Exercises in Global Analysis II

University of Bonn, Winter Semester 2018-2019
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Sheet 8: due on Friday 7 December at 12:00 in Room 1.032.

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1 Intersections and unions of Sobolev spaces [5 points]

Show that we have the following canonical isomorphisms of locally convex spaces:

$$\mathscr{E}(\Omega, \mathbb{C}^l) \cong \bigcap_{r\in\mathbb{Z}} H^r_{\mathrm{loc}}(\Omega, \mathbb{C}^l), \qquad \mathscr{E}'(\Omega, \mathbb{C}^l) \cong \bigcup_{r\in\mathbb{Z}} H^r_c(\Omega, \mathbb{C}^l).$$

2 Symbols and kernels of smoothing operators [5 points]

(a) Show that for every $p \in S^{-\infty}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ there exists a unique

$$A_p \in \mathscr{D}(\Omega \times \Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})))$$

with $\mathrm{Op}(p) = Q_{A_p}$, so in particular we have $\mathrm{Op}(p) \in \Psi^{-\infty}(\Omega; \mathbb{C}^l, \mathbb{C}^{l'})$.

Furthermore, if $\mathrm{Op}(p)|_{\mathscr{D}(\Omega\setminus K;\mathbb{C}^l)} = 0$ for some compact $K \subset \Omega$, then show that one has $\mathrm{supp}(A) \subset \Omega \times K$.

(b) Show that if

$$A \in \mathscr{E}(\Omega \times \Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})))$$

satisfies $\mathrm{supp}(A) \subset \Omega \times K$ for some compact $K \subset \Omega$, then there exists a

$$p \in S^{-\infty}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

such that $Q_A = \mathrm{Op}(p)$.

3 The Schwartz kernel of the Laplacian [5 points]

Consider the Laplacian $\Delta = -\sum_{j=1}^{3} \partial_j^2$ on $\mathbb{R}^3$.

(a) Compute the Schwartz kernel of the differential operator $1 + \Delta$.

(b) Compute the Schwartz kernel of $\mathrm{Op}(p)$, where $p(x, \zeta) := (1 + |\zeta|^2)^{-1}$ for $x, \zeta \in \mathbb{R}^3$.

(c) Show that on $C_c^\infty(\mathbb{R}^3)$ one has

$$\mathrm{Op}(p)(1 + \Delta) = (1 + \Delta)\mathrm{Op}(p) = 1.$$

4 A differentiable map into Sobolev spaces [5 points]

For $x \in \Omega \subset \mathbb{R}^m$ and $j \in \mathbb{N}_{\geq 0}$, consider the compactly supported distribution $\delta_x \otimes e_j \in \mathscr{E}'(\Omega)$ given by $\delta_x \otimes e_j(\phi) := \phi^{(j)}(x)$ for $\phi \in \mathscr{E}(\Omega)$.

(a) For $k \in \mathbb{N}_{\geq 0}$ and $t \in \mathbb{R}$ such that $t > m/2 + k$, show that the map

$$\Omega \longrightarrow H_c^{-t}(\Omega, \mathbb{C}^l), \quad x \longmapsto \delta_x \otimes e_j$$

is well-defined and in C^k.

(b) For any $f \in \mathscr{D}(\Omega, \mathbb{C}^l)$, consider the element $\int_\Omega f^{(j)}(x)\delta_x \otimes e_j dx \in H_c^{-t}(\Omega, \mathbb{C}^l)$ (with $t > m/2 + k$ as above) defined by

$$\left(\int_\Omega f^{(j)}(x)\delta_x \otimes e_j dx \right)(\phi) := \int_\Omega f^{(j)}(x)\delta_x \otimes e_j(\phi) dx,$$

for any $\phi \in \mathscr{D}(\Omega, \mathbb{C}^l)$. Show that the integral converges strongly, that is, for all continuous seminorms p on $H_c^{-t}(\Omega, \mathbb{C}^l)$ one has

$$\int p\big(f^{(j)}(x)\delta_x \otimes e_j\big) dx < \infty.$$

Furthermore, show that in $H_c^{-t}(\Omega, \mathbb{C}^l)$ we have the equality

$$f = \sum_j \int_\Omega f^{(j)}(x)\delta_x \otimes e_j dx.$$