On the geometry of semiclassical limits on Dirichlet spaces

Batu Güneysu

Institut für Mathematik
Humboldt-Universität zu Berlin

A geometry day in Como

January 13, 2017

Compact manifolds and $\mathbb{R}^m$

For the moment, let (X, g) be either the Euclidean $\mathbb{R}^m$ or a compact Riemannian m-manifold

Theorem (Helffer/Robert; early 1980's)

For every "very bounded" smooth potential $w : X \to \mathbb{R}$, one has $Z_{QM}(g; w; \hbar)/Z_{cl}(g; w; \hbar) \to 1$ as $\hbar \to 0+$, where

$$Z_{QM}(g; w; \hbar) := \mathrm{tr}\left(\mathrm{e}^{-(\hbar^2 \Delta_g + w)}\right), \quad ,$$

$$Z_{cl}(g; w; \hbar) := (2\pi\hbar)^{-m} \int_{T^*X} \mathrm{e}^{-(|p|^2_{g^*} + w(q))} \mathrm{d}p \wedge \mathrm{d}q.$$

Note that $Z_{cl}(g; w; \hbar)$ is the integral of a globally defined $2m$-form on T^*X, as $m!\mathrm{d}p \wedge \mathrm{d}q$ is the local representation of the m-th power of the symplectic form on T^*X.

Compact manifolds and $\mathbb{R}^m$

For the moment, let (X, g) be either the Euclidean $\mathbb{R}^m$ or a compact Riemannian m-manifold

Theorem (Helffer/Robert; early 1980's)

For every "very bounded" smooth potential $w : X \to \mathbb{R}$, one has $Z_{QM}(g; w; \hbar)/Z_{cl}(g; w; \hbar) \to 1$ as $\hbar \to 0+$, where

$$Z_{QM}(g; w; \hbar) := \mathrm{tr}\left(\mathrm{e}^{-(\hbar^2 \Delta_g + w)}\right), \quad ,$$

$$Z_{cl}(g; w; \hbar) := (2\pi\hbar)^{-m} \int_{T^*X} \mathrm{e}^{-(|p|^2_{g^*} + w(q))} \mathrm{d}p \wedge \mathrm{d}q.$$

Note that $Z_{cl}(g; w; \hbar)$ is the integral of a globally defined $2m$-form on T^*X, as $m! \mathrm{d}p \wedge \mathrm{d}q$ is the local representation of the m-th power of the symplectic form on T^*X.

• We are interested in generalizations of this results to arbitrary Riemannian manifolds with minimal assumptions on w (no asymptotic expansions at hand), and even more general noncompact "spaces" than Riemannian manifolds

• essential observation for a possible abstract result: integrate out the momentum in HR-formula: HR is equivalent to

$$\lim_{t\to 0+}(2\pi t)^{m/2}\mathrm{tr}\big(\mathrm{e}^{-t(\Delta_g+w/t)}\big)=\int_X \mathrm{e}^{-w}\mathrm{d}\mu_g$$

• nice: no tangent space left! Idea: Consider $\Delta_g \geq 0$ as generator of a semigroup on an L^2-space. But how should we replace $(2\pi t)^{m/2}$? Probably this comes from the heat kernel $p_g(t,x,x)$ $\rightsquigarrow$ Let us build the whole analysis from an abstract heat kernel!

• We are interested in generalizations of this results to arbitrary Riemannian manifolds with minimal assumptions on w (no asymptotic expansions at hand), and even more general noncompact "spaces" than Riemannian manifolds

• essential observation for a possible abstract result: integrate out the momentum in HR-formula: HR is equivalent to

$$\lim_{t\to 0+}(2\pi t)^{m/2}\mathrm{tr}\big(\mathrm{e}^{-t(\Delta_g+w/t)}\big)=\int_X \mathrm{e}^{-w}\mathrm{d}\mu_g$$

• nice: no tangent space left! Idea: Consider $\Delta_g \geq 0$ as generator of a semigroup on an L^2-space. But how should we replace $(2\pi t)^{m/2}$? Probably this comes from the heat kernel $p_g(t,x,x)$ $\rightsquigarrow$ Let us build the whole analysis from an abstract heat kernel!

