Functions with bounded variation on Riemannian manifolds with Ricci curvature unbounded from below

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This talk is about the paper

Batu Güneysu & Diego Pallara: *Functions with bounded variation on a class of Riemannian manifolds with Ricci curvature unbounded from below*. Preprint (2013).

For a detailed treatment of the *local* theory:

L. Ambrosio & N. Fusco & D. Pallara: *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

Review of Euclidean $\mathbb{R}^1$

- Recall that the variation of some $f : \mathbb{R}^1 \to \mathbb{C}$ is defined by

$$\tilde{\mathrm{Var}}(f) = \sup \left\{ \sum_{j=1}^{n-1} |f(x_{j+1}) - f(x_j)| \;\middle|\; n \geq 2,\, x_1 < x_2 \cdots < x_n \right\}.$$

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- Fortunately, de Giorgi realized (1954): For *equivalence class* $f \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}^1)$ set $\mathrm{Var}(f) := \inf_{f(\bullet)\in f} \tilde{\mathrm{Var}}(f)$. Then

$$\mathrm{Var}(f) = \sup\left\{ \Big| \int_{\mathbb{R}^1} f(x)\alpha'(x)\mathrm{d}x \Big| \ \Bigg| \ \alpha \in \mathsf{C}_0^\infty(\mathbb{R}^1), \|\alpha\|_\infty \leq 1 \right\}$$

- $\mathrm{Var}(\bullet)$ is a Riemannian object: De Giorgi also showed that for $f \in \mathsf{L}^1(\mathbb{R}^m)$ one has

$$\lim_{t\to 0+} \int_{\mathbb{R}^m} |\mathrm{grad}(\mathrm{e}^{t\Delta} f)|(x)\mathrm{d}x \tag{1}$$

$$= \sup\left\{ \Big| \int_{\mathbb{R}^m} f(x)\mathrm{div}\alpha(x)\mathrm{d}x \Big| \ \Bigg| \ \alpha \in [\mathsf{C}_0^\infty(\mathbb{R}^m)]^m, \|\alpha\|_\infty \leq 1 \right\}$$

- and, defining $\mathrm{Var}(f)$ for $f \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}^m)$ by rhs of (1), one has $\mathrm{Var}(f) < \infty$ if and only $\mathrm{grad}(f)$ defines a finite $\mathbb{C}^m$-valued Borel measure

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Setting

- $M \equiv (M, g)$: connected possibly noncompact Riemannian m-manifold
- $\mathrm{B}_r(x)$: open geodesic balls
- $\mathbb{P}^x$ resp. ζ: Brownian motion resp. Alexandroff explosion time
- $p(t, x, y)$: minimal positive heat kernel
- Δ resp. Δ_1: (negative) Laplace-Beltrami operator acting on functions resp. 1-forms
- $H \geq 0$ resp. $H_1 \geq 0$: Friedrichs realization of $-\Delta/2$ resp. $-\Delta_1/2$ in $\mathsf{L}^2(M)$ resp. $\Omega^1_{\mathsf{L}^2}(X)$
- ∇: Levi-Civita connection

Everything will be complexified

Definition of $\mathrm{Var}(f)$ on Riemannian manifolds

Being motivated by de Giorgi's observations we define:

Definition

Let $f \in \mathsf{L}^1_{\mathrm{loc}}(M)$. Then the quantity

$$\mathrm{Var}(f)$$
$$:= \sup\left\{ \left| \int_M \overline{f(x)} \mathrm{d}^\dagger \alpha(x) \mathrm{vol}(\mathrm{d}x) \right| \,\middle|\, \alpha \in \Omega^1_{\mathsf{C}^\infty_0}(M), \|\alpha\|_\infty \leq 1 \right\}$$
$$\in [0, \infty]$$

is called the *variation* of f, and f is said to have *bounded variation* if $\mathrm{Var}(f) < \infty$.

Simple generally valid facts

- If $f \in \mathsf{C}^1(M)$, then $\mathrm{Var}(f) = \|\mathrm{d}f\|_1$.
- For any $q \in [1, \infty)$ the maps

$$\mathsf{L}^q_{\mathrm{loc}}(M) \longrightarrow [0, \infty], \;\; f \longmapsto \mathrm{Var}(f)$$
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  are lower semicontinuous
- The space

$$\mathsf{BV}(M) := \left\{ f \,\middle|\, f \in \mathsf{L}^1(M), \mathrm{Var}(f) < \infty \right\}$$

  is a complex Banach space with respect to the norm $\|f\|_{\mathsf{BV}} := \|f\|_1 + \mathrm{Var}(f)$.
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A 'global' characterization of $\mathrm{Var}(\bullet)$

Let $\mathscr{M}(M)$ be the space of equivalence classes $[(\mu,\sigma)]$ of pairs (μ,σ) with μ a finite positive Borel measure on M and σ a Borel section in T^*M with $|\sigma|=1$ μ-a.e. in M, where $(\mu,\sigma)\sim(\mu',\sigma') :\Leftrightarrow \mu=\mu'$ as Borel measures and $\sigma(x)=\sigma'(x)$ for μ/μ' a.e. $x\in M$.

