Generalized Schrödinger semigroups on infinite weighted graphs

Batu Güneysu

Institut für Mathematik
Humboldt-Universität zu Berlin

QMATH 12

Berlin, September 11, 2013

B.G, Ognjen Milatovic, Francoise Truc: *Generalized Schrödinger semigroups on infinite graphs*. Preprint (2013).

Partially:

B.G., Matthias Keller, Marcel Schmidt: *A Feynman-Kac-Ito formula for magnetic Schrödinger operators on graphs.* Preprint (2013).

Motivation

- Classical fact: If H is the self-adjoint operator corresponding to a regular Dirichlet form on some locally compact space X with a Borel measure m (for example the canonical Dirichlet form on a weighted graph), then for appropriate $v : X \to \mathbb{R}$ one has the Feynman-Kac formula

$$\mathrm{e}^{-t(H+v)}f(x) = \mathbb{E}^x\left[\mathrm{e}^{-\int_0^t v(\mathbb{X}_s)\mathrm{d}s}f(\mathbb{X}_t)\right], \quad f \in L^2(X,m). \quad (1)$$

- Generalizations of these probabilistic formulae also hold for covariant Schrödinger type operators acting on sections in Hermitian vector bundles over noncompact Riemannian manifolds
- On the other hand, natural extensions of the latter covariant operators ($\widehat{=}$ LHS of (1)) exist also on weighted graphs. **What about the RHS and the validity of (1) in the covariant weighted discrete graph setting?**

Motivation

- Classical fact: If H is the self-adjoint operator corresponding to a regular Dirichlet form on some locally compact space X with a Borel measure m (for example the canonical Dirichlet form on a weighted graph), then for appropriate $v : X \to \mathbb{R}$ one has the Feynman-Kac formula

$$\mathrm{e}^{-t(H+v)}f(x) = \mathbb{E}^x\left[\mathrm{e}^{-\int_0^t v(\mathbb{X}_s)\mathrm{d}s} f(\mathbb{X}_t)\right], \quad f \in L^2(X,m). \quad (1)$$

- Generalizations of these probabilistic formulae also hold for covariant Schrödinger type operators acting on sections in Hermitian vector bundles over noncompact Riemannian manifolds
- On the other hand, natural extensions of the latter covariant operators ($\widehat{=}$ LHS of (1)) exist also on weighted graphs. **What about the RHS and the validity of (1) in the covariant weighted discrete graph setting?**

Motivation

- Classical fact: If H is the self-adjoint operator corresponding to a regular Dirichlet form on some locally compact space X with a Borel measure m (for example the canonical Dirichlet form on a weighted graph), then for appropriate $v : X \to \mathbb{R}$ one has the Feynman-Kac formula

$$\mathrm{e}^{-t(H+v)}f(x) = \mathbb{E}^x\left[\mathrm{e}^{-\int_0^t v(\mathbb{X}_s)\mathrm{d}s} f(\mathbb{X}_t)\right], \quad f \in L^2(X,m). \quad (1)$$

- Generalizations of these probabilistic formulae also hold for covariant Schrödinger type operators acting on sections in Hermitian vector bundles over noncompact Riemannian manifolds
- On the other hand, natural extensions of the latter covariant operators ($\widehat{=}$ LHS of (1)) exist also on weighted graphs. **What about the RHS and the validity of (1) in the covariant weighted discrete graph setting?**

Setting I

i) Let (X, b, m) be a possibly locally unbounded weighted graph, that is,

- X is a countable set (the *vertices*)
- $b : X \times X \to [0, \infty)$ is symmetric with $b(x, x) = 0$ and $\sum_{y \in X} b(x, y) < \infty$ (the set of $x, y \in X$ with $x \sim_b y :\Leftrightarrow b(x, y) > 0$ are the *neighbors*).
- $m : X \to (0, \infty)$ some function (the *weight function*)

ii) Let $F \to X$ be a rank-ν *vector bundle*, that is, $F = \bigsqcup_{x \in X} F_x$ with each F_x a complex linear space with $F_x \cong \mathbb{C}^\nu$

iii) Let

$$(\bullet, \bullet)_x : F_x \times F_x \longrightarrow \mathbb{C},\ x \in X,$$

be a *Hermitian structure* on $F \to X$, that is, each $(\bullet, \bullet)_x$ is a Hermitian product

