KAC REGULAR SETS AND SOBOLEV SPACES IN GEOMETRY, PROBABILITY AND QUANTUM PHYSICS

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Abstract. Let $\Omega \subset M$ be an open subset of a Riemannian manifold M and let $V \in L^2_{\mathrm{loc}}(M)$ be a potential such that $-\Delta + V$ is bounded from below in $L^2(M)$. With $W^{1,2}_0(M;V)$ the natural form domain of $-\Delta + V$, in this paper we study systematically the following question: Under which assumption on Ω is the statement

$$\text{for all } f \in W^{1,2}_0(M;V) \text{ with } f = 0 \text{ a.e. in } M \setminus \Omega \text{ one has } f|_\Omega \in W^{1,2}_0(\Omega;V)$$

true for every such V? The validity of this statement turns out to provide a highly subtle local boundary regularity assumption on Ω. We prove that without any further assumptions on V, the above property is satisfied, if Ω is Kac regular, a probabilistic property which means that the first exit time of Brownian motion on M from Ω is equal to its first penetration time to $M \setminus \Omega$. In fact, we treat more general covariant Schrödinger operators acting on sections in metric vector bundles, allowing new results concerning the harmonicity of Dirac spinors on singular subsets. Finally, we prove that Lipschitz regular Ω's are Kac regular.

1. Introduction

Consider the following three properties that an open subset Ω of a noncompact Riemannian manifold M may or may not have:

(A) Let $V \in L^2_{\mathrm{loc}}(M)$ be a potential such that $-\Delta + V$ is bounded from below in $L^2(M)$ on smooth compactly supported functions. Let $H_M(V)$ denote the Friedrichs realization of $-\Delta + V$ in $L^2(M)$ and let $H_\Omega(V)$ denote the self-adjoint realization of $-\Delta + V$ in $L^2(\Omega)$ subject to Dirichlet boundary conditions (also a Friedrichs realization). *For every such V one has*

$$\exp\big(-t(H_M(V) + \infty \cdot 1_{M\setminus\Omega})\big) = \exp(-tH_\Omega(V))P_\Omega \quad \text{for all } t > 0, \tag{1}$$

were, $P_\Omega : L^2(M) \to L^2(\Omega)$ denotes the natural projection.

(B) Assume M is a geodesically complete Riemannian spin manifold. *For every spin bundle $\mathscr{S} \to M$ with corresponding Dirac operator D acting on sections of $\mathscr{S} \to M$, and every spinor $\Psi \in \Gamma_{L^2}(M,\mathscr{S})$ with $D\Psi \in \Gamma_{L^2}(M,\mathscr{S})$ and $\Psi = 0$ almost everywhere in $M \setminus \Omega$, one has the implication*

$$D^2\Psi = 0 \text{ in } \Omega \Rightarrow D\Psi = 0 \text{ in } \Omega\ .$$

(C) *The first exit time of Brownian motion from Ω equals the first penetration time of Brownian motion to $M \setminus \Omega$.* This property in particular implies[1] that with a standard

[1] We will note assume M to be stochastically complete

notion (which will also be explained in the next section) one has

$$\begin{aligned}&\{\mathbb{X}_s \in \Omega \textit{ for a.e. } s \in [0,t] \textit{ and } \mathbb{X}_s \in M \textit{ for all } s \in [0,t]\}\\ &=_{\mathbb{P}^x} \{\mathbb{X}_s \in \Omega \textit{ for all } s \in [0,t]\} \quad \textit{for all } x \in \Omega,\ t > 0.\end{aligned}$$

To the best of our knowledge, property (C) has been introduced by D. Stroock in the Euclidean $\mathbb{R}^m$, and is usually referred to as the *Kac regularity* of Ω.

While (A), (B) and (C) seem unrelated at first glance, it should be nevertheless intuitively clear that these properties provide restrictions on the boundary regularity of Ω. The main results of this paper show that the above three problems are much more correlated than one might expect: Indeed we prove that given an arbitrary open subset $\Omega \subset M$,

(i) (A) is equivalent to

(A') for all $f \in W_0^{1,2}(M;V)$ with $f = 0$ a.e. in $M \setminus \Omega$ one has $f|_\Omega \in W_0^{1,2}(\Omega;V)$

for every V as in (A); here $W_0^{1,2}(M;V)$ and $W_0^{1,2}(\Omega;V)$ denote, respectively, the form domain of $H_M(V)$ and $H_\Omega(V)$,

(ii) (C) is equivalent to (A),

(iii) (C) implies (B),

(iv) If $\partial\Omega$ is Lipschitz regular (cf. Definition 2.12), then one has (C).

While (i), (ii) and a variant of (iv) have been studied by I.W. Herbst and Z.X. Zhao in the Euclidean $\mathbb{R}^m$ [8] for $V = 0$ (cf. Remark 2.15), our analysis is the first systematic treatment of these questions on manifolds, allowing in addition potentials and in fact covariant Schrödinger operators. We would like to stress the fact that, in our eyes, none of the equivalent properties (A), (A') and (C) makes it obvious that a boundary regularity implying them should be of an entirely local nature. In that sense, (iv) is certainly a very subtle result. As we only require an L^2_{loc}-regularity on the potential V, we can also treat Schrödinger operators that appear naturally in quantum mechanics, having potentials with Coulomb type singularities. In fact, we are going to treat these problems within the much more general class of covariant Schrödinger operators that act on sections of metric vector bundles over M, allowing to treat magnetic fields or squares of geometric Dirac operators simulatinously, making contact with (B). While our proofs rely on various (partially new) covariant Feynman-Kac formulas, we will not assume any global condition on the negative part of the potential, other than making the underlying Schrödinger operator bounded from below. This is remarkable, as covariant Feynman-Kac formulas seem to require such a (Kato-type or Dynkin-type) condition to hold on a dense set of L^2-sections (the heuristic reason for this is that Brownian paths have an infinite speed). We can avoid such assumptions by making use of a highly technical combination of (partially nonstandard) convergence results for quadratic forms on Hilbert spaces.

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2. Main result

In the sequel, we work in the smooth category, that is, all differential, topological and geometric data (like manifolds, bundles, metrics and covariant derivatives) are understood to be smooth. In addition, any manifold is understood to be without boundary, unless otherwise stated. Without loss of generality, we will consider our function spaces to be over $\mathbb{C}$.

Let M be an arbitrary Riemannian m-manifold. As such, M is equipped with its Riemannian volume measure μ. Given a complex metric vector bundle $E \to M$, the scalar product on the complex Hilbert space $\Gamma_{L^2}(M,E)$ of Borel equivalence classes of square integrable sections of $E \to M$ is simply denoted with

$$\langle f_1, f_2 \rangle = \int (f_1, f_2) d\mu,$$

with

$$\|f\|^2 = \int |f|^2 d\mu$$

the induced norm. Given two such bundles $E_1 \to M$, $E_2 \to M$, the formal adjoint of a smooth linear partial differential operator D from $E_1 \to M$ to $E_2 \to M$ with respect to $\langle \cdot, \cdot \rangle$ is simply denoted with $D^\dagger$. The following conventions will be very convenient in the sequel:

Notation 2.1. If $\Omega \subset M$ is an open subset and $E \to M$ is a complex vector bundle, we define

$$\Gamma_{C_c^\infty}(\Omega, E) := \big\{ f \in \Gamma_{C_c^\infty}(M,E) : \ f \text{ is compactly supported in } \Omega \big\} \subset \Gamma_{C_c^\infty}(M,E),$$

and if $E \to M$ is equipped with a metric, then

$$\Gamma_{L^2}(\Omega, E) := \{ f \in \Gamma_{L^2}(M,E) : \ f = 0 \ \mu\text{-a.e. in } M \setminus \Omega \} \subset \Gamma_{L^2}(M,E).$$

Then $\Gamma_{L^2}(\Omega,E)$ is a closed subspace of $\Gamma_{L^2}(M,E)$, thus a Hilbert space in itself. We denote with

$$P_\Omega : \Gamma_{L^2}(M,E) \longrightarrow \Gamma_{L^2}(\Omega,E)$$

the orthogonal projection onto $\Gamma_{L^2}(\Omega,E)$. Note that P_Ω is nothing but the restriction map $f \mapsto f|_\Omega$.

