The Chern Character of ϑ-summable Fredholm Modules over dg Algebras and the Supersymmetric Path Integral

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January 14, 2019

Abstract

We introduce the notion of a ϑ-summable Fredholm module over a locally convex dg algebra Ω and construct its Chern character as an entire cyclic cocyle in the entire cyclic complex of Ω, leading to a cohomology class in the entire cyclic cohomology of Ω. This extends the cocycle of Jaffe, Lesniewski and Osterwalder to the differential graded case. Using this Chern character, we prove an index theorem involving an abstract version of a Bismut-Chern-character constructed by Getzler, Jones and Petrack. Our theory leads to a rigorous construction of the path integral for $N = 1/2$ supersymmetry and provides a framework that allows to conduct the proof of the Atiyah-Singer theorem by means of a well-defined Duistermaat-Heckman type localization formula on loop space.

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1 Introduction

In [11], [12], Connes introduces the notion of a ϑ-summable Fredholm module over a unital Banach algebra $\mathcal{A}$. Such a Fredholm module $\mathcal{M} = (\mathcal{H}, \mathbf{c}, Q)$ consists of a representation $\mathbf{c}$ of $\mathcal{A}$ on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$ together with an odd self-adjoint unbounded operator Q on $\mathcal{H}$.[1] $\mathcal{M}$ has a *Chern character* $\mathrm{ch}(\mathcal{M})$, which is an even linear functional on the chain complex $\mathsf{C}_\epsilon(\mathcal{A})$ associated to $\mathcal{A}$, and which defines an element of the *entire cyclic cohomology* of $\mathcal{A}$. Using the geometric description of the Chern character of Jaffe, Lesniewski and Osterwalder [21], Getzler and Szenes [18] then show that for every idempotent $p \in \mathrm{Mat}_n(\mathcal{A})$, one has the index theorem

$$\mathrm{ind}(Q_p) = \big\langle \mathrm{ch}(\mathcal{M}), \mathrm{ch}(p) \big\rangle, \tag{1.1}$$

where $Q_p = \mathbf{c}(p) Q \mathbf{c}(p)$ (considered as an operator on $\mathbf{c}(p)\mathcal{H}^n$) and the $\mathrm{ch}(p)$ of $\mathsf{C}_\epsilon(\mathcal{A})$ is the Chern character of p, introduced by Connes [10] and suitably modified by [18].

The goal of this paper is to generalize these constructions to the case where $\mathcal{A}$ is replaced by a locally convex *differential graded* algebra Ω. When defining a ϑ-summable Fredholm module $\mathcal{M} = (\mathcal{H}, \mathbf{c}, Q)$ in this setting, it turns out that requiring $\mathbf{c}$ to be a representation of Ω turns out to be too restrictive for the applications we have in mind. Instead, we just assume that $\mathbf{c}$ is a degree-preserving linear map from Ω to $\mathcal{L}(\mathcal{H})$ such that

$$[Q, \mathbf{c}(f)] = \mathbf{c}(df), \quad \text{and} \quad \mathbf{c}(f\theta) = \mathbf{c}(f)\mathbf{c}(\theta), \quad \mathbf{c}(\theta f) = \mathbf{c}(\theta)\mathbf{c}(f). \tag{1.2}$$

for all $f \in \Omega^0$ and all $\theta \in \Omega$; in particular, $\mathbf{c}$ is a representation only on the subalgebra Ω^0.[2] The space $\mathsf{C}(\Omega)$ of cyclic chains on Ω carries three differentials: the differential $\underline{d}$ which is induced from the differential d of Ω, the *Hochschild differential* $\underline{b}$ and the *Connes differential* $\underline{B}$, so that $\underline{d} + \underline{b} + \underline{B}$ turns $\mathsf{C}(\Omega)$ into a $\mathbb{Z}_2$ graded complex, the *cyclic complex* of Ω. Then the Chern character $\mathrm{Ch}(\mathcal{M})$ is an element of the dual complex $\mathrm{Hom}(\mathsf{C}(\Omega), \mathbb{C})$. The Chern character now has the following fundamental properties.

Theorem A. *The Chern character* $\mathrm{Ch}(\mathcal{M})$ *is even and closed, that is,*

$$(\underline{d} + \underline{b} + \underline{B})\mathrm{Ch}(\mathcal{M}) = 0. \tag{1.3}$$

[1] Of course, these data are required to satisfy certain analytic conditions, most importantly that the heat semigroup of Q^2 is trace class.

[2] In fact, not even these algebraic properties are needed; dropping these assumptions leads to what we call a *weak* ϑ-summable Fredholm module, c.f. Section 2.

Our index theorem requires to pair $\mathrm{Ch}(\mathcal{M})$ with certain elements of the *entire cyclic complex* $\mathsf{C}_\epsilon(\Omega)$ of Ω (see Def. 3.3 below), and in fact we prove the following.

Theorem B. *The Chern character* $\mathrm{Ch}(\mathcal{M})$ *is an* entire cochain*, that is,* $\mathrm{Ch}(\mathcal{M})$ *can be uniquely extended to a closed and even element of the entire cyclic cocomplex* $\mathcal{L}(\mathsf{C}_\epsilon(\Omega), \mathbb{C})$*, the topological dual complex to* $\mathsf{C}_\epsilon(\Omega)$.

Our generalization of the index formula (1.1) of Getzler and Szenes involves the *Bismut-Chern character* $\mathrm{Ch}(p)$ of Getzler, Jones and Petrack [17, Section 6]. In contrast to the ungraded case, $\mathrm{Ch}(p)$ is not an element of $\mathsf{C}_\epsilon(\Omega)$, but an element of $\mathsf{C}_\epsilon(\Omega_\mathbb{T})$, the entire cyclic complex of the *acyclic extension* $\Omega_\mathbb{T} := \Omega[\sigma]$ of Ω, where σ is a formal variable satisfying $\sigma^2 = 0$ (c.f. Example 2.6 below). Given a ϑ-summable Fredholm module $\mathcal{M} = (\mathcal{H}, \mathbf{c}, Q)$ over Ω, one can extend it to a ϑ-summable Fredholm module[3] $\mathcal{M}_\mathbb{T} = (\mathcal{H}, \mathbf{c}_\mathbb{T}, Q)$ over $\Omega_\mathbb{T}$, so that $\mathrm{Ch}(\mathcal{M}_\mathbb{T})$ is defined by our previous construction. We then have the following index theorem.

Theorem C. *For every idempotent* $p \in \mathrm{Mat}_n(\Omega^0)$*, we have the index formula*

$$\mathrm{ind}(Q_p) = \big\langle \mathrm{Ch}(\mathcal{M}_\mathbb{T}), \mathrm{Ch}(p) \big\rangle. \tag{1.4}$$

While $\mathrm{Ch}(p)$ is not closed in the complex $\mathsf{C}_\epsilon(\Omega_\mathbb{T})$, it is closed in the quotient complex $\mathsf{N}_{\mathbb{T},\epsilon}(\Omega)$, the *(extended) Chen normalized entire cyclic complex* (Def. 3.7 below). Moreover, it turns out that $\mathrm{Ch}(\mathcal{M}_\mathbb{T})$ also descends to an even closed element of $\mathcal{L}(\mathsf{N}_{\mathbb{T},\epsilon}(\Omega), \mathbb{C})$ (Thm. 5.5 below). This leads to a homological index theorem involving the K-theory of Ω^0 (Corollary 8.3).

Let us briefly outline our construction of the Chern character, which generalizes that of Jaffe, Lesniewski and Osterwalder [21] in the ungraded case[4], in a presentation similar to that of Quillen in [29, Section 9]. Following Quillen, an odd bar cochain ω on Ω (c.f. (3.1) below) with values in an algebra L can be seen as a *connection form* on the space of L-valued bar cochains. Its *curvature* is then given by

$$F = \delta\omega + \omega^2,$$

where δ is the codifferential on $\mathsf{B}(\Omega)$. The observation is now that a ϑ-summable Fredholm module $\mathcal{M}$ over Ω determines such a cochain $\omega_\mathcal{M}$ taking values in operators on $\mathcal{H}$, with curvature $F_\mathcal{M}$ (c.f. (4.3) and (4.4) below); the Chern character is then defined by

$$\big\langle \mathrm{Ch}(\mathcal{M}), (\theta_0, \dots, \theta_N) \big\rangle := \mathrm{Str}\Big(\mathbf{c}(\theta_0)\Phi_\mathcal{M}(\theta_1, \dots, \theta_N)\Big), \qquad \text{with} \qquad \Phi_\mathcal{M} = \exp(-F_\mathcal{M}).$$

Closedness of $\mathrm{Ch}(\mathcal{M})$, Thm. A, is then essentially a consequence of the Bianchi identity for $F_\mathcal{M}$. However, the fact that $\omega_\mathcal{M}$ and $F_\mathcal{M}$ takes values in *unbounded* operators, not

[3] In fact, this will be only a *weak* ϑ-summable Fredholm module.

[4] In the sense that if Ω is concentrated in degree zero, then $\mathrm{Ch}(\mathcal{M})$ is equal to the JLO cocycle for the algebra Ω^0.

in $\mathcal{L}(\mathcal{H})$, poses severe analytical problems. Solving these, and obtaining the estimates necessary to show that $\mathrm{Ch}(\mathcal{M})$ is entire (Thm. B) are the main tasks of this paper.
We emphasize that the Chern-Weil type method outlined above constructs the Chern character as a *cochain*, not only as a cohomology class. A remarkable feature of this cochain is that the pairing on the right hand side of (1.4) is *exactly equal* to the supertrace of the heat semigroup $e^{-Q_p^2}$ (c.f. Prop. 8.1); using this, the proof of Thm. C does not need the homotopy invariance of $\mathrm{Ch}(\mathcal{M}_{\mathbb{T}})$, and follows straightforwardly from the McKean-Singer formula, at least in the self-adjoint case. Nevertheless, we show that our Chern character is homotopy invariant, in the following sense.

Theorem D. *As a cohomology class of* $\mathcal{L}(\mathsf{N}_{\epsilon,\mathbb{T}}(\Omega), \mathbb{C})$*, the Chern character* $\mathrm{Ch}(\mathcal{M}_{\mathbb{T}})$ *is independent under homotopies of* $\mathcal{M}$*.*

Our main application is the special case where $\Omega = \Omega(X)$, the algebra differential forms on a compact even-dimensional spin manifold. In this case, our construction provides an integration map for differential forms on the loop space $\mathsf{L}X$ of X. Such a *supersymmetric path integral* has been looked for since the early 80's when Alvarez-Gaumé [1], motivated by observations of Witten [31], noticed that such a path integral map would provide a short and conceptual proof of the Atiyah-Singer index theorem; see also (see also [26] [2] [4]. Our theory leads to a rigorous construction of this path integral and provides a framework that allows to conduct the proof of the Atiyah-Singer theorem by means of a Duistermaat-Heckman type localization formula. We outline this below; a careful exposition of these results will be given in [19], [20], [27].
The way our theory connects to the loop space of X is via *Chen's iterated integral map* and its variations [7] [8] [22]. In particular, the *extended* iterated integral map considered by [17] is a map[5]

$$\rho : \mathsf{C}_\epsilon\big(\Omega(X)_{\mathbb{T}}\big) \longrightarrow \Omega(\mathsf{L}X),$$

through which Chen normalized entire cochains provide a combinatorial model for differential forms on the loop space. In fact, this map is the main motivation to consider the Chen normalized complex in our theory, as in this case, we have precisely

$$\mathsf{N}_{\mathbb{T},\epsilon}\big(\Omega(X)\big) = \mathsf{C}_\epsilon\big(\Omega(X)_{\mathbb{T}}\big) \big/ \ker(\rho). \tag{1.5}$$

In other words, the iterated integral map descends to the quotient $\mathsf{N}_{\mathbb{T},\epsilon}(\Omega(X))$ and is injective there.
Now there is a standard ϑ-summable Fredholm module $\mathcal{M}_X = (\mathcal{H}, \mathbf{c}, Q)$ over $\Omega(X)$. Here $Q = \mathsf{D}$, the Dirac operator acting on the Hilbert space $\mathcal{H} = L^2(X, \Sigma)$ of square-integrable sections of the spinor bundle, while $\mathbf{c}$ is the *quantization map*, which maps $\Omega(X)$ isomorphically to sections of the Clifford algebra bundle over X; this is *not* an algebra homomorphism, but satisfies (1.2). The corresponding Chern character $\mathrm{Ch}(\mathcal{M}_X)$ provides

[5] In fact (with a view on Remark 2.7), the map considered by Getzler, Jones and Petrack is defined only on the smaller complex $\mathsf{C}(\Omega(X)_{\mathbb{T}})$, but it can be continuously extended to the entire complex.

a continuous linear functional on $\mathsf{C}_\epsilon(\Omega(X)_\mathbb{T})$. The statement that $\mathrm{Ch}(\mathcal{M}_X)$ descends to the quotient (1.5) means that it can be pushed forward via ρ to give a linear functional

$$\mu := \rho_* \mathrm{Ch}(\mathcal{M}_X)$$

on $\Omega(\mathsf{L}X)$ (i.e. a *current* on the loop space) with domain $\mathrm{im}(\rho)$.[6] This is the desired path integral map. One shows that ρ is compatible with the various differentials, which implies that μ is closed with respect to the equivariant differential on $\Omega(\mathsf{L}X)$; in other words, μ is *supersymmetric*.

Particularly interesting integrands for the path integral map are the Bismut-Chern characters $\mathrm{BCh}(E, \nabla)$ defined in [4], which are equivariantly closed differential forms on $\mathsf{L}X$ associated to a vector bundle E with connection ∇ over X. By the results of [17], one has $\mathrm{BCh}(E, \nabla) = \rho(\mathrm{Ch}(p))$ for a suitable idempotent p, hence a corollary of Thm. C is the formula

$$\mu\big[\mathrm{BCh}(E, \nabla)\big] = \mathrm{ind}(\mathsf{D}^E), \tag{1.6}$$

where D^E is the Dirac operator twisted by (E, ∇). On the other hand, using a version of Getzler's rescaling trick and the homotopy invariance of the cocycle $\mathrm{Ch}(\mathcal{M}_X)$, Thm. D, one can show that $\mathrm{Ch}(\mathcal{M}_X)$ is in fact cohomologous to the pullback $\rho^*\mu_0$ of the functional $\mu_0 : \Omega(\mathsf{L}X) \to \mathbb{C}$, where μ_0 is given by[7]

$$\mu_0[\xi] = \int_X \widehat{A}(X) \wedge j^*\xi \tag{1.7}$$

for $\xi \in \Omega(\mathsf{L}X)$ (see Example 6.8). As the $\widehat{A}$-genus is the (renormalized) Euler class of the normal bundle of $X \subset \mathsf{L}X$ (c.f. e.g. [2] [23, Section 5]), this result extend the finite-dimensional localization principle of equivariant cohomology [3, Thm. 7.13] to this infinite-dimensional context, a result *bona fide* used in [2], [4]. Since $\mathrm{BCh}(E, \nabla)$ is equivariantly closed and $j^*\mathrm{BCh}(E, \nabla) = \mathrm{ch}(E, \nabla)$ (the usual Chern character of the vector bundle E), combining this localization principle with the index formula (1.6) proves the Atiyah-Singer index theorem for twisted Dirac operators.

These remarks show that our path integral has the desired properties on the cohomological level. On the geometric level, much more can be said. Namely, one can show (at least in special cases) that the loop space current μ obtained from our Bismit-Getzler-Chern character is *exactly equal* to the path integral map considered by Atiyah [2], i.e. it is given by the formula[8]

$$\mu[\xi] = \int_{\mathsf{L}X} e^{-S+\omega} \wedge \xi, \tag{1.8}$$

for where the right hand side is to be interpreted as in [19], [20].

As far as we know, the idea to use cyclic cohomology of Ω and the iterated integral map to construct the supersymmetric path integral is due to Getzler, and this paper can be

[6] Naturally, μ is not defined on all of $\Omega(\mathsf{L}X)$. Even the usual integration functional is not defined on all of $\Omega(Y)$ if Y is finite dimensional *non-compact* manifold.

[7] Here $j : X \to \mathsf{L}X$ is the inclusion as constant loops and $\widehat{A}(X)$ denotes the $\widehat{A}$-genus.

[8] Here S is the energy functional on the loop space and ω is a canonical two-form on $\mathsf{L}X$, c.f. [19].

seen as a completion of this endeavor. Needless to say, our work is largely inspired by the work of Getzler (in particular [18], [17], [14], [13] and [15]).

While the example of $\Omega = \Omega(X)$ is of paramount interest, we develop our theory for general (not necessarily graded commutative) dg algebras Ω. It turns out, that even to discuss the commutative case, it is useful to have the general theory at disposal; see the proof of Thm. 8.1, related to Example 2.3.

The outline of this paper is as follows. First we give the definition of a ϑ-summable Fredholm module over Ω and give several motivating examples. In Section 3, we introduce the algebraic preliminaries for the constructions in Section 4, 5 and 6, where we prove Thm. A, Thm. B and Thm. D. Then in Section 7, we discuss the Bismut-Chern character of an idempotent and subsequently prove Thm. C, in Section 8.

Acknowledgement. It is our pleasure to thank J.-M. Bismut, S. Cacciatori, F. Hanisch, N. Higson, M. Lesch, V. Mathai and S. Shen for helpful discussions. We are further indebted to the Max-Planck-Institute in Bonn, the Humboldt University in Berlin and The University of Adelaide, where parts of the work on this project was conducted. The second-named author was supported by the Max-Planck-Foundation and the ARC Discovery Project grant FL170100020 under Chief Investigator and Australian Laureate Fellow Mathai Varghese.

