L^p-INTERPOLATION INEQUALITIES AND GLOBAL SOBOLEV REGULARITY RESULTS (WITH AN APPENDIX BY OGNJEN MILATOVIC)

BATU GÜNEYSU AND STEFANO PIGOLA

ABSTRACT. On any complete Riemannian manifold M and for all $p \in [2,\infty)$, we prove a family of second order L^p-interpolation inequalities that arise from the following simple L^p-estimate valid for every $u \in C^\infty(M)$:

$$\|\nabla u\|_p^p \leq \|u\Delta_p u\|_1 \in [0,\infty],$$

where Δ_p denotes the p-Laplace operator. We show that these inequalities, in combination with abstract functional analytic arguments, allow to establish new global Sobolev regularity results for L^p-solutions of the Poisson equation for all $p \in (1,\infty)$, and new global Sobolev regularity results for the singular magnetic Schrödinger semigroups.

1. Some definitions from analysis on Riemannian manifolds

In the sequel, all manifolds are understood to be without boundary and spaces of functions are understood over $\mathbb{R}$. Let $M = (M,g)$ be a smooth connected Riemann m-manifold. We denote with $d(x,y)$ the geodesic distance of $x,y \in M$, and for all $r > 0$ with $B(x,r)$ the induced open ball with radius r around x. We understand all our function spaces like $C^\infty(M)$ to be real-valued, while complexifications will be denoted with an index '$\mathbb{C}$', like $C^\infty_{\mathbb{C}}(M)$ etc.. For $p \in [1,\infty]$ the Banach space $L^p(M)$ is defined with respect to the Riemannian volume measure μ, with $\|\cdot\|_p$ its norm.
Given a smooth $\mathbb{R}$-metric vector bundle $E \to M$, whenever there is no danger of confusion the underlying fiberwise scalar product will be simply denoted with $(\cdot,\cdot)$, with $|\cdot| := (\cdot,\cdot)^{1/2}$ the induced fiberwise norm. Then one sets

$$\|\Psi\|_p := \||\Psi|\|_p \quad \text{for every Borel section } \Psi \text{ in } E \longrightarrow M,$$

leading to the Banach spaces $\Gamma_{L^p}(M,E)$ and the locally convex spaces $\Gamma_{L^p_{\mathrm{loc}}}(M,E)$ in the usual way. Given another smooth metric $\mathbb{R}$-vector bundle $F \to M$ and a smooth linear partial differential operator P from $E \to M$ to $F \to M$ of order $\leq k$, its adjoint is the uniquely determined smooth linear partial differential operator $P^\dagger$ of order $\leq k$ from $F \to M$ to $E \to M$ which satisfies

$$\int (P^\dagger \psi, \phi) d\mu = \int (\psi, P\phi) d\mu$$

for all $\psi \in \Gamma_{C^\infty}(M,F)$, $\phi \in \Gamma_{C^\infty}(M,E)$, with either ψ or ϕ compactly supported. Given $f \in \Gamma_{L^1_{\mathrm{loc}}}(M,E)$, this allows to define the validity of $Pf \in \Gamma_{L^p_{\mathrm{loc}}}(M,E)$ or $Pf \in \Gamma_{L^p}(M,E)$ in the usual way.
As a particular case of the above constructions, we remark that bundles of the form

$$T^{r,s}M := (T^*M)^{\otimes^r} \otimes (TM)^{\otimes^s} \longrightarrow M$$

canonically become smooth metric $\mathbb{R}$-vector bundles, in view of the Riemannian structure on M. With

$$d : C^\infty(M) \longrightarrow \Gamma_{C^\infty}(M, T^*M)$$

we denote the total derivative, the gradient can be defined by

$$\nabla : C^\infty(M) \to \Gamma_{C^\infty}(M, T^*M), \quad (\nabla u, X) := du(X),$$

where X is an arbitrary vector field on M. The formal adjoint

$$\nabla^\dagger : \Gamma_{C^\infty}(M, T^*M) \longrightarrow C^\infty(M)$$

of ∇ is (-1) times the divergence operator (cf. Theorem 3.14 in [Gri]), and with the usual abuse of notation, the Hessian can be defined by

$$\nabla^2 : C^\infty(M) \longrightarrow \Gamma_{C^\infty}(M, T^{0,2}M), \quad \nabla^2 u(X,Y) := (\nabla^{TM}_X \nabla u, Y),$$

where X, Y are arbitrary vector fields on M, and ∇^{TM} the Levi-Civita connection on M.

We further recall that for $p \in [2, \infty)$, the p-Laplacian is the nonlinear differential operator defined by

$$\Delta_p : C^\infty(M) \longrightarrow C^0(M), \quad \Delta_p u := \nabla^\dagger \left(|\nabla u|^{p-2} \nabla u\right).$$

In particular, one finds that $\Delta_2 = \Delta := \nabla^\dagger \nabla$ is the usual scalar Laplace-Beltrami operator.

Following [Gue, GP], we will call $(\psi_k)_{k\in\mathbb{N}} \subset C^\infty_c(M)$

- *a sequence of first order cut-off functions*, if $0 \le \psi_k \le 1$ pointwise for all k, $\psi_k \nearrow 1$ pointwise, and $\|\nabla \psi_k\|_\infty \to 0$ as $k \to \infty$,
- *a sequence of Hessian cut-off functions*, if it is a sequence of first order cut-off functions such that in addition $\|\nabla^2 \psi_k\|_\infty \to 0$ as $k \to \infty$,
- *a sequence of Laplacian cut-off functions*, if it is a sequence of first order cut-off functions such that in addition $\|\Delta \psi_k\|_\infty \to 0$ as $k \to \infty$.

Note that in view of $|\Delta \psi_k| \le \sqrt{m}|\nabla^2 \psi_k|$, every sequence of Hessian cut-off functions is also a sequence of Laplacian cut-off functions. Moreover, M admits a sequence of first order cut-off functions, if and only if M is geodesically complete, [PS]. The state of the art concerning the existence of Laplacian cut-off functions is contained in [BS]: there the authors have shown that Laplacian cut-off functions exist on M, if M is geodesically complete and there exists a point $o \in M$, and constants $\kappa \in [0, \infty)$, $\widetilde{\kappa} \in [-2, \infty)$, such that

$$\mathrm{Ric} \ge -\kappa(1 + d(\cdot, o)^2)^{-\widetilde{\kappa}/2}. \tag{1}$$

Furthermore, if M is geodesically complete, then M admits a sequence of Hessian cut-off functions, for example if M has absolutely bounded sectional curvatures [GP], or if M has a bounded Ricci curvature and a positive injectivity radius [RV].