Definition 2.2. By a *regular Schrödinger bundle over M*, we will understand a datum

$$(E, \nabla, V) \longrightarrow M$$

with

- $E \to M$ a complex metric vector bundle
- ∇ a metric covariant derivative on $E \to M$,
- $V : M \to \mathrm{End}(E)$ is an L^2_{loc}-potential,

such that there exists $C \in \mathbb{R}$ with

$$\big\langle (\nabla^\dagger \nabla + V) f, f \big\rangle \geq C \|f\|^2 \quad \text{for all } f \in \Gamma_{C_c^\infty}(M,E).$$

Notation 2.3. In the situation of Definition 2.2, let $\Omega \subset M$ be an open subset. Then $\Gamma_{W_0^{1,2}}(\Omega, E; \nabla, V)$ is defined to be the space of all $f \in \Gamma_{L^2}(\Omega, E; \nabla, V)$ which admit a sequence $(f_n) \subset \Gamma_{C_c^\infty}(\Omega, E; \nabla, V)$ such that

$$\|f_n - f\| \to 0 \quad \text{as } n \to \infty, \tag{2}$$

$$\big\langle (\nabla^\dagger \nabla + V)(f_n - f_m), (f_n - f_m) \big\rangle \to 0 \quad \text{as } n, m \to \infty. \tag{3}$$

As $\nabla^\dagger \nabla + V$ with domain of definition $\Gamma_{C_c^\infty}(\Omega, E; \nabla, V)$ is a symmetric semibounded and densely defined operator in $\Gamma_{L^2}(\Omega, E)$, given $f, \tilde{f} \in \Gamma_{W_0^{1,2}}(\Omega, E; \nabla, V)$ with defining sequences $(f_n), (\tilde{f}_n) \subset \Gamma_{C_c^\infty}(\Omega, E; \nabla, V)$, the number

$$\langle f, \tilde{f} \rangle_{\nabla, V, *} := \lim_{n \to \infty} \langle (\nabla^\dagger \nabla + V) f_n, \tilde{f}_n \rangle$$

is well-defined, giving rise to a symmetric semibounded closed sesquilinear form which is densely defined in $\Gamma_{L^2}(\Omega, E)$, thus inducing a nonnegative self-adjoint realization $H_\Omega(\nabla, V)$ of $\nabla^\dagger \nabla + V$ in $\Gamma_{L^2}(\Omega, E)$. In addition, $\Gamma_{W_0^{1,2}}(\Omega, E; \nabla)$ becomes a Hilbert space with its natural scalar product

$$\langle f_1, f_2 \rangle_{\nabla, V} := \langle f_1, f_2 \rangle + \langle f_1, f_2 \rangle_{\nabla, V, *} .$$

Note that $H_\Omega(\nabla, V)$ is nothing but the Friedrichs realization of $\nabla^\dagger \nabla + V$ in $\Gamma_{L^2}(\Omega, E)$. Traditionally, if $\Omega \neq M$, one says that $H_\Omega(\nabla, V)$ is the self-adjoint realization of $\nabla^\dagger \nabla + V$ in $\Gamma_{L^2}(\Omega, E)$ subject to Dirichlet boundary conditions.

To simplify the notation in some special cases, we add:

Remark 2.4. If $E \to M$ is the trivial complex line bundle and ∇ the exterior derivative mapping functions to 1-forms, then we simply omit $\nabla = d$ and E everywhere in the notation. Note that $d^\dagger d = -\Delta$ is the Laplace-Beltrami operator on functions. In case $V = 0$ we will in addition omit V everywhere in the notation. These conventions lead to natural notations such as $W_0^{1,2}(\Omega; V)$, $H_\Omega(V)$ and $W_0^{1,2}(\Omega)$, H_Ω.

Here is an example from quantum mechanics:

Example 2.5. If M is the Euclidean $\mathbb{R}^3$ and $\eta = (\eta_1, \eta_2, \eta_3) \in \Gamma_{C^\infty}(\mathbb{R}^3, T^*\mathbb{R}^3)$ is a smooth 1-form (considered as a magnetic potential), $Z \in \mathbb{N}$, and $V : \mathbb{R}^3 \to \mathbb{R}$ is the Coulomb potential $V(x) := -Z/|x|$, then with

$$\nabla^\eta : C^\infty(\mathbb{R}^3) \longrightarrow \Gamma_{C^\infty}(\mathbb{R}^3, T^*\mathbb{R}^3), \quad \nabla^\eta f := (d + \sqrt{-1}\eta) f = \sum_{j=1}^{3} (\partial_j f + \sqrt{-1} f \eta_j) dx^j,$$

the datum

$$(\mathbb{R}^3 \times \mathbb{C}, \nabla^\eta, V) \longrightarrow \mathbb{R}^3$$

is a regular Schrödinger bundle (clearly, $V \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ and the required semiboundedness follows, e.g., from the Sobolev inequality). In this case, identifying sections in $\mathbb{R}^3 \times \mathbb{C} \to \mathbb{R}^3$ with functions on M the semibounded self-adjoint operator

$$H_{\mathbb{R}^3}(\eta, V) := H_{\mathbb{R}^3}(\nabla^\eta, V) \quad \text{in} \quad L^2(\mathbb{R}^3)$$

is the Hamilton operator of an atom having one electron and a nucleus having Z protons, in the magnetic field given by

$$d_1\eta = \sum_{i<j} (\partial_i \eta_j - \partial_j \eta_i) dx^i \wedge dx^j.$$

More generally, one could replace in this example $\mathbb{R}^3$ with a Riemannian 3-fold M whose minimal nonnegative heat kernel $\exp(-tH_M)(x,y)$ satisfies a Gaussian upper bound of the form

$$\exp(-tH_M)(x,y) \leq C_1 t^{-3/2} \exp\left(-\frac{d(x,y)^2}{C_2 t}\right), \quad (t,x,y) \in (0,\infty) \times M \times M,$$

and one could even take the electron's spin into account, see [6].

Here is an example from geometry:

Example 2.6. A *geometric Dirac bundle over M* is understood to be a datum

$$(E; c, \nabla) \longrightarrow M$$

such that

- $E \to M$ is a complex metric vector bundle
- c is a Clifford multiplication on $E \to M$, in the sense that c is a homomorphism of real (!) vector bundles $c : TM \to \mathrm{End}(E)$, such that for all $X \in \Gamma_{C^\infty}(M, TM)$ one has $c(X) = -c(X)^*$ and $c(X)^* c(X) = |X|^2$
- ∇ is a Clifford connection on $(E; c) \to M$, that is, ∇ is a metric covariant derivative on $E \to M$ such that for all $X, Y \in \Gamma_{C^\infty}(M, TM)$, $\Psi \in \Gamma_{C^\infty}(M, E)$ one has
$$\nabla_X (c(Y)\Psi) = c(\nabla^{TM}_X Y)\Psi + c(Y)\nabla_X \Psi.$$

Then the associated geometric Dirac operator is the first order linear partial differential operator on $E \to M$ defined by $D(c, \nabla) := \sum_{j=1}^m c(e_j)\nabla_{e_j}$, where e_j is any local orthonormal-frame for $TM \to M$ and $m = \dim M$. The operator $D(c, \nabla)$ is formally self-adjoint with symbol c, in particular elliptic. By a Lichnerowicz formula

$$V(c, \nabla) := D(c, \nabla)^2 - \nabla^\dagger \nabla$$

is a smooth potential $V(c, \nabla) : M \to \mathrm{End}(E)$, and we get the regular Schrödinger bundle

$$(E; \nabla, V(c, \nabla)) \longrightarrow M.$$

In this case, $\Gamma_{W^{1,2}_0}(\Omega, E; \nabla, V(c, \nabla))$ is given by all $f \in \Gamma_{L^2}(\Omega, E)$ with $D(c, \nabla) f \in \Gamma_{L^2}(\Omega, E)$ in the sense of distributions, which admit a sequence $f_n \in \Gamma_{C^\infty_c}(\Omega, E)$, $n \in \mathbb{N}$, with

$$\|f_n - f\| + \|D(c, \nabla) f_n - D(c, \nabla) f\| \to 0.$$

We have

$$\langle f_1, f_2 \rangle_{\nabla, V(c,\nabla),*} = \langle D(c, \nabla) f_1, D(c, \nabla) f_2 \rangle,$$

so that in particular the scalar product on $\Gamma_{W^{1,2}_0}(\Omega, E; \nabla, V(c, \nabla))$ is given by

$$\langle f_1, f_2 \rangle_{\nabla, V(c,\nabla)} = \langle f_1, f_2 \rangle + \langle D(c, \nabla) f_1, D(c, \nabla) f_2 \rangle.$$

Given a regular Schrödinger bundle

$$(E,\nabla,V)\longrightarrow M,$$

the inclusion

$$\Gamma_{W^{1,2}_0}(\Omega,E;\nabla,V)\subset\big\{f\in\Gamma_{W^{1,2}_0}(M,E;\nabla,V):f|_{M\setminus\Omega}=0\ \ \mu\text{-a.e. }\big\}$$

is easily seen to be satisfied without any further assumptions on the underlying data: Indeed, given $\psi\in\Gamma_{W^{1,2}_0}(\Omega,E,\nabla,V)$ we can per definitionem pick a sequence $\psi_n\in\Gamma_{C^\infty_c}(\Omega,E)$ with $\|\psi_n-\psi\|_{\nabla,V}\to 0$, so that $\|\psi_n-\psi\|\to 0$ and so $\psi=0$ μ-a.e. in $M\setminus\Omega$.

