The Calderón-Zygmund inequality on noncompact Riemannian manifolds

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Geometric Structures and Spectral Invariants

Berlin, May 16, 2014

This talk is about the paper:

Batu Güneysu & Stefano Pigola: *The Calderón-Zygmund inequality on noncompact Riemannian manifolds*. Preprint (2014).

Partially also:

Batu Güneysu: *Sequences of Riemannian second order cut-off functions*. Preprint (2014).

For a possibly *noncompact* smooth Riemannian m-manifold $M \equiv (M, g)$, consider the following global problems for second order Sobolev spaces on M, on the L^p-scale, $1 < p < \infty$:

- **Problem 1 (denseness):** Under which assumptions on M does one have $\mathsf{H}_0^{2,p}(M) = \mathsf{H}^{2,p}(M)$ (without $r_{\mathrm{inj}}(M) > 0$)?
- **Problem 2 (Poisson's equation):** Under which assumptions on M does one have the implication

$$f \in \mathsf{L}^p(M) \cap \mathsf{C}^2(M), \Delta f \in \mathsf{L}^p(M) \Rightarrow f \in \mathsf{H}^{2,p}(M)$$

  (that is, $|\mathrm{Hess}(f)| \in \mathsf{L}^p(M)$)?
- **Problem 3 (gradient estimate):** Under which assumptions on M does one have an inequality of the form

$$\|\mathrm{grad}(f)\|_p \leq C(\|\Delta f\|_p + \|f\|_p) \quad \text{for all } f \in \mathsf{C}_{\mathrm{c}}^\infty(M)?$$

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- **Problem 1 (denseness):** Under which assumptions on M does one have $\mathsf{H}^{2,p}_0(M) = \mathsf{H}^{2,p}(M)$ (without $r_{\mathrm{inj}}(M) > 0$)?
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As we will see in a moment, there is a *common inequality* behind these types of problems:

Definition

Let $1 < p < \infty$. We say that M *satisfies the* L^p*-Calderón-Zygmund inequality (or in short* $\mathrm{CZ}(p)$*)*, if there are $C_1 \geq 0$, $C_2 > 0$, such that for all $u \in \mathrm{C}_\mathrm{c}^\infty(M)$ one has

$$\left\|\mathrm{Hess}\,(u)\right\|_p \leq C_1 \left\|u\right\|_p + C_2 \left\|\Delta u\right\|_p . \tag{1}$$

- In $\mathbb{R}^m$, $\mathrm{CZ}(p)$ with $C_1 = 0$ follows e.g. from estimates on singular integral operators that go back to Calderón and Zygmund (1950's). Note that in $\mathbb{R}^m$: $\left\|\mathrm{Hess}\,(u)\right\|_2 = \left\|\Delta u\right\|_2$ ("Bochner's formula")
- In general, $\mathrm{CZ}(p)$ depends very sensitively on the curvature and there has been no systematic treatement so far
- $\mathrm{CZ}(p)$ always extends automatically from $\mathrm{C}_\mathrm{c}^\infty(M)$ to $\mathrm{H}_0^{2,p}(M)$

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What is the precise connection between $\mathrm{CZ}(p)$ and the problems 1, 2, 3?

Definition

a) M is said to admit a sequence $(\chi_n) \subset \mathrm{C}_{\mathrm{c}}^{\infty}(M)$ of Laplacian cut-off functions, if (χ_n) has the following properties:

(C1) $0 \leq \chi_n(x) \leq 1$ for all $n \in \mathbb{N}$, $x \in M$,

(C2) for all compact $K \subset M$, there is an $n_0(K) \in \mathbb{N}$ such that for all $n \geq n_0(K)$ one has $\chi_n \mid_K = 1$,

(C3) $\sup_{x \in M} |\mathrm{d}\chi_n(x)|_x \to 0$ as $n \to \infty$,

(C4) $\sup_{x \in M} |\Delta\chi_n(x)| \to 0$ as $n \to \infty$.

b) M is said to admit a sequence $(\chi_n) \subset \mathrm{C}_{\mathrm{c}}^{\infty}(M)$ of Hessian cut-off functions, if (χ_n) has the above properties (C1), (C2), (C3), and in addition

(C4') $\sup_{x \in M} |\mathrm{Hess}(\chi_n)(x)|_x \to 0$ as $n \to \infty$.

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Theorem (G.; G. & P.)

a) If M is complete with $\mathrm{Ric} \geq 0$, then M admits L-C.O.F.'s
b) If M is complete with $\|\mathrm{R}\|_{\infty} < \infty$ (with R the curvature tensor), then M admits H-C.O.F.'s.