• We are interested in generalizations of this results to arbitrary Riemannian manifolds with minimal assumptions on w (no asymptotic expansions at hand), and even more general noncompact "spaces" than Riemannian manifolds
• essential observation for a possible abstract result: integrate out the momentum in HR-formula: HR is equivalent to

$$\lim_{t\to 0+}(2\pi t)^{m/2}\mathrm{tr}\big(\mathrm{e}^{-t(\Delta_g+w/t)}\big)=\int_X \mathrm{e}^{-w}\mathrm{d}\mu_g$$

• nice: no tangent space left! Idea: Consider $\Delta_g \geq 0$ as generator of a semigroup on an L^2-space. But how should we replace $(2\pi t)^{m/2}$? Probably this comes from the heat kernel $p_g(t,x,x)$
$\rightsquigarrow$ Let us build the whole analysis from an abstract heat kernel!

After some thinking (and taking my previous result for infinite weighted graphs into account), I finally had the essential insight: Rewrite the RHS of

$$\lim_{t\to 0+}(2\pi t)^{m/2}\mathrm{tr}\left(\mathrm{e}^{-t(\Delta_g+w/t)}\right)=\int_X \mathrm{e}^{-w}\mathrm{d}\mu_g$$

according to

$$\int_X \mathrm{e}^{-w}\mathrm{d}\mu_g=\int_X \underbrace{\lim_{t\to 0+}(2\pi t)^{m/2}p_g(t,x,x)}_{=1}\mathrm{e}^{-w(x)}\mathrm{d}\mu_g(x)$$

with $p_g(t,x,y)$ the minimal heat kernel. Everything that follows is based on this trivial observation....

μ-heat kernels I

- X: seperable metrizable locally cpt. space; μ: Radon measure on X with full support
- a Borel function $(t,x,y)\mapsto p(t,x,y)$ from $(0,\infty)\times X\times X$ to $(0,\infty)$ is called a *sppc μ-heat kernel*, if it is symmetric in (x,y) with

$$\int p(t,x,y)\mathrm{d}\mu(y)\leq 1,\ p(t+s,x,y)=\int p(t,x,z)p(s,z,y)\mathrm{d}\mu(z)$$

and if $P_t f := \int p(t,\cdot,y)f(y)\mathrm{d}\mu(y)$ is well-defined and strongly continuous at $t=0+$ in $L^2(X,\mu)$

- let $H_p\geq 0$ be the self-adjoint generator in $L^2(X,\mu)$ of $(P_t)_{t>0}$, and let Q_p be the quadratic form of H_p,

$\rightsquigarrow$ we fix such a triple (X,μ,p) in the sequel

μ-heat kernels I

- X: seperable metrizable locally cpt. space; μ: Radon measure on X with full support
- a Borel function $(t,x,y) \mapsto p(t,x,y)$ from $(0,\infty)\times X \times X$ to $(0,\infty)$ is called a *sppc μ-heat kernel*, if it is symmetric in (x,y) with

$$\int p(t,x,y)\mathrm{d}\mu(y) \leq 1,\ p(t+s,x,y) = \int p(t,x,z)p(s,z,y)\mathrm{d}\mu(z)$$

and if $P_t f := \int p(t,\cdot,y)f(y)\mathrm{d}\mu(y)$ is well-defined and strongly continuous at $t=0+$ in $L^2(X,\mu)$

- let $H_p \geq 0$ be the self-adjoint generator in $L^2(X,\mu)$ of $(P_t)_{t>0}$, and let Q_p be the quadratic form of H_p,

$\rightsquigarrow$ we fix such a triple (X,μ,p) in the sequel

μ-heat kernels I

• X: seperable metrizable locally cpt. space; μ: Radon measure on X with full support

• a Borel function $(t,x,y)\mapsto p(t,x,y)$ from $(0,\infty)\times X\times X$ to $(0,\infty)$ is called a *sppc μ-heat kernel*, if it is symmetric in (x,y) with

$$\int p(t,x,y)\mathrm{d}\mu(y)\leq 1,\ p(t+s,x,y)=\int p(t,x,z)p(s,z,y)\mathrm{d}\mu(z)$$

and if $P_t f := \int p(t,\cdot,y)f(y)\mathrm{d}\mu(y)$ is well-defined and strongly continuous at $t=0+$ in $L^2(X,\mu)$

• let $H_p\geq 0$ be the self-adjoint generator in $L^2(X,\mu)$ of $(P_t)_{t>0}$, and let Q_p be the quadratic form of H_p,

$\rightsquigarrow$ we fix such a triple (X,μ,p) in the sequel

μ-heat kernels II

• p is called *regular*, if $C_{\mathrm{cpt}}(X) \cap \mathrm{Dom}(Q_p)$ is dense in $C_{\mathrm{cpt}}(X)$ and dense in $\mathrm{Dom}(Q_p)$ (resp. natural norms) $\rightsquigarrow$ then Q_p automatically is a regular Dirichlet form