Next, we recall that M is said to *satisfy the L^p-Calderón-Zygmund inequality $CZ(p)$* (where $p \in (1, \infty)$), if there exist constants $C_1 \in (0, \infty)$, $C_2 \in [0, \infty)$ such that

$$\left\|\nabla^2 u\right\|_p \le C_1 \left\|\Delta u\right\|_p + C_2 \left\|u\right\|_p \quad \text{for all } u \in C^\infty_c(M).$$

A simple consequence of Bochner's inequality (cf. Appendix C, equation (26)) is that $CZ(2)$ is satisfied if M has Ricci curvature bounded from below by a constant. Moreover, there exist geodesically complete smooth Riemann manifolds which do not satisfy $CZ(2)$

[GP]. The validity of $CZ(p)$ with $p \neq 2$ is a highly delicate business, which has also been addressed in [GP]. For example, M satisfies $CZ(p)$ for every $p \in (1, \infty)$, if M has a positive injectivity radius and a bounded Ricci curvature. For $p \in (1, 2]$, using covariant Riesz-transform techniques it is shown in [GP] that M satisfies $CZ(p)$ under geodesic completeness, a C^1-boundedness of the curvature, and a rather subtle volume doubling condition (but no assumption on the injectivity radius!).

2. Main results

A classical regularity result by Strichartz [St, Corollary 3.5] states that if M is geodesically complete and if $u, f \in L^2(M)$ and if u is a solution of the Poisson equation $\Delta u = f$, then one has $\nabla u \in \Gamma_{L^2}(M, TM)$. The question we will be concerned in this paper is:

Are there natural extensions of Strichartz' result at an L^p-scale?

To begin with, we remark that Strichartz' proof for $p = 2$ uses Hilbert space arguments, in that it relies on the essential self-adjointness of Δ. In particular, it is clear that the examination of the latter question will require new ideas for $p \neq 2$. In our study of this problem for $p > 2$, we found the following very natural result, our first main result:

Theorem 1. *Let M be geodesically complete, let $p \in [2, \infty)$ and let $u \in L^p(M) \cap C^\infty(M)$. Then one has*

$$\|\nabla u\|_p^p \leq \|u \Delta_p u\|_1 \in [0, \infty], \tag{2}$$

and, for all

$$a_1, a_2, a_3, b_1, b_2, b_3 \in [1, \infty] \quad \text{with} \quad 1/a_1 + 1/a_2 + 1/a_3 = 1 = 1/b_1 + 1/b_2 + 1/b_3,$$

one has

$$\|\nabla u\|_p^p \leq \|u\|_{a_1} \|\nabla u\|_{(p-2)a_3}^{p-2} \|\Delta u\|_{a_2} + (p-2) \|u\|_{b_1} \|\nabla u\|_{(p-2)b_3}^{p-2} \left\|\nabla^2 u\right\|_{b_2} \in [0, \infty]. \tag{3}$$

The proof of (2) is based on an integration by parts machinery that relies on the existence of a sequence of first order cut-off functions. In particular, our proof is completely different from Strichartz' proof for $p = 2$. Then, as we will show, (3) follows straightforwardly from (2) in view of an explicit calculation for the p-Laplacian and Hölder's inquality. Inequality (2) itself can be considered as a generalization to $p > 2$ of Strichartz' result: indeed, (2) and Hölder's inequality imply that for all smooth u we have $\nabla u \in \Gamma_{L^p}(M, TM)$, whenever $u, f \in L^p(M)$ and u solves $\Delta_p u = f$. Here, $q \in (1, \infty)$ is defined by $1/p + 1/q = 1$. However a genuine L^p-extension of Strichartz result is contained in part a) of the following result, which was the main motivation of this paper:

Theorem 2. *Let $p \in (1, \infty)$, let $f \in L^p(M)$, and let $u \in L^p(M)$ be a (distributional) solution of the Poisson equation $\Delta u = f$.*
a) *There exists a constant $C \in (0, \infty)$, which only depends on p, with the following property: if M is geodesically complete and if*

$$\max\{p-2, 0\} \nabla^2 u \in \Gamma_{L^p}(M, T^{0,2}M), \tag{4}$$

then one has

$$\|\nabla u\|_p^2 \leq C \|u\|_p \|f\|_p + \max\{p-2, 0\} \|u\|_p \left\|\nabla^2 u\right\|_p < \infty. \tag{5}$$

b) *Assume that M satisfies $CZ(p)$ and admits a sequence of Hessian cut-off functions. Then the following statemens are equivalent:*

- $\nabla u \in \Gamma_{L^p}(M, TM)$,
- $\nabla^2 u \in \Gamma_{L^p}(M, T^{0,2}M)$.

Moreover, there exists a constant $C \in (0, \infty)$, which only depends on p and on the constants from $CZ(p)$, such that if $\nabla u \in \Gamma_{L^p}(M, TM)$ (or equivalently $\nabla^2 u \in \Gamma_{L^p}(M, T^{0,2}M)$), then one has

$$\|\nabla u\|_p + \left\|\nabla^2 u\right\|_p \le C\,\|u\|_p + C\,\|f\|_p \,. \tag{6}$$

Concerning part a) of Theorem 2: for $p > 2$ this result is a simple consequence of (3) and some standard Meyers-Serrin type smoothing argument, while for $p < 2$ it relies on an inequality of Coulhon/Duong [CD] for smooth compactly supported functions and a nonstandard smoothing procedure, which is based on a new functional fact proved in Appendix A of this paper: namely, the minimal and maximal L^p-realization of Δ coincide under geodesic completeness (for all $p \in (1, \infty)$), a result that so far was only known under a C^∞-boundedness assumption on the geometry of M [Sh, Mi] (which by definition means that the curvature tensor of M and all its derivates are bounded and in addition that M has a positive injectivity radius). Note that, for $1 < p \le 2$, condition (4) is trivially satisfied hence no L^p-assumption on the Hessian is required to conclude $\nabla u \in \Gamma_{L^p}(M, TM)$. In particular, the case $p = 2$ is precisely Strichartz' result.