No $r_{\mathrm{inj}}(M) > 0$ required! Proofs are subtle and rely on a highly nontrivial rigidity result by Cheeger-Colding (1996) for a), and a smoothing result by B. Chow et. al. (Ricci flow II, 2008) for b). The connection between $\#$-C.O.F.'s, $\mathrm{CZ}(p)$ and **problems 1, 2** is the following elementary result:

Proposition

Assume that M satisfies $\mathrm{CZ}(p)$.
a) If M admits L-C.O.F.'s, then $\mathsf{H}_0^{2,p}(M) = \mathsf{H}^{2,p}(M)$.
b) Assume that M admits H-C.O.F's. Then for any $u \in \mathsf{C}^2(M)$ with $u, |\mathrm{grad}(u)|, \Delta u \in \mathsf{L}^p(M)$, one has $|\mathrm{Hess}(u)| \in \mathsf{L}^p(M)$.

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The key to **problem 3** (and many other results in the sequel!) is the following interpolation result:

Theorem (G. & P.; Coulhon & Duong 2003 for $1 < p \leq 2$)

Assume that either $2 \leq p < \infty$, or in case $1 < p < 2$ that either M is complete or a relatively compact open subset of an arbitrary smooth Riemannian manifold. Then there is a $C = C(m,p) > 0$ s.t. for all $\varepsilon > 0$, $u \in \mathrm{C}^{\infty}_{\mathrm{c}}(M)$ one has the ***interpolation inequality***

$$\left\|\mathrm{grad}(u)\right\|_p \leq C\varepsilon^{-1}\left\|u\right\|_p + C\varepsilon\left\|\mathrm{Hess}\,(u)\right\|_p,$$

in particular, under $\mathrm{CZ}(p)$ one has the ***gradient estimate from problem 3****: $\left\|\mathrm{grad}(u)\right\|_p \leq C(\left\|\Delta u\right\|_p + \left\|u\right\|_p)$.*

Proof: The $2 \leq p < \infty$ case: Apply the divergence theorem to $X := u \cdot \left(|\mathrm{grad}(u)|^2 + \alpha\right)^{\frac{p-2}{2}} \mathrm{grad}(u)$, $\alpha > 0$, and take $\alpha \to 0+$.

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In what sense do we have $\mathrm{CZ}(p)$ locally?

A Hessian (key-)estimate on balls: Using harmonic coordinates and an appropriate local elliptic estimate, we get (with $r_{Q,k,\alpha}(x)$ the $\mathrm{C}^{k,\alpha}$-harmonic radius at x with $Q^{-1}(\delta_{ij}) \leq (g_{ij}) \leq Q(\delta_{ij})$):

Theorem (G. & P.)

Fix an arbitrary $x \in M$. Then for all $1 < p < \infty$, all $0 < r < r_{2,1,1/2}(x)/2$, and all real numbers D with $\inf_{\mathrm{B}_{r_{2,1,1/2}(x)}(x)} r_{2,1,1/2}(\bullet) \geq D > 0$, there is a $C = C(r,p,m,D) > 0$, such that for all $u \in \mathrm{C}^{\infty}_{\mathrm{c}}(M)$ one has

$$\left\| 1_{\mathrm{B}_{r/2}(x)} \mathrm{Hess}\,(u) \right\|_p \leq C \left(\left\| 1_{\mathrm{B}_{2r}(x)} u \right\|_p + \left\| 1_{\mathrm{B}_{2r}(x)} \Delta u \right\|_p + \left\| 1_{\mathrm{B}_{2r}(x)} \mathrm{grad}(u) \right\|_p \right).$$

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Theorem (G. & P.)

a) $\mathrm{CZ}(p)$ *holds on any relatively compact open subset* $\Omega \subset M$. *Moreover, if* $\Omega \subset M$ *is a relatively compact domain with smooth boundary* $\partial\Omega$, *then* $\mathrm{CZ}(p)$ *holds in the stronger form* $\|\mathrm{Hess}\,(u)\|_p \leq C\,\|\Delta u\|_p$.
b) Assume that either $p \geq 2$ *or that* $1 < p < 2$ *and that* M *is complete. If there is a relatively compact domain* $\Omega \subset M$ *such that* $\mathrm{CZ}(p)$ *holds on* $M \setminus \overline{\Omega}$, *then* $\mathrm{CZ}(p)$ *also holds on* M.

The weak $\mathrm{CZ}(p)$ of part a) follows e.g. from the previous Hessian estimate on balls and interpolation. The strong form follows from the weak form and the (resulting!) gradient estimate, and elliptic regularity.
The *topological stability* from part b) follows from part a) and the interpolation result: we pick up gradient terms from gluing!

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Which noncompact M's admit $\mathrm{CZ}(p)$?

One can give an essentially complete answer for the Hilbert space case $p = 2$:

Theorem (G. & P.)

a) Assume that $\mathrm{Ric} \geq -C^2$. Then $\mathrm{CZ}(2)$ holds in the following "infinitesimal" way: For every $\varepsilon > 0$ and every $u \in \mathrm{C}_{\mathrm{c}}^{\infty}(M)$ one has

$$\|\mathrm{Hess}\,(u)\|_2^2 \leq \frac{C\varepsilon^2}{2}\,\|u\|_2^2 + \left(1 + \frac{C^2}{2\varepsilon^2}\right)\|\Delta u\|_2^2\,.$$

b) There exists a smooth 2-dimensional, complete Riemannian manifold N with unbounded curvature, such that $\mathrm{CZ}(2)$ fails on N.