• Fukushima (1970's): regular Dirichlet forms are in 1:1 correspondence with Hunt processes having càdlàg paths

• thus every regular p induces a Wiener measure $\mathbb{P}^x_t$ with starting point $x \in X$ and terminal time $t > 0$ on the space $\Omega(X, t)$ of càdlàg paths $\gamma : [0, t] \to X$

• Using a construction by Fitzsimmons/Pitman/Yor, we can construct the pinned Wiener measure $\mathbb{P}^{x,y}_t$ with terminal point y; a *very subtle point:* unlike $\mathbb{P}^x_t$, the measure $\mathbb{P}^{x,y}_t$ lives on the smaller σ-algebra $\mathscr{F}^-(X, t) \subsetneq \mathscr{F}(X, t)$....annoying or interesting

μ-heat kernels II

• p is called *regular*, if $C_{\mathrm{cpt}}(X) \cap \mathrm{Dom}(Q_p)$ is dense in $C_{\mathrm{cpt}}(X)$ and dense in $\mathrm{Dom}(Q_p)$ (resp. natural norms) $\rightsquigarrow$ then Q_p automatically is a regular Dirichlet form

• Fukushima (1970's): regular Dirichlet forms are in 1:1 correspondence with Hunt processes having càdlàg paths

• thus every regular p induces a Wiener measure $\mathbb{P}^x_t$ with starting point $x \in X$ and terminal time $t > 0$ on the space $\Omega(X,t)$ of càdlàg paths $\gamma : [0,t] \to X$

• Using a construction by Fitzsimmons/Pitman/Yor, we can construct the pinned Wiener measure $\mathbb{P}^{x,y}_t$ with terminal point y; a *very subtle point:* unlike $\mathbb{P}^x_t$, the measure $\mathbb{P}^{x,y}_t$ lives on the smaller σ-algebra $\mathscr{F}^-(X,t) \subsetneq \mathscr{F}(X,t)$....annoying or interesting

μ-heat kernels II

• p is called *regular*, if $C_{\mathrm{cpt}}(X) \cap \mathrm{Dom}(Q_p)$ is dense in $C_{\mathrm{cpt}}(X)$ and dense in $\mathrm{Dom}(Q_p)$ (resp. natural norms) $\rightsquigarrow$ then Q_p automatically is a regular Dirichlet form

• Fukushima (1970's): regular Dirichlet forms are in 1:1 correspondence with Hunt processes having càdlàg paths

• thus every regular p induces a Wiener measure $\mathbb{P}^x_t$ with starting point $x \in X$ and terminal time $t > 0$ on the space $\Omega(X, t)$ of càdlàg paths $\gamma : [0, t] \to X$

• Using a construction by Fitzsimmons/Pitman/Yor, we can construct the pinned Wiener measure $\mathbb{P}^{x,y}_t$ with terminal point y; a *very subtle point:* unlike $\mathbb{P}^x_t$, the measure $\mathbb{P}^{x,y}_t$ lives on the smaller σ-algebra $\mathscr{F}^-(X, t) \subsetneq \mathscr{F}(X, t)$....annoying or interesting

μ-heat kernels II

- p is called *regular*, if $C_{\mathrm{cpt}}(X)\cap\mathrm{Dom}(Q_p)$ is dense in $C_{\mathrm{cpt}}(X)$ and dense in $\mathrm{Dom}(Q_p)$ (resp. natural norms) $\rightsquigarrow$ then Q_p automatically is a regular Dirichlet form
- Fukushima (1970's): regular Dirichlet forms are in 1:1 correspondence with Hunt processes having càdlàg paths
- thus every regular p induces a Wiener measure $\mathbb{P}^x_t$ with starting point $x\in X$ and terminal time $t>0$ on the space $\Omega(X,t)$ of càdlàg paths $\gamma:[0,t]\to X$
- Using a construction by Fitzsimmons/Pitman/Yor, we can construct the pinned Wiener measure $\mathbb{P}^{x,y}_t$ with terminal point y; a *very subtle point:* unlike $\mathbb{P}^x_t$, the measure $\mathbb{P}^{x,y}_t$ lives on the smaller σ-algebra $\mathscr{F}^-(X,t)\subsetneqq\mathscr{F}(X,t)$....annoying or interesting

