Exercises in Global Analysis II

University of Bonn, Winter Semester 2018-2019
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Sheet 3: due on Friday 2 November at 12:00 in Room 1.032.

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1 Friedrichs mollification of L^∞ functions [5 points]

Let $0 \leq \rho \in C_c^\infty(\mathbb{R}^m)$ be such that $\mathrm{supp}(\rho) \subset B_1(0)$ and $\int_{\mathbb{R}^m} \rho = 1$. For all $\lambda > 0$ define $0 \leq \rho_\lambda \in C_c^\infty(\mathbb{R}^m)$ by the scaling $\rho_\lambda(x) := \lambda^{-m}\rho(x/\lambda)$ (cf. Definition 3.14 of the lecture notes). Give an example of a function $f \in L^\infty(\mathbb{R}^m)$ such that $\rho_\lambda * f$ does *not* converge to f (as $\lambda \to 0+$) in the topology of $L^\infty(\mathbb{R}^m)$.
(Note: this exercise shows that the statement of Proposition 3.18 of the lecture notes fails for $p = \infty$.)

2 Proof of Theorem 3.21 [5 points]

Let $\Omega \subset \mathbb{R}^m$ be an open subset and fix $l \in \mathbb{N}$. Consider $p \in [1, \infty)$ and $k \in \mathbb{N}_{\geq 0}$, and let X be one of the following locally convex spaces: $L^p_{\mathrm{loc}}(\Omega, \mathbb{C}^l)$, $L^p_c(\Omega, \mathbb{C}^l)$, $C^k_c(\Omega, \mathbb{C}^l)$, $\mathcal{D}(\Omega, \mathbb{C}^l)$, $C^k(\Omega, \mathbb{C}^l)$, $\mathcal{E}(\Omega, \mathbb{C}^l)$, or $\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$. In each case, prove the following statement:
For every $f \in X$ one has $(\rho_\lambda * \underline{f})|_\Omega \to f$ as $\lambda \to 0+$ in the topology of X.

3 L^p-functions and tempered distributions [5 points]

(a) Prove that the space of Schwartz functions $\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ embeds continuously into $L^p(\mathbb{R}^m, \mathbb{C}^l)$ for any $1 \leq p \leq \infty$.

(b) Prove that $L^p(\mathbb{R}^m, \mathbb{C}^l)$ embeds continuously into the space of tempered distributions $\mathscr{S}'(\mathbb{R}^m, \mathbb{C}^l)$ for any $1 \leq p \leq \infty$.

4 Faà di Bruno's formula [5 points]

For $n \in \mathbb{N}_{\geq 0}$ and functions $f, g \in C^n(\mathbb{R}, \mathbb{R})$, show that the following formula holds:

$$(f \circ g)^{(n)} = \sum_{k=0}^{n} \sum_{\substack{k_1+k_2+\ldots+k_n=k, \\ k_1+2k_2+\ldots+nk_n=n}} \frac{n!}{k_1!k_2!\cdots k_n!} \left(f^{(k)} \circ g\right) \left(\frac{g'}{1!}\right)^{k_1} \left(\frac{g''}{2!}\right)^{k_2} \cdots \left(\frac{g^{(n)}}{n!}\right)^{k_n}.$$