Setting I

i) Let (X, b, m) be a possibly locally unbounded weighted graph, that is,

- X is a countable set (the *vertices*)
- $b : X \times X \to [0, \infty)$ is symmetric with $b(x, x) = 0$ and $\sum_{y \in X} b(x, y) < \infty$ (the set of $x, y \in X$ with $x \sim_b y :\Leftrightarrow b(x, y) > 0$ are the *neighbors*).
- $m : X \to (0, \infty)$ some function (the *weight function*)

ii) Let $F \to X$ be a rank-ν *vector bundle*, that is, $F = \bigsqcup_{x \in X} F_x$ with each F_x a complex linear space with $F_x \cong \mathbb{C}^\nu$

iii) Let

$$(\bullet, \bullet)_x : F_x \times F_x \longrightarrow \mathbb{C},\ x \in X,$$

be a *Hermitian structure* on $F \to X$, that is, each $(\bullet, \bullet)_x$ is a Hermitian product

Setting I

i) Let (X, b, m) be a possibly locally unbounded weighted graph, that is,

- X is a countable set (the *vertices*)
- $b : X \times X \to [0, \infty)$ is symmetric with $b(x, x) = 0$ and $\sum_{y \in X} b(x, y) < \infty$ (the set of $x, y \in X$ with $x \sim_b y :\Leftrightarrow b(x, y) > 0$ are the *neighbors*).
- $m : X \to (0, \infty)$ some function (the *weight function*)

ii) Let $F \to X$ be a rank-ν *vector bundle*, that is, $F = \bigsqcup_{x \in X} F_x$ with each F_x a complex linear space with $F_x \cong \mathbb{C}^\nu$

iii) Let

$$(\bullet, \bullet)_x : F_x \times F_x \longrightarrow \mathbb{C}, \; x \in X,$$

be a *Hermitian structure* on $F \to X$, that is, each $(\bullet, \bullet)_x$ is a Hermitian product

Setting II

$|\bullet|_x$ denotes the norm and operator norm on F_x given by $(\bullet,\bullet)_x$
$\rightsquigarrow$ we get the corresponding spaces $\Gamma_{\ell^p_m}(X,F)$ of ℓ^p_m-sections (depend on $(\bullet,\bullet)$):

$$\Gamma_{\ell^p_m}(X,F) = \big\{ f : X \to F \mid f(x) \in F_x \text{ for all } x, |f| \in \ell^p(X,m) \big\}$$

$\rightsquigarrow$ $\Gamma_{\ell^2_m}(X,F)$ is our Hilbert space

iv) Let Φ be a *unitary b-connection* on $F \to X$, that is, Φ assigns to any $x \sim_b y$ some unitary map $\Phi_{x,y} : F_x \to F_y$ such that $\Phi_{y,x} = \Phi^{-1}_{x,y}$ (depends on $(\bullet,\bullet)$)

Setting II

$|\bullet|_x$ denotes the norm and operator norm on F_x given by $(\bullet,\bullet)_x$
$\rightsquigarrow$ we get the corresponding spaces $\Gamma_{\ell^p_m}(X,F)$ of ℓ^p_m-sections (depend on $(\bullet,\bullet)$):

$$\Gamma_{\ell^p_m}(X,F) = \big\{ f : X \to F \mid f(x) \in F_x \text{ for all } x, |f| \in \ell^p(X,m) \big\}$$

$\rightsquigarrow$ $\Gamma_{\ell^2_m}(X,F)$ is our Hilbert space

iv) Let Φ be a *unitary b-connection* on $F \to X$, that is, Φ assigns to any $x \sim_b y$ some unitary map $\Phi_{x,y} : F_x \to F_y$ such that $\Phi_{y,x} = \Phi^{-1}_{x,y}$ (depends on $(\bullet,\bullet)$)

Operators

- Define a sesqui-linear form $Q'_{\Phi,0}$ in $\Gamma_{\ell^2_m}(X,F)$ with $D(Q'_{\Phi,0}) :=$ compactly supported sections in $F \to X$, and

$$Q'_{\Phi,0}(f_1,f_2) := \frac{1}{2}\sum_{x\sim_b y} b(x,y)\Big(f_1(x)-\Phi_{y,x}f_1(y), f_2(x)-\Phi_{y,x}f_2(y)\Big)_x$$