Concerning part b) of Theorem 2: note first that this statement can be considered as partially inverse to part a). In fact, it was proved in [GP], under the stated assumptions on M, that for every $f \in L^p(M)$ and every solution $u \in L^p(M)$ of the Poisson equation $\Delta u = f$ with $\nabla u \in \Gamma_{L^p}(M, TM)$ one has $\nabla^2 u \in \Gamma_{L^p}(M, T^{0,2}M)$, leaving the question open whether the assumption $\nabla u \in \Gamma_{L^p}(M, TM)$ was just a technical relict of the proof. Theorem 2 b) shows that the assumption $\nabla u \in \Gamma_{L^p}(M, TM)$ is actually necessary in this context. We also emphasize that, thanks to the abstract formulation of b), the result is so flexible to provide L^p Hessian estimates for the Poisson equation under different geometric conditions on the underlying manifold. We already recalled how the validity of $CZ(p)$ and the existence of Hessian cut-off functions can be related to the geometry of the manifold. Concerning the L^p-integrability of the gradient we mention the interesting paper by E. Amar, [Am], where the case of complete manifolds with $\|\operatorname{Ric}\|_\infty < +\infty$ and $r_{\mathrm{inj}}(M) > 0$ is considered, and the recent preprint by L.-J. Cheng, A. Thalmaier and J. Thompson, [CTT], where the geometric assumptions are strongly relaxed to $\operatorname{Ric} \ge -K^2$ for some $K \ge 0$. Furthermore, we point out that global $W^{2,p}$-estimates of the type (6) for solutions of the Poisson equation have been used in [RV] to produce gradient Ricci soliton structures via log-Sobolev inequalities.

Finally, we present an application of Theorem 1 concerning the global regularity of the semigroups associated with magnetic Schrödinger operators whose potentials are allowed to have local singularities. To this end, we recall that if M is geodesically complete, given an electric potential $0 \le V \in L^2_{\mathrm{loc}}(M)$ and a magnetic potential $A \in \Gamma_{L^4_{\mathrm{loc}}}(M, TM)$ with $\nabla^\dagger A \in L^2_{\mathrm{loc}}(M)$, then the magnetic Schrödinger operator $\Delta_{A,V}$ in $L^2_{\mathbb{C}}(M)$, defined initially on $\Psi \in C^\infty_{c,\mathbb{C}}(M)$ by

$$\Delta_{A,V}\Psi := (\nabla - \sqrt{-1}A)^\dagger(\nabla - \sqrt{-1}A)\Psi \tag{7}$$

$$= \Delta\Psi - 2\sqrt{-1}(A, \nabla\Psi) + \sqrt{-1}(\nabla^\dagger A)\Psi + |A|^2\Psi + V\Psi, \tag{8}$$

is a well-defined nonnegative symmetric operator, which is essentially self-adjoint [GK, LS]. Its self-adjoint closure $H_{A,V}$ is semibounded from below and we can consider its associated magnetic Schrödinger semigroup

$$[0,\infty) \ni t \longmapsto \mathrm{e}^{-tH_{A,V}} \in \mathscr{L}(L^2_{\mathbb{C}}(M))$$

defined by the spectral calculus. In fact, a certain self-adjoint extension of $\Delta_{A,V}$ can be defined using quadratic form methods (even without assuming that M is complete), and it is much more convenient to prove [GK, LS] that $C^\infty_{c,\mathbb{C}}(M)$ is an operator core for this extension, rather than proving directly that $\Delta_{A,V}$ is essentially self-adjoint. To do so, the crucial step in the proof is to show the local regularity

$$\Delta \mathrm{e}^{-tH_{A,V}} f \in L^2_{\mathrm{loc},\mathbb{C}}(M), \quad \nabla \mathrm{e}^{-tH_{A,V}} f \in \Gamma_{L^4_{\mathrm{loc},\mathbb{C}}}(M,TM), \tag{9}$$

for all $f \in L^2_{\mathbb{C}}(M)$, $t \in (0,\infty)$. This result is needed in the above context to make the machinery of Friedrichs mollifiers work. While the latter local regularity does not need any control on the geometry of M, we realized that the inequality (3) from Theorem 1 can be used to answer the following regularity question: *Assume*

$$0 \leq V \in L^2(M), \quad A \in \Gamma_{L^2}(M,TM), \quad \nabla^\dagger A \in L^4(M). \tag{10}$$

Under which geometric assumptions on M do we have the global regularity

$$\Delta \mathrm{e}^{-tH_{A,V}} f \in L^2_{\mathbb{C}}(M), \quad \nabla \mathrm{e}^{-tH_{A,V}} f \in \Gamma_{L^4_{\mathbb{C}}}(M,TM) \tag{11}$$

for all $f \in L^2_{\mathbb{C}}(M)$, $t \in (0,\infty)$? Towards this aim, we recall that M is called *ultracontractive*[1], if the jointly smooth integral of $\mathrm{e}^{-tH_{0,0}}$ satisfies

$$\sup_{x\in M} \mathrm{e}^{-tH_{0,0}}(x,x) < \infty \quad \text{for all } t \in (0,\infty).$$

We are going to use (3) to prove the following result, which seems even new for the Euclidean $\mathbb{R}^m$:

Theorem 3. *Assume M admits a sequence of Laplacian cut-off functions and satisfies $CZ(2)$. Then for all V and A with (10), and all $f \in L^2_{\mathbb{C}}(M) \cap L^\infty(M)$, $t \in (0,\infty)$ one has (11). If in addition M is ultracontractive, then one has (11) for all $f \in L^2_{\mathbb{C}}(M)$, $t \in (0,\infty)$.*

As we have already observed, M admits a sequence of Laplacian cut-off functions and satisfies $CZ(2)$, if M is geodesically complete with Ricci curvature bounded from below by a constant. If in addition to geodesic completeness and a lower Ricci bound M satisfies the volume non-collapsing condition

$$\inf_{x\in M} \mu(B(x,r)) > 0 \quad \text{for all } r \in (0,\infty),$$

then M is even ultracontractive. This follows from Li-Yau's heat kernel estimates, which state that if M is geodesically complete with Ricci curvature bounded from below by a constant, there are constants $C_1, C_2, C_3 \in (0,\infty)$ such that

$$\mathrm{e}^{-tH_{0,0}}(x,y) \leq C_1 \mathrm{e}^{tC_2} \mathrm{e}^{-C_3 \frac{d(x,y)^2}{t}} \mu(B(x,\sqrt{t}))^{-1} \quad \text{for all } t \in (0,\infty)$$

(with an analogous lower bound).