The question we address in this paper is: Under which condition on Ω does the reverse inclusion

$$\Gamma_{W^{1,2}_0}(\Omega,E;\nabla,V)\supset\big\{f\in\Gamma_{W^{1,2}_0}(M,E;\nabla,V):f|_{M\setminus\Omega}=0\ \ \mu\text{-a.e. }\big\}\tag{4}$$

hold true?

By the above, (4) is equivalent to

$$\Gamma_{W^{1,2}_0}(\Omega,E;\nabla,V)=\big\{f\in\Gamma_{W^{1,2}_0}(M,E;\nabla,V):f|_{M\setminus\Omega}=0\ \ \mu\text{-a.e. }\big\}.\tag{5}$$

The validity of (5) turns out to be equivalent to the corresponding Dirichlet heat-flow of $H_\Omega(\nabla,V)$ on Ω being realized as the heat-flow of $H_M(\nabla,V)$ on M perturbed by the 'potential' $\infty\cdot 1_{M\setminus\Omega}$, for one has:

Proposition 2.7. *Let $\Omega\subset M$ be an arbitrary open subset and let $(E,\nabla,V)\to M$ be a regular Schrödinger bundle over M. Then one has (5), if and only if for all $t\geq 0$ one has*

$$\lim_{n\to\infty}\exp\big(-tH_M(\nabla,V+n1_{M\setminus\Omega})\big)=\exp\big(-tH_\Omega(\nabla,V)\big)P_\Omega\tag{6}$$

strongly as bounded operators in $\Gamma_{L^2}(M,E)$.

The proof of Proposition 2.7 will be given in Section 4. It is a consequence of monotone convergence results for sesquilinear forms.

As one might guess, the validity of (5) or (6) should depend on the local regularity of $\partial\Omega$. The reader should recall the simple fact that even if f is continuous, the assumption $f|_{M\setminus\Omega}=0$ a.e. does not imply that $f(x)=0$ for all $x\in M\setminus\Omega$, a fact which already indicates that the above question actually is very subtle. (cf. Proposition 2.10 and Remark 2.11 for much more refined statements of this type).

In order to make precise how (5) or (6) depend on the local regularity of $\partial\Omega$, the probabilistic Definition 2.9 below will turn out to be crucial. To this end, let $W(\hat M)$ denote the Wiener space of continuous paths $\omega:[0,\infty)\to\hat M$ with its Borel-sigma-algebra $\mathcal{F}$, where $\hat M$ is defined to be the (essentially uniquely determined) Alexandroff-compactification of M in case M is noncompact, and $\hat M:=M$ if M is compact.[2] In any case, $W(\hat M)$ is

[2]Note that every open subset of M is also open in $\hat M$ and that for every subset $A\subset M$ one has $\overline{A}^M=\overline{A}^{\hat M}=:\overline{A}$, as M is dense in $\hat M$.

equipped with the topology of locally uniform convergence. Let

$$\mathbb{X} : [0,\infty) \times W(\hat{M}) \longrightarrow \hat{M}, \quad \mathbb{X}_t(\omega) := \omega(t)$$

denote the coordinate process, and for every $x \in M$, the symbol $\mathbb{P}^x$ stands for the Riemannian Brownian motion measure on $\mathcal{F}$ with

$$\mathbb{P}^x\{\mathbb{X}_0 = x\} = 1.$$

With $\mathcal{F}_*$ the filtration of $\mathcal{F}$ which is generated by $\mathbb{X}$, the family $\mathbb{P}^x$, $x \in M$, has the strong Markov property for stopping times in $\mathcal{F}_*$. If $\Omega \subset \hat{M}$ is an open set, then[3]

$$\alpha_\Omega : W(\hat{M}) \longrightarrow [0,\infty], \quad \alpha_\Omega := \inf\big\{t > 0 : \mathbb{X}_t \in \hat{M} \setminus \Omega\big\}$$

denotes the first exit time of $\mathbb{X}$ from Ω. It is well-known that α_Ω is an $\mathcal{F}_*$-stopping time. Likewise,

$$\beta_\Omega := \inf\Big\{t > 0 : \int_0^t 1_{\hat{M}\setminus\Omega}(\mathbb{X}_s)ds > 0\Big\} : W(\hat{M}) \longrightarrow [0,\infty]$$

denotes the first penetrating time of $\mathbb{X}$ into $M \setminus \Omega$. One trivially has

$$\alpha_\Omega \leq \beta_\Omega, \tag{7}$$

and again β_Ω induces an $\mathcal{F}_*$-stopping time. Note that both α and β are pathwise monotonely increasing in Ω, and that one pathwise has

$$\beta_\Omega = \inf\Big\{t \geq 0 : \int_0^t 1_{\hat{M}\setminus\Omega}(\mathbb{X}_s)ds > 0\Big\} = \inf\big\{t \geq 0 : \mathbb{X}_s \in \Omega \text{ for a.e. } s \in [0,t]\big\},$$

and if $\omega \in W(\hat{M})$ is such that $\omega(0) \in \Omega$, then one also gets

$$\beta_\Omega(\omega) \geq \alpha_\Omega(\omega) = \inf\big\{t \geq 0 : \omega(t) \in \hat{M} \setminus \Omega\big\} > 0.$$

The following regularity result will be very useful in the sequel:

Lemma 2.8. *Let $\omega \in W(\hat{M})$, let Ω be an open subset of M, and let $\Omega_n \subset \Omega$, $n \in \mathbb{N}$, be open with $\Omega_n \subset \Omega_{n+1}$ for all n, $\omega(0) \in \Omega_1$ and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$.*
a) *One has $\alpha_{\Omega_n}(\omega) \nearrow \alpha_\Omega(\omega)$ as $n \to \infty$.*
b) *If in addition it holds that $\bigcup_{n\in\mathbb{N}} \overline{\Omega_n} = \overline{\Omega}$, then one has $\beta_{\Omega_n}(\omega) \nearrow \beta_\Omega(\omega)$ as $n \to \infty$.*

The proof of Lemma 2.8, whose part a) is well-known, will be given in Section 3. The following definition is motivated by Herbst/Zhao [8] and Stroock [15], who treat the Euclidean $\mathbb{R}^m$:

Definition 2.9. An open set $\Omega \subset M$ is called *Kac regular*, if one has

$$\mathbb{P}^x\{\alpha_\Omega = \min(\beta_\Omega, \alpha_M)\} = 1 \quad \text{for all } x \in \Omega. \tag{8}$$

[3]In the sequel the infimum of an empty set is understood to be ∞.

Note that we do not assume M to be stochastically complete, noting that in the latter case one has

$$\mathbb{P}^x\{\alpha_\Omega = \min(\beta_\Omega, \alpha_M)\} = \mathbb{P}^x\{\alpha_\Omega = \beta_\Omega\},$$

and some technical problems are not present, that we will have to deal with. The connection between problems such as (5) and Kac regularity is clarified in the following result, which has been established in the Euclidean $\mathbb{R}^m$ by Herbst/Zhao [8]:

Proposition 2.10. *Let $\Omega \subset M$ be an arbitrary open subset. The following properties are equivalent:*

- *Ω is Kac regular*
- *one has*

$$W_0^{1,2}(\Omega) = \big\{ f \in W_0^{1,2}(M) : f|_{M\setminus\Omega} = 0 \ \mu\text{-a.e. } \big\}. \tag{9}$$

- *for all $t \geq 0$ one has*

$$\lim_{n\to\infty} \exp\big(- tH_M(n1_{M\setminus\Omega})\big) = \exp\big(- tH_\Omega\big) P_\Omega \tag{10}$$

  strongly as bounded operators in $L^2(M)$.

The proof of Proposition 2.10 will be given in Section 4. Note that above $H_M(n1_{M\setminus\Omega})$ is the Friedrichs realization of $-\Delta + n1_{M\setminus\Omega}$ in $L^2(M)$, and H_Ω is the Dirichlet realization of $-\Delta$ in $L^2(\Omega)$. The equivalence of (9) and (10) follows immediately from specializing Proposition 2.7 to the scalar case. The equivalence of (10) and Kac-regularity will be established by using a Feynman-Kac formula.