Conventions. In this paper, all vector spaces and algebras are over $\mathbb{C}$, and we always work in the category of graded vector spaces. Every $\mathbb{Z}$-graded vector space is considered as $\mathbb{Z}_2$-graded in terms of the induced even/odd grading, and tensor products and direct sums of graded vector spaces are equipped with their canonically given $\mathbb{Z}$-grading (and thus the induced $\mathbb{Z}_2$-grading). An ungraded vector space will be considered to be $\mathbb{Z}$-graded (and thus $\mathbb{Z}_2$ graded) by declaring all its elements to have degree zero.

If V, W are $\mathbb{Z}_2$-graded vector spaces, the vector space $\mathrm{Hom}(V, W)$ is $\mathbb{Z}_2$-graded in the usual way. Given coefficients W, any $A \in \mathrm{End}(V)$ has a $\mathbb{Z}_2$-graded dual map $A^\vee \in \mathrm{Hom}(V, W)$ given by

$$(A^\vee \ell)(v) := (-1)^{|A||\ell|} \ell(A(v)), \quad \ell \in \mathrm{Hom}(V,W), \quad v \in V. \tag{1.9}$$

We always use this dual (instead of the ungraded version), and usually, we just write A again instead of $A^\vee$. Throughout, given $A, B \in \mathrm{End}(V)$, the bracket $[A, B] \in \mathrm{End}(V)$ denotes the graded commutator (or supercommutator)

$$[A,B] := AB - (-1)^{|A||B|} BA.$$

There are no ungraded commutators in this paper.

Finally, recall that a differential graded algebra Ω (or dg algebra for short) is an algebra that is the direct sum of subspaces $\Omega^j \subset \Omega$, $j \in \mathbb{Z}$, such that $\Omega^i \Omega^j \subset \Omega^{i+j}$ for all $i, j \in \mathbb{Z}$, together with a degree $+1$ differential d which satisfies the graded Leibniz rule. We always assume that Ω has a unit $\mathbf{1} \in \Omega$.

2 ϑ-summable Fredholm Modules over Locally Convex dg Algebras

Recall that a *locally convex dg algebra* is a dg algebra Ω, where the underlying vector space carries the structure of a locally convex vector space such that the differential is continuous and such that the product map is jointly continuous. Explicitly, these two conditions mean that for each continuous seminorm ν on Ω, there exists a continuous seminorm ν' on Ω such that

$$\nu(d\theta) \leq \nu'(\theta), \quad \nu(\theta_1\theta_2) \leq \nu'(\theta_1)\nu'(\theta_2) \quad \text{for all } \theta, \theta_1, \theta_2 \in \Omega. \tag{2.1}$$

Moreover, we require that Ω is the *topological* direct sum of its homogenous summands Ω^j; in particular, each of these is closed in Ω.

Definition 2.1 (ϑ-summable Fredholm module). An *(even) ϑ-summable Fredholm module* over a locally convex dg algebra Ω is a triple $\mathcal{M} = (\mathcal{H}, \mathbf{c}, Q)$, where

(i) $\mathcal{H}$ is a $\mathbb{Z}_2$-graded Hilbert space;

(ii) $\mathbf{c} : \Omega \to \mathcal{L}(\mathcal{H})$ is an even linear map;

(iii) Q is an odd self-adjoint unbounded operator on $\mathcal{H}$,

such that we have

$$[Q, \mathbf{c}(f)] = \mathbf{c}(df), \quad \text{and} \quad \mathbf{c}(f\theta) = \mathbf{c}(f)\mathbf{c}(\theta), \quad \mathbf{c}(\theta f) = \mathbf{c}(\theta)\mathbf{c}(f) \tag{2.2}$$

for all $f \in \Omega^0$ and all $\theta \in \Omega$. Moreover, we have the following analytic requirements.

(A1) There exists a continuous seminorm ν on Ω such that $\|\mathbf{c}(\theta)\| \leq \nu(\theta)$ for all $\theta \in \Omega$;

(A2) For each $\theta \in \Omega$ the operator $\Delta^{1/2}\mathbf{c}(\theta)\Delta^{-1/2}$ is densely defined and bounded, where $\Delta = Q^2 + 1$; moreover, there exists a continuous seminorm ν on Ω such that

$$\left\|\Delta^{1/2}\mathbf{c}(\theta)\Delta^{-1/2}\right\| \leq \nu(\theta) \quad \text{for all } \theta \in \Omega, \tag{2.3}$$

and the same is true for the adjoint $\mathbf{c}(\theta)^*$;

(A3) For each $T > 0$ the operator $e^{-TQ^2} \in \mathcal{L}(\mathcal{H})$ is trace-class.

If instead of (2.2) $\mathcal{M}$ only satisfies the weaker property $\mathbf{c}(\mathbf{1}) = 1$ (the identity operator), we call $\mathcal{M}$ a *weak ϑ-summable Fredholm module.*

Remark 2.2. It is easy to see that condition (A2) is equivalent to requiring that $\mathbf{c}(\theta)$ and $\mathbf{c}(\theta)^*$ map $\mathrm{dom}(\Delta) = \mathrm{dom}(Q)$ to itself, and are moreover bounded with respect to the graph norm on $\mathrm{dom}(Q)$.

In the remainder of this section we give several examples of Fredholm modules. The following example highlights the relation to the usual notion of a ϑ-summable Fredholm module over an algebra.

Example 2.3 (Non-commutative differential forms). Let $\mathcal{A}$ be an ungraded locally convex algebra, together with a ϑ-summable Fredholm module over $\mathcal{A}$ in the sense of Connes [12]. That is, we are given a triple $\mathcal{M}_0 = (\mathcal{H}, \mathbf{c}_0, Q)$, where $\mathcal{H}$ is a $\mathbb{Z}_2$-graded Hilbert space, Q is an odd selfadjoint unbounded operator satisfying (A3) and $\mathbf{c}_0 : \mathcal{A} \to \mathcal{L}^+(\mathcal{H})$ a representation such that there exists a continuous seminorm ν on $\mathcal{A}$ with

$$\|\mathbf{c}_0(a)\| + \|[Q, \mathbf{c}_0(a)]\| \leq \nu(a) \tag{2.4}$$

for all $a \in \mathcal{A}$. Under the additional assumption that there exists a continuous seminorm τ on $\mathcal{A}$ such that

$$\|\Delta^{1/2}[Q, \mathbf{c}_0(a)]\Delta^{-1/2}\| + \|\Delta^{1/2}[Q, \mathbf{c}_0(a)^*]\Delta^{-1/2}\| \leq \tau(a), \tag{2.5}$$

where $\Delta = Q^2 + 1$, there exists a canonical extension of $\mathcal{M}_0$ to a Fredholm module $\mathcal{M} = (\mathcal{H}, \mathbf{c}, Q)$ over the differential graded algebra of non-commutative differential forms $\Omega_{\mathcal{A}}$, as we now explain.

Let us start be reviewing the construction of $\Omega_{\mathcal{A}}$. First let $\Omega^0_{\mathcal{A}} := \mathcal{A}$ and let $\Omega^1_{\mathcal{A}}$ be the bimodule of *one-forms on* $\mathcal{A}$, defined as the quotient of the free bimodule on generators da, $a \in \mathcal{A}$, by the relations

$$\begin{aligned} d(\lambda a + \mu b) &= \lambda da + \mu db \\ d(ab) &= a\, db + da\, b, \end{aligned} \qquad a, b \in \mathcal{A}, \quad \lambda, \mu \in \mathbb{C}.$$

Now we set

$$\Omega^n_{\mathcal{A}} := \underbrace{\Omega^1_{\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1_{\mathcal{A}}}_{n} \qquad \text{and} \qquad \Omega_{\mathcal{A}} := \bigoplus_{n=0}^{\infty} \Omega^n_{\mathcal{A}}.$$

The product in $\Omega_{\mathcal{A}}$ is the obvious one, induced by the tensor algebra, while the formula

$$d(a_0 da_1 \cdots da_n) = da_0 da_1 \cdots da_n$$

gives is a well-defined differential which turns $\Omega_{\mathcal{A}}$ into a dg algebra. Here we use that using the second relation above, any element $\theta \in \Omega^n_{\mathcal{A}}$ can be written as a sum of elements of the form $a_0 da_1 \cdots da_n$ for $a_0, \dots, a_n \in \mathcal{A}$ (for a reference, see e.g. [25, 2.6.4]).

Now it can be checked that setting

$$\mathbf{c}(a_0 da_1 \cdots da_n) := \mathbf{c}_0(a_0)[Q, \mathbf{c}_0(a_1)] \cdots [Q, \mathbf{c}_0(a_n)] \tag{2.6}$$

is a well-defined linear map $\Omega_{\mathcal{A}} \to \mathcal{L}(\mathcal{H})$ which is obviously grading preserving (as Q is odd) and satisfies (2.2). Clearly, the condition (A1) follows from (2.4) and (A2) follows since

$$\Delta^{1/2}\mathbf{c}(a_0 da_1 \cdots da_n)\Delta^{-1/2} = (\Delta^{1/2}\mathbf{c}_0(a_0)\Delta^{-1/2}) \prod_{k=1}^{n} (\Delta^{1/2}[Q, \mathbf{c}_0(a_k)]\Delta^{-1/2}),$$

which can be estimated using (2.4) respectively (2.5).

The following two examples of our construction are the main motivation of this paper, as explained in the introduction.

Example 2.4 (Differential forms and spinors). Let X be an even-dimensional compact spin (or spin^c) manifold and let $\Sigma \to X$ be the corresponding complex spinor bundle with the Dirac operator D. There is a standard way to construct a ϑ-summable Fredholm module over $\Omega(X)$, the locally convex dg algebra of differential forms on X. Here $\mathcal{H} = L^2(X, \Sigma)$, the space of Borel equivalence classes of square-integrable sections $\Sigma \to X$ and $Q = \mathsf{D}$, the Dirac operator, while $\mathbf{c}$ is the so-called *quantization map* (c.f. [3, Prop. 3.5]): For differential one forms $\theta_1, \dots, \theta_k \in \Omega^1(X)$, the endomorphism of the spinor bundle $\mathbf{c}(\theta_1 \wedge \cdots \wedge \theta_k)$ is given by

$$\mathbf{c}(\theta_1 \wedge \cdots \wedge \theta_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathbf{c}_1(\theta_{\sigma_1}) \cdots \mathbf{c}_1(\theta_{\sigma_k}), \tag{2.7}$$

where on the right hand side, $\mathbf{c}_1(\theta_j)$ denotes Clifford multiplication by the vector field which corresponds to θ_j via the Riemannian structure on X. Each $\mathbf{c}(\theta_1 \wedge \cdots \wedge \theta_k)$ acts as multiplication operator on $L^2(X, \Sigma)$, and then $\mathcal{M}_X = (L^2(X, \Sigma), \mathbf{c}, \mathsf{D})$ is a ϑ-summable Fredholm module over $\Omega(X)$: The required algebraic properties are well-known to hold in this case. Moreover, (A1) is satisfied because D^2 is a non-negative elliptic differential operator which thus has a smooth heat kernel; the estimate (A2) is trivial since $\mathbf{c}$ is a fiberwise isometry; finally, (A3) follows from elliptic estimates, as $[\mathsf{D}, \mathbf{c}(\theta)]$ is a first order differential operator for each θ. Note that

$$\mathbf{c} : \Omega(X) \longrightarrow \mathcal{L}\big(L^2(X, \Sigma)\big)$$

is *not* multiplicative (in particular, not a representation), if $\dim(X) \geq 2$.
This example has obvious generalizations to general unitary Clifford modules in the sense of [3, Def. 3.32].

Example 2.5 (Cycles in K-homology). Generalizing the previous example, if X is any smooth manifold (not necessarily compact or spin), compact spin (or spin^c) manifolds M over X give rise to a ϑ-summable Fredholm module $\mathcal{M} = (\mathcal{H}, \mathbf{c}, Q)$ over the locally convex algebra $\Omega(X)$. More precisely, suppose we are given an even-dimensional smooth spin^c manifold M with spinor bundle Σ_M, together with a smooth map $\varphi : M \to X$. Then we can set $\mathcal{H} = L^2(M, \Sigma_M)$, $Q = \mathsf{D}_M$ (the Dirac operator on M) and $\mathbf{c}(\theta) = \mathbf{c}_M(\varphi^*\theta)$, where $\mathbf{c}_M$ is the quanization map on M as discussed in Example 2.4. Remember that such tuples (M, Σ_M, φ) represent cycles in the K-homology of X.

We now describe an important construction in this paper, which produces a new Fredholm module $\mathcal{M}_{\mathbb{T}}$ from a given Fredholm module $\mathcal{M}$; in fact, this will only be a *weak* Fredholm module, which is our main reason to introduce this weaker notion in the first place. The motivation for this construction is its connection to equivariant homology on loop spaces in the special case of Example 2.4.

Example 2.6 (Acyclic extension of a Fredholm module). Given a locally convex dg algebra Ω, we can form the dg algebra

$$\Omega_{\mathbb{T}} := \Omega[\sigma],$$

where σ is a formal variable of degree -1 with $\sigma^2 = 0$. Elements $\theta \in \Omega_{\mathbb{T}}$ can be written uniquely in the form $\theta = \theta' + \sigma\theta''$, where $\theta', \theta'' \in \Omega$, and the differential is

$$d_{\mathbb{T}} = d - \iota \qquad \text{with} \qquad d\theta = d\theta' - \sigma d\theta'' \quad \text{and} \quad \iota\theta = \theta''. \tag{2.8}$$

Then $\Omega_{\mathbb{T}}$ becomes a locally convex dg algebra in view of the vector space isomorphism $\Omega_{\mathbb{T}} \cong \Omega \oplus \Omega[1]$, called the *acyclic extension* of Ω. $\Omega_{\mathbb{T}}$ is indeed acyclic, as all closed elements have the form $\theta' + \sigma d\theta' = -d_{\mathbb{T}}(\sigma\theta')$, hence are exact.
Now given a (weak) ϑ-summable Fredholm module $\mathcal{M} = (\mathcal{H}, \mathbf{c}, Q)$, the *acyclic extension of* $\mathcal{M}$ is the weak ϑ-summable Fredholm module $\mathcal{M}_{\mathbb{T}} = (\mathcal{H}, \mathbf{c}_{\mathbb{T}}, Q)$ over $\Omega_{\mathbb{T}}$, where $\mathbf{c}_{\mathbb{T}}$ is given by

$$\mathbf{c}_{\mathbb{T}}(\theta' + \sigma\theta'') := \mathbf{c}(\theta').$$

We will often write $\mathbf{c}$ again instead of $\mathbf{c}_{\mathbb{T}}$. Notice that usually, this is indeed only a weak Fredholm module, since $(\Omega_{\mathbb{T}})^0$ is strictly larger than Ω^0 unless $\Omega^1 = \{0\}$.

Remark 2.7. In the special case that $\Omega = \Omega(X)$, sending σ to dt gives an isomorphism of algebras

$$\Omega_{\mathbb{T}} \cong \Omega(X \times S^1)^{S^1}, \tag{2.9}$$

the algebra of differential forms on $X \times S^1$ that are invariant under the S^1-action on the second factor. Under this isomorphsim, the differential ι on $\Omega_{\mathbb{T}}$ corresponds to the insertion of the vector field ∂_t on S^1. Note that the isomorphism (2.9) does not preserve degrees, as we declared σ to have degree -1 in order to make the differential $d_{\mathbb{T}}$ homogeneous. Alternatively, this can be fixed by introducing a formal variable of degree two, but we shall avoid that in this paper.

We will see in Section 7 that the Bismut-Chern character $\mathrm{Ch}(p)$ of an idempotent p in Ω^0 is not a chain over Ω but rather a chain over the acyclic extension $\Omega_{\mathbb{T}}$; hence we are going to need the Chern character of $\mathcal{M}_{\mathbb{T}}$ in order to formulate our index theorem.

3 Algebraic Preliminaries

In this section, we discuss the algebraic preliminaries needed for the construction of our Chern character. In particular, we introduce the chain and cochain complexes that we will be working on: The bar complex and the cyclic complex, as well as its entire and Chen normalized versions. Throughout this section, let Ω be a dg algebra.

3.1 Bar Chains and Cochains

The *bar construction* on Ω is the graded vector space

$$\mathsf{B}(\Omega) = \bigoplus_{N=0}^{\infty} \Omega[1]^{\otimes N}. \tag{3.1}$$

Here $\Omega[1]$ denotes the $\mathbb{Z}$-graded space with $\Omega[1]^k = \Omega^{k+1}$. The grading on $\mathsf{B}(\Omega)$ is then explicitly given by

$$\mathsf{B}^n(\Omega) = \bigoplus_{N=0}^{\infty} \bigoplus_{\ell_1+\cdots+\ell_N=n+N} \Omega^{\ell_1} \otimes \cdots \otimes \Omega^{\ell_N}.$$

The space $\mathsf{B}(\Omega)$ carries the two differentials d and b', given by[9]

$$d(\theta_1, \ldots, \theta_N) = \sum_{k=1}^{N} (-1)^{n_{k-1}} (\theta_1, \ldots, \theta_{k-1}, d\theta_k, \ldots, \theta_N)$$

$$b'(\theta_1, \ldots, \theta_N) = -\sum_{k=1}^{N-1} (-1)^{n_k} (\theta_1, \ldots, \theta_{k-1}, \theta_k\theta_{k+1}, \theta_{k+2}, \ldots, \theta_N)$$

where $n_k = |\theta_1| + \cdots + |\theta_k| - k$. The above differentials satisfy $db' + b'd = 0$, hence turn $\mathsf{B}(\Omega)$ into a bi-complex with total differential $d + b'$.