- $Q'_{\Phi,0}$ is densely defined, symmetric, closable, and nonnegative: $Q_{\Phi,0} := \overline{Q'_{\Phi,0}}$
- Let V be a *potential* on $F \to X$, that is, V is a pointwise self-adjoint section in $\mathrm{End}(F) \to X$ (depends on $(\bullet,\bullet)$) $\rightsquigarrow V$ determines a maximally defined quadratic form Q_V in $\Gamma_{\ell^2_m}(X,F)$
- If $V = V^+ - V^-$ for some potentials $V^\pm \geq 0$ such that Q_{V^-} is $Q_{\Phi,0}$-bounded with bound < 1, then $Q_{\Phi,V} := Q_{\Phi,0} + Q_V$ is densely defined, symmetric, closed and bounded from below $\rightsquigarrow H_{\Phi,V} :=$ the corresponding (Schrödinger type!) operator

Question

Precise version of our initial question: **Is there a probabilistic representation of the Schrödinger SG**

$$(\mathrm{e}^{-tH_{\Phi,V}})_{t\geq 0} \subset \mathscr{L}(\Gamma_{\ell^2_m}(X,F)) \ ?$$

Underlying processes I

i) Let H be the operator corresponding to the canonical regular Dirichlet form Q in $\ell^2(X,m)$ given by (X,b,m), and let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},(\mathbb{X}_t)_{t\geq 0},(\mathbb{P}^x)_{x\in X})$ be the corresponding right Markoff process $\rightsquigarrow$ it is in fact a jump process $\rightsquigarrow$ let $\tau_n:\Omega\to[0,\infty]$ be its jump times, $N(t):\Omega\to\mathbb{N}_0\cup\{\infty\}$ its number of jumps until $t\geq 0$, and $\tau=\sup_n\tau$ its explosion time

ii) The (pathwise unitary!!!) process

$$//^{\Phi}:[0,\tau)\times\Omega\longrightarrow F\boxtimes F^*:=\bigcup_{(x,y)\in X\times X}\mathrm{Hom}(F_y,F_x)$$

$$//^{\Phi}_t:=\begin{cases}\mathbf{1}_{F_{\mathbb{X}_0}}, & \text{if } N(t)=0\\ \overleftarrow{\prod}_{1\leq j\leq N(t)}\Phi_{\mathbb{X}_{\tau_{j-1}},\mathbb{X}_{\tau_j}} & \text{else}\end{cases}\in\mathrm{Hom}(F_{\mathbb{X}_0},F_{\mathbb{X}_t})$$

is called the Φ-*parallel transport* along the paths of $\mathbb{X}$.

Underlying processes I

i) Let H be the operator corresponding to the canonical regular Dirichlet form Q in $\ell^2(X,m)$ given by (X,b,m), and let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},(\mathbb{X}_t)_{t\geq 0},(\mathbb{P}^x)_{x\in X})$ be the corresponding right Markoff process $\rightsquigarrow$ it is in fact a jump process $\rightsquigarrow$ let $\tau_n:\Omega\to[0,\infty]$ be its jump times, $N(t):\Omega\to\mathbb{N}_0\cup\{\infty\}$ its number of jumps until $t\geq 0$, and $\tau=\sup_n\tau$ its explosion time

ii) The (pathwise unitary!!!) process

$$//^{\Phi}:[0,\tau)\times\Omega\longrightarrow F\boxtimes F^*:=\bigcup_{(x,y)\in X\times X}\mathrm{Hom}(F_y,F_x)$$

$$//_t^{\Phi}:=\begin{cases}\mathbf{1}_{F_{\mathbb{X}_0}}, & \text{if } N(t)=0\\ \overleftarrow{\prod}_{1\leq j\leq N(t)}\Phi_{\mathbb{X}_{\tau_{j-1}},\mathbb{X}_{\tau_j}} & \text{else}\end{cases}\in\mathrm{Hom}(F_{\mathbb{X}_0},F_{\mathbb{X}_t})$$

is called the Φ-*parallel transport* along the paths of $\mathbb{X}$.

