{W^{-1,2}(G[x,r])} \le c_8 \,\|h\|_{L^{2(n-1)/n}(M[x,r])}. \tag{3.4}$$

Using Theorem 2.3 and Theorem 2.9 from (3.2), (3.3) and (3.4) the result follows. □

Remark 3.10. To underline the relevance of the preceding theorem we want to clarify the connections to usual Lebesgue and Campanato spaces.

(i) Note, that for $p = 2n/(n-\omega)$ and $0 \le \omega < n$ we have the continuous embedding

$$L^p(G^\circ) \hookrightarrow \mathfrak{L}^{2,\omega}(G^\circ).$$

(ii) Furthermore, for $p = 2n/(n-\omega+2)$ and $0 \le \omega < n+2$ we can state

$$L^p(G^\circ) \hookrightarrow \mathfrak{L}^{2n/(n+2),\omega n/(n+2)}(G^\circ), \quad \mathfrak{L}^{2,\omega-2}(G^\circ) \hookrightarrow \mathfrak{L}^{2n/(n+2),\omega n/(n+2)}(G^\circ).$$

(iii) Additionally, for $p = 2(n-1)/(n-\omega)$ and $0 \le \omega < n$ the following holds

$$L^p(M) \hookrightarrow \mathfrak{L}^{2(n-1)/n,\omega(n-1)/n}(M), \quad \mathfrak{L}^{2,\omega-1}(M) \hookrightarrow \mathfrak{L}^{2(n-1)/n,\omega(n-1)/n}(M).$$

(iv) Let $0 \le \omega < n$. We define the subspace $W^{-1,2,\omega}(G)$ of $Y^{-1,2,\omega}(G)$ as

$$W^{-1,2,\omega}(G) := \big\{F(f,g,0) \in W^{-1,2}(G) : f \in \mathfrak{L}^{2,\omega}(G^\circ;\mathbb{R}^n),\ g \in \mathfrak{L}^{2,\omega-2}(G^\circ)\big\},$$

and the norm of an element $F \in W^{-1,2,\omega}(G)$ as the infimum over all sums

$$\|f\|_{\mathfrak{L}^{2,\omega}(G^\circ;\mathbb{R}^n)} + \|g\|_{\mathfrak{L}^{2,\omega-2}(G^\circ)}, \quad f \in \mathfrak{L}^{2,\omega}(G^\circ;\mathbb{R}^n),\ g \in \mathfrak{L}^{2,\omega-2}(G^\circ),\ F = F(f,g,0).$$

Then, $W^{-1,2,\omega}(G)$ is continuously embedded into $Y^{-1,2,\omega}(G)$.

4. Regularity theory

Let $G \subset \mathbb{R}^n$ be a regular set and $0 < \varepsilon \le 1$. Remembering the notation $\mathfrak{S}(\varepsilon,n)$ for the space of real positive definite $(n\times n)$-matrices having the spectrum in the interval $[\,\varepsilon, 1/\varepsilon\,]$, the Lax-Milgram Lemma yields that for all coefficients (A,d) which belong to $L^\infty(G^\circ;\mathfrak{S}(\varepsilon,n)\times\mathfrak{S}(\varepsilon,1))$ the operator $L_G(A,d)$ defined as

$$\langle L_G(A,d)u, w\rangle_G := \int_G (A\nabla u\cdot\nabla w + duw)\,\mathrm{d}\lambda^n, \quad u,w \in W_0^{1,2}(G),$$

is an isomorphism from $W_0^{1,2}(G)$ onto $W^{-1,2}(G)$. Hence, the mixed boundary value problem $L_G(A,d)u = F$ has a uniquely defined solution $u \in W_0^{1,2}(G)$ for every functional $F \in W^{-1,2}(G)$. In Recke [15] and Griepentrog, Recke [9] was proved the following regularity theorem:

Theorem 4.1. *Under the above assumptions there exists a constant $\overline{\mu}(\varepsilon,G) > n-2$ such that for all $0 \le \mu < \overline{\mu}(\varepsilon,G)$ the operator $L_G(A,d)$ is an isomorphism from $W_0^{1,2,\mu}(G)$ onto $W^{-1,2,\mu}(G)$.*

Applying Theorem 3.9 the image of $W_0^{1,2,\omega}(G)$ under the operator $L_G(A,d)$ is continuously embedded into $W^{-1,2,\omega}(G) \hookrightarrow Y^{-1,2,\omega}(G)$ for all $0 \le \omega < n$. In this section we will prove the existence of a constant $\overline{\omega}(\varepsilon, G) > n-2$ such that for all $0 \le \omega < \overline{\omega}(\varepsilon, G)$ the operator $L_G(A,d)$ has the isomorphism property from $W_0^{1,2,\omega}(G)$ onto $Y^{-1,2,\omega}(G)$. Hence, we will get the desired coincidence of the spaces $W^{-1,2,\omega}(G)$ and $Y^{-1,2,\omega}(G)$ for all $0 \le \omega < \overline{\omega}(\varepsilon, G)$ as conjectured by Rakotoson [13, 14], where the result was shown for the case $G = G^\circ$, $n-2 < \omega < \overline{\omega}(\varepsilon, G)$.

4.1. Admissible sets

We will formulate and prove our regularity results using the concept of admissibility of regular sets which is essentially due to Recke [15].