Definition 3.1 (Bar chains and cochains). Elements of $\mathsf{B}(\Omega)$ will be called *bar chains on* Ω. Given a $\mathbb{Z}_2$-graded algebra L of coefficients, we define an *L-valued bar cochain on* Ω to be an element of the space $\mathrm{Hom}(\mathsf{B}(\Omega), L)$.

An L-valued bar cochain can be identified with a sequence $\ell^{(0)}, \ell^{(1)}, \ell^{(2)}, \ldots$, consisting of its arity N components; each component is a multi-linear map

$$\ell^{(N)} : \underbrace{\Omega \times \cdots \times \Omega}_{N} \longrightarrow L.$$

In particular, the component $\ell^{(0)}$ is a linear map from $\mathbb{C}$ to L, which can be identified with an element of L.

Following Quillen [29] we introduce another sign in the definition of the *codifferential*

$$\delta := -(d + b') : \mathrm{Hom}(\mathsf{B}(\Omega), L) \longrightarrow \mathrm{Hom}(\mathsf{B}(\Omega), L), \tag{3.2}$$

which is defined as in (1.9). It will be important for our constructions that the space of L-valued bar cochains is a $\mathbb{Z}_2$-graded algebra with respect to the product given by

$$\ell_1\ell_2(\theta_1, \ldots, \theta_N) = \sum_{k=1}^{N} (-1)^{|\ell_2|(|\theta_1|+\cdots+|\theta_k|)} \ell_1(\theta_1, \ldots, \theta_k)\ell_2(\theta_{k+1}, \ldots, \theta_N), \tag{3.3}$$

[9] The sign of b' coincides with that of Quillen's b' [29], who has considered the case of ungraded algebras.

in a way that δ satisfies the $\mathbb{Z}_2$-graded Leibniz rule

$$\delta(\ell_1\ell_2) = (\delta\ell_1)\ell_2 + (-1)^{|\ell_1|}\ell_1(\delta\ell_2).$$

Thus δ turns $\mathrm{Hom}(\mathsf{B}(\Omega), L)$ into a differential $\mathbb{Z}_2$-graded algebra.

3.2 Cyclic Chains and Cochains

Throughout, we denote by $\underline{\Omega}$ the quotient space $\underline{\Omega} := \Omega/\mathbb{C}\mathbf{1}$. Note that this is *not* an algebra again, unless we are provided with an augmentation. The *(reduced) cyclic complex* on Ω is the graded vector space

$$\mathsf{C}(\Omega) = \bigoplus_{N=0}^{\infty} \Omega \otimes \underline{\Omega}[1]^{\otimes N}. \tag{3.4}$$

Explicitly, the grading is given by

$$\mathsf{C}^n(\Omega) = \bigoplus_{M=0}^{\infty} \bigoplus_{\ell_0+\cdots+\ell_M=n+M} \Omega^{\ell_0} \otimes \underline{\Omega}^{\ell_1} \otimes \cdots \otimes \underline{\Omega}^{\ell_M}.$$

The complex $\mathsf{C}(\Omega)$ carries two degree $+1$ differentials, given by the formulas

$$\underline{d}(\theta_0, \dots, \theta_N) = (d\theta_0, \theta_1, \dots, \theta_N) - \sum_{k=1}^{N} (-1)^{m_{k-1}} (\theta_0, \dots \theta_{k-1}, d\theta_k, \theta_{k+1}, \dots, \theta_N)$$

$$\underline{b}(\theta_0, \dots, \theta_N) = \sum_{k=0}^{N-1} (-1)^{m_k} (\theta_0, \dots, \theta_k\theta_{k+1}, \dots, \theta_N) - (-1)^{(|\theta_N|-1)m_{N-1}} (\theta_N\theta_0, \theta_1, \dots, \theta_{N-1}).$$

where $m_k = |\theta_0| + \cdots + |\theta_k| - k$. It is easy to check that these formulas make sense on the quotient space $\Omega \otimes \underline{\Omega}[1]^{\otimes N}$. There is another differential, of degree -1, the *Connes operator* $\underline{B}$, which is given by the formula

$$\underline{B}(\theta_0, \dots, \theta_N) = \sum_{k=0}^{N} (-1)^{(m_k+1)(m_N-m_k)} (\mathbf{1}, \theta_{k+1}, \dots, \theta_N, \theta_0, \dots, \theta_k)$$

with m_k as before. We remark that the Connes differential is one reason to restrict discussion to the reduced case, since its formula becomes more complicated otherwise.

Definition 3.2 (Cyclic chains and cochains). A *cyclic chain* is an element of $\mathsf{C}(\Omega)$. A *cyclic cochain* is an element of the space $\mathrm{Hom}(\mathsf{C}(\Omega), \mathbb{C})$ of linear forms on $\mathsf{C}(\Omega)$.

The three differentials $\underline{d}$, $\underline{b}$ and $\underline{B}$ pairwise anti-commute, hence we get the $\mathbb{Z}_2$-graded complex (resp. cocomplex)

$$\mathsf{C}^+(\Omega) \underset{\underline{d}+\underline{b}+\underline{B}}{\overset{\underline{d}+\underline{b}+\underline{B}}{\rightleftarrows}} \mathsf{C}^-(\Omega), \qquad \mathrm{Hom}_+(\mathsf{C}(\Omega), \mathbb{C}) \underset{\underline{d}+\underline{b}+\underline{B}}{\overset{\underline{d}+\underline{b}+\underline{B}}{\rightleftarrows}} \mathrm{Hom}_-(\mathsf{C}(\Omega), \mathbb{C}), \tag{3.5}$$

where the spaces $\mathsf{C}^{\pm}(\Omega)$ are given by reducing the $\mathbb{Z}$-grading of $\mathsf{C}(\Omega)$ modulo two. Note that the above are not $\mathbb{Z}$-graded (co)complexes, as the differentials are inhomogeneous. We call (3.5) the *cyclic complex*, respectively the *cyclic cocomplex* of Ω.

3.3 Entire Cyclic Chains and Cochains

In this section, we consider the setting where our dg algebra Ω carries a topology. Hence throughout this subsection, we assume that Ω is a locally convex dg algebra. The space $\mathsf{C}(\Omega)$ then carries the induced *projective tensor product topology*. Remember that this topology is generated by the family of seminorms

$$\nu_N(c) = \inf\left\{\sum_\alpha \nu\big(\theta_0^{(\alpha)}\big)\cdots\nu\big(\theta_N^{(\alpha)}\big) : c = \sum_\alpha \theta_0^{(\alpha)}\otimes\cdots\otimes\theta_N^{(\alpha)}\right\}, \tag{3.6}$$

on $\Omega\otimes\underline{\Omega}^{\otimes N}$ for continuous seminorms ν on Ω, where the infimum is taken over all representations of $c \in \mathsf{C}(\Omega)$ as a finite sum of elementary tensors.

Definition 3.3 (Entire cyclic chains and cochains). For each continuous seminorm ν on Ω, we define a seminorm ν_ϵ on $\mathsf{C}(\Omega)$ by

$$\nu_\epsilon(c) := \sum_{N=0}^\infty \frac{\nu_N(c_N)}{\sqrt{N!}} \qquad \text{for} \quad c = \sum_{N=0}^\infty c_N.$$

The space $\mathsf{C}_\epsilon(\Omega)$ of *entire cyclic chains on* Ω is then defined to be the closure of $\mathsf{C}(\Omega)$ with respect to this collection of seminorms. An *entire cyclic cochain* is an element of the space $\mathcal{L}(\mathsf{C}_\epsilon(\Omega), \mathbb{C})$ of continuous linear functionals on $\mathsf{C}_\epsilon(\Omega)$.

Remark 3.4. These growth conditions were previously considered e.g. in [24] or [22].

In view of the infinite product appearing in the definition of $\mathsf{C}_\epsilon(\Omega)$, this vector space is not $\mathbb{Z}$-graded anymore, but the $\mathbb{Z}_2$-grading prevails. As the differentials $\underline{d}$, $\underline{b}$, $\underline{B}$ are easily checked to be continuous on $\mathsf{C}(\Omega)$ with respect to the seminorms ν_ϵ, they extend to odd parity differentials on $\mathsf{C}_\epsilon(\Omega)$; we denote these extensions by the same symbols again, leading to the *entire cyclic complex* and the *entire cyclic cocomplex* of Ω.

As for bar cochains, a cyclic cochain $\ell \in \mathrm{Hom}(\mathsf{C}(\Omega), \mathbb{C})$ can be identified with the sequence $\ell^{(0)}, \ell^{(1)}, \ell^{(2)}, \dots$ of its arity N components, each component being a multi-linear map

$$\ell^{(N)} : \Omega \times \underbrace{\underline{\Omega}[1]\times\cdots\times\underline{\Omega}[1]}_{N} \longrightarrow \mathbb{C}.$$

If there exists a continuous seminorm ν on Ω, and constants $C, z > 0$, such that for each $N \in \mathbb{N}_0$, we have

$$|\ell^{(N)}(\theta_0,\dots,\theta_N)| \leq z^N \frac{\nu(\theta_0)\cdots\nu(\theta_N)}{\sqrt{N!}} \tag{3.7}$$

for all $\theta_0 \in \Omega$, $\theta_1,\dots,\theta_N \in \underline{\Omega}^{\otimes N}$, then ℓ has a unique extension to an entire cochain, i.e. an element of $\mathcal{L}(\mathsf{C}_\epsilon(\Omega), \mathbb{C})$. Note that in the estimate, the factor z^N in fact be absorbed in the seminorm ν.

3.4 Chen Normalization

In this section, we construct the (co)complexes that our constructions will ultimately live on. To this end, we consider a certain subcomplex $\mathsf{D}(\Omega)$ of the complex $\mathsf{C}(\Omega)$, first considered in [16], based on the original idea of [9]. We then extend these ideas to entire chains; this was first considered in [24]. Again, we assume that Ω is a locally convex dg algebra.

To define the subcomplex $\mathsf{D}(\Omega)$ of $\mathsf{C}(\Omega)$ consider for all $f \in \Omega^0$ and $i \in \mathbb{N}_0$ the continuous linear map

$$S_i^{(f)} : \mathsf{C}(\Omega) \longrightarrow \mathsf{C}(\Omega)$$

defined by

$$S_i^{(f)}(\theta_0, \ldots, \theta_N) = \begin{cases} (\theta_0, \ldots, \theta_i, f, \theta_{i+1}, \ldots, \theta_N) & 0 \leq i \leq N \\ 0 & \text{otherwise} \end{cases}. \tag{3.8}$$

Denote by $\mathsf{D}(\Omega)$ the subspace of $\mathsf{C}(\Omega)$ spanned by the images of the operators $S_i^{(f)}$ and $[\underline{d} + \underline{b}, S_i^{(f)}]$. It is straightforward to check that both $\underline{d} + \underline{b}$ and $\underline{B}$ preserve $\mathsf{D}(\Omega)$.

Remark 3.5. The subcomplex $\mathsf{D}(\Omega)$ was first considered by Chen in [9] in the case that $\Omega = \Omega(X)$ for a manifold X. He also showed that $\mathsf{D}(\Omega(X))$ is acyclic with respect to the differential $\underline{d} + \underline{b}$ in the case that X is connected, meaning that the projection from $\mathsf{C}(\Omega(X))$ to the quotient complex

$$\mathsf{N}\big(\Omega(X)\big) := \mathsf{C}\big(\Omega(X)\big)/\mathsf{D}\big(\Omega(X)\big)$$

is homotopy equivalent to $\mathsf{C}(\Omega(X))$ in this case (with respect to $\underline{d} + \underline{b}$). This was suitably generalized by Getzler and Jones to general dg algebras [16, Section 5].

Definition 3.6 (Chen normalized entire cyclic (co)-chains). The space of *Chen normalized entire cyclic chains* is the quotient space

$$\mathsf{N}_\epsilon(\Omega) := \mathsf{C}_\epsilon(\Omega)/\overline{\mathsf{D}(\Omega)},$$

where $\overline{\mathsf{D}(\Omega)}$ denotes the closure of $\mathsf{D}(\Omega)$ in $\mathsf{C}_\epsilon(\Omega)$. The space of *Chen normalized entire cyclic cochains* is the space $\mathcal{L}(\mathsf{N}_\epsilon(\Omega),\mathbb{C})$. This leads to the *Chen normalized entire cyclic complex* and, respectively, the *Chen normalized entire cyclic cocomplex*

$$\mathsf{N}_\epsilon^+(\Omega) \underset{\underline{d}+\underline{b}+\underline{B}}{\overset{\underline{d}+\underline{b}+\underline{B}}{\rightleftarrows}} \mathsf{N}_\epsilon^-(\Omega), \qquad \text{and} \qquad \mathcal{L}_+\big(\mathsf{N}_\epsilon(\Omega), \mathbb{C}\big) \underset{\underline{d}+\underline{b}+\underline{B}}{\overset{\underline{d}+\underline{b}+\underline{B}}{\rightleftarrows}} \mathcal{L}_-\big(\mathsf{N}_\epsilon(\Omega), \mathbb{C}\big).$$

We denote the corresponding homology groups by $\mathsf{h}_\epsilon^\pm(\Omega)$ respectively $\mathsf{h}_\pm^\epsilon(\Omega)$.

Clearly, by passing to the quotient, any chain $c \in \mathsf{C}^{\pm}_{\epsilon}(\Omega)$ defines an element in $\mathsf{N}^{\pm}_{\epsilon}(\Omega)$. Notice that in order for c to define an element in $\mathsf{h}^{\pm}_{\epsilon}(\Omega)$, it has to be closed only up to $\mathsf{D}(\Omega)$, in other words, it just has to satisfy $(\underline{d} + \underline{b} + \underline{B})c \in \mathsf{D}(\Omega)$. On the other hand, a cochain ℓ on $\mathsf{C}_{\epsilon}(\Omega)$ descends to a cochain on $\mathsf{N}^{\pm}_{\epsilon}(\Omega)$ only if it vanishes on $\mathsf{D}(\Omega)$.

We finish by discussing a suitable extension of this construction to the acyclic extension $\Omega_{\mathbb{T}}$ of Ω, discussed in Example 2.6. In this case, we consider the maps

$$S_i^{(f)} : \mathsf{C}(\Omega_{\mathbb{T}}) \longrightarrow \mathsf{C}(\Omega_{\mathbb{T}})$$

defined by the same formula (3.8). However, we emphasize that we still take f to be in Ω^0, not in the (generally larger) subspace $(\Omega_{\mathbb{T}})^0$. We then define $\mathsf{D}_{\mathbb{T}}(\Omega)$ to be the span of the images of all operators $S_i^{(f)}$ and $[\underline{d} + \underline{b}, S_i^{(f)}]$, for $f \in \Omega^0$ and $i \in \mathbb{N}_0$. Again, note that $\mathsf{D}_{\mathbb{T}}(\Omega) \neq \mathsf{D}(\Omega_{\mathbb{T}})$. It turns out that in this context, it is suitable to also consider the operator

$$S : \mathsf{C}(\Omega_{\mathbb{T}}) \longrightarrow \mathsf{C}(\Omega_{\mathbb{T}}), \qquad S(\theta_0, \dots, \theta_N) = \sum_{k=0}^{N} (\theta_0, \dots, \theta_k, \sigma, \theta_{k+1}, \dots, \theta_N), \tag{3.9}$$

which appears in the following definition.

Definition 3.7 (Extended Chen normalized entire cyclic (co-)chains). The space of *extended Chen normalized entire cyclic chains* is the quotient complex

$$\mathsf{N}_{\mathbb{T},\epsilon}(\Omega) := \mathsf{C}_{\epsilon}(\Omega_{\mathbb{T}}) / \overline{\big(\mathsf{D}_{\mathbb{T}}(\Omega) + \mathrm{im}(S - \mathrm{id})\big)},$$

where we take the closure inside $\mathsf{C}_{\epsilon}(\Omega_{\mathbb{T}})$. The space of *extended Chen normalized entire cyclic cochains* is the space $\mathcal{L}(\mathsf{N}_{\mathbb{T},\epsilon}(\Omega), \mathbb{C})$. We get the *extended Chen normalized cyclic complex* and the *extended Chen normalized cyclic cocomplex*

$$\mathsf{N}^{+}_{\mathbb{T},\epsilon}(\Omega) \underset{\underline{d}+\underline{b}+\underline{B}}{\overset{\underline{d}+\underline{b}+\underline{B}}{\rightleftarrows}} \mathsf{N}^{-}_{\mathbb{T},\epsilon}(\Omega), \qquad \text{and} \qquad \mathcal{L}_{+}\big(\mathsf{N}_{\mathbb{T},\epsilon}(\Omega), \mathbb{C}\big) \underset{\underline{d}+\underline{b}+\underline{B}}{\overset{\underline{d}+\underline{b}+\underline{B}}{\rightleftarrows}} \mathcal{L}_{-}\big(\mathsf{N}_{\mathbb{T},\epsilon}(\Omega), \mathbb{C}\big).$$

The corresponding homology groups will be denoted by $\mathsf{h}^{\pm}_{\mathbb{T},\epsilon}(\Omega)$ respectively $\mathsf{h}^{\mathbb{T},\epsilon}_{\pm}(\Omega)$.

Again, any chain $c \in \mathsf{C}^{\pm}_{\epsilon}(\Omega)$ defines an element in $\mathsf{N}^{\pm}_{\epsilon}(\Omega)$, while a cochain $\ell \in \mathcal{L}(\mathsf{C}_{\epsilon}(\Omega), \mathbb{C})$ descends to a cochain in $\mathcal{L}(\mathsf{C}_{\epsilon}(\Omega), \mathbb{C})$ if and only if it vanishes on $\mathsf{D}_{\mathbb{T}}(\Omega)$ and on $\mathrm{im}(S - \mathrm{id})$.

