THE BMS CONJECTURE

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Abstract. I explain an open conjecture by Braverman/Milatovic/Shubin (BMS) on the positivity of square integrable solutions f of $(-\Delta+1)f \geq 0$ on a complete Riemannian manifold, and its connection to essential self-adjointness problems of covariant Schrödinger operators. The latter conjecture has remained open for more than 14 years now.

Let M be a smooth connected Riemannian manifold, equipped with its usual Riemannian volume measure $\mathrm{d}\mu$. We denote the scalar Laplace-Beltrami operator with[1] $-\Delta = \mathrm{d}^{\dagger}\mathrm{d}$, and its Friedrichs realization in $\mathsf{L}^2(M)$ with $H \geq 0$, where we understand all our spaces of functions to be complex-valued, unless otherwise stated. Let $E \to M$ be a smooth complex metric vector bundle with a smooth metric covariant derivative ∇ thereon. Spaces of sections having a certain global or regularty $*$ will be denoted with Γ_*. For any Borel section f of $E \to M$, the section $\mathrm{sign}(f) \in \Gamma_{\mathsf{L}^\infty}(M,E)$ is defined by

$$\mathrm{sign}(f)(x) := \begin{cases} \frac{f(x)}{|f(x)|}, & \text{if } f(x) \neq 0 \\ 0, & \text{else.} \end{cases}$$

The central result behind anything that follows is the following geometric variant of a classical distributional inequality by Kato [6] ("covariant Kato inequality"): *For all $f \in \Gamma_{\mathsf{L}^1_{\mathrm{loc}}}(M,E)$ with $\nabla^{\dagger}\nabla f \in \Gamma_{\mathsf{L}^1_{\mathrm{loc}}}(M,E)$ weakly, one has the weak inequality*

$$-\Delta |f| \leq \Re\left(\nabla^{\dagger}\nabla f, \mathrm{sign}(f)\right). \tag{1}$$

A proof of the latter inequality can be found in [2]. It is in fact a local result which therefore holds without any further assumptions on M. Let us now pick a potential

$$0 \leq V \in \Gamma_{\mathsf{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E)) \tag{2}$$

and assume we want to prove that the symmetric nonnegative operator $(\nabla^{\dagger}\nabla + V)|_{\Gamma_{\mathsf{C}^\infty_c}(M,E)}$ in $\Gamma_{\mathsf{L}^2}(M,E)$ is essentially self-adjoint. By an abstract functional analytic fact and some simple distribution theory, the latter essential self-adjoint is equivalent to the following implication:

$$\begin{aligned} & f \in \Gamma_{\mathsf{L}^2}(M,E),\ (\nabla^{\dagger}\nabla + V + 1)f = 0 \ \text{ weakly} \\ & \Rightarrow\ f = 0. \end{aligned} \tag{3}$$

[1] A "†" always stands for the formal adjoint of a differential operator acting between sections of metric vector bundles over M; it depends on the fixed Riemannian metric on M and the underlying metrics on the bundles (which are trivial in the scalar case).

So let f be given with (3). In order to prove $f = 0$, following Kato's original approach for $M = \mathbb{R}^m$, it is tempting to use the covariant Kato inequality, which in combination with $V \geq 0$ immediately implies

$$(-\Delta + 1)(-|f|) \geq 0 \quad \text{weakly.}$$

This motivates the $\mathscr{C} = \mathsf{L}^2_{\mathbb{R}}(M)$ case of following definition, which is taken from [3]:

Definition 1.1. Let[2] $\mathscr{C} \subset \mathsf{L}^1_{\mathrm{loc},\mathbb{R}}(M)$ be an arbitrary subset. Then the Riemannian manifold M is called *$\mathscr{C}$-positivity preserving* (PP), if the following implication of weak inequalities holds true for every $\phi \in \mathscr{C}$,

$$(-\Delta + 1)\phi \geq 0 \Rightarrow \phi \geq 0. \tag{4}$$

Assume now M is $\mathsf{L}^2_{\mathbb{R}}(M)$-positivity preserving. Then in the above situation we can conclude $-|f| \geq 0$, thus $f = 0$, and we have shown:

Proposition 1.2. *If M is $\mathsf{L}^2_{\mathbb{R}}(M)$-positivity preserving, then for every potential V with (2), the operator $(\nabla^\dagger\nabla + V)|_{\Gamma_{\mathsf{C}^\infty_c}(M,E)}$ in $\Gamma_{\mathsf{L}^2}(M, E)$ is essentially self-adjoint.*