Definition 4.2. Let $G \subset \mathbb{R}^n$ be a regular set. A regular subset G_0 of G is called *admissible with respect to* G, if and only if for every $0 < \varepsilon \le 1$ there exists $\overline{\omega} > n-2$ such that for all $0 \le \omega < \overline{\omega}$ one can find a positive constant $c_1 = c_1(n, \varepsilon, \omega, G, G_0) > 0$ such that for all coefficients $(A,d) \in L^\infty(G^\circ; \mathfrak{S}(\varepsilon, n) \times \mathfrak{S}(\varepsilon, 1))$ and $F \in Y^{-1,2,\omega}(G)$ the solution $u \in W_0^{1,2}(G)$ of $L_G(A,d)\, u = F$ satisfies $\nabla u|_{G_0^\circ} \in \mathfrak{L}^{2,\omega}(G_0^\circ; \mathbb{R}^n)$ and, additionally,

$$\|\nabla u\|_{\mathfrak{L}^{2,\omega}(G_0^\circ;\mathbb{R}^n)} \le c_1 \left\{ \|F\|_{Y^{-1,2,\omega}(G)} + \|u\|_{W_0^{1,2}(G)} \right\}.$$

If the set G is admissible with respect to itself, then we will call it *admissible.* In that case we denote by $\overline{\omega}(\varepsilon, G)$ the supremum of all real numbers $n-2 < \overline{\omega} < n$, such that for all $0 \le \omega < \overline{\omega}$ there exists a positive constant $c_2 = c_2(n, \varepsilon, \omega, G) > 0$, such that for all functionals $F \in Y^{-1,2,\omega}(G)$ and coefficients $(A,d) \in L^\infty(G^\circ; \mathfrak{S}(\varepsilon, n) \times \mathfrak{S}(\varepsilon, 1))$ the solution $u \in W_0^{1,2}(G)$ of $L_G(A,d)\, u = F$ satisfies $\nabla u \in \mathfrak{L}^{2,\omega}(G^\circ; \mathbb{R}^n)$ and, furthermore,

$$\|\nabla u\|_{\mathfrak{L}^{2,\omega}(G^\circ;\mathbb{R}^n)} \le c_2 \left\{ \|F\|_{Y^{-1,2,\omega}(G)} + \|u\|_{W_0^{1,2}(G)} \right\}.$$

The aim of this section is to prove that every regular set $G \subset \mathbb{R}^n$ is admissible, which is in fact the desired regularity result announced in our introduction. To do so, first of all we show certain properties of admissible sets.

Lemma 4.3. *Let $G \subset \mathbb{R}^n$ be a regular set and $\{U_1, \dots, U_m\}$, $\{V_1, \dots, V_m\}$ open coverings of $\overline{G}$ such that for all $j \in \{1, \dots, m\}$ $V_j \subset U_j$ and $V_j \cap G$ is admissible with respect to $U_j \cap G$. Then G is admissible.*

Proof. Let $0 < \varepsilon \le 1$ and consider $(A,d) \in L^\infty(G^\circ; \mathfrak{S}(\varepsilon, n) \times \mathfrak{S}(\varepsilon, 1))$. For every $j \in \{1, \dots, m\}$ we construct bounded linear operators $L_j(A,d) : W_0^{1,2}(U_j \cap G) \to W^{-1,2}(U_j \cap G)$ by

$$\langle L_j(A,d)\, v, w \rangle_{U_j \cap G} := \int_{U_j \cap G} (A \nabla v \cdot \nabla w + dvw)\, \mathrm{d}\lambda^n, \quad v, w \in W_0^{1,2}(U_j \cap G).$$

Because of the admissibility of $V_j \cap G$ with respect to $U_j \cap G$ there exists a parameter $n-2 < \overline{\omega} < n$ such that for all $0 \le \omega < \overline{\omega}$ one can find a constant $c_1 > 0$ depending on $n, \varepsilon, \omega, G$ and $\{U_1, \dots, U_m\}$, $\{V_1, \dots, V_m\}$ such that for every index $j \in \{1, \dots, m\}$, every $F_j \in Y^{-1,2,\omega}(U_j \cap G)$ and all coefficients $(A, d) \in L^\infty(G^\circ; \mathfrak{S}(\varepsilon, n) \times \mathfrak{S}(\varepsilon, 1))$ the gradient $\nabla u_j|_{V_j \cap G^\circ}$ of the solution $u_j \in W_0^{1,2}(U_j \cap G)$ to $L_j(A, d)\, u_j = F_j$ belongs to $\mathfrak{L}^{2,\omega}(V_j \cap G^\circ; \mathbb{R}^n)$ and, furthermore,

$$\|\nabla u_j\|_{\mathfrak{L}^{2,\omega}(V_j \cap G^\circ; \mathbb{R}^n)} \le c_1 \left\{ \|F_j\|_{Y^{-1,2,\omega}(U_j \cap G)} + \|u_j\|_{W_0^{1,2}(U_j \cap G)} \right\}. \tag{4.1}$$

If $\{\chi_1, \dots, \chi_m\} \subset C_0^\infty(\mathbb{R}^n)$ is a partition of unity subordinate to $\{V_1, \dots, V_m\}$ then

$$\delta = \min_{1 \le j \le m} \operatorname{dist}(\operatorname{supp}(\chi_j), \partial V_j) > 0.$$