Underlying processes II

iii) For any $\omega \in \Omega$ there is a unique solution

$$\mathscr{V}^{\Phi}_{\bullet}(\omega) : [0, \tau(\omega)) \longrightarrow \mathrm{End}(F_{\mathbb{X}_0(\omega)})$$

of the initial value problem

$$\mathscr{V}^{\Phi}_{t}(\omega) = \mathbf{1}_{F_{\mathbb{X}_0(\omega)}} - \int_0^t \mathscr{V}^{\Phi}_{s}(\omega) \; //^{\Phi,-1}_{s} V(\mathbb{X}_s) //^{\Phi}_{s} \Big|_{\omega} \mathrm{d}s \quad \text{in} \quad \mathrm{End}(F_{\mathbb{X}_0(\omega)}).$$

$\rightsquigarrow$ we have canonically associated the process

$$\mathscr{V}^{\Phi} : [0, \tau) \times \Omega \longrightarrow \mathrm{End}(F)$$

with Φ and V.

Feynman-Kac formula

Note that for any $x \in X$, $t \geq 0$, one has

$$//_t^{\Phi}(\omega) \in \mathrm{Hom}(F_x, F_{\mathbb{X}_t(\omega)}),\ \mathscr{V}_t^{\Phi}(\omega) \in \mathrm{End}(F_x) \ \text{ for } \mathbb{P}^x\text{-a.e. } \omega \in \{t < \tau\}.$$

Main result:

Theorem (B.G., O. Milatovic, F. Truc)

Assume that V admits a decomposition $V = V^+ - V^-$ into potentials $V^{\pm} \geq 0$ such that $Q_{|V^-|}$ is Q-bounded with bound < 1. Then Q_{V^-} is $Q_{\Phi,0}$-bounded with bound < 1, and for any $f \in \Gamma_{\ell^2_m}(X, F)$, $t \geq 0$, $x \in X$ one has

$$\mathrm{e}^{-tH_{\Phi,V}} f(x) = \mathbb{E}^x \left[1_{\{t<\tau\}} \mathscr{V}_t^{\Phi} //_t^{\Phi,-1} f(\mathbb{X}_t) \right]. \qquad (2)$$

Generalizes main result of B.G/M. Keller/M. Schmidt away from scalar magnetic Schrödinger SG's on (X, b, m) (the latter however deals with larger class of V's)

Feynman-Kac formula

Note that for any $x \in X$, $t \geq 0$, one has

$$//_t^{\Phi}(\omega) \in \mathrm{Hom}(F_x, F_{\mathbb{X}_t(\omega)}),\ \mathscr{V}_t^{\Phi}(\omega) \in \mathrm{End}(F_x) \text{ for } \mathbb{P}^x\text{-a.e. } \omega \in \{t < \tau\}.$$

Main result:

Theorem (B.G., O. Milatovic, F. Truc)

Assume that V admits a decomposition $V = V^+ - V^-$ into potentials $V^{\pm} \geq 0$ such that $Q_{|V^-|}$ is Q-bounded with bound < 1. Then Q_{V^-} is $Q_{\Phi,0}$-bounded with bound < 1, and for any $f \in \Gamma_{\ell^2_m}(X, F)$, $t \geq 0$, $x \in X$ one has

$$\mathrm{e}^{-tH_{\Phi,V}} f(x) = \mathbb{E}^x\left[1_{\{t<\tau\}} \mathscr{V}_t^{\Phi} //_t^{\Phi,-1} f(\mathbb{X}_t)\right]. \qquad (2)$$

Generalizes main result of B.G/M. Keller/M. Schmidt away from scalar magnetic Schrödinger SG's on (X, b, m) (the latter however deals with larger class of V's)

Some applications of Feynman-Kac formula I

- For any $(x, y) \in X \times X$, $t > 0$, the *integral kernel*

$$\mathrm{e}^{-tH_{\Phi,V}}(x,y) \in \mathrm{End}(E_y, E_x)$$

of $\mathrm{e}^{-tH_{\Phi,V}}$ is given by

$$\mathrm{e}^{-tH_{\Phi,V}}(x,y) = \frac{1}{m(y)}\mathbb{E}^x\left[1_{\{\mathbb{X}_t = y\}}\mathscr{V}_t^{\Phi}//_t^{\Phi,-1}\right];$$

$\rightsquigarrow$ formula for $\mathrm{tr}(\mathrm{e}^{-tH_{\Phi,V}}) \in [0, \infty]$