Remark 2.11. 1. The potential theoretic variant of (9) does not see anything from the geometry of Ω, namely [4], for *every open* $\Omega \subset M$ one has

$$W_0^{1,2}(\Omega) = \big\{ f \in W_0^{1,2}(M) : \breve{f}|_{M\setminus\Omega} = 0 \ \text{q.e.}\big\},$$

where 'q.e.' (quasi everywhere) is understood with respect to the capacity associated with the regular strongly local Dirichlet form

$$\langle f_1, f_2\rangle_* = \int (df_1, df_2) d\mu$$

in $L^2(M)$ with domain of definition $W_0^{1,2}(M)$, and where $\breve{f}$ denotes the quasi-continuous representative of f. This again reflects the subtlety of the questions under investigation.
2. It was conjectured in [12] by B. Simon that open subsets $\Omega \subset \mathbb{R}^m$ of the Euclidean space with $\mathbb{R}^m \setminus \Omega$ perfect satisfy (10). A counterexample to this conjecture was given by L.I. Hedberg [7] and P. Stollmann [13] (cf. [14], p.127), who even show that there exists $\Omega \subset \mathbb{R}^m$ open with $\mathbb{R}^m \setminus \Omega$ compact and

$$\mathbb{R}^m \setminus \Omega = \overline{\mathbb{R}^m \overset{\circ}{\setminus} \Omega}$$

such that (9) and thus (10) fails. Note that this counterexample also entails that even for quasi-continuous $f's$ the assumption $f|_{M\setminus\Omega} = 0$ a.e. does not imply that $f|_{M\setminus\Omega} = 0$ q.e.

In order to formulate our main result, we add:

Definition 2.12. An open subset $\Omega \subset M$ is called *Lipschitz regular*, if there exists an exhaustion $\Omega_n \subset \Omega$, $n \in \mathbb{N}$, with open relatively compact subsets having a Lipschitz boundary, such that $\overline{\Omega_n}$ is an exhaustion of $\overline{\Omega}$.

Here comes our main result:

Theorem 2.13. *Let Ω be an arbitrary open subset of M.*
a) *If Ω is Kac regular, then for every regular Schrödinger bundle $(E, \nabla, V) \to M$ one has (5) and (6).*
b) *If Ω is Lipschitz regular, then Ω is Kac regular.*

Using transversality theory one finds that every open subset $\Omega \subset M$ having a smooth boundary is Lipschitz regular, implying:

Corollary 2.14. *Every open subset $\Omega \subset M$ having a smooth boundary is Kac regular.*

We **conjecture** that every open subset $\Omega \subset M$ having a locally Lipschitz boundary is Lipschitz regular.

Theorem 2.13 a) is new even in the Euclidean case, where so far only $-\Delta$ has been treated, and not even Schrödinger operators $-\Delta + V$. The proof of Theorem 2.13 will be given in Section 4. We continue with remarks on this proof: The proof of Theorem 2.13 a) relies on Proposition 2.7 and Proposition 2.10, while both worlds are linked through various partially completely new covariant Feynman-Kac formula. We believe it is a very surprising fact, that no conditions on the negative part of V is needed in Theorem 2.13 a), as such a condition is certainly required for the covariant Feynman-Kac formula to hold (roughly speaking, because Brownian paths have an infinite speed). The point here is that using rather subtle approximation arguments, one can reduce everything to the case of V's that are bounded from below by a constant.
The proof of Theorem 2.13 b) relies again on Proposition 2.10: In a first step , we will give an analytic proof of the stronger statement

$$W_0^{1,2}(\Omega) = \left\{ f \in W^{1,2}(M) : f|_{M \setminus \Omega} = 0 \ \ \mu\text{-a.e. } \right\}, \tag{11}$$

under the assumption that Ω is relatively compact with $\partial\Omega$ Lipschitz, directly verifying (9). As usual, $W^{1,2}(M)$ is defined to be the space of all $f \in L^2(M)$ such that $df \in \Gamma_{L^2}(M, T^*M)$ in the sense of distributions. In a second step we will then combine this local result with a probabilistic approximation argument, directly confirming (8).

Remark 2.15. 1. It is false that every open subset $\Omega \subset M$ with smooth boundary satisfies[4] (11). As a counterexample, one can consider the Euclidean ball and its open upper half

$$M := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}, \quad \Omega := \{(x, y, z) \in M : z > 0\},$$

with the Euclidean metric.
2. It is shown in [8] that in the Euclidean case Ω's with $\partial\Omega$ having the segment property are

[4] But of course this statement is true by Proposition 2.10 and Theorem 2.13 b), if M is geodesically complete, as then $W^{1,2}(M) = W_0^{1,2}(M)$ [1].

Kac regular. In fact, many comparable Euclidean results can be found in P. Stollmann's diploma thesis [13]. The segment property is more general then being locally Lipschitz, but is also a concept that does not apply to manifolds.

Corollary 2.16. *Let M be geodesically complete and let $(E;c,\nabla)\to M$ be a geometric Dirac bundle. Then for every open subset $\Omega\subset M$ which is Kac regular, and every Ψ with*

$$\Psi\in\Gamma_{L^2}(M,E),\quad D(c,\nabla)\Psi\in\Gamma_{L^2}(M,E),\quad \Psi=0\ \mu\text{-a.e. in } M\setminus\Omega,$$

one has the implication

$$D(c,\nabla)^2\Psi=0\ \text{ in }\Omega\quad\Rightarrow\quad D(c,\nabla)\Psi=0\text{ in }\Omega.$$

Proof. Recall that $(E;c,\nabla)\to M$ induces the regular Schrödinger bundle

$$(E,\nabla,V(c,\nabla))\longrightarrow M,$$

As M is geodesically complete one has

$$\Psi\in\Gamma_{W^{1,2}_0}\big(M,E;\nabla,V(c,\nabla)\big),$$

so that as Ω is Kac regular we have

$$\Psi\in\Gamma_{W^{1,2}_0}\big(\Omega,E,\nabla,V(c,\nabla)\big),$$

and we can pick a sequence $\Psi_n\in\Gamma_{C^\infty_c}(\Omega,E)$ with

$$\lim_{n\to\infty}\|\Psi_n-\Psi\|_{\nabla,V(c,\nabla)}=0.$$

It follows that

$$\begin{aligned}\big\|D(c,\nabla)\Psi\big\|^2&=\lim_{n\to\infty}\big\langle D(c,\nabla)\Psi_n,D(c,\nabla)\Psi\big\rangle\\ &=\int_\Omega(D(c,\nabla)\Psi_n,D(c,\nabla)\Psi)d\mu=\int_\Omega(\Psi_n,D(c,\nabla)^2\Psi)d\mu=0,\end{aligned}$$

where the integration by parts is justified as Ψ is smooth in Ω by local elliptic regularity, and Ψ_n is smooth and compactly supported in Ω. ∎

Note that in Corollary 2.16 the geodesic completeness of M was only assumed to guarantee

$$\Psi\in\Gamma_{W^{1,2}_0}(M,E;\nabla,V(c,\nabla)),$$

which one could assume instead.

3. Proof of Lemma 2.8

Proof of Lemma 2.8. a) This statement is well-known and elementary to check.
b) As this observation seems to be new, we have decided to add its proof (which is rather technical): Assume first that there exists $t_0\in(0,\infty)$ such that $\beta_\Omega(\omega)=t_0$. This means that the Lebesgue measure of the set $\{s\in[0,t_0]:\omega(s)\notin\Omega\}$ is zero, briefly

$$|\{s\in[0,t_0]:\omega(s)\notin\Omega\}|=0$$

and moreover that for each $\epsilon>0$ we have

$$|\{s\in[0,t_0+\epsilon]:\omega(s)\notin\Omega\}|>0.$$

Let $n_0 > 0$ be such that

$$\omega([0,t_0]) \cap \Omega = \omega([0,t_0]) \cap \Omega_{n_0}.$$

In this way we have

$$|\{s \in [0,t_0] : \omega(s) \notin \Omega_n\}| = 0$$

for each $n \geq n_0$, because

$$\{s \in [0,t_0] : \omega(s) \notin \Omega_n\} = \{s \in [0,t_0] : \omega(s) \notin \Omega_{n_0}\} = \{s \in [0,t_0] : \omega(s) \notin \Omega\}$$

and the latter has Lebesgue measure 0. Moreover we have

$$|\{s \in [0,t_0+\delta] : \omega(s) \notin \Omega_n\}| > 0$$

for each $\delta > 0$ and for each $n \geq n_0$, because

$$\{s \in [0,t_0+\delta] : \omega(s) \notin \Omega\} \subset \{s \in [0,t_0+\delta] : \omega(s) \notin \Omega_n\} \subset \{s \in [0,t_0+\delta] : \omega(s) \notin \Omega_{n_0}\}$$

and

$$|\{s \in [0,t_0+\delta] : \omega(s) \notin \Omega\}|) > 0.$$

Therefore $\beta_{\Omega_n}(\omega) = t_0$ for each $n \geq n_0$ and hence we can conclude that $\lim_n \beta_{\Omega_n}(\omega) = \beta_\Omega(\omega)$.
Assume now that there is no $t_0 \in (0,\infty)$ such that $\beta_\Omega \omega = t_0$. Therefore