Let $F \in Y^{-1,2,\omega}(G)$ be a functional and $u \in W_0^{1,2}(G)$ the solution of $L_G(A, d)\, u = F$. Now, we define for all $j \in \{1, \dots, m\}$ the functions

$$u_j = (u\chi_j)|_{U_j \cap G^\circ} \in W_0^{1,2}(U_j \cap G)$$

and the functionals $F_{0j} \in W^{-1,2}(U_j \cap G)$ by

$$\langle F_{0j}, w \rangle_{U_j \cap G} := \int_{U_j \cap G} \big(uA\nabla\chi_j \cdot \nabla w - A\nabla u \cdot \nabla \chi_j \cdot w \big)\, \mathrm{d}\lambda^n, \quad w \in W_0^{1,2}(U_j \cap G),$$

respectively. Hence, for all $w \in W_0^{1,2}(U_j \cap G)$ we get the identity

$$\begin{aligned} \langle L_j(A, d)\, u_j, w \rangle_{U_j \cap G} &= \langle L_G(A, d)\, u, R_G(w\chi_j) \rangle_G + \langle F_{0j}, w \rangle_{U_j \cap G} \\ &= \langle F, R_G(w\chi_j) \rangle_G + \langle F_{0j}, w \rangle_{U_j \cap G}. \end{aligned}$$

Therefore, $u_j \in W_0^{1,2}(U_j \cap G)$ is the solution of the variational problem

$$\langle L_j(A, d)\, u_j, w \rangle_{U_j \cap G} = \langle (\chi_j F)|_{U_j \cap G} + F_{0j}, w \rangle_{U_j \cap G}, \quad w \in W_0^{1,2}(U_j \cap G). \tag{4.2}$$

Because of the embedding $W_0^{1,2}(G) \hookrightarrow \mathfrak{L}^{2,2}(G^\circ)$ for $\mu = \min\{\omega, 2\}$ the following holds

$$uA\nabla\chi_j \in \mathfrak{L}^{2,\mu}(G^\circ; \mathbb{R}^n) \quad \text{and} \quad -A\nabla u \cdot \nabla\chi_j \in \mathfrak{L}^{2,\mu-2}(G^\circ).$$

Hence, by Theorem 3.9 we get $F_{0j} \in Y^{-1,2,\mu}(U_j \cap G)$ and there exists a constant $c_2 > 0$ depending on ε, μ, G and the above partition of unity such that

$$\|F_{0j}\|_{Y^{-1,2,\mu}(U_j \cap G)} \le c_2\, \|u\|_{W_0^{1,2}(G)} \quad \text{for all } j \in \{1, \dots, m\}.$$

On the other hand, $(\chi_j F)|_{U_j \cap G}$ belongs to $Y^{-1,2,\mu}(U_j \cap G)$, too, and we have

$$\big\|(\chi_j F)|_{U_j \cap G}\big\|_{Y^{-1,2,\mu}(U_j \cap G)} \le c_3\, \|F\|_{Y^{-1,2,\mu}(G)} \quad \text{for all } j \in \{1, \dots, m\},$$

where $c_3 > 0$ is a positive constant depending on μ and the above partition of unity. Applying relation (4.1) to the functionals

$$F_j = (\chi_j F)|_{U_j \cap G} + F_{0j} \in Y^{-1,2,\mu}(U_j \cap G),$$

we get the estimate

$$\|\nabla u_j\|_{\mathfrak{L}^{2,\mu}(V_j\cap G^\circ;\mathbb{R}^n)} \le c_1 \left\{ \left\|(\chi_j F)|_{U_j\cap G} + F_{0j}\right\|_{Y^{-1,2,\mu}(U_j\cap G)} + \|u_j\|_{W_0^{1,2}(U_j\cap G)} \right\}.$$

Hence, there exists a constant $c_4 = c_4(c_1, c_2, c_3) > 0$ such that for all $j \in \{1, \ldots, m\}$

$$\|\nabla u_j\|_{\mathfrak{L}^{2,\mu}(V_j\cap G^\circ;\mathbb{R}^n)} \le c_4 \left\{ \|F\|_{Y^{-1,2,\mu}(G)} + \|u\|_{W_0^{1,2}(G)} \right\}.$$

Summing up the results we get

$$u = \sum_{j=1}^{m} u\chi_j = \sum_{j=1}^{m} R_G u_j \in W_0^{1,2,\mu}(G),$$

and, moreover,

$$\|u\|_{W_0^{1,2,\mu}(G)} \le c_5 \left\{ \|F\|_{Y^{-1,2,\mu}(G)} + \|u\|_{W_0^{1,2}(G)} \right\},$$

where $c_5 = c_5(c_4, m, \delta) > 0$ is a positive constant. Because of the continuity of the embedding $W_0^{1,2,\mu}(G) \hookrightarrow \mathfrak{L}^{2,\mu+2}(G^\circ)$ there exists a constant $c_6 = c_6(c_5, \mu, G) > 0$ with

$$\|u\|_{\mathfrak{L}^{2,\mu+2}(G^\circ)} \le c_6 \left\{ \|F\|_{Y^{-1,2,\mu}(G)} + \|u\|_{W_0^{1,2}(G)} \right\}.$$