μ-heat kernels III

• If p is regular, we say it satisfies *the principle of not feeling the boundary*, if for all compact subsets $K \subset X$ with $\mathring{K} \neq \emptyset$ and all $x \in \mathring{K}$, one has

$$\lim_{t\to 0+} \mathbb{P}^{x,x}_t\{\gamma : \ \gamma(s) \in K \text{ for all } s \in [0,t)\} = 1.$$

• We call a pair (ϱ_1, ϱ_2) of Borel functions $\varrho_1 : (0,1) \to (0,\infty)$, $\varrho_2 : X \to [0,\infty)$ an *asymptotic control pair for p*, if:

- the limit $\lim_{t\to 0+} p(t,x,x)\varrho_1(t)$ exists for all $x \in X$
- there exists a Borel function $\phi : (0,1) \to (0,\infty)$ such that

$$p(t,x,x) \lesssim \varrho_2(x)\phi(t), \qquad \sup_{0<t<1} \phi(t)\varrho_1(t) < \infty.$$

μ-heat kernels III

• If p is regular, we say it satisfies *the principle of not feeling the boundary*, if for all compact subsets $K\subset X$ with $\mathring{K}\neq\emptyset$ and all $x\in\mathring{K}$, one has

$$\lim_{t\to 0+}\mathbb{P}^{x,x}_t\{\gamma:\ \gamma(s)\in K \text{ for all } s\in[0,t)\}=1.$$

• We call a pair (ϱ_1,ϱ_2) of Borel functions $\varrho_1:(0,1)\to(0,\infty)$, $\varrho_2:X\to[0,\infty)$ an *asymptotic control pair for p*, if:

- the limit $\lim_{t\to 0+}p(t,x,x)\varrho_1(t)$ exists for all $x\in X$
- there exists a Borel function $\phi:(0,1)\to(0,\infty)$ such that

$$p(t,x,x)\lesssim\varrho_2(x)\phi(t),\qquad \sup_{0<t<1}\phi(t)\varrho_1(t)<\infty.$$

Example I: Complete Riemannian manifolds

• (X, g) a geodesically complete connected Riemannian m-manifold, μ_g the volume measure, $\Delta_g \geq 0$ the Laplace-Beltrami operator, $p_g(t, x, y) > 0$ the minimal nonnegative heat kernel
• the principle of not feeling the boundary is equivalent to

$$p_g^U(t, x, x)/p_g(t, x, x) \to 0 \quad \text{as } t \to 0+,$$

a classical fact (Kac, Varadhan, Hsu)!
• asymptotic expansion of heat kernel implies

$$\lim_{t \to 0+} p_g(t, x, x)\varrho^{(m)}(t) = 1,$$

where $\varrho^{(m)}(t) := (2\pi t)^{m/2}$

Example I: Complete Riemannian manifolds

• (X, g) a geodesically complete connected Riemannian m-manifold, μ_g the volume measure, $\Delta_g \geq 0$ the Laplace-Beltrami operator, $p_g(t, x, y) > 0$ the minimal nonnegative heat kernel

• the principle of not feeling the boundary is equivalent to

$$p_g^U(t, x, x)/p_g(t, x, x) \to 0 \quad \text{as } t \to 0+,$$

a classical fact (Kac, Varadhan, Hsu)!

• asymptotic expansion of heat kernel implies

$$\lim_{t \to 0+} p_g(t, x, x)\varrho^{(m)}(t) = 1,$$

where $\varrho^{(m)}(t) := (2\pi t)^{m/2}$

Example I: Complete Riemannian manifolds

• (X, g) a geodesically complete connected Riemannian m-manifold, μ_g the volume measure, $\Delta_g \geq 0$ the Laplace-Beltrami operator, $p_g(t, x, y) > 0$ the minimal nonnegative heat kernel

• the principle of not feeling the boundary is equivalent to

$$p_g^U(t, x, x)/p_g(t, x, x) \to 0 \quad \text{as } t \to 0+,$$

a classical fact (Kac, Varadhan, Hsu)!

• asymptotic expansion of heat kernel implies

$$\lim_{t \to 0+} p_g(t, x, x)\varrho^{(m)}(t) = 1,$$

where $\varrho^{(m)}(t) := (2\pi t)^{m/2}$

Example I: Complete Riemannian manifolds

- Given $x \in X$, and $b > 1$, let $r_{g,b}(x)$ be the supremum of all $r > 0$ such that $B_g(x, r)$ admits a centered chart with $(1/b)(\delta_{ij}) \leq (g_{ij}) \leq b(\delta_{ij})$; then for all $b > 1$ the function

$$\varrho_g(x) := 1/\min(r_{g,b}(x, b), 1)^m$$

turns $(\varrho^{(m)}, \varrho_g)$ into an asymptotic control function for p_g (Grigor'yan or G. in Potential Analysis 2016);

- if $\mathrm{Ric}_g \geq -C^2$, then $\varrho'_g(x) := 1/\mu_g(B_g(x, 1))$ turns $(\varrho^{(m)}, \varrho'_g)$ into an asymptotic control function for p_g
- set $H_g := H_{p_g}$ in $L^2(X, \mu_g)$, $Q_g := Q_{p_g}$ in $L^2(X, \mu_g)$.