$$|\{s \in [0,\infty) : \omega(s) \notin \Omega\}| = 0.$$

In this case we set $\beta_\Omega(\omega) := \infty$. Consider first the case where $\overline{\omega([0,\infty))}$ is compact. Let

$$A := \{s \in [0,\infty) : \omega(s) \notin \Omega\}$$

and let $B := [0,\infty) \setminus A$. Let n_0 be such that $\omega(B) \subset \Omega_{n_0}$. Since $\omega(B) \subset \Omega_{n_0}$ we have

$$\omega([0,\infty)) \cap \Omega = \omega([0,\infty)) \cap \Omega_n$$

for each $n \geq n_0$. Therefore

$$|\{s \in [0,\infty) : \omega(s) \notin \Omega_n\}| = 0$$

for any $n \geq n_0$ and hence we can conclude that $\lim_n \beta_{\Omega_n}(\omega) = \beta_\Omega(\omega)$. Finally consider the case where $\overline{\omega([0,\infty))}$ is not compact. The sequence $\beta_{\Omega_n}(\omega)$ is increasing and unbounded, because given an arbitrarily big t_0 we can find an integer n_0 such that $\omega([0,t_0]) \cap \Omega = \omega([0,t_0]) \cap \Omega_{n_0}$ and $\omega(t_1) \notin F_{n_0}$ for some $t_1 > t_0$. Hence $\beta_{\Omega_{n_0}}(\omega) \geq t_0$ and therefore $\lim_n \beta_{\Omega_n}(\omega) = \infty$. ∎

4. Proofs of main results

We first introduce some notation that will be relevant in the sequel. Let

$$(E, V, \nabla) \longrightarrow M$$

be a regular Schrödinger bundle.

Notation 4.1. Given an open subset $\Omega \subset M$ let

(12)
$$\Gamma_{\widetilde{W}_0^{1,2}}(\Omega, E; \nabla, V) := \big\{ f \in \Gamma_{W_0^{1,2}}(M, E; \nabla, V) : f|_{M\setminus\Omega} = 0 \quad \mu\text{-a.e. } \big\} \subset \Gamma_{W_0^{1,2}}(M, E; \nabla, V),$$

and let $\widetilde{H}_\Omega(\nabla, V)$ denote the self-adjoint nonnegative operator in $\Gamma_{L^2}(\Omega, E)$ which corresponds to $\langle\cdot,\cdot\rangle_{\nabla,V,*}$ with domain of definition $\Gamma_{\widetilde{W}_0^{1,2}}(\Omega, E; \nabla, V)$, a closed densely defined symmetric sesquilinear form in $\Gamma_{L^2}(\Omega, E)$. Again we will follow the conventions from (2.4) analogously.

The above auxiliary operator $\widetilde{H}_\Omega(\nabla, V)$ satisfies the following two important results, without any further assumptions on Ω:

Proposition 4.2. *For every open subset Ω of M and every $t \geq 0$ one has*
$$\lim_{n\to\infty} \exp\big(-tH_M(\nabla, V + n1_{M\setminus\Omega})\big) = \exp\big(-t\widetilde{H}_\Omega(\nabla, V)\big)P_\Omega$$
strongly as bounded operators in $\Gamma_{L^2}(M, E)$.

Proof. This is a simple application of Theorem 5.1: Regarding $n1_{M\setminus\Omega}$ as a bounded multiplication operator it follows that with
$$Q_n := \langle\cdot,\cdot\rangle_{\nabla,V+n1_{M\setminus\Omega},*} = \langle\cdot,\cdot\rangle_{\nabla,V,*} + n\int_{M\setminus\Omega} (\cdot,\cdot)d\mu, \quad \mathrm{Dom}(Q_n) := \Gamma_{W_0^{1,2}}(M, E; \nabla, V),$$
one has
$$S_{Q_n} = H_M(\nabla, V + n1_{M\setminus\Omega}).$$
It follows that $f \in \Gamma_{W_0^{1,2}}(M, E; \nabla, V)$ is in $\mathrm{Dom}(Q_\infty)$, if and only if
$$\sup_{n\in\mathbb{N}} n\int_{M\setminus\Omega} |f|^2 d\mu < \infty,$$
which again is equivalent to $f = 0$ in $M \setminus \Omega$ μ-a.e., and for such f's one has
$$Q_\infty(f, f) = \lim_{n\to\infty} Q_n(f, f) = \langle f, f\rangle_{\nabla,V,*},$$
proving $S_{Q_\infty} = \widetilde{H}_\Omega(\nabla, V)$. ■

Suppose that for every $x \in M$ we are given a strongly adapted Riemannian Brownian motion $\mathbb{X}(x)$ on M with
$$\mathbb{P}\{\mathbb{X}_0(x) = x\} = 1,$$
which is defined on some fixed filtered probability space which satisfies the usual assumptions of completeness and right-continuity. In other words, $\mathbb{X}(x)$ is an M-valued semimartingale up to its explosion time, such that the law of $\mathbb{X}(x)$ is $\mathbb{P}^x$. Given an open subset $\Omega \subset M$, let $\alpha_\Omega(x) := \alpha_\Omega(\mathbb{X}(x))$ denote the first exit time of $\mathbb{X}(x)$ from Ω, and let $\beta_\Omega(x) := \beta_\Omega(\mathbb{X}(x))$ denote the penetration time of $\mathbb{X}(x)$ to $M \setminus \Omega$. Let $//_t^{\nabla}(x)$, $t < \alpha_M(x)$, be the Stratonovic parallel transport with respect to ∇ along the paths of $\mathbb{X}(x)$, which, ∇ being metric, is a random variable taking values in the unitary maps $E_x \to E_{\mathbb{X}(t)}$. So far $x \in M$ was arbitrary. On the other hand, for μ-a.e. $x \in M$, we can define pathwise for

$t < \alpha_M(x)$ a random variable $\mathscr{A}_V^{\nabla}(x,t)$ taking values in $\mathrm{End}(E_x)$, to be the unique locally absolutely continuous solution of

$$(d/dt)\mathscr{A}_V^{\nabla}(x,t) = -\mathscr{A}_V^{\nabla}(x,t)\big(//_t^{\nabla}(x)^{-1}V(\mathbb{X}_t(x))//_t^{\nabla}(x)\big),$$
$$\mathscr{A}_V^{\nabla}(x,0) = 1_{\mathrm{End}(E_x)}.$$

The existence and uniqueness of $\mathscr{A}_V^{\nabla}(x,t)$ is guaranteed by the L^2_{loc}-assumption on V (L^1_{loc} would suffice), which implies

$$\mathbb{P}\left\{|V(\mathbb{X}(x))| \in L^1_{\mathrm{loc}}\big[0,\alpha_M(x)\big)\right\} = 1 \quad \text{for } \mu\text{-a.e. } x \in M,$$

and so

$$\mathbb{P}\left\{\big|//_t^{\nabla}(x)^{-1}V(\mathbb{X}_t(x))//_t^{\nabla}(x)\big| \in L^1_{\mathrm{loc}}\big[0,\alpha_M(x)\big)\right\} = 1 \quad \text{for } \mu\text{-a.e. } x \in M.$$

We refer the reader to [5] and the references therein for the details of these constructions which are by now standard in stochastic analysis.

Proposition 4.3. *Assume there exists a constant $C \in \mathbb{R}$ such that for all $x \in M$ all eigenvalues of $V(x) \in \mathrm{End}(E_x)$ are $\geq C$. Then for every $\Omega \subset M$ open, $t \geq 0$, $f \in \Gamma_{L^2}(\Omega, E)$, and μ-a.e. $x \in M$ one has*

$$\exp\big(-t\widetilde{H}_\Omega(\nabla,V)\big)f(x) = \int_{\{t<\min(\beta_\Omega(x),\alpha_M(x))\}} //_t^{\nabla}(x)^{-1}\mathscr{A}_V^{\nabla}(x,t)f(\mathbb{X}_t(x))d\mathbb{P}. \tag{13}$$

Note that the expectation in (13) is on the fiber E_x, as $f(\mathbb{X}_t(x)) \in E_{\mathbb{X}_t(x)}$ in $\{t < \alpha_M(x)\}$. Following well-known approximation procedures, one could remove the made lower boundedness assumption on V considerably, but we will not need this generalization in the sequel. Anyway, it is clear that *some* condition on some negative part of V is needed to guarantee

$$\int_{\{t<\alpha_M(x)\}} \big|//_t^{\nabla}(x)^{-1}\mathscr{A}_V^{\nabla}(x,t)f(\mathbb{X}_t(x))\big|d\mathbb{P} < \infty.$$