On the other hand, either using refined integration by parts techniques [2] or using wave equation techniques [4], one can prove:

Theorem 1.3. *If M is geodesically complete, then for every potential every potential V with (2), the operator $(\nabla^\dagger\nabla + V)|_{\Gamma_{\mathsf{C}^\infty_c}(M,E)}$ in $\Gamma_{\mathsf{L}^2}(M, E)$ is essentially self-adjoint.*

This lead M. Braverman, O. Milatovic and M. Shubin to the following conjecture from 2002, which I formulate for convenience in the language of Definition 1.1[3]:

Conjecture 1.4 (BMS-conjecture)**.** *If M is geodesically complete, then M is $\mathsf{L}^2_{\mathbb{R}}(M)$-PP.*

I invite the interested reader to attack this problem, which is still open in this generality!

It is instructive in this context to explain Kato's simple and elegant proof of the fact that the Euclidean $M = \mathbb{R}^m$ is $\mathsf{L}^2_{\mathbb{R}}(\mathbb{R}^m)$-PP: In this case, $-\Delta + 1$ induces an isomorphism (of topological linear spaces)

$$-\Delta + 1 : \mathscr{S}(\mathbb{R}^m)' \xrightarrow{\sim} \mathscr{S}(\mathbb{R}^m)'$$

on the space of Schwartz distributions, whose inverse is positivity preserving. Thus, if a real-valued $\phi \in \mathsf{L}^2(\mathbb{R}^m) \subset \mathscr{S}(\mathbb{R}^m)'$ satisfies (4), then we can immeadiately conclude $\phi \geq 0$.

On a general Riemannian manifold there seems to be no appropriate substitute for the space of Schwartz distributions, and so one needs a new idea. The best result known so far on general Riemannian manifolds is the following one from [5] (which slightly generalizes [3]), that requires an additional lower bound on the Ricci curvature, but then also has a stronger conclusion on the full L^q-scale:

[2] $\mathsf{L}^1_{\mathrm{loc},\mathbb{R}}$ stands for the space of *real-valued* locally integrable functions, and likewise for $\mathsf{L}^2_{\mathbb{R}}(M)$.

[3] Note that the BMS-conjecture is much older than Definition 1.1, which was in fact modelled on the conjecture.

Theorem 1.5. *If M is geodesically complete with a Ricci curvature bounded from below by a constant, then M is $\mathsf{L}^q_{\mathbb{R}}(M)$-PP for all $q \in [1,\infty]$.*

The reader may find the following final remarks helpful:

Remark 1.6. 1. The proof of Theorem 1.5 is based on the construction of a sequence of *Laplacian cut-off functions* (cf. [3] for a precise definition), which leads to the assumption on the Ricci curvature. Once one has such a sequence, at least the $q = 2$ case follows easily using that $(H+1)^{-1}$ is positivity preserving on $\mathsf{L}^2(M)$ in combination with simple integration by parts arguments (the $q \neq 2$ case requires an additional argument to prove the boundedness of the gradient $\mathrm{d}(H+1)^{-1}$ of the resolvent from $\mathsf{L}^q(M)$ to $\Omega^1_{\mathsf{L}^q}(M)$, which again leads to the curvature assumption). For $M = \mathbb{R}^m$ such a sequence of Laplacian cut-off functions is readily obtained using the distance function and scaling.
2. At least for $q = 2$, it has been communicated to me by A. Setti that Theorem 1.5 can be generalized to allow a Ricci curvature having an appropriate *variable* lower bound, as then one can still prove the existence of a sequence of Laplacian cut-off functions (cf. [1]).
3. It really makes sense to consider the positivity preservation property on a full L^q-scale: For example, it is easy to check [3] that every $(\mathsf{C}^\infty \cap \mathsf{L}^\infty_{\mathbb{R}})$-PP Riemannian manifold is stochastically complete, meaning that

$$\int_M \mathrm{e}^{-tH}(x,y)\mathrm{d}\mu(y) = 1 \quad \text{for all } t > 0,\ x \in M,$$

or in other words, that Brownian motions on M cannot explode in a finite time. So for example, Theorem 1.5 provides an independent proof of S.T. Yau's classical result which states that geodesically complete Riemannian manifolds with a Ricci curvature bounded from below by a constant are stochastically complete. This was my original motivation for the general form of Definition 1.1, that is, the definition should be flexible enough to deal with problems such as stochastic completeness and essential self-adjointness simultaniously.

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