Part a) is really just Bochner's formula (remember the $\mathbb{R}^m$ case!):

$$\int |\mathrm{Hess}(u)|^2 = \int (\mathrm{d}u, \mathrm{d}\Delta u) - \int \mathrm{Ric}(\mathrm{grad}(u), \mathrm{grad}(u)).$$

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A result for arbitrary p, but positive injectivity radius:

Theorem (G. & P.)

Let $1 < p < \infty$ and assume $\|\mathrm{Ric}\|_\infty < \infty$, $r_{\mathrm{inj}}(M) > 0$. Then there is a $C = C(m, p, \|\mathrm{Ric}\|_\infty, r_{\mathrm{inj}}(M)) > 0$ such that for all $u \in \mathsf{C}_\mathrm{c}^\infty(M)$ one has

$$\|\mathrm{Hess}\,(u)\|_p \leq C(\|u\|_p + \|\Delta u\|_p).$$

Idea of proof: By harmonic radius estimates there is a $D = D(m, r_{\mathrm{inj}}(M), \|\mathrm{Ric}\|_\infty) > 0$ such that $r_{2,1,1/2}(M) \geq D$. Let $r := D/2$. By the Hessian estimate on balls we have a $c = c(r, p, m, D) > 0$ such that, for all (x_i), all $u \in \mathsf{C}_\mathrm{c}^\infty(M)$,

$$\int_{\mathrm{B}_{r/2}(x_i)} |\mathrm{Hess}\,(u)|^p \leq c \int_{\mathrm{B}_{2r}(x_i)} (|\Delta u|^p + |\mathrm{grad}\,(u)|^p + |u|^p).$$

Take appropriate sequence of points (x_i), sum over i and use interpolation.

A result for arbitrary p, but positive injectivity radius:

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Take appropriate sequence of points (x_i), sum over i and use interpolation.

The latter theorem applies to give:

Theorem

Let $f : M \to N$ be an isometric immersion of M into a smooth complete, simply connected Riemannian manifold N with $-\tilde{C}^2 \leq \mathrm{Sec}_N \leq 0$. If $\|\mathbf{II}_f\|_\infty < \infty$, then $\mathrm{CZ}(p)$ holds on M for every $1 < p < \infty$.

A result for small p, but without positive injectivity radius assumption

Theorem (G. & P.)

Let $1 < p \leq 2$, *and assume that* M *is complete, that* $\max(\|\mathrm{R}\|_\infty, \|\nabla \mathrm{R}\|_\infty) < \infty$, *and that there are* $D \geq 1$, $0 \leq \delta < 2$ *with the volume growth*

$$\mathrm{vol}(\mathrm{B}_{tr}(x)) \leq D t^D \mathrm{e}^{t^\delta + r^\delta} \mathrm{vol}(\mathrm{B}_r(x)) \ \textit{ for all } x \in M,\ r > 0,\ t \geq 1.$$

Then there is a

$$C = C(m, p, \|\mathrm{R}\|_\infty, \|\nabla \mathrm{R}\|_\infty, D, \delta) > 0,$$

such that for all $u \in \mathrm{C}_\mathrm{c}^\infty(M)$ *one has*

$$\|\mathrm{Hess}\,(u)\|_p \leq C(\|u\|_p + \|\Delta u\|_p).$$

- The proof of the latter result shows a deep connection between *boundedness of covariant* L^p*-Riesz transforms* and $\mathrm{CZ}(p)$: The asserted inequality follows immediately, once we have
$$\left\| \nabla(\Delta_1 + a_1)^{-1/2} \mathrm{d}(\Delta_0 + a_1)^{-1/2} u \right\|_p \leq a_2 \left\| u \right\|_p .$$
But $\left\| \mathrm{d}(\Delta_0 + a_1)^{-1/2} \right\|_{p,p} \leq C(p)$ for all p is a classical result by Bakry (1987), and one can use probabilistic heat equation derivative formula by Thalmaier/F. Y. Wang (2003) for $\nabla \mathrm{e}^{-t\Delta_1}$ to estimate $\left\| \nabla(\Delta_1 + a_1)^{-1/2} \right\|_{p,p}$ for $1 < p \leq 2$
- The volume growth assumption is subtle: It is satisfied under completeness and $\mathrm{Ric} \geq 0$, but a negative lower bound is not enough. Indeed, if M is complete with $\mathrm{Ric} \geq (-C)(m-1)$ for some $C \geq 0$, then one has Gromov's estimate
$$\mathrm{vol}(\mathrm{B}_{tr}(x)) \leq \mathrm{vol}(\mathrm{B}_r(x)) t^m \mathrm{e}^{(m-1)\sqrt{C}(t-1)r} \text{ for all } t > 1,\ r > 0.$$
This inequality is sufficient, only if we can pick $C = 0$!

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Thank you!