[1] If M is not geodesically complete, $H_{0,0}$ has to be replaced with the Friedrichs realization of Δ in the definition of ultracontractivity.

This paper is organized as follows: in section 3 we prove Theorem 1, section 4 is devoted to the proof of Theorem 2, and section 5 to the proof of Theorem 3. In section A of the appendix the aforementioned result on the equality of the minimal and maximal L^p-realization of Δ under geodesic completeness is proved (cf. Theorem 5). In section B of the appendix we have recorded a Meyers-Serrin smooting result for Riemannian manifolds, which will be used at several places, and finally section C of the appendix contains a list of standard formulae from calculus on Riemannian manifolds that are used throughout the paper.

3. Proof of Theorem 1

We first prove the formula

$$\Delta_p u = |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}\nabla^2 u(\nabla u, \nabla u). \tag{12}$$

Indeed, one has

$$\begin{aligned}
\Delta_p u &= \nabla^\dagger\ (|\nabla u|^{p-2}\nabla u) = -(\nabla|\nabla u|^{p-2}, \nabla u) + |\nabla u|^{p-2}\nabla^\dagger\nabla u\\
&= -\big(\nabla(\nabla u, \nabla u)^{p/2-1}, \nabla u\big) + |\nabla u|^{p-2}\Delta u\\
&= -(p/2-1)(\nabla u, \nabla u)^{p/2-2}\big(\nabla(\nabla u, \nabla u), \nabla u\big) + |\nabla u|^{p-2}\Delta u\\
&= -(p/2-1)(\nabla u, \nabla u)^{p/2-2}2(\nabla^{TM}_{\nabla u}\nabla u, \nabla u) + |\nabla u|^{p-2}\Delta u\\
&= -(p-2)|\nabla u|^{p-4}\nabla^2 u(\nabla u, \nabla u) + |\nabla u|^{p-2}\Delta u,
\end{aligned}$$

where we have used (in this order) the product rule, the chain rule, the compatibility of the Levi-Civita connection with the Riemannian metric and, finally, the definition of ∇^2. Now (12) implies

$$|u\Delta_p u| \leq |u||\nabla u|^{p-2}|\Delta u| + (p-2)|u||\nabla u|^{p-2}|\nabla^2 u|, \tag{13}$$

so that (3) follows from (2) and Hölder's inequality (as $p \geq 2$).
It remains to prove (2), fix $0 \leq \varphi \in C^\infty_c(M)$ and define the vector field

$$X = \varphi^p u|\nabla u|^{p-2}\nabla u \in \Gamma_{C^1_c}(M, TM). \tag{14}$$

Using again the product rule and the definition of the p-Laplacian we can calculate

$$\begin{aligned}
\nabla^\dagger X &= \varphi^p u\Delta_p u - (\nabla(\varphi^p u), |\nabla u|^{p-2}\nabla u)\\
&= \varphi^p u\Delta_p u - u|\nabla u|^{p-2}(\nabla\varphi^p, \nabla u) - \varphi^p|\nabla u|^p,
\end{aligned}$$

so that using the divergence theorem we have

$$\int |\nabla u|^p\varphi^p d\mu = -\int \varphi^p u\Delta_p u\ d\mu - \int u(|\nabla u|^{p-2}\nabla u, \nabla\varphi^p)d\mu =: J_1 + J_2. \tag{15}$$

Clearly, one has

$$|J_1| \leq \int \varphi^p|u\Delta_p u|d\mu. \tag{16}$$

On the other hand, with $\nabla\varphi^p = p\varphi^{p-1}\nabla\varphi$ and $1/q := 1 - 1/p$, Young's inequality implies that for all $\epsilon \in (0, \infty)$ we have

$$
\begin{aligned}
|J_2| &\leq p \int (|u||\nabla\varphi|)\,(\varphi|\nabla u|)^{p-1}\, d\mu \\
&\leq \frac{p\epsilon^q}{q} \int |\nabla u|^p \varphi^p d\mu + \frac{1}{\epsilon^p} \int |u|^p |\nabla\varphi|^p d\mu
\end{aligned} \tag{17}
$$

Using (16), (17) and (15) it follows that, for $0 < \epsilon < (q/p)^{1/q}$, the term $\int |\nabla u|^p \varphi^p d\mu$ on the RHS can be absorbed into the LHS and we get:

$$
\int |\nabla u|^p \varphi^p d\mu \leq \frac{1}{1 - p\epsilon^q/q} \int \varphi^p |u\Delta_p u| d\mu + \frac{1}{\epsilon^p(1 - p\epsilon^q/q)} \int |u|^p |\nabla\varphi|^p d\mu. \tag{18}
$$

As M is geodesically complete, we can pick a sequence $\varphi = \psi_k \in C_c^\infty(M)$ of first order cut-off functions. Taking limits as $k \to \infty$, using monotone and dominated convergence theorems, and taking $\epsilon \to 0+$ afterwards, we finally obtain the desired estimate (2).

4. Proof of Theorem 2

a) If $p \geq 2$, and $u \in C^\infty(M)$, by applying Theorem 1 b) with $a_1 = a_2 = b_1 = b_2 = p$ and $a_3 = b_3 = p/(p-2)$ we obtain

$$
\|\nabla u\|_p^2 \leq \|u\|_p \|f\|_p + (p-2)\|u\|_p \|\nabla^2 u\|_p \tag{19}
$$

which is precisely (5) with $C = 1$. In the general case, by a Meyers-Serrin's theorem (cf. Theorem 7 in Appendix B), we can pick a sequence $(u_k) \subset C^\infty(M)$ with

$$
\left\|\nabla^2 u_k - \nabla^2 u\right\|_p \to 0, \quad \|\Delta u_k - f\|_p \to 0, \quad \|u_k - u\|_p \to 0.
$$

Then (19) shows that ∇u_k is a Cauchy sequence in $\Gamma_{L^p}(M, TM)$, which necessarily converges to ∇u. Therefore, evaluating (19) along u_k and taking the limit as $k \to +\infty$ completes the proof.