Remark 3.8. This quotient was first considered by Getzler, Jones and Petrack in the special case $\Omega = \Omega(X)$ for a manifold X: In fact, the image of $S - \mathrm{id}$ can be written as an ideal with respect to the shuffle product, as defined in Section 4 of [17]; it is generated by the chain $(1, \sigma) - (1) \in \mathsf{C}(\Omega)$. Cast in these terms, such a quotient is considered on p. 28 of [17]. Clearly, there is a map of chain complexes

$$\mathsf{N}_{\epsilon}(\Omega) \longrightarrow \mathsf{N}_{\mathbb{T},\epsilon}(\Omega),$$

natural in Ω. This map is neither surjective nor injective in general, but the results of Getzler, Jones and Petrack seem to indicate that one might suspect this map to be a quasi-isomorphism, at least in the special case $\Omega = \Omega(X)$.

4 The Quantization Map

In this section, we start with the construction of the Chern character for a weak ϑ-summable Fredholm module $\mathcal{M} := (\mathcal{H}, \mathbf{c}, Q)$ over a locally convex dg algebra, which we assume fixed throughout this section.

As observed by Quillen [29], an odd bar cochain ω with values in a $\mathbb{Z}_2$-graded algebra L of coefficients can be seen as a *connection form* on the space of L-valued bar cochains, as we can consider the *connection*[10] $\nabla := \delta + \omega$. Its *curvature* is given by

$$F := \nabla^2 = \delta\omega + \omega^2. \tag{4.1}$$

Note that since $\delta^2 = 0$, F is an L-valued bar cochain itself, as opposed to merely an operator acting on cochains. It satisfies the *Bianchi identity*

$$0 = [\nabla, F] = \delta F + [\omega, F], \tag{4.2}$$

since $[\nabla, F] = \nabla^3 - \nabla^3 = 0$. These two formulas lie at the heart of our investigations.

To apply this, observe that the Fredholm module $\mathcal{M}$ naturally provides such a connection form $\omega_{\mathcal{M}}$ with values in the algebra of operators on $\mathcal{H}$. It is defined by

$$\omega_{\mathcal{M}}^{(0)} = -Q, \qquad \omega_{\mathcal{M}}^{(1)}(\theta) = \mathbf{c}(\theta), \qquad \omega_{\mathcal{M}}^{(N)}(\theta_1, \dots, \theta_N) = 0, \quad N \geq 2, \tag{4.3}$$

and it is odd due to the grading shift in the definition of $\mathsf{B}(\Omega)$. Let

$$F_{\mathcal{M}} = \delta\omega_{\mathcal{M}} + \omega_{\mathcal{M}}^2$$

be the curvature of the connection ω. Explicity, the components of $F_{\mathcal{M}}$ can be easily worked out to be given by

$$\begin{aligned} F_{\mathcal{M}}^{(0)} &= Q^2, \\ F_{\mathcal{M}}^{(1)}(\theta) &= \mathbf{c}(d\theta) - [Q, \mathbf{c}(\theta)] \\ F_{\mathcal{M}}^{(2)}(\theta_1, \theta_2) &= (-1)^{|\theta_1|}\big(\mathbf{c}(\theta_1\theta_2) - \mathbf{c}(\theta_1)\mathbf{c}(\theta_2)\big). \end{aligned} \tag{4.4}$$

with all higher components zero. Notice that the first component essentially is the curvature of the derivation on $\mathcal{L}(\mathcal{H})$ given by taking the commutator with Q, the second component measures the failure of $\mathbf{c}$ and Q to satisfy the relation $\mathbf{c}(d\theta) = [Q, \mathbf{c}(\theta)]$ for $\theta \in \Omega$, while the third component measures the failure of $\mathbf{c}$ to be multiplicative.
What causes complications is that $\omega_{\mathcal{M}}$ does not take values in bounded operators (hence does not actually take values in an algebra). Hence Quillen's constructions are more of a guiding principle, so we have to put in some effort to make them work.

[10] Since ω does not necessarily have arity one, this should rather be called a *superconnection* in the sense of Quillen.

In order to define the Chern character of our Fredholm module, we now aim to define a bar cochain $\Phi_T(\mathcal{M})$ with values in $\mathcal{L}(\mathcal{H})$, the *quantization map*, by exponentiating the curvature $F_{\mathcal{M}}$ in the algebra $\mathcal{L}(\mathsf{B}(\Omega), \mathcal{L}(\mathcal{H}))$, i.e. we would like to set

$$\Phi_T(\mathcal{M}) \overset{\text{formally}}{=} e^{-TF_{\mathcal{M}}}, \tag{4.5}$$

where $T > 0$ is some parameter. If Q was a bounded operator, this is well-defined using the exponential series; however, the fact that Q is unbounded leads to analytical difficulties. Before we discuss the actual definition (see Def. 4.6 below), let us first state the main results of this section. From now on, we drop the dependence on $\mathcal{M}$ in notation, in order to increase readability.

Theorem 4.1 (Fundamental estimate). *For $T > 0$ and all $\theta_1, \dots, \theta_N \in \Omega$, the operator $\Phi_T(\theta_1, \dots, \theta_N)$ is well-defined and trace-class. Moreover, there exists a continuous seminorm ν on Ω, independent of T, such that we have the estimate*

$$\big\|\Phi_T(\theta_1, \dots, \theta_N)\big\|_1 \leq e^{T/2}\mathrm{Tr}(e^{-TQ^2/2})\frac{T^N}{\sqrt{N!}}\nu(\theta_1)\cdots\nu(\theta_N) \tag{4.6}$$

for its trace class norm. The same statement is true for $Q\Phi_T$ and $\Phi_T Q$ instead of Φ_T.

Theorem 4.2 (Bianchi identity). *The map Φ_T satisfies*

$$\delta\Phi_T + \big[\omega, \Phi_T\big] = 0.$$

Thm. 4.2 is obvious at a formal level, using the formal equality (4.5) together with the Bianchi identity (4.2) for $F_{\mathcal{M}}$.

Remark 4.3. Explicitly, on elements, this identity can be worked out to state

$$\begin{aligned}\Phi_T\big((d + b')(\theta_1, \dots, \theta_N)\big) = \mathbf{c}(\theta_1)\Phi_T(\theta_2, \dots, \theta_N) - (-1)^{n_{N-1}}\Phi_T(\theta_1, \dots, \theta_{N-1})\mathbf{c}(\theta_N) \\ - [Q, \Phi_T(\theta_1, \dots, \theta_N)],\end{aligned} \tag{4.7}$$

where $n_k = |\theta_1| + \cdots + |\theta_k| - k$. Here both sides are well-defined by Thm. 4.1.

In order to actually define Φ_T, the plan is to split

$$F_{\mathcal{M}} = Q^2 + F_{\mathcal{M}}^{\geq 1}, \tag{4.8}$$

where $F_{\mathcal{M}}^{\geq 1}$ is the part containing the cochains of arity at least one (which takes values in bounded operators on $\mathcal{H}$). Using this, we have (formally)

$$e^{-TF_{\mathcal{M}}} = e^{-T(Q^2 + F_{\mathcal{M}}^{\geq 1})}$$

which can be rewritten into a perturbation series with respect to the heat operator e^{-TQ^2}. To facilitate this, we first introduce the following notation.

Notation 4.4 (Bracket). Given suitable[11] operators $A_1, \dots, A_N$ on $\mathcal{H}$, we set

$$\{A_1, \dots, A_N\}_Q := \int_{\Delta_N} e^{-\tau_1 Q^2} A_1 e^{-(\tau_2-\tau_1)Q^2} A_2 \cdots e^{-(\tau_N-\tau_{N-1})Q^2} A_N e^{-(1-\tau_N)Q^2} \mathrm{d}\tau \tag{4.9}$$

where $\Delta_N = \{\tau \in \mathbb{R}^n \mid 0 \leq \tau_1 \leq \cdots \leq \tau_N \leq 1\}$ is the standard simplex. If there is no danger of confusion, we drop the subscript Q in notation.

If $A_1, \dots, A_N$ are all bounded, it is clear that $\{A_1, \dots, A_N\}_Q$ is a well-defined bounded operator on $\mathcal{H}$. Since we need to allow unbounded operators for the A_j as well, the following lemma is necessary.

Lemma 4.5. *For $0 \leq a_0, \dots, a_N < 1$ let $A_0, \dots, A_N$ be densely defined operators on $\mathcal{H}$ such that $A_j \Delta^{-a_j}$ is densely defined and extends to a bounded operator on $\mathcal{H}$, where $\Delta = Q^2 + 1$. Then for each $T > 0$, $A_0\{A_1, \dots, A_N\}_{TQ}$ is a well-defined trace-class operator, and we have the estimate*

$$\big\|A_0\{A_1, \dots, A_N\}_{TQ}\big\|_1 \leq \frac{e^{T/2}\,\mathrm{Tr}\big(e^{-TQ^2/2}\big)}{\Gamma\big(N+1-(a_0+\cdots+a_N)\big)} \prod_{j=0}^{N} \|A_j \Delta^{-a_j}\| \left(\frac{2a_j\Gamma(1-a_j)}{eT}\right)^{a_j},$$

for its nuclear norm, where Γ denotes the gamma function.

Proof. By the assumptions on A_j, for each $\varepsilon > 0$, the operator

$$A_j e^{-\varepsilon Q^2} = (A_j \Delta^{-a_j})\big(\Delta^{a_j} e^{-\varepsilon/2Q^2}\big)e^{-\varepsilon/2Q^2}$$

is trace-class, hence for each fixed $\tau \in \Delta_N$ such that $\tau_j - \tau_{j-1} > 0$, the integrand in (4.9) is a trace-class operator, by the assumptions on the A_j. In order to show that the integral over Δ_N gives a well-defined trace-class operator, it then suffices to show that the nuclear norm as a function of τ is integrable.
To this end, set $s_j = \tau_{j+1} - \tau_j$, $j = 0, \dots, N$, where $\tau_{N+1} = 1$, $\tau_0 = 0$. Since $\sum_{j=0}^N s_j = 1$, we have using Hölder's inequality for Schatten norms

$$\begin{aligned}\left\|\prod_{j=0}^N A_j e^{-T(\tau_j-\tau_{j-1})Q^2}\right\|_1 = \left\|\prod_{j=0}^N A_j e^{-Ts_jQ^2}\right\|_1 &\leq \prod_{j=0}^N \|A_j\Delta^{-a_j}\| \|\Delta^{a_j} e^{-Ts_jQ^2}\|_{1/s_j} \\ &\leq \prod_{j=0}^N \|A_j\Delta^{-a_j}\| \prod_{j=0}^N \|\Delta^{a_j} e^{-Ts_jQ^2/2}\| \prod_{j=0}^N \|e^{-Ts_jQ^2/2}\|_{1/s_j}.\end{aligned}$$

Now for the last product, we get

$$\prod_{j=0}^N \|e^{-Ts_jQ^2/2}\|_{1/s_j} = \prod_{j=0}^\infty \left(\sum_{k=1}^\infty \big(e^{-Ts_j\lambda_j/2}\big)^{1/s_j}\right)^{s_j} = \prod_{j=0}^N \mathrm{Tr}\big(e^{-TQ^2/2}\big)^{s_j} = \mathrm{Tr}(e^{-TQ^2/2}),$$

[11] In the sense that they satisfy the assumptions of Lemma 4.5 below.

where λ_j, for $j = 1, 2, \dots$, are the eigenvalues of Q^2. To estimate the second product, observe that

$$(x^2+1)^a e^{-\tau x^2/2} \leq \left(\frac{2a}{s}\right)^a e^{\tau/2-a}.$$

for all $x \in \mathbb{R}$. Hence

$$\prod_{j=0}^{N} \|\Delta^{a_j} e^{-Ts_j Q^2/2}\| \leq e^{T/2} \prod_{j=0}^{N} \left(\frac{2a_j}{eT}\right)^{a_j} (\tau_{j+1}-\tau_j)^{-a_j}$$

and the lemma now follows from the formula

$$\int_{\Delta_N} \tau_1^{-a_0} (\tau_2-\tau_1)^{-a_1} \dots (1-\tau_N)^{-a_N} \, d\tau = \frac{\Gamma(1-a_0)\cdots\Gamma(1-a_N)}{\Gamma\big(N+1-(a_0+\cdots+a_N)\big)};$$

in the case $N = 1$, this latter formula is the usual integral formula for the beta function, while the case for general N can be easily shown by induction. □

In general, the operators $F(\theta)$ and $F(\theta_1, \theta_2)$ are unbounded operators. However, by the assumption (A2) of the Fredholm module, they satisfy the assumptions of Lemma 4.5. Therefore, using the algebra structure on the space of $\mathcal{L}(\mathcal{H})$-valued bar cochains, the expression $\{F^{\geq 1}, \dots, F^{\geq 1}\}_{TQ}$ defines a well-defined cochain, used in the definition below.

Definition 4.6 (Quantization Map). For $T > 0$ fixed, the *quantization map* Φ_T is the $\mathcal{L}(\mathcal{H})$-valued bar cochain defined by the formula

$$\Phi_T := \sum_{N=0}^{\infty} (-T)^N \{\underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{N}\}_{TQ}. \tag{4.10}$$

For an explicit formula for the components of Φ_T see (4.11) below.

Remark 4.7. Φ_T is an even cochain, in the sense that it maps even elements of $\mathsf{B}(\Omega)$ to even operators on $\mathcal{H}$ and odd elements to odd operators. This follows from the fact that F has this property (which in turn follows since ω is odd, or from an inspection of (4.4) above), and the fact that the bracket is also even in the obvious sense (as it depends only on the square of Q).

To state explicitly how Φ_T evaluates on chains, we need the following notation.

Notation 4.8 (Partitions). Denote by $\mathcal{P}_{M,N}$ the set of ordered partitions of the set $\{1, \dots, N\}$ of length M, i.e. collections of nonempty subsets $I = (I_1, \dots, I_M)$ such that $I_1 \cup \dots \cup I_M = \{1 \dots, N\}$ and such that each element of I_a is smaller than any element of I_b whenever $a < b$.

Using this notation, we get the formula

$$\Phi_T(\theta_1, \dots, \theta_N) = \sum_{M=1}^{N} (-T)^M \sum_{I \in \mathcal{P}_{M,N}} \big\{F(\theta_{I_1}), \dots, F(\theta_{I_M})\big\}_{TQ}, \tag{4.11}$$

where we put

$$\theta_{I_a} := (\theta_{i+1}, \dots, \theta_{i+m})$$

if $I_a = \{j \mid i < j \leq i+m\}$ for some i, m. This formula follows from the product formula (3.3) for cochains. Notice that a summand corresponding to $I = (I_1, \dots, I_M) \in \mathcal{P}_{M,N}$ is zero as soon as I contains an I_a with $|I_a| \geq 3$.

Proof (of Thm. 4.1). First consider the cochain Φ_T. The operator $\Phi_T(\theta_1, \dots, \theta_N)$ is a sum of operators $A_0\{A_1, \dots, A_N\}_{QT}$, with $A_0 = 0$ and each A_j, $j \geq 1$ equalling on of the terms in (4.4). Now by the properties (2.1) of the locally convex topology and the assumptions (A1)-(A2) on the Fredholm module, there exists a continuous seminorm ν on Ω such that

$$\|F(\theta)\Delta^{-1/2}\| \leq \nu(\theta) \quad \text{and} \quad \|F(\theta_1, \theta_2)\| \leq \nu(\theta_1)\nu(\theta_2). \tag{4.12}$$

Hence Lemma 4.5 implies is a well-defined trace-class operator for each N. The same is true for $Q\Phi_T(\theta_1, \dots, \theta_N)$, only that in this case, $A_0 = Q$. For $\Phi_T(\theta_1, \dots, \theta_N)Q$, we can argue by passing to the adjoint.
It therefore remains to show the estimate (4.6). Here we consider the cochain Φ_T, the proof for $Q\Phi_T$ and $\Phi_T Q$ is similar. Now notice that since F has only components of arity less than two, for any partition $I \in \mathcal{P}_{M,N}$, we have

$$\{F(\theta_{I_1}), \dots, F(\theta_{I_M})\}_{TQ} = 0$$

as soon as I contains a subset I_j with $|I_j| \geq 2$. For all other partitions, we have $|I_j| = 1$ for $2M - N$ indices j and $|I_j| = 2$ for $N - M$ indices j. Hence by (4.12), we can apply Lemma 4.5 with $a_j = 1/2$ for $2M - N$ and $a_j = 0$ for $N - M$ indices j. This gives

$$\big\|\{F(\theta_{I_1}), \dots, F(\theta_{I_M})\}_T\big\|_1 \leq \frac{e^{T/2}\operatorname{Tr}(e^{-TQ^2/2})}{\Gamma\left(\frac{3}{2}N + 1 - M\right)} \left(\frac{\sqrt{\pi}}{eT}\right)^{(2M-N)/2} \nu(\theta_1)\cdots\nu(\theta_N),$$

as $\Gamma(1/2) = \sqrt{\pi}$. With a view on (4.11), we therefore get

$$\begin{aligned} \big\|\Phi_T(\theta_1, \dots, \theta_N)\big\|_1 &\leq \sum_{M=\lceil N/2 \rceil}^{N} T^M \sum_{I \in \mathcal{P}_{M,N}} \frac{e^{T/2}\operatorname{Tr}(e^{-TQ^2/2})}{\Gamma\left(\frac{3}{2}N + 1 - M\right)} \left(\frac{\sqrt{\pi}}{eT}\right)^{(2M-N)/2} \nu(\theta_1)\cdots\nu(\theta_N) \\ &\leq T^{N/2}\frac{e^{T/2}\operatorname{Tr}(e^{-TQ^2/2})}{\Gamma(N/2)} \sum_{M=\lceil N/2 \rceil}^{N} \sum_{I \in \mathcal{P}_{M,N}} \left(\frac{\sqrt{\pi}}{e}\right)^{(2M-N)/2} \nu(\theta_1)\cdots\nu(\theta_N) \end{aligned}$$

The sum over M growth at most like c^N with N, where c is some constant, hence it can be absorbed into the seminorm (by replacing it by a larger one). Finally, using the Sterling formula, this quotient $\Gamma(N/2)$ can be replaced by $\sqrt{N!}$, again at the cost of replacing ν by a larger seminorm. □

We are now in the position to prove Thm. 4.2.