Now, we can complete the proof by a recursive argumentation. Because of the continuous embedding $W_0^{1,2,\mu}(G) \hookrightarrow \mathfrak{L}^{2,\mu+2}(G^\circ)$ for $\mu = \min\{\omega, 4\}$ we have

$$uA\nabla\chi_j \in \mathfrak{L}^{2,\mu}(G^\circ;\mathbb{R}^n) \quad \text{and} \quad -A\nabla u \cdot \nabla\chi_j \in \mathfrak{L}^{2,\mu-2}(G^\circ).$$

Repeating the above arguments, we get $u \in W_0^{1,2,\mu}(G)$ and the corresponding norm estimate. After a finite number of analogous steps, we arrive at the desired result for $\mu = \omega$, in other words, $u \in W_0^{1,2,\omega}(G)$ and there exists a positive constant $c_7 = c_7(n, \omega, G) > 0$ such that

$$\|u\|_{W_0^{1,2,\omega}(G)} \le c_7 \left\{ \|F\|_{Y^{-1,2,\omega}(G)} + \|u\|_{W_0^{1,2}(G)} \right\},$$

which proves the admissibility of G. □

Lemma 4.4. *Let $G_0 \subset G \subset \mathbb{R}^n$ be two regular sets and Ψ be a Lipschitz transformation from an open neighborhood of $\overline{G}$ onto another open subset of $\mathbb{R}^n$. If $H_0 = \Psi(G_0)$ is admissible with respect to $H = \Psi(G)$, then G_0 is admissible with respect to G.*

Proof. Let $0 < \varepsilon \le 1$ and $L \ge 1$ be a Lipschitz constant of the transformation Ψ. Furthermore, we consider coefficients $(A, d) \in L^\infty(G^\circ; \mathfrak{S}(\varepsilon, n) \times \mathfrak{S}(\varepsilon, 1))$.

Because of the properties of the Jacobian matrix $D\Psi$ and the Jacobian $J\Psi^{-1}$, respectively, for the transformed coefficients

$$(A_H, d_H) := \big((D\Psi)(\Psi_*^{-1}A)(D\Psi)^* \cdot J\Psi^{-1}, \Psi_*^{-1}d \cdot J\Psi^{-1}\big)$$

we have the relation $(A_H, d_H) \in L^\infty\big(H^\circ; \mathfrak{S}(L^{-n-2}\varepsilon, n) \times \mathfrak{S}(L^{-n-2}\varepsilon, 1)\big)$.

We construct the bounded linear operator $L_H(A_H, d_H) : W_0^{1,2}(H) \to W^{-1,2}(H)$ by

$$\langle L_H(A_H, d_H)\, v, w\rangle_H := \int_H (A_H \nabla v \cdot \nabla w + d_H v w)\, \mathrm{d}\lambda^n, \quad v, w \in W_0^{1,2}(H).$$

Because of the transformation invariance of $Y^{-1,2,\omega}(G)$ (Lemma 3.8) the admissibility of H_0 with respect to H yields the existence of a parameter $n-2 < \overline{\omega} < n$ such that for all $0 \le \omega < \overline{\omega}$ one can find a constant $c_1 > 0$ depending on $n, \varepsilon, \omega, \Psi, G$ and H only such that for all $(A, d) \in L^\infty(G^\circ; \mathfrak{S}(\varepsilon, n) \times \mathfrak{S}(\varepsilon, 1))$ and every $F \in Y^{-1,2,\omega}(G)$ the gradient $\nabla v|_{H_0^\circ}$ of the solution $v \in W_0^{1,2}(H)$ to $L_H(A_H, d_H)\, v = \Psi^* F$ belongs to $\mathfrak{L}^{2,\omega}(H_0^\circ; \mathbb{R}^n)$ and, furthermore,

$$\|\nabla v\|_{\mathfrak{L}^{2,\omega}(H_0^\circ;\mathbb{R}^n)} \le c_1 \left\{ \|\Psi^* F\|_{Y^{-1,2,\omega}(H)} + \|v\|_{W_0^{1,2}(H)} \right\}. \tag{4.3}$$

Let $u \in W_0^{1,2}(G)$ be the uniquely determined solution to $L(A, d)\, u = F$. Then, by the chain rule and the transformation formula for all $w \in W_0^{1,2}(H)$ we get the identity

$$\langle L_H(A_H, d_H)\, \Psi_*^{-1} u, w\rangle_H = \langle L(A, d)\, u, \Psi_* w\rangle_G = \langle F, \Psi_* w\rangle_G = \langle \Psi^* F, w\rangle_H.$$

Hence, $v = \Psi_*^{-1} u \in W_0^{1,2}(H)$ is the solution to $L_H(A_H, d_H)\, v = \Psi^* F$. By (4.3) and Lemma 3.8 we get the existence of a positive constant $c_2 = c_2(c_1, \Psi, G) > 0$ such that

$$\|\nabla(\Psi_*^{-1} u)\|_{\mathfrak{L}^{2,\omega}(H_0^\circ;\mathbb{R}^n)} \le c_2 \left\{ \|F\|_{Y^{-1,2,\omega}(G)} + \|\Psi_*^{-1} u\|_{W_0^{1,2}(H)} \right\}.$$

