Exercises in Global Analysis II

University of Bonn, Winter Semester 2018-2019
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Sheet 4: due on Friday 9 November at 12:00 in Room 1.032.

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1 Proof of Proposition 3.30 [5 points]

For any fixed $a \in \mathbb{R}^m$ consider the net of bounded functions $f_R \colon \mathbb{R}^m \to \mathbb{C}$ (with $R > 0$) given by

$$f_R(y) := \int_{B_R(0)} e^{-i(x,a-y)} dx.$$

Let $T_R := T_{f_R}$ denote the corresponding distributions. Show that in the topology of $\mathscr{S}'(\mathbb{R}^m)$ one has

$$\lim_{R\to\infty} T_R = C\delta_a,$$

and determine the constant C.

2 Distributional derivatives [5 points]

Consider the distribution $T_f \in \mathscr{D}'((-1,1))$ which is induced by the locally integrable function $f(x) := \log(x)$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$ (cf. Example 3.33.2). Show that the derivative of T_f is given by

$$\partial T_f(\phi) = \int_0^1 \frac{\phi(x) - \phi(0)}{x} dx,$$

for any $\phi \in \mathscr{D}((-1,1))$. Give a similar expression for the higher derivatives $\partial^n T_f$ for $n = 2$ and $n = 3$, such that the integrand on the right-hand-side has only terms containing $\phi(x)$, $\phi(0)$, $\phi'(0)$, and $\phi''(0)$ (i.e. no terms containing $\phi'(x)$ or $\phi''(x)$). Can you guess a formula for $\partial^n T_f$ for any $n \in \mathbb{N}$?

3 Convergence in $\mathcal{D}'(\mathbb{R})$ [5 points]

In this exercise we denote the distribution T_f which is induced by a locally integrable function f simply with f again. Let $f_\epsilon(x) := \frac{1}{x+i\epsilon}$ for $x \in \mathbb{R}$ and $0 \neq \epsilon \in \mathbb{R}$.

(a) Show that in the weak-$*$-topology of $\mathcal{D}'(\mathbb{R})$ one has

$$\lim_{\epsilon\to 0+} \big(f_\epsilon - f_{-\epsilon}\big) = -2\pi i\delta_0.$$

In particular, this shows that $\lim_{\epsilon\to 0} f_\epsilon$ does not exist.

(b) Show that in the weak-$*$-topology of $\mathcal{D}'(\mathbb{R})$ one has

$$\lim_{\epsilon\to 0+} f_\epsilon = \partial \log|\cdot| - i\pi\delta_0, \qquad \lim_{\epsilon\to 0-} f_\epsilon = \partial \log|\cdot| + i\pi\delta_0,$$

where $\partial \log|\cdot|$ denotes the distributional derivative of the logarithm function.

4 Moments of the Fourier transform [5 points]

Consider the Schwartz function $f \in \mathscr{S}(\mathbb{R})$ given by

$$f(x) := \begin{cases} 0, & x \leq 0, \\ e^{-(x^2+x^{-2})}, & x > 0. \end{cases}$$

Show that the Fourier transform $\widehat{f}$ of f has trivial moments, i.e. for each $k \in \mathbb{N}$ we have

$$\int_{\mathbb{R}} \zeta^k \widehat{f}(\zeta) d\zeta = 0.$$