Proof (of Thm. 4.2). For the proof, we introduce for $\varepsilon \geq 0$ the Bracket $\{A_1, \dots, A_N\}_Q^\varepsilon$, which is defined just as the usual bracket by formula (4.9), but instead of Δ_N we integrate over

$$\Delta_N^\varepsilon = \big\{\tau \in \Delta_N \mid \tau_j - \tau_{j-1} \geq \varepsilon \ \text{ for all } j\big\},$$

the simplex with the ε-neighborhood of the diagonals removed. Due to the fact that for any m, the norm of the operator $Q^k e^{-tQ^2}$ is bounded uniformly on $[\varepsilon, 1]$, the expression $\{A_1, \dots, A_N\}_Q^\epsilon$ gives a well-defined operator for collection of operators A_j of the form $A_j = Q^n A_j' Q^m$, with A_j' bounded. We denote by Φ_T^ε the cochain defined by the same procedure as above, but using this bracket.
Now for any $\varepsilon \geq 0$, we have

$$\delta\Phi_T^\varepsilon = \sum_{N=0}^\infty (-T)^N \sum_{k=1}^N \{\underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{N-k}, \delta F^{\geq 1}, \underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{k-1}\}_{TQ}^\varepsilon. \tag{4.13}$$

By the Bianchi identity (4.2) for F, we have

$$\delta F^{\geq 1} = \delta F = -[\omega, F] = [Q, F^{\geq 1}] - [\mathbf{c}, F^{\geq 1}] - [\mathbf{c}, Q^2]. \tag{4.14}$$

At this point, this identity is just formal as both side are cochains with values in unbounded operators and we do not have enough control over their domains; however, it is easy to verify that

$$\begin{aligned}
\delta\Phi_T^\varepsilon &= \sum_{N=1}^\infty (-T)^N \sum_{k=1}^N \{\underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{k-1}, \delta F^{\geq 1}, \underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{N-k}\}_{TQ}^\varepsilon \\
&= \sum_{N=1}^\infty (-T)^N \Bigg(\sum_{k=1}^N \{\underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{k-1}, [Q, F^{\geq 1}], \underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{N-k}\}_{TQ}^\varepsilon \\
&\qquad - \sum_{k=1}^N \{\underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{k-1}, [\mathbf{c}, F^{\geq 1}], \underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{N-k}\}_{TQ}^\varepsilon \\
&\qquad - \sum_{k=1}^N \{\underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{k-1}, [\mathbf{c}, Q^2], \underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{N-k}\}_{TQ}^\varepsilon \Bigg),
\end{aligned} \tag{4.15}$$

formally obtained by plugging the right hand of (4.14) side into (4.13), is in fact an identity of well-defined cochains provided that $\varepsilon > 0$, by the discussion of the ε-bracket above. Notice that since taking the commutator with Q is a derivation, we have

$$\{\underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{k-1}, [Q, F^{\geq 1}], \underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{N-k}\}_{TQ}^\varepsilon = \Big[Q, \{\underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{N}\}_{TQ}^\varepsilon\Big],$$

which in fact makes sense even for $\varepsilon = 0$, by Lemma 4.1 above. Also the second term on the right hand side of (4.15) makes sense for $\varepsilon = 0$; for the third term, we use the following lemma.

Lemma 4.9. *Let $A_1, \ldots, A_N$ be a collection of operators of the form $A_j = Q^m A_j' Q^n$ with A_j' bounded and assume that $A_k = A_k'$ for some k. Then for $\varepsilon > 0$, we have the identity*

$$T\big\{A_1, \ldots, [Q^2, A_k], \ldots, A_N\big\}_{TQ}^{\varepsilon} \\ = \big\{A_1, \ldots, A_k e^{-\varepsilon Q^2} A_{k+1}, \ldots, A_N\big\}_{TQ}^{\varepsilon} - \big\{A_1, \ldots, A_{k-1} e^{-\varepsilon Q^2} A_k, \ldots, A_N\big\}_{TQ}^{\varepsilon}.$$

Here, if $k = 1$, the second term on the right-hand-side is $A_1\{A_2, \ldots, A_N\}_{TQ}^{\varepsilon}$ instead, while if $k = N$, the first term is $\{A_1, \ldots, A_{N-1}\}_T^{\varepsilon} A_N$.

Proof. This follows from integration by parts, after realizing that the term on the left hand side equals

$$\int_{\Delta_N^{\varepsilon}} \frac{\partial}{\partial \tau_k} \left[e^{-T(1-\tau_N)Q^2} \prod_{j=1}^{N} A_j e^{-T(\tau_j - \tau_{j-1})Q^2} \right] \mathrm{d}\tau. \qquad \square$$

Hence for any $\varepsilon > 0$, we have

$$\{\underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{k-1}, [\mathbf{c}, Q^2], \underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{N-k}\}_{TQ} = \{\underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{k-1}, \mathbf{c} e^{-\varepsilon Q^2} F^{\geq 1}, \underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{N-k-1}\}_{TQ} \\ - \{\underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{k-2}, F^{\geq 1} e^{-\varepsilon Q^2} \mathbf{c}, \underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{N-k}\}_{TQ}.$$

Now notice that the right hand side of the above identity makes sense for $\varepsilon = 0$ as well. Hence after replacing the last term in (4.15) with this term, we may take the limit $\varepsilon \to 0$ in the identity (4.15). This gives

$$\begin{aligned} \delta\Phi_T = [Q, \Phi_T] &- \sum_{N=1}^{\infty} (-T)^N \sum_{k=1}^{N} \{\underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{k-1}, [\mathbf{c}, F^{\geq 1}], \underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{N-k}\}_{TQ} \\ &- \sum_{N=1}^{\infty} (-T)^{N-1} \Bigg(\mathbf{c}\{\underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{N-1}\}_{TQ} - \{\underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{N-1}\}_{TQ}\mathbf{c} \\ &\qquad + \sum_{k=2}^{N} \{\underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{k-2}, F^{\geq 1}\mathbf{c}, \underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{N-k}\}_{TQ} \\ &\qquad - \sum_{k=1}^{N-1} \{\underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{k-1}, \mathbf{c}F^{\geq 1}, \underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{N-k-1}\}_{TQ} \Bigg). \end{aligned}$$

Finally, after an index shift, we obtain that the third term on the right hand side equals

$$\begin{aligned} - \sum_{N=1}^{\infty} (-T)^{N-1} \Bigg(&\mathbf{c}\{\underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{N-1}\}_{T} - \{\underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{N-1}\}_{TQ}\mathbf{c} \\ &+ \sum_{k=1}^{N-1} \{\underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{k-1}, F^{k\geq 1}\mathbf{c} - \mathbf{c}F^{\geq 1}, \underbrace{F^{\geq 1}, \ldots, F^{\geq 1}}_{N-k-1}\}_{TQ} \Bigg) \end{aligned}$$

$$= -[\mathbf{c}, \Phi_T] + \sum_{N=2}^{\infty} (-T)^{N-1} \sum_{k=1}^{N-1} \{\underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{k-1}, [\mathbf{c}, F^{\geq 1}], \underbrace{F^{\geq 1}, \dots, F^{\geq 1}}_{N-k-1}\}_{TQ}$$

Combining this with the identity before, we obtain

$$\delta\Phi_T = [Q, \Phi_T] - [\mathbf{c}, \Phi_T] = -[\omega, \Phi_T],$$

which was the claim. □

5 The Chern Character

Throughout this section, let Ω be a locally convex dg algebra.

Definition 5.1 (Chern character). For any weak ϑ-summable Fredholm module $\mathcal{M} := (\mathcal{H}, \mathbf{c}, Q)$, we define an even cyclic cochain $\mathrm{Ch}(\mathcal{M}) \in \mathrm{Hom}^+(\mathsf{C}(\Omega), \mathbb{C})$ by

$$\mathrm{Ch}(\mathcal{M})(\theta_0, \dots, \theta_N) = \mathrm{Str}\Big(\mathbf{c}(\theta_0)\Phi_1(\mathcal{M})(\theta_1, \dots, \theta_N)\Big)$$

for $\theta_0 \in \Omega$, $\theta_1, \dots, \theta_N \in \underline{\Omega}[1]$. Here $\Phi_T(\mathcal{M})$ is the quantization map defined in Section 4 and Str denotes the supertrace for trace-class operators on the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$. The cochain $\mathrm{Ch}(\mathcal{M})$ is called the *Chern character* of $\mathcal{M}$.

Note that $\mathrm{Ch}(\mathcal{M})$ is well-defined: First, that the operator $\mathbf{c}(\theta_0)\Phi_1(\mathcal{M})(\theta_1, \dots, \theta_N)$ is trace-class follows from the fundamental estimate from Thm. 4.1. Secondly, $\mathrm{Ch}(\mathcal{M})$ is a well-defined element of $\mathrm{Hom}(\mathsf{C}(\Omega), \mathbb{C})$. To see this, one has to observe that

$$\Phi_1(\mathcal{M})(\theta_1, \dots, \theta_j, \mathbf{1}, \theta_{j+1}, \dots, \theta_N) = 0$$

for all $\theta_1, \dots, \theta_N \in \Omega$ and all $0 \leq j \leq N$. However, this follows from the assumption that $\mathbf{c}(\mathbf{1}) = 1$, with a view on the explicit formula (4.4) for $F_{\mathcal{M}}$. Finally, $\mathrm{Ch}(\mathcal{M})$ is even, that is, it vanishes on odd elements of $\mathsf{C}(\Omega)$. This follows directly from the fact that both $\mathbf{c}$ and $\Phi_1(\mathcal{M})$ are parity-preserving (c.f. 4.7), since the supertrace vanishes on odd operators. The following was stated as Thm. B in the introduction.

Theorem 5.2 (The Chern character is entire). *The Chern character* $\mathrm{Ch}(\mathcal{M})$ *has a unique extension to an element of* $\mathcal{L}^+(\mathsf{C}_\epsilon(\Omega), \mathbb{C})$*, again denoted by* $\mathrm{Ch}(\mathcal{M})$*.*

Proof. This follows directly from the fundamental estimate, Thm. 4.1. □

We now proceed to show the closedness of $\mathrm{Ch}(\mathcal{M})$, stated as Thm. A above.

Theorem 5.3 (Closedness). *The Chern character* $\mathrm{Ch}(\mathcal{M})$ *is closed, that is,*

$$(\underline{d} + \underline{b} + \underline{B})\mathrm{Ch}(\mathcal{M}) = 0.$$

A conceptual way to prove Thm. 5.3 is to exploit the coalgebra structure on the space $\mathrm{Hom}(\mathsf{B}(\Omega), \mathcal{L}(\mathcal{H}))$ of $\mathcal{L}(\mathcal{H})$-valued bar cochains, together with derived constructions. The result then follows from calculations similar to those of Quillen [29, Section 8]. However, instead of introducing the algebraic machinery to adapt Quillen's arguments, we use the trick of passing to the acyclic extension of $\mathcal{M}$ as explained in Example 2.6.

Proof (of Thm. 5.3). Let us form the extended ϑ-summable Fredholm module $\mathcal{M}_{\mathbb{T}} := (\mathcal{H}, \mathbf{c}_{\mathbb{T}}, Q)$ over the dg algebra $\Omega_{\mathbb{T}} := \Omega[\sigma]$ with differential $d_{\mathbb{T}}$ (given by formula (2.8)), where σ is a formal variable of degree -1 with $\sigma^2 = 0$. Observe that it suffices to prove Thm. 5.3 for $\mathcal{M}_{\mathbb{T}}$ over $\Omega_{\mathbb{T}}$: Indeed, the inclusion inclusion $\Omega \subset \Omega_{\mathbb{T}}$ induces a map

$$j : \mathsf{C}(\Omega) \longrightarrow \mathsf{C}(\Omega_{\mathbb{T}}),$$

that commutes with all differentials, and one has $\mathrm{Ch}(\mathcal{M}) = j^*\mathrm{Ch}(\mathcal{M}_{\mathbb{T}})$. Hence the result for $\mathrm{Ch}(\mathcal{M}_{\mathbb{T}})$ implies that for $\mathrm{Ch}(\mathcal{M})$.
We abbreviate

$$\Phi_1^{\mathbb{T}} := \Phi_1(\mathcal{M}_{\mathbb{T}}), \quad \text{and} \quad F_{\mathbb{T}} := F_{\mathcal{M}_{\mathbb{T}}},$$

and note that these maps descend to the quotient $\mathsf{B}(\underline{\Omega}_{\mathbb{T}})$; let us denote by $\underline{\Phi}_1^{\mathbb{T}}$ the quotient map induced by $\Phi_1^{\mathbb{T}}$. For the acyclic extension, the nonzero components of the curvature are given by

$$\begin{aligned} F_{\mathbb{T}}^{(0)} &= Q^2, \\ F_{\mathbb{T}}^{(1)}(\theta) &= \mathbf{c}(d\theta') - [Q, \mathbf{c}(\theta')] - \mathbf{c}(\theta'') \\ F_{\mathbb{T}}^{(2)}(\theta_1, \theta_2) &= (-1)^{|\theta_1'|}\big(\mathbf{c}(\theta_1'\theta_2') - \mathbf{c}(\theta_1')\mathbf{c}(\theta_2')\big), \end{aligned} \tag{5.1}$$

where again the elements of $\theta \in \Omega_{\mathbb{T}}$ have been written as $\theta = \theta' + \sigma\theta''$. Consider the parity-preserving map[12]

$$\alpha : \mathsf{C}(\Omega_{\mathbb{T}}) \longrightarrow \mathsf{B}(\underline{\Omega}_{\mathbb{T}}), \qquad (\theta_0, \dots, \theta_N) \longmapsto \underline{\mathbf{N}}(\sigma\theta_0, \theta_1, \dots, \theta_N),$$

where

$$\underline{\mathbf{N}} : \mathsf{B}(\underline{\Omega}_{\mathbb{T}}) \longrightarrow \mathsf{B}(\underline{\Omega}_{\mathbb{T}})$$

is the quotient map of the *averaging operator*, given by

$$\mathbf{N} : \mathsf{B}(\Omega_{\mathbb{T}}) \longrightarrow \mathsf{B}(\Omega_{\mathbb{T}}), \qquad (\theta_1, \dots, \theta_N) \longmapsto \sum_{k=1}^{N} (-1)^{n_k(n_N - n_k)}(\theta_{k+1}, \dots, \theta_N, \theta_1, \dots, \theta_k),$$

with $n_k = |\theta_1| + \cdots + |\theta_k| - k$. By (5.1), we have $F_{\mathbb{T}}(\sigma) = -1$; therefore an application of Lemma 5.4 below shows that (up to a sign) $\mathrm{Ch}(\mathcal{M}_{\mathbb{T}})$ is the pullback of $\underline{\Phi}_1^{\mathbb{T}}$ under α, more precisely

$$\alpha^* \,\mathrm{Str}\big(\underline{\Phi}_1^{\mathbb{T}}\big) = \mathrm{Str}\big(\underline{\Phi}_1^{\mathbb{T}}\alpha\big) = -\mathrm{Ch}(\mathcal{M}_{\mathbb{T}}). \tag{5.2}$$

[12] Here of course, $\mathsf{B}(\underline{\Omega}_{\mathbb{T}})$ denotes the space given by formula (3.1), but with the *space* $\underline{\Omega}_{\mathbb{T}}$ instead of the *algebra* $\Omega_{\mathbb{T}}$. Of course, the differentials d and b' descend to this quotient.

Denote by $\mathsf{B}^\sharp(\Omega_{\mathbb{T}}) \subset \mathsf{B}(\Omega_{\mathbb{T}})$ the image of $\mathbf{N}$, which is the space of cyclic bar chains. On this subspace, $\mathrm{Str}(\Phi_1^{\mathbb{T}})$ is coclosed by Thm. 4.2, meaning that

$$\mathrm{Str}\Big(\Phi_1^{\mathbb{T}}\big((d_{\mathbb{T}} + b')c\big)\Big) = 0 \tag{5.3}$$

for all bar chains $c \in \mathsf{B}^\sharp(\Omega_{\mathbb{T}})$. This follows from the fact that $\Phi_1((d_{\mathbb{T}} + b')\theta)$ is a sum of super-commutators of operators on $\mathcal{H}$ in this case, as can be seen from the explicit formula (4.7).