Finally, the transformation invariance of $W_0^{1,2,\omega}(G)$ yields the existence of a constant $c_3 = c_3(c_2, \Psi, G) > 0$ such that

$$\|\nabla u\|_{\mathfrak{L}^{2,\omega}(G_0^\circ;\mathbb{R}^n)} \le c_3 \left\{ \|F\|_{Y^{-1,2,\omega}(G)} + \|u\|_{W_0^{1,2}(G)} \right\},$$

which proves the admissibility of G_0 with respect to G. □

4.2. Local estimates on concentric balls

For the proof of admissibility of the standard sets B, E_1, E_2 and E_3 we want to utilize local estimates for the gradient of the solution to elliptic problems on concentric balls and halfballs, respectively. We start with the so called Campanato inequality (see De Giorgi [5], Campanato [4] or Troianiello [17]).

Lemma 4.5. *Let $0 < \varepsilon \le 1$. Then there exist positive constants $n-2 < \overline{\omega} < n$ and $c = c(n, \varepsilon, \overline{\omega}) > 0$, such that for all $x \in \mathbb{R}^n$, $0 < \varrho \le r < 1$, coefficients $A \in L^\infty(B(x,r); \mathfrak{S}(\varepsilon, n))$, functionals $F \in W^{-1,2}(B(x,r))$ and functions $u \in W^{1,2}(B(x,r))$ satisfying*

$$\int_{B(x,r)} A \nabla u \cdot \nabla w \, \mathrm{d}\lambda^n = \langle F, w\rangle_{B(x,r)} \quad \textit{for all } w \in W_0^{1,2}(B(x,r)),$$

the following estimate holds

$$\|\nabla u\|^2_{L^2(B(x,\varrho);\mathbb{R}^n)} \le c\left\{\left(\frac{\varrho}{r}\right)^{\overline{\omega}} \|\nabla u\|^2_{L^2(B(x,r);\mathbb{R}^n)} + \|F\|^2_{W^{-1,2}(B(x,r))}\right\}.$$

Remark 4.6. For every number $0 < \varepsilon \le 1$ we define the supremum $\overline{\omega}(\varepsilon)$ of all parameters $n-2 < \overline{\omega} < n$, for which Lemma 4.5 holds true. Obviously, that supremum depends on n and ε only, and the map $\varepsilon \mapsto \overline{\omega}(\varepsilon)$ is non-decreasing.

Lemma 4.7. *For every $0 < R < 1$ the ball $B(0,R)$ is admissible with respect to B.*

Proof. Let $0 < \varepsilon \le 1$, $0 < R < 1$ and $n-2 < \overline{\omega} < \overline{\omega}(\varepsilon)$ be given. Now, we define the decreasing sequence $\{r_k\}_{k\in\mathbb{N}}$ by

$$R < r_k := R + 2^{-k}(1-R) \le 1, \quad k \in \mathbb{N}.$$

We fix a radius $0 < r_B \le 4^{-n}\min\{R, 1-R\}$ and consider $x \in B(0,r_1)$, $0 < r \le r_B$, coefficients $(A,d) \in L^\infty(B;\mathfrak{S}(\varepsilon,n)\times\mathfrak{S}(\varepsilon,1))$ and functionals $F \in W^{-1,2}(B)$.

Let $u \in W^{1,2}_0(B)$ be the uniquely determined solution to $L_B(A,d)\,u = F$. If we define the functional $F_d \in W^{-1,2}(B(x,r))$ by

$$\langle F_d, w\rangle_{B(x,r)} := -\int_{B(x,r)} duw\,\mathrm{d}\lambda^n, \quad w \in W^{1,2}_0(B(x,r)),$$

then $u|_{B(x,r)} \in W^{1,2}(B(x,r))$ satisfies the identity

$$\int_{B(x,r)} A\nabla u\cdot\nabla w\,\mathrm{d}\lambda^n = \langle F_d + F|_{B(x,r)}, w\rangle_{B(x,r)} \quad \text{for all } w \in W^{1,2}_0(B(x,r)).$$

Hence, Lemma 4.5 yields the existence of a constant $c_1 = c_1(n,\varepsilon,\overline{\omega},R) > 0$, such that for all $0 < \varrho \le r \le r_B$, coefficients $(A,d) \in L^\infty(B;\mathfrak{S}(\varepsilon,n)\times\mathfrak{S}(\varepsilon,1))$ and functionals $F \in Y^{-1,2,\overline{\omega}}(B)$ for the gradient ∇u the following holds

$$\begin{aligned}\|\nabla u\|^2_{L^2(B(x,\varrho);\mathbb{R}^n)} &\le c_1\left\{\left(\frac{\varrho}{r}\right)^{\overline{\omega}}\|\nabla u\|^2_{L^2(B(x,r);\mathbb{R}^n)} + \|F_d + F|_{B(x,r)}\|^2_{W^{-1,2}(B(x,r))}\right\}\\ &\le 2c_1\left\{\left(\frac{\varrho}{r}\right)^{\overline{\omega}}\|\nabla u\|^2_{L^2(B(x,r);\mathbb{R}^n)} + \frac{1}{\varepsilon^2}\|u\|^2_{L^2(B(x,r))} + \|F|_{B(x,r)}\|^2_{W^{-1,2}(B(x,r))}\right\}.\end{aligned}$$