In the scalar case we get

$$\exp\big(-t\widetilde{H}_\Omega(w)\big)f(x) = \int_{\{t<\min(\beta_\Omega(x),\alpha_M(x))\}} \exp\Big(-\int_0^t w(\mathbb{X}_s)\Big)f(\mathbb{X}_t)d\mathbb{P}. \tag{14}$$

Proof of Proposition 4.3. By Proposition 4.2 and the covariant Feynman-Kac formula for

$$\exp\big(-tH_M(\nabla, V + n1_{M\setminus\Omega})\big)f,$$

we have, possibly by taking a subsequence of

$$\exp\big(-tH_M(\nabla, V + n1_{M\setminus\Omega})\big)f$$

if necessary, that for μ-a.e. $x \in M$ it holds that

$$\exp\big(-t\widetilde{H}_\Omega(\nabla,V)\big)f(x) = \lim_{n\to\infty}\exp\big(-tH_M(\nabla, V + n1_{M\setminus\Omega})\big)f(x)$$
$$= \lim_{n\to\infty}\int_{\{t<\alpha_M(x)\}} \exp\Big(-n\int_0^t 1_{M\setminus\Omega}(\mathbb{X}_s(x))ds\Big)//_t^{\nabla}(x)^{-1}\mathscr{A}_V^{\nabla}(x,t)f(\mathbb{X}_t(x))d\mathbb{P}.$$

For paths from the set

$$\{t < \alpha_M(x)\} =_{\mathbb{P}} \{\mathbb{X}_s(x) \in M \text{ for all } s \in [0,t]\}$$

one has

$$\int_0^t 1_{\hat{M}\setminus\Omega}(\mathbb{X}_s(x))ds = \int_0^t 1_{M\setminus\Omega}(\mathbb{X}_s(x))ds,$$

and $\int_0^t 1_{M\setminus\Omega}(\mathbb{X}_s(x))ds = 0$ is equivalent to $t < \beta_\Omega(x)$. Thus, $\mathbb{P}$-a.s. in $\{t < \alpha_M(x)\}$ one has

$$\lim_{n\to\infty} \exp\Big(- n\int_0^t 1_{M\setminus\Omega}(\mathbb{X}_s(x))ds\Big) = 1_{\{t<\beta_\Omega(x)\}}$$

and

$$\lim_{n\to\infty}\int_{\{t<\alpha_M(x)\}} \exp\Big(- n\int_0^t 1_{M\setminus\Omega}(\mathbb{X}_s(x))ds\Big)//_t^\nabla(x)^{-1}\mathscr{A}_V^\nabla(x,t)f(\mathbb{X}_t(x))d\mathbb{P}$$
$$= \int_{\{t<\min(\beta_\Omega(x),\alpha_M(x))\}} //_t^\nabla(x)^{-1}\mathscr{A}_V^\nabla(x,t)f(\mathbb{X}_t(x))d\mathbb{P}$$

follows from dominated convergence. ■

4.1. **Proof of Proposition 2.7.**

Proof of Proposition 2.7. With the preparations made, there is almost nothing left to prove: Note first that (5) is equivalent to

$$H_\Omega(\nabla,V) = \widetilde{H}_\Omega(\nabla,V). \tag{15}$$

(5) ⇒ (6): We have (15), so that (6) follows from Proposition 4.2.
(6) ⇒ (5): (6) and Proposition 4.2 imply

$$\exp(-tH_\Omega(\nabla,V)) = \exp(-t\tilde{H}_\Omega(\nabla,V)) \quad \text{for all } t>0,$$

showing (15).

■

4.2. **Proof of Proposition 2.10.**

Proof of Proposition 2.10. Note first that one has the Feynman-Kac formula

$$\exp(-tH_\Omega)f(x) = \int_{\{t<\alpha_\Omega\}} f(\mathbb{X}_t)d\mathbb{P}^x$$

for all $f\in L^2(\Omega)$, $t>0$, $x\in\Omega$, meaning also that the right hand side is the smooth representative of $\exp(-tH_\Omega)f$. In case Ω is relatively compact with smooth boundary this is well-known. For the general case we can assume $f\geq 0$. Fix $x\in\Omega$ and exhaust Ω with a sequence of relatively compact smooth Ω_n's with $x\in\Omega_0$. Then the latter formula is valid with Ω replaced with Ω_n, and we can take $n\to\infty$ using monotone convergence for integrals on the probabilistic side, noting that $\alpha_{\Omega_n}\nearrow\alpha_\Omega$, and

$$\exp(-tH_\Omega)f(x) \nearrow \exp(-tH_{\Omega_n})f(x)$$

by the parabolic maximum principle. Let us come to the actual proof of Proposition 2.10. Being equipped with the previously established results, we can follow the Euclidean proof from [8] from here on:
As we have already remarked, the equivalence of (9) and (10) follows immediately from Proposition 2.7.

(9) ⇒ Kac-regularity: (9) implies

$$\widetilde{H}_\Omega = H_\Omega, \tag{16}$$

by comparing the associated forms. In particular, the semigroups coincide and the Feynman-Kac formulae for the semigroups imply the first identity in

$$0 = \int (1_{\{t<\min(\beta_\Omega,\alpha_M)\}} - 1_{\{t<\alpha_\Omega\}}) f(\mathbb{X}_t) d\mathbb{P}^x = \int_{\{\alpha_\Omega \le t < \beta_\Omega\}} f(\mathbb{X}_t) d\mathbb{P}^x$$

for all $f \in L^2(\Omega)$, $t > 0$, μ-a.e. $x \in \Omega$, where the second equality follows from

$$1_{\{\alpha_\Omega \le t < \min(\beta_\Omega,\alpha_M)\}} = 1_{\{t<\min(\beta_\Omega,\alpha_M))\}} - 1_{\{t<\alpha_\Omega\}\cap\{t<\min(\beta_\Omega,\alpha_M)\}}$$

and $\alpha_\Omega \le \beta_\Omega$ and $\alpha_\Omega \le \alpha_M$. Letting f run through $f_n := 1_{A_n}$, where A_n is an exhaustion on M by compact sets, we get

$$\mathbb{P}^x\{\alpha_\Omega \le t < \min(\beta_\Omega, \alpha_M)\} = 0 \quad \text{for all } t > 0,\ \mu\text{-a.e. } x \in \Omega,$$

and letting t run through $\mathbb{Q}_{>0}$,

$$\mathbb{P}^x\{\alpha_\Omega < \min(\beta_\Omega, \alpha_M)\} = 0 \quad \text{for } \mu\text{-a.e. } x \in \Omega. \tag{17}$$

In view of $\alpha_\Omega \le \beta_\Omega$, the proof of the Kac-regularity of Ω is complete, once we have shown that the bounded function $h(x) := \mathbb{P}^x\{\alpha_\Omega < \min(\beta_\Omega, \alpha_M)\}$ is continuous in Ω. We are going to prove that

$$\exp(-tH_\Omega)h(x) = h(x) \text{ for all } t > 0,\ x \in \Omega, \tag{18}$$

which implies that $x \mapsto \mathbb{P}^x\{\alpha_\Omega < \min(\beta_\Omega, \alpha_M)\}$ is smooth (and harmonic) in Ω. To see (18), note that

$$\exp(-tH_\Omega)h(x) = \int_{\{t<\alpha_\Omega\}} \mathbb{P}^{\mathbb{X}_t}\{\alpha_\Omega < \min(\beta_\Omega, \alpha_M)\} d\mathbb{P}^x \tag{19}$$

$$= \mathbb{P}^x\{\alpha_\Omega < \min(\beta_\Omega, \alpha_M)\} = h(x), \tag{20}$$

where we have used the Feynman-Kac formula for the first equality and the Markov property of Brownian motion for the second equality.

Kac-regularity ⇒ (9): By the corresponding Feynman-Kac formulae we immediately find that Kac-regularity implies

$$\exp(-t\widetilde{H}_\Omega) = \exp(-tH_\Omega) \quad \text{for all } t > 0,$$

so that $\widetilde{H}_\Omega = H_\Omega$. ∎

4.3. **Proof of Theorem 2.13 a).** We will need the following covariant-Feynman-Kac formula:

Proposition 4.4. *Assume there exists a constant* $C \in \mathbb{R}$ *such that for all* $x \in M$ *the smallest eigenvalue of* $V(x) \in \mathrm{End}(E_x)$ *is* $\geq C$. *Then for every* $\Omega \subset M$ *open,* $t \geq 0$, $f \in \Gamma_{L^2}(\Omega, E)$, *and* μ*-a.e.* $x \in M$ *one has the covariant Feynman-Kac formula*

$$\exp\big(-tH_\Omega(\nabla, V)\big)f(x) = \int_{\{t<\alpha_\Omega(x)\}} //_t^\nabla(x)^{-1}\mathscr{A}_V^\nabla(x,t)f(\mathbb{X}_t(x))d\mathbb{P}. \tag{21}$$

At least if the boundary of Ω is not smooth, formula (21) seems to be new in a geometric context.