We are therefore interested in the interaction of α with respect to the various differentials. Here straightforward calculations show that

$$\underline{b}'\alpha + \alpha\underline{b} = 0, \qquad \underline{d}_{\mathbb{T}}\alpha + \alpha\underline{d}_{\mathbb{T}} = -h, \qquad \alpha\underline{B} = \underline{S}h, \tag{5.4}$$

where $\underline{b}'$, $\underline{d}_{\mathbb{T}}$ are the differentials on the quotient $\mathsf{B}(\underline{\Omega}_{\mathbb{T}})$, h is given by

$$h : \mathsf{C}(\Omega_{\mathbb{T}}) \longrightarrow \mathsf{B}(\underline{\Omega}_{\mathbb{T}}), \qquad (\theta_0, \dots, \theta_N) \longmapsto \underline{\mathbf{N}}(\theta_0, \theta_1, \dots, \theta_N),$$

and

$$\underline{S} : \mathsf{B}(\underline{\Omega}_{\mathbb{T}}) \longrightarrow \mathsf{B}(\underline{\Omega}_{\mathbb{T}})$$

is the quotient map of the map S considered in (3.9). We obtain that

$$\alpha(\underline{d}_{\mathbb{T}} + \underline{b} + \underline{B}) = (\underline{d}_{\mathbb{T}} + \underline{b}')\alpha + (\underline{S} - 1)h, \tag{5.5}$$

so with a view on (5.2), we have that

$$-(\underline{d}_{\mathbb{T}} + \underline{b} + \underline{B})\mathrm{Ch}^{\mathbb{T}} = \mathrm{Str}\left(\Phi_1^{\mathbb{T}}(\underline{d}_{\mathbb{T}} + \underline{b}')\alpha\right) + \mathrm{Str}(\Phi_1^{\mathbb{T}}(\underline{S} - 1)h) \tag{5.6}$$

The first summand on the right hand side is zero by (5.3), since α takes values in $\mathsf{B}^\sharp(\underline{\Omega}_{\mathbb{T}})$. On the other hand, it follows from the special case (5.8) of Lemma 5.4 below that on the subspace $\mathsf{B}^\sharp(\Omega_{\mathbb{T}})$, we have

$$\begin{aligned}\mathrm{Str}(\Phi_1^{\mathbb{T}} S) &= -\sum_{N=0}^{\infty}(-1)^{N+1}\sum_{k=0}^{N}\mathrm{Str}\{\underbrace{F_{\geq 1}^{\mathbb{T}}, \dots, F_{\geq 1}^{\mathbb{T}}}_{k}, 1, \underbrace{F_{\geq 1}^{\mathbb{T}}, \dots, F_{\geq 1}^{\mathbb{T}}}_{N-k}\}_Q \\ &= -\sum_{N=0}^{\infty}(-1)^{N+1}\mathrm{Str}\{\underbrace{F_{\geq 1}^{\mathbb{T}}, \dots, F_{\geq 1}^{\mathbb{T}}}_{N}\}_Q = \mathrm{Str}(\Phi_1^{\mathbb{T}}).\end{aligned} \tag{5.7}$$

Since h takes values in $\mathsf{B}^\sharp(\underline{\Omega}_{\mathbb{T}})$, this shows that the second summand on the right hand side of (5.6) vanishes. This finishes the proof. □

In the proof above, we used the following lemma.

Lemma 5.4. *For operators $A_0, \dots, A_N$ satisfying the assumptions of Lemma 4.5, we have*

$$\sum_{j=0}^{N}(-1)^{k_j}\,\mathrm{Str}\big(\{A_{j+1}, \dots, A_N, A_0, \dots, A_j\}_Q\big) = \mathrm{Str}\big(A_0\{A_1, \dots, A_N\}_Q\big),$$

where $k_j = (|A_0| + \dots + |A_j|)(|A_{j+1}| + \dots + |A_N|)$.

Proof. First notice that by the cyclic permutation property of the trace, we have after a substitution in Δ_N that

$$\mathrm{Str}\big(A_0\{A_1, \dots, A_N\}_Q\big) = (-1)^{k_j}\,\mathrm{Str}\big(A_{j+1}\{A_{j+2}, \dots, A_N, A_0, \dots, A_j\}_Q\big)$$

for any $j = 0, \dots, N$. A special case of this is

$$\mathrm{Str}\big(A_0\{A_1, \dots, A_j, 1, A_{j+1}, \dots, A_N\}_Q\big) = (-1)^{k_j}\,\mathrm{Str}\big(\{A_{j+1}, \dots, A_N, A_0, \dots, A_j\}_Q\big).$$

On the other hand, the semigroup property and integration by parts yields

$$\{A_1, \dots, A_j, 1, A_{j+1}, \dots, A_N\}_Q = \int_{\Delta_N} (\tau_{j+1} - \tau_j) e^{-\tau_1 Q^2} \prod_{j=1}^{N} A_j e^{-(\tau_{j+1}-\tau_j)Q^2} \mathrm{d}\tau$$

Therefore, using that $\sum_{j=0}^{N}(\tau_{j+1} - \tau_j) = 1$, we obtain

$$\sum_{j=0}^{N} \{A_1, \dots, A_j, 1, A_{j+1}, \dots, A_N\}_Q = \{A_1, \dots, A_N\}_Q. \tag{5.8}$$

Combining this with the observation before yields the claim. □

We close this section by showing that the Chern character descends to the Chen normalized complxes in the case that $\mathcal{M}$ is a ϑ-summable Fredholm module, not only a *weak* Fredholm module, i.e. when $\mathcal{M}$ satisfies the identities (2.2).

Theorem 5.5 (Chen normalization). *Let $\mathcal{M}$ be a ϑ-summable Fredholm module. Then the Chern character* $\mathrm{Ch}(\mathcal{M})$ *decsends to an element of* $\mathcal{L}(\mathsf{N}_+^{\epsilon}(\Omega), \mathbb{C})$ *and thus defines an element of* $\mathsf{h}_+^{\epsilon}(\Omega)$*. Likewise,* $\mathrm{Ch}(\mathcal{M}_{\mathbb{T}})$ *decends to an element of* $\mathcal{L}(\mathsf{N}_+^{\epsilon,\mathbb{T}}(\Omega), \mathbb{C})$ *and thus defines an element of* $\mathsf{h}_+^{\mathbb{T},\epsilon}(\Omega)$*.*

Proof. It suffices to consider $\mathrm{Ch}(\mathcal{M}_{\mathbb{T}})$ since this implies the result for $\mathrm{Ch}(\mathcal{M})$. First, the fact that $\mathrm{Ch}(\mathcal{M})$ vanishes on the image on $S - \mathrm{id}$ is true also for weak Fredholm modules, as it follows from the calculation (5.7) above.
It remains to show that if (2.2) holds, then for all $f \in \Omega^0$ and each $i \geq 1$, we have $\mathrm{Ch}(\mathcal{M})S_i^{(f)} = 0$ as well as $\mathrm{Ch}(\mathcal{M})[\underline{d} + \underline{b}, S_i^{(f)}] = 0$. The first identity follows directly from the definition of $\Phi_T(\mathcal{M})$, since the assumptions (2.2) imply that

$$F(f) = 0 \quad \text{and} \quad F(f, \theta) = F(\theta, f) = 0$$

for any $\theta \in \Omega$. To see the second identity, observe first that

$$\begin{aligned} [(\underline{d} + \underline{b}), S_i^{(f)}](\theta_0, \dots, \theta_N) =& (\theta_0, \dots, \theta_i, df, \theta_{i+1}, \dots, \theta_N) \\ &- (\theta_0, \dots, \theta_i f, \theta_{i+1}, \dots, \theta_N) \\ &+ (\theta_0, \dots, \theta_i, f\theta_{i+1}, \dots, \theta_N) \end{aligned} \tag{5.9}$$

for $0 \leq i \leq N-1$, while for $i = N$, we have

$$[(\underline{d} + \underline{b}), S_N^{(f)}](\theta_0, \dots, \theta_N) = (\theta_0, \dots, \theta_N, df) - (\theta_0, \dots, \theta_N f) + (f\theta_0, \dots, \theta_N). \tag{5.10}$$

To compute $\Phi_1(\mathcal{M})$ of this, we use the identities

$$\begin{aligned} F(df) &= -[Q^2, \mathbf{c}(f)], & F(f\theta_1, \theta_2) &= \mathbf{c}(f)F(\theta_1, \theta_2), \\ F(df, \theta) + F(f\theta) &= \mathbf{c}(f)F(\theta), & F(\theta_1 f, \theta_2) &= \mathbf{c}(f)F(\theta_1, \theta_2), \\ F(\theta, df) - F(\theta f) &= F(\theta)\mathbf{c}(f), & F(\theta_1 f, \theta_2) &= F(\theta_1, f\theta_2), \end{aligned}$$

which follow directly from (2.2). The theorem is then a consequence of the integration by parts lemma, Lemma 4.9, after a careful investigation of the terms appearing in the formula (4.11). □

6 Homotopy Invariance of the Chern Character

In this section, we show that the Chern character defined above is invariant under suitable deformations of Fredholm modules. While the results of this section are not necessary for the remainder of the paper, we believe them to be of independent interest, in particular for the proof of the localization formula for the Chern character in the case of Example 2.4, see Example 6.8 below. Throughout, let Ω be a locally convex dg algebra.

Definition 6.1 (Homotopy of Fredholm modules). A *homotopy* of (weak) ϑ-summable Fredholm modules over Ω is a family $\mathcal{M}^s = (\mathcal{H}, \mathbf{c}^s, Q_s)$, $s \in [0,1]$, of (weak) ϑ-summable Fredholm modules, satisfying the following conditions.

(H1) The seminorms appearing in (A1), (A2) of Definition 2.1 can be chosen independently of s, and for all $T > 0$, we have

$$\sup_{s \in [0,1]} \mathrm{Tr}(e^{-TQ_s^2}) < \infty;$$

(H2) For all $s \in [0,1]$, the operators Q_s have the same domain of definition, and for each element $h \in \mathrm{dom}(Q_s)$, the map $s \mapsto Q_s h$ is a continuously differentiable curve in $\mathcal{H}$; in particular, the derivative $\dot{Q}_s$ is a densely defined operator on $\mathcal{H}$. Moreover, we require that $\dot{Q}_s \Delta_s^{-1/2}$ and $\Delta_s^{-1/2}\dot{Q}_s$ are bounded, where $\Delta_s = Q_s^2 + 1$, with uniform norm bound

$$\sup_{s \in [0,1]} \left\| \Delta_s^{-1/2} \dot{Q}_s \right\| + \sup_{s \in [0,1]} \left\| \dot{Q}_s \Delta_s^{-1/2} \right\| < \infty;$$

(H3) For all $\theta \in \Omega$, the map $s \mapsto \mathbf{c}^s(\theta)$ is continuously differentiable with respect to the strong operator topology.

The main result of this section is the following.

Theorem 6.2 (Homotopy invariance). *If $\mathcal{M}^s = (\mathcal{H}, \mathbf{c}^s, Q_s)$, $s \in [0,1]$, is an homotopy of ϑ-summable Fredholm modules, then the Chern characters* $\mathrm{Ch}(\mathcal{M}^0)$ *and* $\mathrm{Ch}(\mathcal{M}^1)$ *define the same element of* $\mathsf{h}^{\epsilon}_{+}(\Omega)$. *Similarly, the Chern characters* $\mathrm{Ch}(\mathcal{M}^0_{\mathbb{T}})$ *and* $\mathrm{Ch}(\mathcal{M}^1_{\mathbb{T}})$ *define the same element of* $\mathsf{h}^{\mathbb{T},\epsilon}_{+}(\Omega)$.

We are going to prove this theorem by constructing a "Chern-Simons form" on the acyclicly extended Fredholm module over $\Omega_{\mathbb{T}}$, as in the proof of Thm. 5.3. To this end, we fix a homotopy $\mathcal{M}^s = (\mathcal{H}, \mathbf{c}^s, Q_s)$, $s \in [0,1]$, of weak Fredholm modules. We now define a family of auxiliary $\mathcal{L}(\mathcal{H})$-valued bar cochains

$$\Psi^s_T := -T \int_0^1 \Phi^s_{uT}\, \dot{\omega}_s \Phi^s_{(1-u)T}\, \mathrm{d}u, \tag{6.1}$$

where we abbreviated $\Phi^s_T := \Phi_T(\mathcal{M}^s)$.

Proposition 6.3. *There exists a continuous seminorm ν on Ω such that for each $s \in [0,1]$, each $T > 0$, and all $\theta_1, \dots, \theta_N \in \Omega$, one has the estimate*

$$\big\|\Psi^s_T(\theta_1, \dots, \theta_N)\big\|_1 \leq e^{T/2}\mathrm{Tr}[e^{-TQ_s^2/2}]\frac{T^N}{\sqrt{N!}}\nu(\theta_1)\cdots\nu(\theta_N)$$

for the trace class norms. The same is true for $\Psi_T Q_s$ and $Q_s \Psi_T$ instead of Ψ_T.

Proof. Again, the argument for $\Psi^s_T Q_s$ and $Q_s \Psi^s_T$ is similar to the one for Ψ^s_T, so we restrict ourselves to the discussion of the latter case.
For suitable operators $A_1, \dots, A_N, B$ on $\mathcal{H}$, we have the identity

$$\begin{aligned}\{A_1, \dots, A_k, B, A_{k+1}, \dots, A_N\}_{TQ_s}& \\ = \int_0^1 u^k (1-u)^{N-k} \{A_1, \dots, A_k\}^s_{uTQ} B \{A_{k+1}, \dots, A_N\}_{(1-u)TQ_s}\, \mathrm{d}u.&\end{aligned} \tag{6.2}$$

Using this and formula (4.11) for the quantization map, we obtain that $\Psi^s_T(\theta_1, \dots, \theta_N)$ is given by the formula

$$\sum_{k=1}^N \sum_{M=1}^N (-T)^{M+1} \sum_{j=1}^M \Bigg(\sum_{I \in \mathcal{P}_{M,N}} \big\{F_s(\theta_{I_1}), \dots, F_s(\theta_{I_{j-1}}), \dot{Q}_s, F_s(\theta_{I_j}), \dots, F_s(\theta_{I_M})\big\}_{TQ_s}$$
$$\sum_{\substack{I \in \mathcal{P}_{M,N} \\ I_j = \{k\}}} \big\{F_s(\theta_{I_1}), \dots, F_s(\theta_{I_{j-1}}), \dot{\mathbf{c}}^s(\theta_k), F_s(\theta_{I_{j+1}}), \dots, F_s(\theta_{I_M})\big\}_{TQ_s}\Bigg).$$

Now each of the brackets can be estimated using Lemma 4.5, where to estimate the first bracket, we also use the bound on $\dot{Q}_s \Delta_s^{-1/2}$. We can then proceed as in the proof of Thm. 4.1. The uniformity in s follows from the assumptions on the admissible pair. □

Proposition 6.4. *We have*

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi_T^s = \delta\Psi_T^s + [\omega_s, \Psi_T^s].$$

For the proof, we need the following lemma, which is proved similar to Lemma 2.2 (5) in the paper of Getzler and Szenes [18].

Lemma 6.5. *We have*

$$\frac{\mathrm{d}}{\mathrm{d}s}\{A_1,\dots,A_N\}_{TQ_s} = -T\sum_{k=0}^{N}\big\{A_1,\dots,A_k,[Q_s,\dot{Q}_s],A_{k+1},\dots,A_N\big\}_{TQ_s},$$

assuming both sides are well-defined.

Proof (of Prop. 6.4). Using the lemma and shifting an index, we obtain, similar to the proof of Thm. 4.1 that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Phi_T^s &= \sum_{N=0}^{\infty}(-T)^{N+1}\sum_{k=1}^{N+1}\big\{\underbrace{F_s^{\geq 1},\dots,F_s^{\geq 1}}_{k},[Q_s,\dot{Q}_s]+\dot{F}_s^{\geq 1},\underbrace{F_s^{\geq 1},\dots,F_s^{\geq 1}}_{N-k}\big\}_{TQ_s}\\ &= -T\int_0^1 \Phi_{uT}^s\dot{F}_s\Phi_{(1-u)T}^s\mathrm{d}u,\end{aligned}$$

where we used that $\dot{F}_s = [Q_s,\dot{Q}_s]+\dot{F}_s^{\geq 1}$, as well as the identity (6.2). Now

$$\delta\dot{\omega}_s + [\omega_s,\dot{\omega}_s] = \frac{\mathrm{d}}{\mathrm{d}s}\big\{\delta\omega_s+\omega_s^2\big\} = \dot{F}_s.$$

Therefore, using that $\delta\Phi_T^s + [\omega_s,\Phi_T^s]=0$ by Thm. 4.2, we obtain

$$\int_0^1 \Phi_{sT}^s\dot{F}_s\Phi_{(1-s)T}^s\mathrm{d}s = \delta\int_0^1 \Phi_{sT}^s\dot{\omega}_s\Phi_{(1-s)T}^s\mathrm{d}s + \Big[\omega_s,\int_0^1 \Phi_{sT}^s\dot{\omega}_s\Phi_{(1-s)T}^2\mathrm{d}s\Big],$$

as requested. □

In order to prove Thm. 6.2, fix a homotopy $\mathcal{M}^s := (\mathcal{H},\mathbf{c}^s,Q_s)$, $s\in[0,1]$, of ϑ-summable Fredholm modules. To define a Chern-Simons form, we again use the trick to pass to the extended Fredholm module $\Omega_{\mathbb{T}}$, as in the proof of Thm. 5.3. Then $\mathcal{M}_{\mathbb{T}}^s := (\mathcal{H},\mathbf{c}_{\mathbb{T}}^s,Q_s)$, $s\in[0,1]$ is a homotopy of *weak* ϑ-summable Fredholm modules over $\Omega_{\mathbb{T}}$. Now using the map

$$\alpha : \mathsf{C}(\Omega_{\mathbb{T}}) \longrightarrow \mathsf{B}(\underline{\Omega_{\mathbb{T}}}), \qquad (\theta_0,\dots,\theta_N)\longmapsto \underline{\mathbf{N}}(\sigma\theta_0,\theta_1,\dots,\theta_N),$$

from the proof of Thm. 5.3, we set

$$\mathrm{CS}\big((\mathcal{M}_{\mathbb{T}}^s)_{s\in[0,1]}\big) := -\int_0^1 \alpha^*\,\mathrm{Str}(\Psi_1^{\mathbb{T},s})\mathrm{d}s. \tag{6.3}$$

By Prop 6.3, $\mathrm{CS}((\mathcal{M}_{\mathbb{T}}^s)_{s\in[0,1]})$ extends to an odd entire cochain, and similar to the proof of Thm. 5.5, one shows that it is Chen normalized. Thm. 6.2 is then a consequence of the following result.