Let us define for all $0 \le \mu < \overline{\omega}$ the quantity

$$\kappa_\mu(u,F) := \|u\|^2_{W^{1,2}_0(B)} + \|F\|^2_{Y^{-1,2,\mu}(B)},$$

and let $0 \le \omega < \overline{\omega}$ be a fixed. Because of the embedding $W^{1,2}(B) \hookrightarrow \mathfrak{L}^{2,\mu}(B)$ for $\mu = \min\{\omega, 2\}$, there exists a constant $c_2 = c_2(n,\varepsilon,\mu) > 0$ such that

$$\|u\|^2_{\mathfrak{L}^{2,\mu}(B)} \le c_2\,\kappa_\mu(u,F).$$

Now, the last two estimates yield the existence of a constant $c_3 = c_3(n, \varepsilon, \mu, \overline{\omega}, R) > 0$, such that for all $0 < \varrho \le r \le r_B$ we have

$$\|\nabla u\|^2_{L^2(B(x,\varrho);\mathbb{R}^n)} \le c_3 \left\{ \left(\frac{\varrho}{r}\right)^{\overline{\omega}} \|\nabla u\|^2_{L^2(B(x,r);\mathbb{R}^n)} + r^\mu \kappa_\mu(u,F) \right\}.$$

Having in mind $0 \le \mu = \min\{\omega, 2\} < \overline{\omega}$, we can apply an elementary lemma (see, for instance, Giaquinta [7]) to get

$$\|\nabla u\|^2_{L^2(B(x,\varrho);\mathbb{R}^n)} \le c_4 \left\{ \left(\frac{\varrho}{r}\right)^{\mu} \|\nabla u\|^2_{L^2(B(x,r);\mathbb{R}^n)} + \varrho^\mu \kappa_\mu(u,F) \right\}$$

for all $0 < \varrho \le r \le r_B$, where $c_4 = c_4(n, \varepsilon, \mu, \overline{\omega}, R) > 0$ is a positive constant. Now, by specifying $r = r_B$, we arrive at $\nabla u|_{B(0,r_1)} \in \mathfrak{L}^{2,\mu}(B(0,r_1);\mathbb{R}^n)$. Moreover, there exists a constant $c_5 = c_5(n, \varepsilon, \mu, \overline{\omega}, R) > 0$ such that

$$\|\nabla u\|^2_{\mathfrak{L}^{2,\mu}(B(0,r_1);\mathbb{R}^n)} \le c_5\, \kappa_\mu(u,F).$$

We want to complete the proof by a recursive argumentation. Because of the continuous embedding $W^{1,2,\mu-2}(B(0,r_1)) \hookrightarrow \mathfrak{L}^{2,\mu}(B(0,r_1))$ for $\mu = \min\{\omega, 4\}$ one can find a positive constant $c_6 = c_6(n, \varepsilon, \mu, \overline{\omega}, R) > 0$ such that

$$\|u\|^2_{\mathfrak{L}^{2,\mu}(B(0,r_1))} \le c_6\, \kappa_\mu(u,F).$$

Then, we repeat the above arguments to get $u|_{B(0,r_2)} \in W^{1,2,\mu}(B(0,r_2))$ and the corresponding norm estimate. Because of

$$R < r_k \le 1 \quad \text{and} \quad r_B < r_k - r_{k+1} \quad \text{for all } k = \{0, 1, \ldots, n\},$$

after at most n analogous steps we arrive at the desired result for $\mu = \omega$, in other words, there exists a constant $c_7 = c_7(n, \varepsilon, \omega, \overline{\omega}, R) > 0$ such that

$$\|\nabla u\|^2_{\mathfrak{L}^{2,\omega}(B(0,R);\mathbb{R}^n)} \le c_7\, \kappa_\omega(u,F),$$

which proves the admissibility of $B(0,R)$ with respect to B. □

Lemma 4.8. *For every $0 < R < 1$ and $k \in \{1,2\}$ the set $E_k(0,R)$ is admissible with respect to E_k.*

Proof. Let $0 < \varepsilon \le 1$, $0 < R < 1$ and $n - 2 < \overline{\omega} < \overline{\omega}(\varepsilon)$ be fixed. Because of Lemma 4.7 and the reflection invariance of the coefficients and functionals (see Lemma 3.7) for all $0 \le \omega < \overline{\omega}$ one can find a constant $c_1 = c_1(n, \varepsilon, \omega, R) > 0$, such that for all coefficients $(A, d) \in L^\infty(E_1; \mathfrak{S}(\varepsilon, n) \times \mathfrak{S}(\varepsilon, 1))$ and $F \in Y^{-1,2,\omega}(E_k)$ the gradient $\nabla v|_{B(0,R)}$ of the solution $v \in W^{1,2}_0(B)$ to the problem $L_B(R_2 A, R_2 d)\, v = R_k F$ belongs to $\mathfrak{L}^{2,\omega}(B(0,R);\mathbb{R}^n)$ and, furthermore,

$$\|\nabla v\|_{\mathfrak{L}^{2,\omega}(B(0,R);\mathbb{R}^n)} \le c_1 \left\{ \|R_k F\|_{Y^{-1,2,\omega}(B)} + \|v\|_{W^{1,2}_0(B)} \right\}. \tag{4.4}$$