Proof of Proposition 4.4. It is enough to prove the formula in case Ω is relatively compact with a smooth boundary. Indeed, in the general case we can then pick a smooth relatively compact exhaustion Ω_n of Ω. Then we have $\alpha_{\Omega_n}(x) \nearrow \alpha_\Omega(x)$ and

$$\int_{\{t<\alpha_{\Omega_n}(x)\}} //_t^\nabla(x)^{-1}\mathscr{A}_V^\nabla(x,t)f(\mathbb{X}_t(x))d\mathbb{P} \to \int_{\{t<\alpha_\Omega(x)\}} //_t^\nabla(x)^{-1}\mathscr{A}_V^\nabla(x,t)f(\mathbb{X}_t(x))d\mathbb{P}. \tag{22}$$

follows from dominated convergence. Likewise,

$$\exp\big(-tH_{\Omega_n}(\nabla, V)\big)f(x) \to \exp\big(-tH_\Omega(\nabla, V)\big)f(x) \tag{23}$$

follows from Theorem 5.1.
Thus we can and we will assume Ω is relatively compact and has a smooth boundary. In case V is smooth and bounded, the asserted formula is a simple consequence of Ito's formula and parabolic regularity up to the boundary (cf. p. 102 in [3]). In case V is bounded, we can use Friedrichs mollifiers to pick a sequence of smooth potentials $V_n : M \to \mathrm{End}(E)$ with $|V_n| \leq |V|$ and $|V_n - V| \to 0$ μ-a.e. in M. Applying (21) with V replaced with V_n and taking $n \to \infty$ gives the result in this case. Finally, if V is bounded from below, we can use the spectral calculus on the fibers of $E \to M$ to define a sequence of potentials

$$V_n : M \longrightarrow \mathrm{End}(E), \quad V_n(x) := \min(n, V(x))$$

then apply (21) with V replaced with V_n and take $n \to \infty$. Each approximation argument can be justified precisely as in the proof of Theorem 2.11 in [5], which treats the $M = \Omega$ situation: In each case on uses convergence results for sesquilinear forms to control the left-hand site of (21) and convergence theorems for integrals to control the right-hand-side. ∎

Finally we can give:

Proof of Theorem 2.13 a). We know from Proposition 4.2 that

$$\lim_{n\to\infty} \exp\big(-tH_M(\nabla, V + n1_{M\setminus\Omega})\big) = \exp\big(-t\widetilde{H}_\Omega(\nabla, V)\big)P_\Omega$$

strongly (without any further assumptions on Ω), so that it remains to prove

$$\widetilde{H}_\Omega(\nabla, V) = H_\Omega(\nabla, V).$$

For every $l \in \mathbb{N}$ we define a potential, which is bounded from below by a constant, by setting (using the spectral calculus fiberwise)

$$V_l : M \longrightarrow \mathrm{End}(E),$$
$$V(x) := \max(-l, V(x)) = \max(V(x), 0) - \min\big(-\min(V(x), 0), l\big).$$

Using that Ω is Kac regular, a comparison of the covariant Feynman-Kac formulae for $\exp(-t\widetilde{H}_\Omega(\nabla, V_l))$ and $\exp(-tH_\Omega(\nabla, V_l))$ immediately gives

$$\widetilde{H}_\Omega(\nabla, V_l) = H_\Omega(\nabla, V_l) \quad \text{for all } l.$$

We are going to use Theorem 5.2 to take $l \to \infty$: Firstly, precisely as in the proof of Lemma 4 from [9] one finds that $Q_\infty = \langle \cdot, \cdot \rangle_{\nabla, V, *}$ with $\mathrm{Dom}(Q_\infty) = \Gamma_{W_0^{1,2}}(\Omega, E; \nabla, V)$, and it remains to prove $\mathrm{Dom}(Q_\infty) = \Gamma_{\widetilde{W}_0^{1,2}}(\Omega, E; \nabla, V)$. This, however, follows from $Q' = \langle \cdot, \cdot \rangle_{\nabla, V, *}$, with $\mathrm{Dom}(Q') = \Gamma_{\widetilde{W}_0^{1,2}}(\Omega, E; \nabla, V)$, which is straightforward to check. ■

4.4. **Proof of Theorem 2.13 b).** We recall that given an open subset $\Omega \subset M$, the space $W^{1,2}(\Omega; d)$ is defined to be the space of all $f \in L^2(\Omega)$ such that $df \in \Gamma_{L^2}(\Omega; T^*M)$ in the sense of distributions. We are going to need a special case of the following result for the proof of Proposition 4.4:

Proposition 4.5. *Let $\Omega \subset M$ be a relatively compact open subset with a Lipschitz boundary. Then one has*

$$W_0^{1,2}(\Omega) = \big\{ f \in W^{1,2}(M) : f|_{M \setminus \Omega} = 0 \;\; \mu\text{-a.e.} \big\},$$

in particular, Ω is Kac-regular.

Proof. Set $m := \dim M$. It remains to prove

$$W_0^{1,2}(\Omega) \supset \{ f \in W^{1,2}(M) : f|_{M \setminus \Omega} = 0 \; \mu\text{-a.e.} \}.$$

To this end, we record that for every $p \in \overline{\Omega} \setminus \Omega = \partial\Omega$ there exists a pair of open neighborhoods V, X of p, and a smooth diffeomorphism $\alpha : X \to A \subset \mathbb{R}^m$ such that:

(1) $\overline{V} \subset X$,
(2) $\overline{V}$ is compact,
(3) $\alpha(V \cap \Omega)$ is a bounded Lipschitz set of $\mathbb{R}^m$.

Let us consider a finite collection of open subsets of M, $\mathcal{W} := \{V_1, ..., V_q, W_1, ..., W_n\}$, such that

- for each $j \in \{1, ..., n\}$ we have $\overline{W_i} \subset \Omega$,
- $\overline{\Omega} \subset V_1 \cup ... \cup V_q \cup W_1 \cup ... \cup W_n$,
- for each $i \in \{1, ..., q\}$ there exists another open subset X_i such that the pair given by V_i and X_i satisfies the properties (1)–(3) required in the statement.

Clearly such a finite collection of open subsets exists. Let I be an open subset of M such that $I \cap \Omega = \emptyset$ and

$$M = V_1 \cup ... \cup V_q \cup W_1 \cup ... \cup W_n \cup I.$$

Let

$$\mathcal{W}' := \{V_1, ..., V_q, W_1, ..., W_n, I\}$$

and let

$$\{\phi_1, ..., \phi_q, \psi_1, ..., \psi_n, \tau\}$$

be a partition of unity subordinated to $\mathcal{W}$. Given now $f \in W^{1,2}(M)$ with $f|_{M\setminus\Omega} = 0$ μ-a.e., for each $i \in \{1, ..., n\}$ we have $\mathrm{supp}(\psi_i f) \subset W_i$. Therefore there exists a sequence $\{\beta_p^i\}_{p\in\mathbb{N}} \subset C_c^\infty(W_i)$ such that $\beta_p^i \to \psi_i f$ in $W_0^{1,2}(W_i)$ as $p \to \infty$. Indeed, by a Meyers-Serrin type theorem [10], we can pick a sequence $\{\overline{\beta}_p^i\}_{p\in\mathbb{N}}$ in $C^\infty(W_i)\cap W^{1,2}(W_i)$ such that $\overline{\beta}_p^i \to f$ in $W^{1,2}(W_i)$. Then, by defining $\beta_p^i := \psi_i\overline{\beta}_p^i$, it is clear that $\{\beta_p^i\}_{p\in\mathbb{N}} \subset C_c^\infty(W_i)$ and that $\beta_p^i \to \psi_i f$ in $W_0^{1,2}(W_i)$ as $p \to \infty$. Let us now consider the other case. For each $i \in \{1, ..., q\}$ let $Y_i := V_i \cap \Omega$. Then we have $\mathrm{supp}(\phi_i f) \subset (\overline{Y_i} \cap V_i)$. Let $A_i := \alpha_i(X_i)$, $E_i := \alpha_i(V_i)$ and $B_i := \alpha_i(Y_i)$. Since we assumed that $\overline{V_i} \subset X_i$ we have that $(\alpha_i^* g_e)|_{V_i}$ is quasi-isometric to $g|_{V_i}$ where g_e is the standard Euclidean metric on $\mathbb{R}^m$ and g the metric on M. In this way we can conclude that $(\phi_i f) \circ (\alpha|_{V_i})^{-1} \in W_{0,e}^{1,2}(E_i)$ and that $(\phi_i f) \circ (\alpha|_{Y_i})^{-1} \in W_e^{1,2}(B_i)$, where of course $W_e^{1,2}$ stands for the various Sobolev spaces that re defined with respect to g_e. Furthermore we know that $(\phi_i f) \circ (\alpha|_{Y_i})^{-1}|_{A_i\setminus B_i} \equiv 0$ because $(\phi_i f)|_{X_i\setminus Y_i} \equiv 0$. Therefore, by extending $(\phi_i f)\circ(\alpha|_{Y_i})^{-1}$ outside its support as the identically zero function, we can say, with a little abuse of notation, that $(\phi_i f) \circ (\alpha|_{Y_i})^{-1} \in W_e^{1,2}(\mathbb{R}^m)$ and that $(\phi_i f)\circ(\alpha|_{Y_i})^{-1}|_{\mathbb{R}^m\setminus B_i} \equiv 0$. As B_i is a bounded Lipschitz set in $\mathbb{R}^m$, it follows from Remark 2.15.2 that there exists a sequence $\{\upsilon_p^i\}_{p\in\mathbb{N}} \subset C_c^\infty(B_i)$ such that $\upsilon_p^i \to (\phi_i f) \circ (\alpha|_{Y_i})^{-1}$ in $W_{0,e}^{1,2}(B_i)$ as $p \to \infty$. Finally, using the fact that $(\alpha_i^* g_e)|_{V_i}$ is quasi-isometric to $g|_{V_i}$, we can conclude that $\gamma_p^i \to \phi_i f$ in $W_0^{1,2}(Y_i)$ as $p \to \infty$ where $\gamma_p^i := \upsilon_p^i \circ (\alpha_i|_{Y_i})$ and clearly $\{\gamma_p^i\}_{p\in\mathbb{N}} \subset C_c^\infty(Y_i)$ by construction. Let us define now the following sequence of functions $\eta_p := \gamma_p^1 + ... + \gamma_p^q + \beta_p^1 + ... + \beta_p^n$. It is clear by construction that $\{\eta_p\}_{p\in\mathbb{N}} \subset C_c^\infty(\Omega)$. Moreover we have