Example I: Complete Riemannian manifolds

• Given $x \in X$, and $b > 1$, let $r_{g,b}(x)$ be the supremum of all $r > 0$ such that $B_g(x,r)$ admits a centered chart with $(1/b)(\delta_{ij}) \leq (g_{ij}) \leq b(\delta_{ij})$; then for all $b > 1$ the function

$$\varrho_g(x) := 1/\min(r_{g,b}(x,b),1)^m$$

turns $(\varrho^{(m)}, \varrho_g)$ into an asymptotic control function for p_g (Grigor'yan or G. in Potential Analysis 2016);

• if $\mathrm{Ric}_g \geq -C^2$, then $\varrho'_g(x) := 1/\mu_g(B_g(x,1))$ turns $(\varrho^{(m)}, \varrho'_g)$ into an asymptotic control function for p_g

• set $H_g := H_{p_g}$ in $L^2(X,\mu_g)$, $Q_g := Q_{p_g}$ in $L^2(X,\mu_g)$.

Example I: Complete Riemannian manifolds

- Given $x \in X$, and $b > 1$, let $r_{g,b}(x)$ be the supremum of all $r > 0$ such that $B_g(x,r)$ admits a centered chart with $(1/b)(\delta_{ij}) \leq (g_{ij}) \leq b(\delta_{ij})$; then for all $b > 1$ the function

$$\varrho_g(x) := 1/\min(r_{g,b}(x,b),1)^m$$

turns $(\varrho^{(m)}, \varrho_g)$ into an asymptotic control function for p_g (Grigor'yan or G. in Potential Analysis 2016);

- if $\mathrm{Ric}_g \geq -C^2$, then $\varrho'_g(x) := 1/\mu_g(B_g(x,1))$ turns $(\varrho^{(m)}, \varrho'_g)$ into an asymptotic control function for p_g
- set $H_g := H_{p_g}$ in $L^2(X,\mu_g)$, $Q_g := Q_{p_g}$ in $L^2(X,\mu_g)$.

Example II: Infinite weighted graphs

• (X, b, μ): a weighted connected graph, that is $b(x, y) \geq 0$ is edge weight with $\sum_y b(x, y) < \infty$ and $\mu(x) > 0$ is vertex weight; X carries discrete topology; for $\psi : X \to \mathbb{C}$ (say bounded) set

$$\Delta_{b,\mu}\psi(x) = -\frac{1}{\mu(x)} \sum_{\{y:y\sim_b x\}} b(x, y)\big(\psi(x) - \psi(y)\big).$$

$\rightsquigarrow$ minimal heat kernel $(t, x, y) \mapsto p_{b,\mu}(t, x, y) > 0$ exists and is a regular sppc μ-heat kernel (Keller/Lenz);

• from discreteness we immediately get

$$p_{b,\mu}(t, x, y) \leq 1/\mu(x) \quad \text{for all } (t, x, y) \in (0, \infty) \times X \times X,$$

and

$$\lim_{t\to 0+} p_{b,\mu}(t, x, x) \cdot 1 = 1/\mu(x) \quad \text{for all } x \in X.$$

Example II: Infinite weighted graphs

• (X, b, μ): a weighted connected graph, that is $b(x, y) \geq 0$ is edge weight with $\sum_y b(x, y) < \infty$ and $\mu(x) > 0$ is vertex weight; X carries discrete topology; for $\psi : X \to \mathbb{C}$ (say bounded) set

$$\Delta_{b,\mu}\psi(x) = -\frac{1}{\mu(x)} \sum_{\{y : y \sim_b x\}} b(x, y)\big(\psi(x) - \psi(y)\big).$$

$\rightsquigarrow$ minimal heat kernel $(t, x, y) \mapsto p_{b,\mu}(t, x, y) > 0$ exists and is a regular sppc μ-heat kernel (Keller/Lenz);