- *Semigroup domination:* If $w : X \to \mathbb{R}$ is such that $V \geq w$ (and s.t. $H \dot{+} w$ is well-defined), then $|\mathrm{e}^{-tH_{\Phi,V}}f(x)|_x \leq \mathrm{e}^{-t(H \dot{+} w)}|f|(x)$, in particular $\min(\sigma(H_{\Phi,V})) \geq \min(\sigma(H \dot{+} w))$

Some applications of Feynman-Kac formula I

- For any $(x,y) \in X \times X$, $t > 0$, the *integral kernel*

$$\mathrm{e}^{-tH_{\Phi,V}}(x,y) \in \mathrm{End}(E_y, E_x)$$

  of $\mathrm{e}^{-tH_{\Phi,V}}$ is given by

$$\mathrm{e}^{-tH_{\Phi,V}}(x,y) = \frac{1}{m(y)} \mathbb{E}^x \left[1_{\{\mathbb{X}_t = y\}} \mathscr{V}_t^{\Phi} //_t^{\Phi,-1} \right];$$

  $\rightsquigarrow$ formula for $\mathrm{tr}(\mathrm{e}^{-tH_{\Phi,V}}) \in [0,\infty]$

- *Semigroup domination:* If $w : X \to \mathbb{R}$ is such that $V \geq w$ (and s.t. $H \dot{+} w$ is well-defined), then $|\mathrm{e}^{-tH_{\Phi,V}} f(x)|_x \leq \mathrm{e}^{-t(H \dot{+} w)}|f|(x)$, in particular $\min(\sigma(H_{\Phi,V})) \geq \min(\sigma(H \dot{+} w))$

- *Golden-Thompson inequalty*: If $w : X \to \mathbb{R}$ is such that $V \geq w$ (and s.t. $H \dot{+} w$ is well-defined), then

$$\sum_x \mathrm{e}^{-tH}(x,x)\mathrm{e}^{-tw(x)}m(x) < \infty$$

$$\Rightarrow \mathrm{tr}(\mathrm{e}^{-tH_{\Phi,V}}) \leq \sum_x \mathrm{e}^{-tH}(x,x)\mathrm{tr}_{F_x}(\mathrm{e}^{-tV(x)})m(x)$$

The latter proof is rather technical: A naive approach only gives the (nevertheless illustrative) bound

$$\mathrm{tr}(\mathrm{e}^{-tH_{\Phi,V}}) \leq \mathrm{rank}(F)\sum_x \mathrm{e}^{-tH}(x,x)\mathrm{e}^{-tw(x)}m(x).$$

- $\Gamma_{\ell^2_m}(X,F) \to \Gamma_{\ell^p_m}(X,F)$, $p \in [2,\infty]$, *smoothing properties* of $\mathrm{e}^{-tH_{\Phi,V}}$, if V^- is Kato and $\sup_{x,y} \mathrm{e}^{-tH}(x,y) < \infty$

- *Golden-Thompson inequalty*: If $w : X \to \mathbb{R}$ is such that $V \geq w$ (and s.t. $H \dot{+} w$ is well-defined), then

$$\sum_x \mathrm{e}^{-tH}(x,x)\mathrm{e}^{-tw(x)} m(x) < \infty$$
$$\Rightarrow \mathrm{tr}(\mathrm{e}^{-tH_{\Phi,V}}) \leq \sum_x \mathrm{e}^{-tH}(x,x)\mathrm{tr}_{F_x}(\mathrm{e}^{-tV(x)})m(x)$$

  The latter proof is rather technical: A naive approach only gives the (nevertheless illustrative) bound

$$\mathrm{tr}(\mathrm{e}^{-tH_{\Phi,V}}) \leq \mathrm{rank}(F) \sum_x \mathrm{e}^{-tH}(x,x)\mathrm{e}^{-tw(x)} m(x).$$

- $\Gamma_{\ell^2_m}(X,F) \to \Gamma_{\ell^p_m}(X,F)$, $p \in [2,\infty]$, *smoothing properties* of $\mathrm{e}^{-tH_{\Phi,V}}$, if V^- is Kato and $\sup_{x,y} \mathrm{e}^{-tH}(x,y) < \infty$

Thank you!