$$\begin{aligned}\|\eta_p - f\| &= \|\gamma_p^1 + ... + \gamma_p^q + \beta_j^1 + ... + \beta_p^n - \sum_{i=1}^q \phi_i f - \sum_{i=1}^n \psi_i f\| \\ &\leq \|\gamma_p^1 - \phi_1 f\| + ... + \|\gamma_p^q - \phi_q f\| + \|\beta_p^1 - \psi_1 f\| + ... + \|\beta_p^n - \psi_n f\|\end{aligned}$$

and by construction all the terms in the second line tend to zero as $p \to \infty$. Hence we have shown that $\eta_j \to f$ in $\Gamma_{L^2}(\Omega, T^*M)$ as $p \to \infty$. Similarly we have

$$\begin{aligned}&\|d\eta_p - df\| = \|d\gamma_p^1 + ... + d\gamma_p^q + d\beta_p^1 + ... + d\beta_p^n - d(\sum_{i=1}^q \phi_i f - \sum_{i=1}^n \psi_i f)\| \\ &\leq \|d\gamma_p^1 - d(\phi_1 f)\| + ... + \|d\gamma_p^q - d(\phi_q f)\| + \|d\beta_p^1 - d(\psi_1 f)\| + ... \\ &+ \|d\beta_p^n - d(\psi_n f)\|\end{aligned}$$

and again by construction we know that all the terms on the right hand side of the inequality tend to zero as $p \to \infty$. This tells us that $d\eta_p \to df$ in $\Gamma_{L^2}(\Omega, T^*M)$. In conclusion we have shown that $\eta_p \to f$ in $W^{1,2}(\Omega)$ and thereby we can conclude that $f \in W_0^{1,2}(\Omega)$ as desired.

■

Proof of Theorem 2.13 b). Fix $x \in \Omega$. Pick a sequence $\Omega_n \subset \Omega$ of relatively compact open sets with Lipschitz boundary, such that $x \in \Omega_1$, $\Omega_n \subset \Omega_{n+1}$, $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$, $\bigcup_{n\in\mathbb{N}} \overline{\Omega_n} = \overline{\Omega}$. Then by Proposition 2.13 and Proposition 4.5 we have

$$\mathbb{P}^x\{\alpha_{\Omega_n} = \min(\beta_{\Omega_n}, \alpha_M)\} = 1 \quad \text{for all } n,$$

so that using $\alpha_{\Omega_n} \to \alpha_\Omega$ and $\beta_{\Omega_n} \to \beta_\Omega$ $\mathbb{P}^x$-a.s. as $n \to \infty$, a consequence of Lemma 2.8, we arrive at

$$\mathbb{P}^x\{\alpha_\Omega \neq \min(\beta_\Omega, \alpha_M\} \subset \mathbb{P}^x \bigcup_{n\in\mathbb{N}} \{\alpha_{\Omega_n} \neq \min(\beta_{\Omega_n}, \alpha_M)\} \leq \sum_{n\in\mathbb{N}} \mathbb{P}^x\{\alpha_{\Omega_n} \neq \min(\beta_{\Omega_n}, \alpha_M)\},$$

which is $= 0$, completing the proof.

■

5. Appendix: Some functional analytic facts

In this section we collect some facts about the monotone convergence of nonnegative closed sesquilinear forms which are possibly not densely defined. Let $\mathscr{H}$ be complex seperable Hilbert space and assume that Q is such a form on $\mathscr{H}$. Then Q is a densely defined nonnegative closed sesquilinear form on the Hilbert subspace $\mathscr{H}_Q := \overline{\mathrm{Dom}(Q)}^{\|\cdot\|_{\mathscr{H}}} \subset \mathscr{H}$ and thus there exists a unique nonnegative self-adjoint operator S_Q on $\mathscr{H}_Q$ such that

$$\mathrm{Dom}(\sqrt{S_Q}) = \mathrm{Dom}(Q), \quad Q(f_1, f_2) = \left\langle \sqrt{S_Q} f_1, \sqrt{S_Q} f_2 \right\rangle_{\mathscr{H}}.$$

Let then $P_Q : \mathscr{H} \to \mathscr{H}_Q$ denote the orthogonal projection. We recall also that given a nonnegative sesquilinear form Q on $\mathscr{H}$ there exists a largest (w.r.t. '$\leq$') closable nonnegative sesquilinear form $\mathrm{reg}(Q)$ on $\mathscr{H}$, the *regular part of* Q, such that $\mathrm{reg}(Q) \leq Q$. Recall that for forms Q_1 and Q_2 one per definitionem has $Q_1 \leq Q_2$ if and only if $\mathrm{Dom}(Q_1) \supset \mathrm{Dom}(Q_2)$ and $Q_1(f, f) \leq Q_2(f, f)$ for all $f \in \mathrm{Dom}(Q_2)$.

Theorem 5.1. *Let $Q_1 \leq Q_2 \leq \dots$ be a sequence of closed nonnegative sesquilinear forms on $\mathscr{H}$. Then*

$$Q_\infty(f_1, f_2) := \lim_{n\to\infty} Q_n(f_1, f_2)$$

with

$$\mathrm{Dom}(Q_\infty) := \Big\{ f \in \bigcap_{n\in\mathbb{N}} \mathrm{Dom}(Q_n) : \sup_{n\in\mathbb{N}} Q_n(f, f) < \infty \Big\}$$

is a closed nonnegative sesquilinear form Q_∞ in $\mathscr{H}$, and one has

$$\mathrm{e}^{-tS_{Q_n}} P_{Q_n} \to \mathrm{e}^{-tS_{Q_\infty}} P_{Q_\infty} \quad \textit{strongly in } \mathscr{L}(\mathscr{H}) \textit{ as } n \to \infty, \textit{ for all } t \geq 0.$$

Theorem 5.2. *Let $Q_1 \geq Q_2 \geq \dots$ be a sequence of closed nonnegative sesquilinear forms on $\mathscr{H}$. Define a nonnegative sesquilinear form in $\mathscr{H}$ given by*

$$Q'(f_1, f_2) := \lim_{n\to\infty} Q_n(f_1, f_2), \quad \mathrm{Dom}(Q') := \bigcup_{n\in\mathbb{N}} \mathrm{Dom}(Q_n)$$

and let Q_∞ *denote the form on* $\mathscr{H}$ *given by the closure of* $\mathrm{reg}(Q')$ *in* $\mathscr{H}$*. Then one has*

$$\mathrm{e}^{-tS_{Q_n}} P_{Q_n} \to \mathrm{e}^{-tS_{Q_\infty}} P_{Q_\infty} \text{ strongly in } \mathscr{L}(\mathscr{H}) \text{ as } n \to \infty, \text{ for all } t \geq 0 \ .$$

Note that $Q_\infty = Q'$ if Q' is closed.

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