• from discreteness we immediately get

$$p_{b,\mu}(t, x, y) \leq 1/\mu(x) \quad \text{for all } (t, x, y) \in (0, \infty) \times X \times X,$$

and

$$\lim_{t \to 0+} p_{b,\mu}(t, x, x) \cdot 1 = 1/\mu(x) \quad \text{for all } x \in X.$$

Example II: Finite weighted graphs

• $p_{b,\mu}$ satisfies principle of not feeling the boundary: $\mathbb{P}_t^{x,x}$ is concentrated on pure jump paths, and

$$\begin{aligned}&\mathbb{P}_t^{x,x}\{\gamma : \gamma(s) \in \{x, x_1, \dots, x_l\} \text{ for all } s \in [0,t)\}\\ &\geq \mathbb{P}_t^{x,x}\{\gamma : \gamma \text{ has not jumped before } t\}\\ &\geq \exp\Big(- \frac{t}{\mu(x)} \sum_y b(x,y)\Big) \big(p_{b,\mu}(t,x,x)\mu(x)\big)^{-1}.\end{aligned}$$

Cf. Norris' book or G./Keller/Schmidt in Probability Theory and Related Fields 2016.

• the operator $H_{b,\mu} := H_{p_{b,\mu}}$ in $L^2(X,b)$ is a restriction of $\Delta_{b,\mu}$; set $Q_{b,\mu} := Q_{p_{b,\mu}}$

Example II: Finite weighted graphs

• $p_{b,\mu}$ satisfies principle of not feeling the boundary: $\mathbb{P}_t^{x,x}$ is concentrated on pure jump paths, and

$$\begin{aligned}
&\mathbb{P}_t^{x,x}\{\gamma : \gamma(s) \in \{x, x_1, \dots, x_l\} \text{ for all } s \in [0,t)\} \\
&\geq \mathbb{P}_t^{x,x}\{\gamma : \gamma \text{ has not jumped before } t\} \\
&\geq \exp\Big(-\frac{t}{\mu(x)}\sum_y b(x,y)\Big)\big(p_{b,\mu}(t,x,x)\mu(x)\big)^{-1}.
\end{aligned}$$

Cf. Norris' book or G./Keller/Schmidt in Probability Theory and Related Fields 2016.

• the operator $H_{b,\mu} := H_{p_{b,\mu}}$ in $L^2(X,b)$ is a restriction of $\Delta_{b,\mu}$; set $Q_{b,\mu} := Q_{p_{b,\mu}}$

The main result

Theorem (B.G.)

Assume that X admits a metric which induces the original topology, such that for every $x \in X$ there is an $r > 0$ with $B(x,r)$ relatively compact. Let p satisfy the principle of not feeling the boundary. Then for every asymptotic control pair (ϱ_1, ϱ_2) for p, and every continuous potential $w : X \to \mathbb{R}$ with w^- being infinitesimally Q_p-bounded and $\int \mathrm{e}^{-w} \varrho_2 \mathrm{d}\mu < \infty$, one has

$$\lim_{t \to 0+} \varrho_1(t) \mathrm{tr}\left(\mathrm{e}^{-t(H_p \dot{+} w/t)}\right) = \int \mathrm{e}^{-w(x)} \lim_{t \to 0+} p(t,x,x) \varrho_1(t) \mathrm{d}\mu(x).$$

For example w^- could be bounded or more generally Kato (in practice an $L^q + L^\infty$ condition).

Complete Riemannian manifolds I

Recall that $\varrho^{(m)}(t) := (2\pi t)^{m/2}$.

Corollary

Assume that (X, g) *is a smooth geodesically complete connected Riemannian m-manifold. Then for every Borel function* $\varrho : X \to [0, \infty)$ *which makes* $(\varrho^{(m)}, \varrho)$ *an asymptotic control pair for p, and for every continuous potential* $w : X \to \mathbb{R}$ *with* w^- *infinitesimally* Q_g*-bounded and* $\int \mathrm{e}^{-w(x)} \varrho(x) \mathrm{d}\mu_g(x) < \infty$*, one has*

$$\lim_{t \to 0+} (2\pi t)^{m/2} \mathrm{tr}\left(\mathrm{e}^{-t(H_g \dot{+} w/t)}\right) = \int \mathrm{e}^{-w} \mathrm{d}\mu_g .$$

Complete Riemannian manifolds II

Corollary

Let (X, g) *be a smooth geodesically complete connected Riemannian m-manifold with* $\mathrm{Ric}_g \geq -A$ *for some constant* $A \geq 0$*, and let* $w : X \to \mathbb{R}$ *be a continuous with* $\inf_X w > -\infty$ *and*

$$\sum_{k=2}^{\infty} \exp\Big(-\inf_{x\in X, k-1<d_g(x,x_0)<k} w(x)\Big) k^m \mathrm{e}^{2k\sqrt{(m-1)A}} < \infty$$

for some $x_0 \in X$*. Then one has*

$$\lim_{t\to 0+} (2\pi t)^{m/2} \mathrm{tr}\big(\mathrm{e}^{-t(H_g \dotplus w/t)}\big) = \int \mathrm{e}^{-w} \mathrm{d}\mu_g.$$

Proof: Use volume doubling machinery to prove $\int \mathrm{e}^{-w} \varrho \mathrm{d}\mu_g < \infty$.