On the other hand, the solution $u \in W^{1,2}_0(E_k)$ to $L_{E_k}(A,d)\, u = F$ for all $w \in W^{1,2}_0(B)$ satisfies the identity

$$\langle L_B(R_2 A, R_2 d)\, R_k u, w\rangle_B = \langle L_{E_k}(A,d)\, u, T_k w\rangle_{E_k} = \langle F, T_k w\rangle_{E_k} = \langle R_k F, w\rangle_B.$$

Hence, $v = R_k u \in W_0^{1,2}(B)$ is the solution of $L_B(R_2A, R_2d)\, v = R_kF$, and by (4.4) we get the estimate

$$\|\nabla(R_k u)\|_{\mathfrak{L}^{2,\omega}(B(0,R);\mathbb{R}^n)} \le c_1 \left\{ \|R_kF\|_{Y^{-1,2,\omega}(B)} + \|R_k u\|_{W_0^{1,2}(B)} \right\}.$$

Finally, the continuity of the extension operator R_k on $Y^{-1,2,\omega}(E_k)$ yields a constant $c_2 = c_2(n, \varepsilon, \omega, R) > 0$, such that

$$\|\nabla u\|_{\mathfrak{L}^{2,\omega}(E_k(0,R);\mathbb{R}^n)} \le c_2 \left\{ \|F\|_{Y^{-1,2,\omega}(E_k)} + \|u\|_{W_0^{1,2}(E_k)} \right\},$$

in other words, $E_k(0, R)$ is admissible with respect to E_k. □

4.3. Global estimates and isomorphism theorem

We proof the global regularity result for the standard sets B, E_1, E_2 and E_3.

Lemma 4.9. *The open unit ball B is an admissible set.*

P r o o f. First of all, we choose an atlas $\{(\Phi_1, U_1), \dots, (\Phi_m, U_m)\}$ of the ball B with the properties (2.9) and (2.10). Then, there exist radii $0 < \delta_1 < \delta_2 < 1$ such that the families $\{V_1, \dots, V_m\}$, $\{W_1, \dots, W_m\}$ are open coverings of $\overline{B}$ if we define

$$V_j := \Phi_j^{-1}(B(0, \delta_1)) \quad \text{and} \quad W_j := \Phi_j^{-1}(B(0, \delta_2)), \quad j \in \{1, \dots, m\}.$$

By Lemma 4.8 the set $E_1(0, \delta_1)$ is admissible with respect to $E_1(0, \delta_2)$. Furthermore, by Lemma 4.7 the ball $B(0, \delta_1)$ is admissible with respect to $B(0, \delta_2)$. Therefore, Lemma 4.4 yields the admissibility of $V_j \cap B$ with respect to $W_j \cap B$ for every index $j \in \{1, \dots, m\}$. Applying Lemma 4.3, finally, it follows the admissibility of B. □

Lemma 4.10. *The sets E_1, E_2 and E_3 are admissible.*

P r o o f. *Case $k \in \{1, 2\}$*: Let $0 < \varepsilon \le 1$ and $n - 2 < \overline{\omega} < \overline{\omega}(\varepsilon, B)$ be fixed. Because of Lemma 4.9 and the reflection invariance of the coefficients and functionals (see Lemma 3.7) for all $0 \le \omega < \overline{\omega}$ we find a positive constant $c_1 = c_1(n, \varepsilon, \omega) > 0$, such that for all coefficients $(A, d) \in L^\infty(E_1; \mathfrak{S}(\varepsilon, n) \times \mathfrak{S}(\varepsilon, 1))$ and functionals $F \in Y^{-1,2,\omega}(E_k)$ the gradient ∇v of the solution $v \in W_0^{1,2}(B)$ to the problem $L_B(R_2A, R_2d)\, v = R_kF$ belongs to $\mathfrak{L}^{2,\omega}(B; \mathbb{R}^n)$ and, furthermore,

$$\|\nabla v\|_{\mathfrak{L}^{2,\omega}(B;\mathbb{R}^n)} \le c_1 \left\{ \|R_kF\|_{Y^{-1,2,\omega}(B)} + \|v\|_{W_0^{1,2}(B)} \right\}. \tag{4.5}$$

Since the solution $u \in W_0^{1,2}(E_k)$ to $L_{E_k}(A, d)\, u = F$ for all $w \in W_0^{1,2}(B)$ satisfies

$$\langle L_B(R_2A, R_2d)\, R_k u, w\rangle_B = \langle L_{E_k}(A, d)\, u, T_k w\rangle_{E_k} = \langle F, T_k w\rangle_{E_k} = \langle R_kF, w\rangle_B,$$

by (4.5) we get an estimate for $v = R_k u \in W_0^{1,2}(B)$:

$$\|\nabla(R_k u)\|_{\mathfrak{L}^{2,\omega}(B;\mathbb{R}^n)} \le c_1 \left\{ \|R_kF\|_{Y^{-1,2,\omega}(B)} + \|R_k u\|_{W_0^{1,2}(B)} \right\}.$$

Finally, the continuity of the extension operator R_k on $Y^{-1,2,\omega}(E_k)$ yields a positive constant $c_2 = c_2(n, \varepsilon, \omega) > 0$, such that

$$\|\nabla u\|_{\mathfrak{L}^{2,\omega}(E_k;\mathbb{R}^n)} \leq c_2 \left\{ \|F\|_{Y^{-1,2,\omega}(E_k)} + \|u\|_{W_0^{1,2}(E_k)} \right\},$$

which proves the admissibility of E_k.