Theorem 6.6 (Transgression formula). *For any $T > 0$, we have the transgression formula*

$$\mathrm{Ch}(\mathcal{M}_1^{\mathbb{T}}) - \mathrm{Ch}(\mathcal{M}_0^{\mathbb{T}}) = (\underline{d}_{\mathbb{T}} + \underline{b} + \underline{B})\mathrm{CS}\big((\mathcal{M}_{\mathbb{T}}^s)_{s\in[0,1]}\big).$$

Proof. Since $-\mathrm{Ch}(\mathcal{M}_s^{\mathbb{T}}) = \alpha^* \operatorname{Str}(\Phi^{\mathbb{T},s}) = \operatorname{Str}(\Phi^{\mathbb{T},s}\alpha)$ by the proof Thm. 5.3, Prop. 6.4 implies

$$-\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Ch}(\mathcal{M}_s^{\mathbb{T}}) = \operatorname{Str}(\delta\Psi_1^{\mathbb{T},s}\alpha) + \operatorname{Str}\Big([\omega^s, \Psi_1^{\mathbb{T},s}]\alpha\Big) = \operatorname{Str}\Big(\Psi_1^{\mathbb{T},s}(d_{\mathbb{T}} + b')\alpha\Big),$$

where we used that α takes values in the space $\mathsf{B}^\sharp(\underline{\Omega}_{\mathbb{T}})$ of cyclic chains, on which $[\omega^s, \Psi_1^{\mathbb{T},s}]$ vanishes. Using (5.5), we then get

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Ch}(\mathcal{M}_s^{\mathbb{T}}) = -(\underline{d}_{\mathbb{T}} + \underline{b} + \underline{B})\alpha^* \operatorname{Str}(\Psi_1^{\mathbb{T},s}) - \operatorname{Str}(\Psi_1^{\mathbb{T},s}(S - \mathrm{id})h).$$

The second summand on the right hand side vanishes by an argument similar to that in the proof Thm. 5.3, which finishes the proof. □

Remark 6.7. Of course, the above argument can be made just as well in the case of a homotopy $\mathcal{M}^s$, $s \in [0, 1]$, of *weak* ϑ-summable Fredholm modules over Ω. In this case, one obtains that $\mathrm{Ch}(\mathcal{M}^0)$ and $\mathrm{Ch}(\mathcal{M}^1)$ define the same class in the entire cyclic homology of Ω, the homology of the complex $\mathcal{L}(\mathsf{C}_\epsilon(\Omega), \mathbb{C})$.

Example 6.8 (The localization principle). As an application of the homotopy invariance, given a Fredholm module $\mathcal{M} = (\mathcal{H}, \mathbf{c}, Q)$, consider the family of Fredholm modules $\mathcal{M}^s = (\mathcal{H}, \mathbf{c}^s, Q_s)$, $s \in \mathbb{R}_+$, where

$$\mathbf{c}^s(\theta) = s^{|\theta|/2}\mathbf{c}(\theta), \qquad Q_s = s^{1/2}Q.$$

Clearly $\mathcal{M}_1 = \mathcal{M}$. It is easy to see that (2.2) is still satisfied, as well as the assumptions of Def. 6.1. Hence Thm. 6.2 implies that the cohomology class in $\mathsf{h}_+^{\epsilon,\mathbb{T}}(\Omega)$ defined by $\mathrm{Ch}(\mathcal{M}_{\mathbb{T}}^s)$ is independent of s. In particular, if $c \in \mathsf{N}_{\epsilon,\mathbb{T}}(\Omega)$ is closed, $(\underline{d}_{\mathbb{T}} + \underline{b} + \underline{B})c = 0$, then the number $\langle \mathrm{Ch}(\mathcal{M}_{\mathbb{T}}^s), c\rangle$ is independent of s.
We can say much more if we specialize to the Example 2.4 where X is a compact spin manifold and we consider the Fredholm module $\mathcal{M}_X$ and its acyclic extension $\mathcal{M}_{X,\mathbb{T}}$ over $\Omega(X)$. From the short time asymptotic expansion of the heat kernel, it follows that the expression $\langle \mathrm{Ch}(\mathcal{M}_{X,\mathbb{T}}^s), c\rangle$ has an asymptotic expansion in s (as s goes to zero) for *any* chain $c \in \mathsf{C}_\epsilon(\Omega_{\mathbb{T}})$, not only the closed ones. In fact, using a souped up version of Getzler's rescaling trick, one can show that it even has a *limit* [27]. Explicitly, the result is

$$\lim_{s\to 0} \big\langle \mathrm{Ch}(\mathcal{M}_{X,\mathbb{T}}^s), (\theta_0, \dots, \theta_N)\big\rangle = \int_X \widehat{A}(X) \wedge \theta_0' \wedge \theta_1'' \wedge \dots \wedge \theta_N'',$$

for $\theta_0 \in \Omega_{\mathbb{T}}$, $\theta_1, \dots, \theta_N \in \Omega_{\mathbb{T}}[1]$, where as usual, we wrote $\theta_j = \theta_j' + \sigma\theta_j''$. Note that the right hand side above defines a cochain $\mathrm{Ch}_0(X)$, given by

$$\big\langle \mathrm{Ch}_0(X), (\theta_0, \dots, \theta_N)\big\rangle := \int_X \widehat{A}(X) \wedge \theta_0' \wedge \theta_1'' \wedge \dots \wedge \theta_N'',$$

which is closed and Chen normalized. The results above show that it is cohomologous to $\mathrm{Ch}(\mathcal{M}_{X,\mathbb{T}})$ in the cocomplex $\mathcal{L}(\mathsf{N}^+_{\epsilon,\mathbb{T}}(\Omega(X)), \mathbb{C})$. It is moreover easy to show that $\mathrm{Ch}_0(X)$ is equal to the pullback of the cochain μ_0 on $\Omega(\mathsf{L}X)$ (c.f. (1.7)) by the extended iterated integral map ρ of Getzler, Jones and Petrack [17].

7 The Bismut-Chern Character of an Idempotent

Let Ω be a locally convex dg algebra. The algebra of $n \times n$ matrices with entries in Ω is denoted by $\mathrm{Mat}_n(\Omega)$, which is then again a locally convex dg algebra, with the differential (also denoted by d) acting entrywise. We can form the acyclic extension $\Omega_{\mathbb{T}}$ of Ω as in Example 2.6 by adjoining a formal variable σ of degree -1. The same can be done for $\mathrm{Mat}_n(\Omega)$; we have $\mathrm{Mat}_n(\Omega)_{\mathbb{T}} \cong \mathrm{Mat}_n(\Omega_{\mathbb{T}})$.
The goal of this section is to construct the Bismut-Chern character associated to an in idempotent p in the algebra $\mathrm{Mat}_n(\Omega^0)$, which will be an even element of $\mathsf{C}_\epsilon(\Omega_{\mathbb{T}})$, closed as an element in the complex $\mathsf{N}_{\mathbb{T},\epsilon}(\Omega)$. To this end, let $p \in \mathrm{Mat}_n(\Omega^0)$ be an idempotent, $p^2 = p$, and write $p^\perp := \mathbf{1} - p$. The formula

$$\nabla\theta = p\, d(p\theta) + p^\perp d(p^\perp\theta). \tag{7.1}$$

then defines an operator ∇ on $\mathrm{Mat}_n(\Omega)$. The corresponding *connection form* $\varrho_p \in \mathrm{Mat}_n(\Omega^1)$ is defined by the equation $\nabla = d + \varrho_p$. Its *curvature* is the element R_p of $\mathrm{Mat}_n(\Omega^2)$ given by $R_p = d\varrho_p + \varrho_p^2$. We have the explicit formulas

$$\varrho_p = p\, dp + p^\perp dp^\perp = (2p - \mathbf{1})dp, \qquad R_p = (dp)^2.$$

for the connection form and the curvature.

Example 7.1 (Vector bundles with connection). Given a manifold X, any smooth function $p : X \to \mathrm{Mat}_n(\mathbb{C})$ taking values in idempotents (equivalently, an idempotent in $\mathrm{Mat}_n(C^\infty(X)) \subset \mathrm{Mat}_n(\Omega(X))$) determines a vector bundle $E := \mathrm{im}(p)$ with connection ∇ as defined above. It is well-known [28, Thm. 1] that any vector bundle with connection on X is isomorphic (through a connection-preserving isomorphism) to one of this form.

Following [17, Section 6], we combine these elements into a single even element $\mathfrak{R}_p \in \mathrm{Mat}_n(\Omega_{\mathbb{T}})$ given by

$$\mathfrak{R}_p := \varrho_p + \sigma R_p.$$

Definition 7.2 (Bismut-Chern character). The *Bismut-Chern character* of p is the even entire chain $\mathrm{Ch}(p) \in \mathsf{C}^+_\epsilon(\Omega_{\mathbb{T}})$ given by

$$\mathrm{Ch}(p) = \sum_{N=0}^{\infty} (-1)^N \mathrm{tr}\big(p, \underbrace{\mathfrak{R}_p, \dots, \mathfrak{R}_p}_{N}\big).$$

In the definition,

$$\mathrm{tr}:\mathsf{C}_\epsilon(\mathrm{Mat}_n(\Omega_{\mathbb{T}}))\longrightarrow\mathsf{C}_\epsilon(\Omega_{\mathbb{T}}),\quad(\Theta_0,\dots,\Theta_N)\longmapsto\sum_{i_0,\dots,i_N=1}^{n}\big((\Theta_0)^{i_1}_{i_0},(\Theta_1)^{i_1}_{i_2},\dots,(\Theta_N)^{i_0}_{i_N}\big)$$

is the *generalized trace map*, which preserves all differentials, see [25, 1.2.1]. Note that $\mathrm{Ch}(p)$ is indeed entire: If ν is a continuous seminorm on $\Omega_{\mathbb{T}}$, then

$$\nu_{N+1}\big(p,\underbrace{\mathfrak{R}_p,\dots,\mathfrak{R}_p}_{N}\big)\leq\nu(p)\nu(\mathfrak{R}_p)^N.$$

Moreover, $\mathrm{Ch}(p)$ is even because $\mathfrak{R}_p$ is odd in $\mathrm{Mat}_n(\Omega_{\mathbb{T}})$, hence even in $\mathrm{Mat}_n(\Omega_{\mathbb{T}})[1]$.

Remark 7.3. The entire chain $\mathrm{Ch}(p)$ was first considered by Getzler, Jones and Petrack in [17, Section 6], in the Example 2.4, where $\Omega=\Omega(X)$, differential forms on a manifold X. In this case, $\Omega_{\mathbb{T}}\cong\Omega(X\times S^1)^{S^1}$, the space of S^1-invariant forms on $X\times S^1$ (c.f. Remark 2.7). The significance of $\mathrm{Ch}(p)$ is that under the extended iterated integral map

$$\rho:\mathsf{C}_\epsilon\big(\Omega(X)_{\mathbb{T}}\big)\longrightarrow\Omega(\mathsf{L}X),$$

$\mathrm{Ch}(p)$ is sent to the *Bismut-Chern character* $\mathrm{BCh}(E,\nabla)$ of the vector bundle with connection (E,∇) corresponding to p as in Example 7.1. This is an equivariantly closed differential form on the loop space $\mathsf{L}X$ of X, first considered by Bismut in [4]. An odd variant of $\mathrm{Ch}(p)$ has been recently constructed by the first named author and S. Cacciatori in [6], producing the odd Bismut-Chern character of Wilson [30].

Theorem 7.4. *In the quotient* $\mathsf{N}_{\mathbb{T},\epsilon}(\Omega)$*, one has*

$$(\underline{d}_{\mathbb{T}}+\underline{b}+\underline{B})\mathrm{Ch}(p)=0,$$

hence $\mathrm{Ch}(p)$ *defines an element of* $\mathsf{h}^+_{\mathbb{T},\epsilon}(\Omega)$.

For the proof of the proof of Thm. 7.4, we need the following Lemma; throughout, we write $\mathfrak{R}$ instead of $\mathfrak{R}_p$.

Lemma 7.5. *We have the identities*

$$d_{\mathbb{T}}p+[\mathfrak{R},p]=0 \tag{7.2}$$

$$d_{\mathbb{T}}\mathfrak{R}+\mathfrak{R}^2=0 \tag{7.3}$$

$$d_{\mathbb{T}}(\sigma p)+[\mathfrak{R},\sigma p]=-p. \tag{7.4}$$

Proof. This follow from straightforward calculation, using the relations

$$[p,\varrho]=dp,\quad[\varrho,dp]=[dp,p],\quad\text{and}\quad[R,p]=0.$$

The first of these latter identities follows from differentiating the equation $p^2=p$ and straightforward calculation using the definition of ϱ, the second identity follows from differentiating the first, and the third is a direct calculation using the previous results. □

Remark 7.6. In fact, the proof shows that the element defined above is even closed *before* taking the trace, i.e. as an element in $\mathsf{C}_\epsilon(\mathrm{Mat}_n(\Omega_{\mathbb{T}}))$.

Proof (of Thm. 7.4). We will show that that $(\underline{d}_{\mathbb{T}}+\underline{b}+\underline{B})\mathrm{Ch}(p)$ is contained in the closure $\overline{\mathsf{D}_{\mathbb{T}}(\Omega)}$ of $\mathsf{D}_{\mathbb{T}}(\Omega)$ in $\mathsf{C}_{\epsilon,\mathbb{T}}(\Omega)$. To this end, calculate

$$\underline{d}_{\mathbb{T}}(p,\underbrace{\mathfrak{R},\dots,\mathfrak{R}}_{N}) = \big(\underline{d}_{\mathbb{T}}p,\underbrace{\mathfrak{R},\dots,\mathfrak{R}}_{N}\big) - \sum_{k=1}^{N}\big(p,\underbrace{\mathfrak{R},\dots,\mathfrak{R}}_{k-1},\underline{d}_{\mathbb{T}}\mathfrak{R},\underbrace{\mathfrak{R},\dots,\mathfrak{R}}_{N-k}\big)$$

$$\underline{b}(p,\underbrace{\mathfrak{R},\dots,\mathfrak{R}}_{N}) = ([p,\mathfrak{R}],\underbrace{\mathfrak{R},\dots,\mathfrak{R}}_{N-1}) + \sum_{k=1}^{N-1}(p,\underbrace{\mathfrak{R},\dots,\mathfrak{R}}_{k-1},\mathfrak{R}^2,\underbrace{\mathfrak{R},\dots,\mathfrak{R}}_{N-k-1}).$$

By Lemma 7.5, the second equation can be rewritten as

$$\underline{b}(p,\underbrace{\mathfrak{R},\dots,\mathfrak{R}}_{N}) = \big(\underline{d}_{\mathbb{T}}p,\underbrace{\mathfrak{R},\dots,\mathfrak{R}}_{N-1}\big) - \sum_{k=1}^{N-1}\big(p,\underbrace{\mathfrak{R},\dots,\mathfrak{R}}_{k-1},\underline{d}_{\mathbb{T}}\mathfrak{R},\underbrace{\mathfrak{R},\dots,\mathfrak{R}}_{N-k-1}\big),$$

which telescopes with the the right hand side of the first equation as we take the alternating sum over N, meaning that

$$(\underline{d}_{\mathbb{T}}+\underline{b})\mathrm{Ch}(p)=0.$$

Now $\underline{B}\mathrm{Ch}(p)$ is a sum of terms of the form $\underline{B}(p,\mathfrak{R},\dots,\mathfrak{R})$, which is contained in $\overline{\mathsf{D}_{\mathbb{T}}(\Omega)}$, since p is of degree zero. □

We proceed by discussing the homotopy invariance of the Bismut-Chern character. First some notation: If $(\theta_j^s)_{s\in[0,1]}$, $j=0,\dots,N$ are continuously differentiable families of elements in Ω, we set

$$\partial_s(\theta_0^s,\dots,\theta_N^s) := -\sum_{k=0}^{N}(-1)^{m_{k-1}}\big(\theta_0^s,\dots,\theta_{k-1}^s,\sigma\frac{\mathrm{d}}{\mathrm{d}s}\theta_k^s,\theta_{k+1}^s,\dots,\theta_N^s\big),$$

where $m_k=|\theta_0|+\dots+|\theta_k|-k$ and $\dot{\theta}_k^s$ denotes the derivative in direction of s. Now given a continuously differentiable family $(p_s)_{s\in[0,1]}$ of idempotents, we define the corresponding *Bismut-Chern-Simons form* by

$$\mathrm{CS}\big((p_s)_{s\in[0,1]}\big) := \int_0^1 \partial_s\mathrm{Ch}(p_s)\mathrm{d}s.$$

As for $\mathrm{Ch}(p)$, it can be easily shown that $\mathrm{Ch}(p_s)$ is a continuously differentiable function of s with values in $\mathsf{C}_\epsilon(\Omega_{\mathbb{T}})$ and that $\partial_s\mathrm{Ch}(p_s)$ is entire for any $s\in[0,1]$. Now since this is a continuous function of s with values in the complete locally convex space $\mathsf{C}_\epsilon(\Omega_{\mathbb{T}})$, its integral $\mathrm{CS}((p_s)_{s\in[0,1]})$ is well-defined.

Theorem 7.7 (Transgression). *In the quotient complex $\mathsf{N}_{\mathbb{T},\epsilon}(\Omega)$, we have the transgression formula*

$$\mathrm{Ch}(p_1) - \mathrm{Ch}(p_0) = (\underline{d}_{\mathbb{T}} + \underline{b} + \underline{B})\mathrm{CS}\big((p_s)_{s\in[0,1]}\big).$$

In particular, $\mathrm{Ch}(p_1)$ *and* $\mathrm{Ch}(p_0)$ *define the same element of* $\mathsf{h}^+_{\mathbb{T},\epsilon}(\Omega)$.