Infinite weighted graphs

We recover the following result (G., Journal of Mathematical Physics 2014 or so)

Corollary

Let (X, b, μ) be a weighted graph which is connected in the graph theoretic sense. Then for every potential $w : X \to \mathbb{R}$ with w^- infinitesimally $Q_{b,\mu}$-bounded and $\sum_{x \in X} \mathrm{e}^{-w(x)} < \infty$, one has

$$\lim_{t \to 0+} \mathrm{tr}\left(\mathrm{e}^{-t(H_{b,\mu} \dot{+} w/t)}\right) = \sum_{x \in X} \mathrm{e}^{-w(x)}.$$

Integration on RHS is not w.r.t. to underlying Hilbert space measure!

Upper bound

- From functional analysis and Chapman-Kolmogorov (and some approximation arguments...) we get the Golden-Thompson inequality:

$$\mathrm{tr}\left(\mathrm{e}^{-t(H_p\dot{+}w)}\right) \leq \int p(t,x,x)\mathrm{e}^{-w(x)}\mathrm{d}\mu(x) \quad \text{for all } t>0, \quad (1)$$

so that

$$\limsup_{t\to 0+} \varrho_1(t)\mathrm{tr}\left(\mathrm{e}^{-tH_p(w/t)}\right) \leq \limsup_{t\to 0+} \int \varrho_1(t)p(t,x,x)\mathrm{e}^{-w(x)}\mathrm{d}\mu(x).$$

- assumptions on the heat kernel control pair (ϱ_1, ϱ_2) and w are precisely such that we can interchange limit with integral using *dominated convergence*

Upper bound

• From functional analysis and Chapman-Kolmogorov (and some approximation arguments...) we get the Golden-Thompson inequality:

$$\mathrm{tr}\left(\mathrm{e}^{-t(H_p\dot{+}w)}\right) \leq \int p(t,x,x)\mathrm{e}^{-w(x)}\mathrm{d}\mu(x) \quad \text{for all } t>0, \quad (1)$$

so that

$$\limsup_{t\to 0+} \varrho_1(t)\mathrm{tr}\left(\mathrm{e}^{-tH_p(w/t)}\right) \leq \limsup_{t\to 0+} \int \varrho_1(t)p(t,x,x)\mathrm{e}^{-w(x)}\mathrm{d}\mu(x).$$

• assumptions on the heat kernel control pair (ϱ_1, ϱ_2) and w are precisely such that we can interchange limit with integral using *dominated convergence*

Lower bound

Let K_n be a rel. cpt. exhaustion of X. For each n pick $\delta_n \in (0, \infty]$ such that for all $0 < \delta < \delta_n$ and all $x \in K_n$ the ball $B(x, \delta)$ is rel. cpt. (always possible)

$$\varrho_1(t)\mathrm{tr}\left(\mathrm{e}^{-tH_p(w/t)}\right) = \int \varrho_1(t)p(t,x,x)\int \mathrm{e}^{-\frac{1}{t}\int_0^t w(\gamma(s))\mathrm{d}s}\mathrm{d}\mathbb{P}_t^{x,x}(\gamma)\,\mathrm{d}\mu(x)$$

$$\geq \int_{K_n}\varrho_1(t)p(t,x,x)\int_{\{\gamma:\ \gamma(s)\in\overline{B(x,\delta)}\ \forall\ s\in[0,t)\}} \mathrm{e}^{-\frac{1}{t}\int_0^t w(\gamma(s))\mathrm{d}s}\mathrm{d}\mathbb{P}_t^{x,x}(\gamma)\,\mathrm{d}\mu$$

$$\geq \int_{K_n}\varrho_1(t)p(t,x,x)\mathbb{P}_t^{x,x}\{\gamma:\ \gamma(s)\in\overline{B(x,\delta)}\ \forall\ s\in[0,t)\}\mathrm{e}^{-w_\delta(x)}\mathrm{d}\mu(x),$$

where $w_\delta(x) := \max_{\overline{B(x,\delta)}} w$; by principle of not feeling the boundary and Fatou's Lemma:

$$\liminf_{t\to 0+}\varrho_1(t)\mathrm{tr}\left(\mathrm{e}^{-tH_p(w/t)}\right) \geq \int_{K_n}\mathrm{e}^{-w_\delta(x)}\liminf_{t\to 0+} p(t,x,x)\varrho_1(t)\mathrm{d}\mu(x).$$