If $1 < p < 2$, and $u \in C_c^\infty(M)$, by Theorem 4.1 in [CD] we have that

$$
\|\nabla u\|_p^2 \leq C_p \|u\|_p \|f\|_p, \tag{20}
$$

for some absolute constant $C_p > 0$. This is precisely what is stated in (5). In the general case, we appeal to Theorem 5 from Appendix A in order to pick a sequence $(u_k) \subset C_c^\infty(M)$ such that

$$
\|\Delta u_k - \Delta u\|_p \to 0, \quad \|u_k - u\|_p \to 0.
$$

By (20), for all $k, h \in \mathbb{N}$, one has

$$
\|\nabla(u_k - u_h)\|_p^2 \leq C_p \|u_k - u_h\|_p \|\Delta(u_k - u_h)\|_p.
$$

Whence, we deduce again that ∇u_k is a Cauchy sequence in $\Gamma_{L^p}(M, TM)$, which necessarily converges to ∇u. To conclude the validity of (5) we now evaluate (20) along u_k and take the limit as $k \to +\infty$.

b) Assume first $\nabla u \in \Gamma_{L^p}(M, TM)$. By Meyers-Serrin we can pick a sequence $(u_k) \subset C^\infty(M)$ with

$$
\|\Delta u_k - f\|_p \to 0, \quad \|u_k - u\|_p \to 0.
$$

Then Proposition 3.8 in [GP] implies

$$
\left\|\nabla^2(u_k - u_h)\right\|_p \leq C \|\Delta(u_k - u_h)\|_p + C\|u_k - u_h\|_p
$$

for every $k, h \in \mathbb{N}$ and for some constant $C > 0$ which only depends on the $CZ(p)$ constants. Therefore, with the same Cauchy-sequence argument as above,

$$\left\|\nabla^2 u\right\|_p \leq C \left\|f\right\|_p + C\|u\|_p.$$

If $\max\{p-2,0\}\nabla^2 u \in \Gamma_{L^p}(M, T^{0,2}M)$, then by part a) for every $\epsilon \in (0,\infty)$ we can pick $C_\epsilon \in (0,\infty)$ such that

$$\|\nabla u\|_p \leq C_p \left\|u\right\|_p + C_p \left\|f\right\|_p + \max\{p-2,0\}C_\epsilon \left\|u\right\|_p + \epsilon \left\|\nabla^2 u\right\|_p.$$

Combining these two estimates yields (6).

5. Proof of Theorem 3

We start with the following result, which is well-known in the Euclidean case, but has only been recorded so far for smooth magnetic potentials in the case of manifolds:

Proposition 4 (Kato-Simon inequality)**.** *Assume M is geodesically complete and*

$$0 \leq V \in L^2_{\mathrm{loc}}(M), \quad A \in \Gamma_{L^2_{\mathrm{loc}}}(M, TM), \quad \nabla^\dagger A \in L^4_{\mathrm{loc}}(M).$$

Then for all $t \in (0,\infty)$, $f \in L^2_{\mathbb{C}}(M)$, and μ-a.e. $x \in M$ one has

$$\left|\mathrm{e}^{-tH_{A,V}} f(x)\right| \leq \left|\mathrm{e}^{-tH_{0,0}} f(x)\right|.$$

Proof. If A is smooth, the asserted inequality follows from Theorem VII.8 in [Gue3] (see also [Gue2]).
In the general case, we pick a sequence $(\psi_k)_{k\in\mathbb{N}} \subset C^\infty_c(M)$ of first order cut-off functions. Then by the Meyers-Serrin theorem, for every $k \in \mathbb{N}$, we can pick a sequence $(A_{k,n})_{n\in\mathbb{N}} \subset \Gamma_{C^\infty}(M, TM)$ such that with

$$A_k := \psi_k A$$

one has

$$\lim_{n\to\infty} A_{k,n} = A_k \text{ in } \Gamma_{L^2}(M, TM) \text{ and } \lim_{n\to\infty} \nabla^\dagger A_{k,n} = \nabla^\dagger A_k \text{ in } L^2(M).$$

In particular, using (7), for all Ψ in the common operator core $C^\infty_{c,\mathbb{C}}(M)$ of $H_{A_{k,n},V}$ and $H_{A_k,V}$ one has

$$\lim_{n\to\infty} \left\|H_{A_{k,n},V}\Psi - H_{A_k,V}\Psi\right\|_2 = 0, \quad \text{so that } \lim_{n\to\infty} \mathrm{e}^{-tH_{A_{k,n},V}} = \mathrm{e}^{-tH_{A_k,V}} \text{ strongly in } L^2_{\mathbb{C}}(M),$$

and so

$$\lim_{n\to\infty} \mathrm{e}^{-tH_{A_{k,n},V}} f(x) = \mathrm{e}^{-tH_{A_k,V}} f(x) \quad \text{for } \mu\text{-a.e. } x \in M,$$

possibly by taking a subsequence.
Likewise, using the product formula

$$\nabla^\dagger A_k = -(\nabla\psi_k, A) + \psi_k \nabla^\dagger A$$

one gets

$$\lim_{k\to\infty} A_k = A \text{ in } \Gamma_{L^2_{\mathrm{loc}}}(M, TM) \text{ and } \lim_{k\to\infty} \nabla^\dagger A_k = \nabla^\dagger A \text{ in } L^2_{\mathrm{loc}}(M),$$

and so, for all Ψ in the common operator core $C^\infty_{c,\mathbb{C}}(M)$ of $H_{A_k,V}$ and $H_{A,V}$ it holds that

$$\lim_{k\to\infty} \left\|H_{A_k,V}\Psi - H_{A,V}\Psi\right\|_2 = 0, \quad \text{so that } \lim_{k\to\infty} \mathrm{e}^{-tH_{A_k,V}} = \mathrm{e}^{-tH_{A,V}} \text{ strongly in } L^2_{\mathbb{C}}(M),$$

and we arrive at (possibly by taking a subsequence)

$$\lim_{k\to\infty}\lim_{n\to\infty} \mathrm{e}^{-tH_{A_{k,n},V}} f(x) = \mathrm{e}^{-tH_{A,V}} f(x) \quad \text{for } \mu\text{-a.e. } x \in M.$$