Case $k = 3$: There exists a Lipschitz transformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$ which maps E_2 onto E_3. Lemma 4.4 and the admissibility of E_2 yields the admissibility of E_3. □

Theorem 4.11. *Every regular set $G \subset \mathbb{R}^n$ is admissible.*

Proof. We take an atlas $\{(\Phi_1, U_1), \ldots, (\Phi_m, U_m)\}$ of G satisfying (2.9) and (2.10). Then there exists a radius $0 < \delta < 1$ such that the family $\{V_1, \ldots, V_m\}$ is an open covering of $\overline{G}$ if we set $V_j := \Phi_j^{-1}(B(0, \delta))$ for $j \in \{1, \ldots, m\}$.

By Lemma 4.10 the set $E_k(0, \delta)$ is admissible for every index $k \in \{1, 2, 3\}$. Furthermore, by Lemma 4.9 the ball $B(0, \delta)$ is an admissible set. Therefore, Lemma 4.4 yields the admissibility of $V_j \cap G$ for all $j \in \{1, \ldots, m\}$. Applying Lemma 4.3, finally, we get the admissibility of G. □

Hence, we are able to prove the main result:

Theorem 4.12. *Let $G \subset \mathbb{R}^n$ be a regular set and $0 < \varepsilon \leq 1$. Then there exists a real constant $n - 2 < \overline{\omega}(\varepsilon, G) < n$ such that for all $0 \leq \omega < \overline{\omega}(\varepsilon, G)$ and all coefficients $(A, d) \in L^\infty(G^\circ; \mathfrak{S}(\varepsilon, n) \times \mathfrak{S}(\varepsilon, 1))$ the elliptic operator $L_G(A, d)$ is a linear isomorphism from $W_0^{1,2,\omega}(G)$ onto $Y^{-1,2,\omega}(G)$.*

Proof. Applying Theorem 4.11, for every $0 \leq \omega < \overline{\omega}(\varepsilon, G)$ one can find a constant $c_1 = c_1(n, \varepsilon, \omega, G) > 0$, such that for all $(A, d) \in L^\infty(G^\circ; \mathfrak{S}(\varepsilon, n) \times \mathfrak{S}(\varepsilon, 1))$ and functionals $F \in Y^{-1,2,\omega}(G)$ the uniquely determined solution $u = L_G(A, d)^{-1}F$ to the problem $L_G(A, d)\, u = F$ belongs to $W_0^{1,2,\omega}(G)$, and, furthermore

$$\|u\|_{W_0^{1,2,\omega}(G)} \leq c_1 \left\{ \|F\|_{Y^{-1,2,\omega}(G)} + \|u\|_{W_0^{1,2}(G)} \right\}.$$

By the isomorphism property of $L_G(A, d)$ between $W_0^{1,2}(G)$ and $W^{-1,2}(G)$ and the continuous embedding $Y^{-1,2,\omega}(G) \hookrightarrow W^{-1,2}(G)$ it follows

$$\|L_G(A, d)^{-1}F\|_{W_0^{1,2,\omega}(G)} \leq c_2 \|F\|_{Y^{-1,2,\omega}(G)} \quad \text{for all } F \in Y^{-1,2,\omega}(G),$$

where $c_2 = c_2(c_1, n, \varepsilon, \omega, G) > 0$ is a positive constant.

Because of embedding theorems for Sobolev–Campanato spaces and Theorem 3.9 the elliptic operator $L_G(A, d)$ is a bounded linear operator from $W_0^{1,2,\omega}(G)$ into $Y^{-1,2,\omega}(G)$ for every $0 \leq \omega < \overline{\omega}(\varepsilon, G)$, which proves the desired regularity result. □

Remark 4.13. We want to emphasize that for $n - 2 < \omega < n$, $\alpha = (\omega - n + 2)/2$ the space $W_0^{1,2,\omega}(G)$ is continuously embedded into the Hölder space $C^{0,\alpha}(\overline{G})$.

By Theorem 3.9 the image of $W_0^{1,2,\omega}(G)$ under $L_G(A, d)$ is continuously embedded into $W^{-1,2,\omega}(G) \hookrightarrow Y^{-1,2,\omega}(G)$ for all $0 \leq \omega < n$. Hence, Theorem 4.12 yields

Corollary 4.14. *Let $G \subset \mathbb{R}^n$ be a regular set and $0 < \varepsilon \leq 1$. Then for every parameter $0 \leq \omega < \overline{\omega}(\varepsilon, G)$ the spaces $W^{-1,2,\omega}(G)$ and $Y^{-1,2,\omega}(G)$ coincide.*

Remark 4.15. The result of Theorem 4.12 can be generalized to the case of linear elliptic systems with diagonal structure and general lower order terms. Then the linear elliptic operator is still a Fredholm operator of index zero from the corresponding vector valued version of the Sobolev–Campanato space into a Campanato space of functionals (see Griepentrog, Recke [9] and Griepentrog [10]).

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