Proof. We have to show that $(\underline{d}_{\mathbb{T}} + \underline{b} + \underline{B})\mathrm{CS}\big((p_s)_{s\in[0,1]}\big)$ is contained in the closure $\overline{\mathsf{D}_{\mathbb{T}}(\Omega)}$ of $\mathsf{D}_{\mathbb{T}}(\Omega)$ in $\mathsf{C}_\epsilon(\Omega_{\mathbb{T}})$, for all $s \in [0,1]$. To this end, it is straightforward to verify that

$$\underline{d}\partial_s + \partial_s\underline{d} = \underline{b}\partial_s + \partial_s\underline{b} = \underline{B}\partial_s + \partial_s\underline{B} = 0, \qquad \text{and} \qquad \underline{\iota}\partial_s + \partial_s\underline{\iota} = \frac{\mathrm{d}}{\mathrm{d}s},$$

where

$$\frac{\mathrm{d}}{\mathrm{d}s}(\theta_0^s, \dots, \theta_N^s) = \sum_{k=0}^N (\theta_0^s, \dots, \frac{\mathrm{d}}{\mathrm{d}s}\theta_k^s, \dots, \theta_N^s)$$

is the derivative in the locally convex space $\mathsf{C}_\epsilon(\Omega_{\mathbb{T}})$. Hence we have

$$\begin{aligned}(\underline{d}_{\mathbb{T}} + \underline{b} + \underline{B})\mathrm{CS}\big((p_s)_{s\in[0,1]}\big) &= -\int_0^1 \partial_s(\underline{d}_{\mathbb{T}} + \underline{b} + \underline{B})\mathrm{Ch}(p_s)\mathrm{d}s + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Ch}(p_s)\mathrm{d}s\\ &= -\int_0^1 \partial_s\underline{B}\mathrm{Ch}(p_s)\mathrm{d}s + \mathrm{Ch}(p_1) - \mathrm{Ch}(p_0),\end{aligned}$$

where in the last step, we used the fundamental theorem of calculus and the fact that $(\underline{d}_{\mathbb{T}} + \underline{b})\mathrm{Ch}(p_s) = 0$ by the proof of Thm. 7.4 above.
It therefore remains to show that $\partial_s\underline{B}\mathrm{Ch}(p_s)$ is contained in $\overline{\mathsf{D}_{\mathbb{T}}(\Omega)}$ for all s, which is not quite obvious as ∂_s does not generally preserve $\overline{\mathsf{D}_{\mathbb{T}}(\Omega)}$. It is easy to see that $\partial_s\underline{B}\mathrm{Ch}(p_s)$ is entire, so we just need an algebraic argument. To this end, observe that modulo $\overline{\mathsf{D}_{\mathbb{T}}(\Omega)}$,

$$\begin{aligned}\underline{B}\mathrm{Ch}(p_s) &= \sum_{N=0}^\infty (-1)^N \sum_{k=0}^N (\mathbf{1}, \underbrace{\mathfrak{R}, \dots, \mathfrak{R}}_{k}, \sigma\dot{p}, \underbrace{\mathfrak{R}, \dots, \mathfrak{R}}_{N-k})\\ &= \underline{B}\sum_{N=0}^\infty (-1)^N(\sigma\dot{p}, \underbrace{\mathfrak{R}, \dots, \mathfrak{R}}_{N}),\end{aligned} \tag{7.5}$$

where we dropped the dependence on s for readability and wrote $\dot{p} = \frac{\mathrm{d}}{\mathrm{d}s}p_s$. Now because of $\dot{p} = \dot{p}p + p\dot{p}$ and the identity (5.9), we have modulo $\overline{\mathsf{D}_{\mathbb{T}}(\Omega)}$

$$(\sigma\dot{p}, \underbrace{\mathfrak{R}, \dots, \mathfrak{R}}_{N}) = (p\sigma\dot{p}, \underbrace{\mathfrak{R}, \dots, \mathfrak{R}}_{N}) + (\sigma\dot{p}, d_{\mathbb{T}}p, \underbrace{\mathfrak{R}, \dots, \mathfrak{R}}_{N}) + (\sigma\dot{p}, p\mathfrak{R}, \underbrace{\mathfrak{R}, \dots, \mathfrak{R}}_{N-1}).$$

After summing over N and shifting an index on the second factor, we can use (7.2) to obtain that modulo $\overline{\mathsf{D}_{\mathbb{T}}(\Omega)}$,

$$\sum_{N=0}^\infty (-1)^N(\sigma\dot{p}, \underbrace{\mathfrak{R}, \dots, \mathfrak{R}}_{N}) = \sum_{N=0}^\infty (-1)^N(p\sigma\dot{p}, \underbrace{\mathfrak{R}, \dots, \mathfrak{R}}_{N}) + \sum_{N=0}^\infty (-1)^N(\sigma\dot{p}, \mathfrak{R}p, \underbrace{\mathfrak{R}, \dots, \mathfrak{R}}_{N-1}).$$

Repeating this, we can shift the factor of p in the second sum all the way to the right and then use (7.2) combined with (5.10), making this term cancels with the first sum on the right hand side of the above equation. This implies

$$\sum_{N=0}^{\infty}(-1)^N(\sigma\dot{p},\underbrace{\mathfrak{R},\dots,\mathfrak{R}}_{N}) \in \overline{\mathsf{D}_{\mathbb{T}}(\Omega)}.$$

The claim now follows from (7.5), bearing in mind that $\underline{B}$ preserves $\overline{\mathsf{D}_{\mathbb{T}}(\Omega)}$. □

8 An Index Theorem

In this section, let Ω be a locally convex dg algebra and let $\mathcal{M} = (\mathcal{H}, \mathbf{c}, Q)$ be a ϑ-summable Fredholm module over Ω. We emphasize that to obtain the results of this section, it is crucial that $\mathcal{M}$ satisfies (2.2).

Given an idempotent $p = p^2 \in \mathrm{Mat}_n(\Omega^0)$, (2.2) implies that the operator $\mathbf{c}(p)$ is an even idempotent acting on the Hilbert space $\mathcal{H}^n$, so that $\mathbf{c}(p)\mathcal{H}^n$ becomes a closed subspace of $\mathcal{H}$ and thus a Hilbert space on its own right. As $\mathbf{c}(p)$ is even, the $\mathbb{Z}_2$-grading restricts to $\mathbf{c}(p)\mathcal{H}^n$. We obtain a well-defined odd unbounded operator

$$Q_p := \mathbf{c}(p)Q\mathbf{c}(p),$$

a densely defined closed operator on the Hilbert space $\mathbf{c}(p)\mathcal{H}^n$. The purpose of this section is to prove the following index theorem.

Theorem 8.1 (Geometric index theorem). *The operator Q_p defined is Fredholm and its $\mathbb{Z}_2$-graded index is given by the formula*

$$\mathrm{ind}(Q_p) = \big\langle \mathrm{Ch}(\mathcal{M}_{\mathbb{T}}), \mathrm{Ch}(p)\big\rangle. \tag{8.1}$$

This theorem can be seen as a "loop space version" of the index theorem for the JLO-cocycle of Getzler-Szenes [18]. While their proof relies on the homotopy invariance of the Chern character, a remarkable observation is that in our setup, the right hand side of (8.1) is *directly* the super trace of a heat operator, without having to defer to any homotopy result. Precisely, we have the following result.

Proposition 8.2 (Perturbation series). *Let $p \in \mathrm{Mat}_n(\Omega^0)$ be an idempotent. Then the unbounded operator*

$$Q_1 := \mathbf{c}(p)Q\mathbf{c}(p) + \mathbf{c}(p^\perp)Q\mathbf{c}(p^\perp),$$

on $\mathcal{H}^n$ is closed on the same domain as Q, and its square generates a strongly continuous semigroup of operators. Moreover, one has

$$e^{-TQ_1^2} = \sum_{N=0}^{\infty}(-1)^N \Phi_T^{\mathbb{T}}(\underbrace{\mathfrak{R}_p,\dots,\mathfrak{R}_p}_{N}), \tag{8.2}$$

for all $T > 0$, where $\Phi_T^{\mathbb{T}}$ is the quantization map of the Fredholm module $\mathcal{M}_{\mathbb{T}} = (Q, \mathbf{c}_{\mathbb{T}}, \mathcal{H}^n)$ over the dg algebra $\mathrm{Mat}_n(\Omega_{\mathbb{T}})$.

Proof. From (2.2), we obtain $[Q, \mathbf{c}(p)] = \mathbf{c}(dp)$ and $\mathbf{c}(p)\mathbf{c}(dp) = \mathbf{c}(p\,dp)$. Therefore

$$Q_1 = Q + \mathbf{c}(2p - \mathbf{1})[Q, \mathbf{c}(p)] = Q + \mathbf{c}\big((2p - \mathbf{1})dp\big).$$

This shows that that Q_1 is in fact a bounded perturbation of Q, hence is a closed operator on the same domain as Q. On the other hand, we have $p\,dp + p^\perp dp^\perp = (2p - \mathbf{1})dp$, so with a view on (5.1),

$$\begin{aligned} F(\mathfrak{R}) &= \mathbf{c}\big(d\{(2p - \mathbf{1})dp\}\big) - \big[Q, \mathbf{c}\big((2p - \mathbf{1})dp\big)\big] - \mathbf{c}\big((dp)^2\big) \\ &= \mathbf{c}\big((dp)^2\big) - \big[Q, \mathbf{c}\big((2p - \mathbf{1})dp\big)\big], \end{aligned}$$

as well as

$$\begin{aligned} F(\mathfrak{R}, \mathfrak{R}) &= \mathbf{c}\big((2p - \mathbf{1})dp\big)^2 - \mathbf{c}\big(((2p - \mathbf{1})dp)^2\big) \\ &= \mathbf{c}\big((2p - \mathbf{1})dp\big)^2 + \mathbf{c}\big((dp)^2\big). \end{aligned}$$

Putting together, we obtain

$$Q^2 - F(\mathfrak{R}) + F(\mathfrak{R}, \mathfrak{R}) = Q^2 + \big[Q, \mathbf{c}\big((2p - \mathbf{1})dp\big)\big] + \mathbf{c}\big((2p - \mathbf{1})dp\big) = Q_1^2.$$

Hence we can write $Q_1^2 = Q^2 + R$, where $R := F(\mathfrak{R}, \mathfrak{R}) - F(\mathfrak{R})$. By the assumptions (A1), (A2) on the Fredholm module, we can use Lemma 4.5 to define $e^{-TQ_1^2} = e^{-T(Q^2+R)}$ using the perturbation series

$$e^{-T(Q^2+R)} = \sum_{M=0}^{\infty} (-T)^M \{\underbrace{R, \dots, R}_{M}\}_{TQ}. \tag{8.3}$$

It is easy to see that this indeed defines a strongly continuous semigroup of operators with infinitesimal generator $Q^2 + P = Q_1^2$. Now we have

$$\sum_{M=0}^{\infty} (-T)^M \{\underbrace{R, \dots, R}_{M}\}_{TQ} = \sum_{M=0}^{\infty} (-T)^M \big\{F(\mathfrak{R}, \mathfrak{R}) - F(\mathfrak{R}), \dots, F(\mathfrak{R}, \mathfrak{R}) - F(\mathfrak{R})\big\}_{TQ}.$$

Expanding each term on the right hand side of this equation by multi-linearity, we see that we obtain an infinite sum of brackets $\{\dots\}_{TQ}$ which contains all possible sequences of operators $F(\mathfrak{R})$ and $F(\mathfrak{R}, \mathfrak{R})$. Observe now that by formula (4.11), this coincides with the right hand side of (8.2). □

Proof (of Thm. 8.1). If $\mathbf{c}(p)$ is self-adjoint, the proof follows directly from Prop. 8.2. Namely, in that case, we get

$$\big\langle \mathrm{Ch}(\mathcal{M}_{\mathbb{T}}), \mathrm{Ch}(p) \big\rangle = \mathrm{Str}_{\mathcal{H}^n}\big(\mathbf{c}(p) e^{-Q_1^2}\big) = \mathrm{Str}_{\mathbf{c}(p)\mathcal{H}^n}(e^{-Q_p^2}),$$

which equals the graded index of Q_p by the usual McKean-Singer formula.
To deal with the general case, we adapt an idea of Getzler-Szenes [18] to our setup, making the following construction. Consider the $*$-subalgebra

$$\mathcal{B}_Q := \big\{A \in \mathcal{L}(\mathcal{H})^+ \mid \Delta^{1/2} A \Delta^{-1/2} \text{ and } \Delta^{1/2} A^* \Delta^{-1/2} \text{ are densely defined and bounded}\big\}, \tag{8.4}$$

of $\mathcal{L}(\mathcal{H})$, where $\Delta = Q^2 + 1$ and $\mathcal{L}(\mathcal{H})^+$ denotes the subalgebra of even operators. Note that we have

$$(Q + A)^2 = Q^2 + [Q, A] + A^2,$$

hence for $A \in \mathcal{B}_Q$, the operator $R = [Q, A] + A^2$ satisfies the assumptions of Lemma 4.5. Hence for any $A \in \mathcal{B}_Q$, the heat semigroup $e^{-T(Q+A)^2}$ can be defined using its perturbation series (8.3).
Furthermore, set

$$\mathcal{A}_Q := \big\{A \in \mathcal{L}(\mathcal{H}) \mid [Q, A] \in \mathcal{B}_Q\big\}$$

and let $\Omega_Q = \Omega_{\mathcal{A}_Q}$ the corresponding dg algebra of non-commutative differential forms, which comes with a canonical map $\mathbf{c}_Q : \Omega_Q \to \mathcal{L}(\mathcal{H})$ so that $\mathcal{M}_Q = (\mathcal{H}, \mathbf{c}_Q, Q)$ is a Fredholm module over Ω_Q, as explained in Example 2.3.
Now the calculations in the proof of Prop. 8.2 show that for an idempotent $p \in \Omega^0$, we have

$$\big\langle \mathrm{Ch}(\mathcal{M}_{\mathbb{T}}), \mathrm{Ch}(p)\big\rangle = \big\langle \mathrm{Ch}(\mathcal{M}_{Q,\mathbb{T}}), \mathrm{Ch}\big(\mathbf{c}(p)\big)\big\rangle,$$

where the right hand side denotes the paring of the extended entire cyclic complex associated to Ω_Q. Therefore, it suffices to prove Thm. 8.1 for the "universal example" Ω_Q.
To this end, we will show that the idempotent $\mathbf{c}(p)$ can be homotoped within $\mathcal{A}_Q$ to a self-adjoint projection P, as then

$$\big\langle \mathrm{Ch}(\mathcal{M}_{Q,\mathbb{T}}), \mathrm{Ch}\big(\mathbf{c}(p)\big)\big\rangle = \big\langle \mathrm{Ch}(\mathcal{M}_{Q,\mathbb{T}}), \mathrm{Ch}(P)\big\rangle$$

by the transgression formula Thm. 7.7 and the closedness of $\mathrm{Ch}(\mathcal{M}_{Q,\mathbb{T}})$, Thm. 5.3 (together with the fact that it is Chen normalized, Thm. 5.5). By Prop. 4.6.2 in [5], this property follows from the fact that $\mathcal{A}_Q$ is a *spectral* subalgebra of $\mathcal{L}(\mathcal{H})$, i.e. that for any $A \in \mathcal{A}_Q$ which is invertible in $\mathcal{L}(\mathcal{H})$, its inverse $A^{-1} \in \mathcal{A}_Q$. To see that the larger algebra $\mathcal{B}_Q$ is spectral, notice that for $A \in \mathcal{B}_Q$,

$$\Delta^{1/2} A^{-1} \Delta^{-1/2} = (\Delta^{1/2} A \Delta^{-1/2})^{-1}.$$

But $\Delta^{1/2} A \Delta^{-1/2}$ is bounded, hence its inverse is bounded as well. Now for $A \in \mathcal{A}_Q$, we have

$$\big[Q, A^{-1}\big] = A^{-1}[Q, A]A^{-1},$$

which is a product of elements in $\mathcal{B}_Q$ by assumption, hence in $\mathcal{B}_Q$. Since these arguments also work for A^* instead of A, this shows that $\mathcal{A}_Q$ is indeed spectral in $\mathcal{L}(\mathcal{H})$ and finishes the proof. □

Finally, we formulate a homological version of the above result. We remind the reader that the K-theory $\mathsf{K}_0(\mathcal{A})$ of an algebra $\mathcal{A}$ can be defined as the Grothendieck group of the monoid $\mathrm{Proj}(\mathcal{A})$ of equivalence classes of idempotents $p = p^2 \in \mathrm{Mat}_\infty(\mathcal{A})$, where two idempotents are called equivalent if they are equal up to conjugation by an element of $\mathrm{GL}_\infty(\mathcal{A})$.

Thm. 8.1 in combination with Thm. 7.7 and homotopy invariance of the index easily implies the following result.

Corollary 8.3 (Homological index theorem). *We have a commutative diagram*

$$\begin{array}{ccc} \mathsf{K}_0(\Omega^0) & \xrightarrow{\ \mathrm{ind}_{\mathcal{M}}\ } & \mathbb{Z} \\ {\scriptstyle \mathrm{Ch}}\downarrow & & \downarrow \\ \mathsf{h}^+_{\mathbb{T},\epsilon}(\Omega) & \xrightarrow[\ \mathrm{Ch}(\mathcal{M}_{\mathbb{T}})\ ]{} & \mathbb{C}, \end{array}$$

where $\mathrm{ind}_{\mathcal{M}}(p) = \mathrm{ind}(Q_p)$.

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