This reduces the proof of the Kato-Simon for nonsmooth A's to the aforementioned smooth case. □

Proof of Theorem 3. Step 1: One has

$$\|\nabla u\|_4^2 \leq C_1 \|u\|_\infty \|\Delta u\|_2 + C_2 \|u\|_\infty \|u\|_2 \quad \textit{for all } u \in L^\infty(M) \cap L^2(M) \textit{ with } \Delta u \in L^2(M), \tag{21}$$

for some constants $C_1, C_2 > 0$ which only depend on the constant from $CZ(2)$. To see this, we can assume u is real-valued (if not, we decompose u into its real-part and its imaginary-part and use the triangle inequality). We first assume that that u is in addition smooth and pick a sequence $(\psi_k) \subset C_c^\infty(M)$ of Laplacian cut-off functions. Then one has (21) with u replaced by $u_k := \psi_k u$ by Theorem 1 and $CZ(2)$. Using the product rules

$$\nabla u_k = u\nabla\psi_k + \psi_k \nabla u$$

and

$$\Delta u_k = \psi_k \Delta u + u\Delta\psi_k + 2(\nabla\psi_k, \nabla u)$$

and that at u, Δu and ∇u are L^2 (the latter follows, for example, from Theorem 2 a)), the inequality extends by Fatou and dominated convergence to u, taking $k \to \infty$. In the general case, by u, $\Delta u \in L^2(M)$ using Meyers-Serrin's theorem we can pick a sequence $(u_k) \subset C^\infty(M)$ with u_k, $\Delta u_k \in L^2(M)$ with

$$\|u_k - u\|_2 \to 0, \quad \|\Delta u_k - \Delta u\|_2 \to 0$$

and in addition

$$\|u_k\|_\infty \leq \|u\|_\infty \quad \text{for all } k.$$

Using (21) with u_k shows that ∇u_k is Cauchy in $\Gamma_{L^4}(M, TM)$ and then one necessarily has

$$\|\nabla u_k - \nabla u\|_4 \to 0.$$

Step 2: For all $f \in L^2_{\mathbb{C}}(M) \cap L^\infty(M)$, $t \in (0,\infty)$ one has (11). To prove that, we set $f_t := \mathrm{e}^{-tH_{A,V}} f$ and record that by the Kato-Simon inequality one has the first inequality in

$$\|f_t\|_\infty \leq \left\|\mathrm{e}^{-tH_{0,0}} f\right\|_\infty \leq \|f\|_\infty < \infty, \tag{22}$$

where the second inequality follows from noting that

$$\int \mathrm{e}^{-tH_{0,0}}(x,y)d\mu(y) \leq 1 \quad \text{for all } x \in M,\, t \in (0,\infty),$$

as $H_{0,0}$ stems from a Dirichlet form. Pick now a sequence $(\psi_k) \subset C_c^\infty(M)$ of Laplacian cut-off functions. Our aim is to prove

$$\sup_{k\in\mathbb{N}} \|\Delta(\psi_k f_t)\|_2 < \infty. \tag{23}$$

Indeed, then $(\Delta(\psi_k f_t))_k$ has a subsequence which converges weakly to some $h \in L^2_{\mathbb{C}}(M)$, but as we have $\|\psi_k f_t - f_t\|_2 \to 0$, we have $\Delta f_t = h \in L^2_{\mathbb{C}}(M)$. Then, applying (21) with

$u = f_t$ using (22) also shows $\nabla f_t \in \Gamma_{L^4_{\mathbb{C}}}(M, TM)$.
Thus it remains to prove (23): To this end, by the spectral calculus we have

$$\mathrm{Dom}(H_{A,V}) \subset \mathrm{Dom}(\sqrt{H_{A,V}})$$

and $f_t \in \mathrm{Dom}(H_{A,V})$, and from essential self-adjointness

$$\mathrm{Dom}(H_{A,V}) = \{u \in L^2_{\mathbb{C}}(M) : \Delta_{A,V} u \in L^2_{\mathbb{C}}(M)\}, \ \ H_{A,V} u = \Delta_{A,V}$$

and

$$\mathrm{Dom}(\sqrt{H_{A,V}}) = \{u \in L^2_{\mathbb{C}}(M) : (\nabla - \sqrt{-1}A) f \in \Gamma_{L^2_{\mathbb{C}}}(M, TM), \ \sqrt{V} f \in L^2_{\mathbb{C}}(M)\}.$$

It follows from a simple calculation that $\psi_k f_t \in \mathrm{Dom}(H_{A,V})$ with

$$H_{A,V}(\psi_k f_t) = \psi_k \Delta_{A,V} f + 2\big(\nabla\psi_k, (\nabla - \sqrt{-1}A) f_t\big) - (\Delta\psi_k) f_t. \tag{24}$$

On the other hand, from $(\psi_k f_t) \in \mathrm{Dom}(\sqrt{H_{A,V}})$ we have

$$(\nabla - \sqrt{-1}A)(\psi_k f_t) \in \Gamma_{L^2_{\mathbb{C}}}(M, TM)$$

which from the assumption on A easily implies

$$\nabla(\psi_k f_t) = (\nabla - \sqrt{-1}A)(\psi_k f_t) - \sqrt{-1}A(\psi_k f_t) \in \Gamma_{L^2_{\mathbb{C}}}(M, TM) \tag{25}$$

as $\psi_k f_t$ is bounded with a compact support. Likewise, it follows from (25) and the assumptions on A and V that

$$\begin{aligned}\Delta(\psi_k f_t) =& \Delta_{A,V}(\psi_k f_t) + 2(A, \nabla(\psi_k f_t)) \\ &- \sqrt{-1}(\nabla^\dagger A)\psi_k f_t - |A|^2 \psi_k f_t - V \psi_k f_t \in L^2_{\mathbb{C}}(M),\end{aligned}$$