Lower bound

Let K_n be a rel. cpt. exhaustion of X. For each n pick $\delta_n \in (0, \infty]$ such that for all $0 < \delta < \delta_n$ and all $x \in K_n$ the ball $B(x, \delta)$ is rel. cpt. (always possible)

$$\varrho_1(t)\mathrm{tr}\left(\mathrm{e}^{-tH_p(w/t)}\right) = \int \varrho_1(t)p(t,x,x) \int \mathrm{e}^{-\frac{1}{t}\int_0^t w(\gamma(s))\mathrm{d}s}\mathrm{d}\mathbb{P}_t^{x,x}(\gamma)\ \mathrm{d}\mu(x)$$

$$\geq \int_{K_n} \varrho_1(t)p(t,x,x) \int_{\{\gamma:\ \gamma(s)\in\overline{B(x,\delta)}\ \forall\ s\in[0,t)\}} \mathrm{e}^{-\frac{1}{t}\int_0^t w(\gamma(s))\mathrm{d}s}\mathrm{d}\mathbb{P}_t^{x,x}(\gamma)\ \mathrm{d}\mu$$

$$\geq \int_{K_n} \varrho_1(t)p(t,x,x)\mathbb{P}_t^{x,x}\{\gamma:\ \gamma(s)\in\overline{B(x,\delta)}\ \forall\ s\in[0,t)\}\mathrm{e}^{-w_\delta(x)}\mathrm{d}\mu(x),$$

where $w_\delta(x) := \max_{\overline{B(x,\delta)}} w$; by principle of not feeling the boundary and Fatou's Lemma:

$$\liminf_{t\to 0+} \varrho_1(t)\mathrm{tr}\left(\mathrm{e}^{-tH_p(w/t)}\right) \geq \int_{K_n} \mathrm{e}^{-w_\delta(x)} \liminf_{t\to 0+} p(t,x,x)\varrho_1(t)\mathrm{d}\mu(x).$$

Lower bound

Let K_n be a rel. cpt. exhaustion of X. For each n pick $\delta_n \in (0,\infty]$ such that for all $0<\delta<\delta_n$ and all $x \in K_n$ the ball $B(x,\delta)$ is rel. cpt. (always possible)

$$\varrho_1(t)\mathrm{tr}\left(\mathrm{e}^{-tH_p(w/t)}\right) = \int \varrho_1(t)p(t,x,x)\int \mathrm{e}^{-\frac{1}{t}\int_0^t w(\gamma(s))\mathrm{d}s}\mathrm{d}\mathbb{P}_t^{x,x}(\gamma)\ \mathrm{d}\mu(x)$$

$$\geq \int_{K_n} \varrho_1(t)p(t,x,x)\int_{\{\gamma:\ \gamma(s)\in\overline{B(x,\delta)}\ \forall\ s\in[0,t)\}} \mathrm{e}^{-\frac{1}{t}\int_0^t w(\gamma(s))\mathrm{d}s}\mathrm{d}\mathbb{P}_t^{x,x}(\gamma)\ \mathrm{d}\mu$$

$$\geq \int_{K_n} \varrho_1(t)p(t,x,x)\mathbb{P}_t^{x,x}\{\gamma:\ \gamma(s)\in\overline{B(x,\delta)}\ \forall\ s\in[0,t)\}\mathrm{e}^{-w_\delta(x)}\mathrm{d}\mu(x),$$

where $w_\delta(x) := \max_{\overline{B(x,\delta)}} w$; by principle of not feeling the boundary and Fatou's Lemma:

$$\liminf_{t\to 0+} \varrho_1(t)\mathrm{tr}\left(\mathrm{e}^{-tH_p(w/t)}\right) \geq \int_{K_n} \mathrm{e}^{-w_\delta(x)} \liminf_{t\to 0+} p(t,x,x)\varrho_1(t)\mathrm{d}\mu(x).$$

A conjecture

Conjecture: The principle of not feeling the boundary holds automatically (at least if Q_p is strongly local).

Thank you for listening!