Theorem (B.G. & D. Pallara: $(\Omega^1_{\mathsf{C}_\infty}(M))^*$ *is the actual space of vector measures*)

a) The map

$$\Psi : \mathscr{M}(M) \longrightarrow (\Omega^1_{\mathsf{C}_\infty}(M))^*, \quad \Psi[(\mu,\sigma)](\alpha) := \int_M (\sigma,\alpha)\mathrm{d}\mu$$

is a well-defined bijection with $\|\Psi[(\mu,\sigma)]\|_{\infty,*} = \mu(M)$.

b) For any $f\in \mathsf{L}^1_{\mathrm{loc}}(M)$ *one has* $\mathrm{Var}(f) = \|\mathrm{d}f\|_{\infty,*}\in[0,\infty]$.

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Application: Characterization of Sobolev space $\mathsf{W}^{1,1}(M)$

If $\mathrm{Var}(f) < \infty$, we write $|\mathrm{D}f|$ for the finite positive Borel measure, and σ_f for the $|\mathrm{D}f|$-equivalence class of 1-forms given by $\Psi^{-1}(\mathrm{d}f)$, so that we have

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Corollary

a) One has $\|f\|_{\mathsf{BV}} = \|f\|_{1,1}$ for all $f \in \mathsf{W}^{1,1}(M)$. In particular, $\mathsf{W}^{1,1}(M)$ is a closed subspace of $\mathsf{BV}(M)$.

b) Some $f \in \mathsf{BV}(M)$ is in $\mathsf{W}^{1,1}(M)$, if and only if one has $|\mathrm{D}f| \ll \mathrm{vol}$ as Borel measures.

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Kato class

Recall that a Borel function $w : M \to \mathbb{C}$ is said to be in the *Kato class* $\mathcal{K}(M)$ of M, if

$$\lim_{t \to 0+} \sup_{x \in M} \int_0^t \mathbb{E}^x \left[1_{\{s<\zeta\}} |w(X_s)| \right] \mathrm{d}s = 0. \tag{2}$$

- For any g: $\mathsf{L}^\infty(M) \subset \mathcal{K}(M) \subset \mathsf{L}^1_{\mathrm{loc}}(M)$, and any $w \in \mathcal{K}(M)$ is infinitesimally H-form bounded (and there is even a rich theory of Kato type *measure perturbations* of H: Stollman/Voigt; Sturm; Kuwae;...)
- Many Theorems (mainly Kuwae/Takahashi; B.G.) of the form: *Some mild control on* $g \Rightarrow \mathsf{L}^q_{\mathrm{u,loc}}(M) \subset \mathcal{K}(M)$ *or at least* $\mathsf{L}^q(M) \subset \mathcal{K}(M)$ *for* $q = q(m)$

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Main result

- Kato class historically comes from dealing with Coulomb type *local singularites* in $\mathbb{R}^3$ (techniques can be extended to nonparabolic M's; e.g. my paper in AHP 13)
- But here we use $\mathcal{K}(M)$ to control smooth geometric objects *globally*:

Theorem (B.G. & D. Pallara)

Let M be geodesically complete and assume that Ric admits a decomposition $\mathrm{Ric} = R_1 - R_2$ into self-adjoint Borel sections $R_1, R_2 \geq 0$ in $\mathrm{End}(\mathrm{T}^* M)$ such that $|R_2| \in \mathcal{K}(M)$. Then for any $f \in \mathsf{L}^1(M)$ one has

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Earlier results on heat kernel characterization of $\mathrm{Var}(\bullet)$

- Miranda/Pallara/Paronetto/Preunkert needed $\mathrm{Ric} > -\infty$ and volume nontrapping (Crelle 2006)
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Tools for the proof

- $\mathrm{Var}(f) \leq \liminf_{t\to 0+}$ is trivial and in fact true for any g
- $\mathrm{Var}(f) \geq \limsup_{t\to 0+}$ relies on the bound

$$\left\| \mathrm{e}^{-tH_1} |_{\Omega^1_{\mathrm{L}^2\cap\mathrm{L}^\infty}(M)} \right\|_{\infty,\infty} \leq \delta \mathrm{e}^{tC(\delta)}, \quad t \geq 0,\ \delta > 1, \qquad (4)$$

which follows from $-\Delta_1/2 = \nabla^\dagger\nabla/2 + \mathrm{Ric}/2$, my results on generalized (= covariant) Schrödinger semigroups (JFA 262), and the following observation: For any $v \in \mathcal{K}(M)$ one has

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A stability result

We can deal with conformal changes: Let $\psi \in \mathsf{C}^{\infty}_{\mathbb{R}}(M)$ be bounded, $g_\psi := \mathrm{e}^{\psi} g$, and $\mathscr{T}_\psi := \mathrm{Ric} - \mathrm{Ric}_\psi$.

Corollary

Let M be geodesically complete and let $q \geq 1$ if $m = 1$, and $q > m/2$ if $m \geq 2$. Assume that there are $C_1, C_2, R > 0$ with the following property: one has $\mathrm{Ric} \geq -C_1$ and

$$\mathrm{vol}(\mathrm{B}_r(x)) \geq C_2 r^m \quad \text{for all } 0 < r \leq R,\ x \in M. \tag{5}$$

If $\mathscr{T}_\psi = \mathscr{T}_1 - \mathscr{T}_2$ with self-adjoint Borel sections $\mathscr{T}_1, \mathscr{T}_2 \geq 0$ in $\mathrm{End}(\mathrm{T}^* M)$ such that $|\mathscr{T}_2| \in \mathsf{L}^q_{\mathrm{u,loc}}(M; g_\psi) + \mathsf{L}^\infty(M; g_\psi)$, then for any $f \in \mathsf{L}^1(M; g_\psi)$ one has the heat kernel characterization of $\mathrm{var}_\psi(\bullet)$.

Thank you!