so that

$$\begin{aligned}\|\Delta(\psi_k f_t)\|_2 \leq& \|\Delta_{A,V}(\psi_k f_t)\|_2 + 2\,\|(A, \nabla(\psi_k f_t))\|_2 + \left\|((\nabla^\dagger A) - |A|^2 - V)\psi_k f_t\right\|_2 \\ \leq& \|\psi_k \Delta_{A,V} f\|_2 + 2\left\|\big(\nabla\psi_k, (\nabla - \sqrt{-1}A) f_t\big)\right\|_2 + \|(\Delta\psi_k) f_t\|_2 \\ &+ 2\,\|(A, \nabla(\psi_k f_t))\|_2 + \left\|((\nabla^\dagger A) - |A|^2 - V)\right\|_2 \|f\|_\infty \\ \leq& \|\Delta_{A,V} f\|_2 + 2\sup_k \|\nabla\psi_k\|_\infty \left\|(\nabla - \sqrt{-1}A) f_t\right\|_2 + \sup_k \|(\Delta\psi_k)\|_\infty \|f_t\|_2 \\ &+ 2\,\|(A, \nabla(\psi_k f_t))\|_2 + \left\|((\nabla^\dagger A) - |A|^2 - V)\right\|_2 \|f\|_\infty .\end{aligned}$$

Finally, using (21), for every $\epsilon \in (0, \infty)$ we have

$$\|(A, \nabla(\psi_k f_t))\|_2 \leq \|A\|_4 \|\nabla(\psi_k f_t)\|_4 \leq \|A\|_4 C(1/\epsilon) \|f_t\|_\infty + \|A\|_4 C\epsilon \|\Delta(\psi_k f_t)\|_2 ,$$

completing the proof of (23).

Step 3: Removal of the assumption $f \in L^\infty(M)$ in the ultracontractive case. If in addition M is ultracontractive, then for all $s \in (0, \infty)$ one has that $\mathrm{e}^{-sH_{0,0}}$ maps $L^2_{\mathbb{C}}(M) \to L^\infty_{\mathbb{C}}(M)$, so by (22) the same is true for $\mathrm{e}^{-sH_{A,V}}$. Thus, for all $f \in L^2_{\mathbb{C}}(M)$, $t \in (0, \infty)$, one has

$$\mathrm{e}^{-tH_{A,V}} f = \mathrm{e}^{-(t/2)H_{A,V}} \tilde{f},$$

where $\tilde{f} := \mathrm{e}^{-(t/2)H_{A,V}} f \in L^2_{\mathbb{C}}(M) \cap L^\infty(M)$, $t \in (0, \infty)$, so the claim follows from Step 2. This completes the proof.

□

Appendix A. $\Delta_{\min,p} = \Delta_{\max,p}$ under geodesic completeness (by Ognjen Milatovic)

Let M be a smooth Riemannian manifold. Given a linear partial differential operator

$$\mathscr{T} : C^\infty(M) \longrightarrow C^\infty(M)$$

with smooth coefficients and $p \in (1,\infty)$, we define a closable operator $\mathscr{T}_p$ in $L^p(M)$ as follows:

$$\mathrm{Dom}(\mathscr{T}_p) = C^\infty_c(M), \quad \mathscr{T}_p f := \mathscr{T} f.$$

Then one can further define two closed extensions in $L^p(M)$ of $\mathscr{T}_p$ as follows: $\mathscr{T}_{\min,p}$ is defined as the closure of $\mathscr{T}_p$ in $L^p(M)$, and $\mathrm{Dom}(\mathscr{T}_{\max,p})$ is defined to be the space of all $f \in L^p(M)$ such that $\mathscr{T} f \in L^p(M)$ (distributionally), with $\mathscr{T}_{\max,p} f := \mathscr{T} f$ for such f's. Assuming M has a C^∞-bounded geometry it has been shown in [Sh, Mi] that $\Delta_{\min,p} = \Delta_{\max,p}$. The main result of this section shows that in fact one can completely remove any curvature and injectivity radius for the equality $\Delta_{\min,p} = \Delta_{\max,p}$:

Theorem 5. *Let M be geodesically complete and let $p \in (1,\infty)$. Then one has $\Delta_{\min,p} = \Delta_{\max,p}$, in other words, $C^\infty_c(M)$ is an operator core for $\Delta_{\max,p}$. Moreover, $\Delta_{\max,p}$ generates a strongly continuous contraction semigroup in $L^p(M)$.*

Proof. It follows from distribution theory (cf. Lemma I.25 in [Gue3]) that under the isometric identification $L^p(M) = L^q(M)^*$ (where $q \in (1,\infty)$ is defined by $1/p + 1/q = 1$), the adjoint $(\mathscr{T}_{\min,q})^*$ for every $\mathscr{T}$ as above is given by $(\mathscr{T}_{\min,q})^* = (\mathscr{T}^\dagger)_{\max,p}$, where

$$\mathscr{T}^\dagger : C^\infty(M) \longrightarrow C^\infty(M)$$

denotes the formal adjoint of $\mathscr{T}^\dagger$. In particular, $(\Delta_{\min,q})^* = \Delta_{\max,p}$. It has been shown in [St] that under geodesic completeness $\Delta_{\min,r}$ is the generator of a strongly continuous contraction semigroup in $L^r(M)$ for all $r \in (1,\infty)$. As adjoints of generators of strongly continuous contraction semigroups in reflexive Banach spaces again generate such semigroups ([ABHN], p. 138), this property remains true for $\Delta_{\max,p}$. As generators of strongly continuous contraction semigroups are maximally accretive ([RS], p. 241), it follows that $\Delta_{\max,p}$ is an accretive extension of the maximally accretive operator $\Delta_{\min,p}$ and so $\Delta_{\min,p} = \Delta_{\max,p}$. □

Note that $\Delta_{\min,p} = \Delta_{\max,p}$ is equivalent to the following density result: For every $f \in L^p(M)$ (w.l.o.g smooth by Meyers-Serrin; cf. Theorem 7 below) with $\Delta f \in L^p(M)$ there exists a sequence $(f_k) \subset C^\infty_c(M)$ such that as $k \to \infty$,

$$\|f_k - f\|_p \to 0, \quad \|\Delta f_k - \Delta f\|_p \to 0.$$

It is remarkable that even assuming the existence of Laplacian cut-off functions, there seems to be no way to prove this density by hand, that is, without using some functional analytic machinery. In fact, this "phenomenon" already occurs for $p = 2$.

Let $p \in (1,\infty)$. The paper [Mi] deals with operator core problems as in Theorem 5 in the situation where Δ is replaced with the Schrödinger operator $\Delta + V$ with $0 \leq V \in L^p_{\mathrm{loc}}(M)$. In fact, the main result therein shows that $C^\infty_c(M)$ is an operator core for $\Delta_{V,\max,p}$, if M has a C^∞-bounded geometry, and if $\Delta_{V,\max,p}$ is the closed operator in $L^p(M)$ defined by

$$\mathrm{Dom}(\Delta_{V,\max,p}) := \{f \in L^p(M) : Vf \in L^1_{\mathrm{loc}}(M), (\Delta + V)f \in L^p(M)\}.$$

The proof given there uses the C^∞-boundedness assumption on M only to prove that M is L^p-positivity preserving in the language of [Gue] and that $\Delta_{\max,p} = \Delta_{\min,p}$, together with some perturbation theory. As by recent results it is known that geodesically complete Riemannian manifolds with a Ricci curvature bounded from below by a constant are L^q-positivity preserving (in fact also for $p = 1$ and $p = \infty$) [Gue, BS], showing the following result which should be of an independent interest:

Theorem 6. *Let M be geodesically complete with a Ricci curvature bounded from below by a constant, and let $p \in (1,\infty)$, $0 \leq V \in L^p_{\mathrm{loc}}(M)$. Then $C^\infty_c(M)$ is an operator core for $\Delta_{V,\max,p}$.*

It is also reasonable to expect that using the techniques fro [Mi2], these results can be extended to covariant Schrödinger operators.

Appendix B. A geometric Meyers-Serrin Theorem

The following result follows from the main result in [GGP] and its proof:

Theorem 7. *Let M be a smooth Riemannian manifold, let $E \to M$ be a smooth metric $\mathbb{K}$-vector bundle (where $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$), and let $p \in (1,\infty)$. Then for every $f \in \Gamma_{L^p}(M,E)$ there exists a sequence $(f_k) \subset \Gamma_{C^\infty}(M,E)$, whose elements can be chosen compactly supported if f has a compact (μ-essential) support, such that*

- $\left\|f_k - f\right\|_p \to 0$ *as* $k \to \infty$,
- $\left\|f_k\right\|_\infty \leq \left\|f\right\|_\infty \in [0,\infty]$ *for all* k,
- *for every smooth metric vector bundle $F \to M$ over $\mathbb{K}$, every $l \in \mathbb{N}_{\geq 1}$, and every smooth $\mathbb{K}$-linear partial differential operator P from $M \to E$ to $M \to F$ of order $\leq l$ with $Pf \in \Gamma_{L^p}(M,F)$, one has $\left\|Pf_k - Pf\right\|_p \to 0$ as $k \to \infty$, if in case $l \geq 2$ one has $f \in \Gamma_{W^{l-1,p}_{\mathrm{loc}}}(M,E)$ (with no further assumption for $l = 1$).*

Appendix C. Some useful formulae

Let us first record that for all vector fields X, Y, Z on M one has

$$X(Y,Z) = (\nabla^{TM}_X Y, Z) + (Y, \nabla^{TM}_X Z),$$

where in the LHS X acts as a derivation on the smooth function $x \mapsto (X(x), Y(x))$ on M. This equation just means that the Levi-Civita connection is compactible with the Riemannian metric. Assume ϕ_1 is a function on M. Recalling that $\nabla^\dagger$ is (-1) times the divergence operator, one finds the product rule

$$\nabla^\dagger(\phi_1 Y) = \phi_1 \nabla^\dagger Y - (\nabla\phi_1, Y).$$

If ϕ_2 is another function on M, then one has the product rule

$$\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1,$$

and

$$\Delta(\phi_1\phi_2) = \phi_1\Delta\phi_2 + \phi_2\Delta\phi_1 + 2(\nabla\phi_1, \nabla\phi_2).$$

For every function f on $\mathbb{R}$ one has the chain rule

$$\nabla f(\phi_1) = f'(\phi_1)\nabla\phi_1.$$

If X is compactly supported, then the divergence theorem holds

$$\int \nabla^\dagger X d\mu = 0,$$

which holds by the definition of $\nabla^\dagger$:

$$\int \nabla^\dagger X d\mu = \int (\nabla^\dagger X)\cdot 1 d\mu = \int (\nabla^\dagger X)\cdot 1 d\mu = \int X\cdot(\nabla 1) d\mu = 0.$$

Finally, we record Bochner's equality:

$$\left|\nabla^2\phi_1\right|^2 = -\frac{1}{2}\Delta|\nabla\phi_1|^2 + (\nabla\phi_1, \nabla\Delta\phi_1) - \mathrm{Ric}(\nabla\phi_1, \nabla\phi_1).$$

In particular, it follows that if $\mathrm{Ric} \geq -C$ for some constant $C \geq 0$ and ϕ_1 is compactly supported, then in view of $\Delta = \nabla^\dagger\nabla$ one has

$$\int \left|\nabla^2\phi_1\right|^2 d\mu \leq -\int \frac{1}{2}(\nabla^\dagger\nabla|\nabla\phi_1|^2)\cdot 1 d\mu + \int (\nabla\phi_1, \nabla\Delta\phi_1) d\mu + C\int (\nabla\phi_1, \nabla\phi_1) d\mu \tag{26}$$

$$= \int |\Delta\phi_1|^2 d\mu + C\int (\nabla\phi_1, \nabla\phi_1) d\mu, \tag{27}$$

which is nothing but the Calderón-Zygmund inequality $CZ(2)$.

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Batu Güneysu, Institut für Mathematik, Humboldt-Universität zu Berlin, 12489 Berlin, Germany

E-mail address: `gueneysu@math.hu-berlin.de`

Stefano Pigola, Dipartimento di Scienza e Alta Tecnologia - Sezione di Matematica, Universit dell'Insubria, 22100 Como, Italy

E-mail address: `stefano.pigola@uninsubria.it`