Uniqueness of solutions to dynamic complementarity problems

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Differential Complementarity Problems

Differential Complementarity Problems (DCPs) have the form

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{f}(\mathbf{x}) + B(\mathbf{x})\mathbf{z}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \\ K \ni \mathbf{z}(t) &\perp \mathbf{G}(\mathbf{x}(t), \mathbf{z}(t)) \in K^* \end{aligned}$$

where K is a closed convex cone, and K^* its dual cone:

$$K^* = \left\{ \mathbf{w} \mid \mathbf{w}^T\mathbf{u} \geq 0 \text{ for all } \mathbf{u} \in K \right\}.$$

Examples: $K = \mathbb{R}^n$ and $K^* = \{0\}$; $K = \mathbb{R}^n_+$ and $K^* = \mathbb{R}^n_+$.

$$\begin{aligned} K &= \left\{ \begin{bmatrix} \mathbf{x} \\ y \end{bmatrix} \mid y \geq \|\mathbf{x}\|_2 \right\}, \\ K^* &= K. \qquad \text{(ice-cream or Lorentz cone).} \end{aligned}$$

Index

If $\mathbf{G}(\mathbf{x}, \mathbf{z})$ is *strongly monotone* in $\mathbf{z}$

$$(\mathbf{G}(\mathbf{x}, \mathbf{z}_2) - \mathbf{G}(\mathbf{x}, \mathbf{z}_1))^T (\mathbf{z}_2 - \mathbf{z}_1) \geq \beta \|\mathbf{z}_2 - \mathbf{z}_1\|^2 \qquad \text{for all } \mathbf{z}_1,\ \mathbf{z}_2,$$

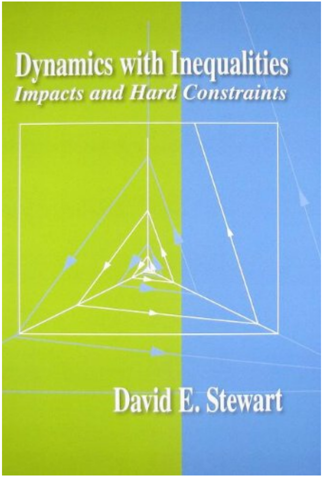
and Lipschitz in $\mathbf{x}$, then we can write $\mathbf{z}$ as a Lipschitz function of $\mathbf{x}$ and we can substitute into the differential equation.

This is *index zero*. The existence/uniqueness theory is the same as for Lipschitz ODEs.

More interesting are *index one* problems, where we assume that $\mathbf{G}(\mathbf{x}, \mathbf{z}) = \mathbf{G}(\mathbf{x})$. We also assume that

$$\nabla \mathbf{G}(\mathbf{x})\, B(\mathbf{x}) \qquad \text{is positive definite for all } \mathbf{x}.$$

A useful book (shameless plug)



Uniqueness for DCPs (index one)

Regularity assumptions: $\mathbf{f}$, $\nabla\mathbf{G}$, B are all Lipschitz

Positivity assumption: $\nabla\mathbf{G}(\mathbf{x})\,B(\mathbf{x})$ is positive definite.

Something extra assumption: $\nabla\mathbf{G}(\mathbf{x})\,B(\mathbf{x})$ is symmetric.

With these assumptions, we can show existence and uniqueness of solutions for DCPs, provided $\mathbf{G}(\mathbf{x}_0) \in K^*$.

The first two assumptions and $\mathbf{G}(\mathbf{x}_0) \in K^*$ are sufficient for ***existence*** of solutions.

Counter-examples

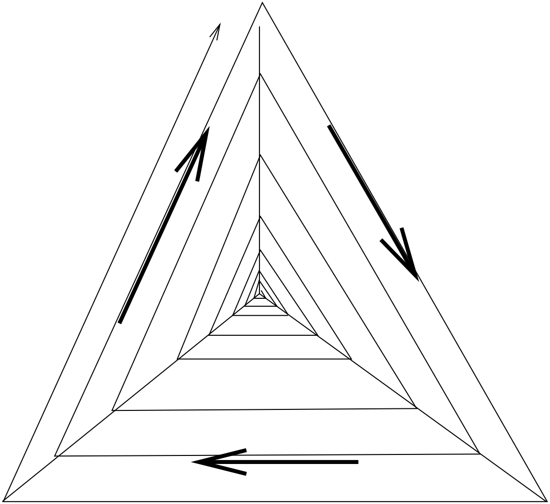
The first counter-example to uniqueness was probably the following (actually $\nabla \mathbf{G}(\mathbf{x})\, B(\mathbf{x})$ is not positive definite, but it is a P-matrix):

$$\begin{aligned} \frac{d\mathbf{w}}{dt} &= R\mathbf{z}(t) + \mathbf{q}, \qquad \mathbf{w}(t_0) = \mathbf{w}_0 \in \mathbb{R}^n_+ \\ 0 \leq \mathbf{w}(t) &\perp \mathbf{z}(t) \geq 0 \qquad \text{for all } t, \end{aligned}$$

with

$$R = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}.$$

See, e.g., Bernard and El–Kharroubi (1990).



Bernard & El–Kharroubi example:
looking at origin from near (1,1,1)

Side remark on Zeno solutions

If $K = \mathbb{R}^n_+$ and $\nabla \mathbf{G}(\mathbf{x})\, B(\mathbf{x})$ is a P-matrix for all $\mathbf{x}$, and all functions analytic, then we have uniqueness amongst non-Zeno solutions.

Examples of non-uniqueness typically have reverse-Zeno type behavior.

Some solutions can have forward Zeno behavior with $\nabla \mathbf{G}(\mathbf{x})\, B(\mathbf{x})$ positive definite for $K = \mathbb{R}^n_+$:

Use the previous example with

$$R = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & \alpha \\ \alpha & 0 & 1 \end{bmatrix}, \qquad 1 < \alpha < 2.$$

Convolution Complementarity Problems

A ***convolution complementarity problems*** (CCP) has the form: Given $m\colon [0,\infty) \to \mathbb{R}^{n\times n}$ and $\mathbf{q}\colon [0,\infty) \to \mathbb{R}^n$ find $\mathbf{z}\colon [0,\infty) \to \mathbb{R}^n$ such that

$$0 \le \mathbf{z}(t) \quad \perp \quad (m * \mathbf{z})(t) + \mathbf{q}(t) \ge 0 \qquad \text{for all } t \ge 0, \text{ where}$$

$$(m * \mathbf{z})(t) \quad = \quad \int_0^t m(t-\tau)\,\mathbf{z}(\tau)\,d\tau.$$

For the DCP

$$\frac{d\mathbf{x}}{dt}(t) = A\mathbf{x}(t) + B\mathbf{z}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0,$$

$$\mathbf{w}(t) = C\mathbf{x}(t) + D\mathbf{z}(t),$$

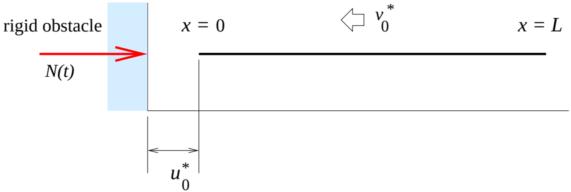
we have an equivalent CCP: $m(t) = C\,e^{At}\,B + D\,\delta(t)$ for $t \ge 0$. We extend $m(t) = 0$ for $t < 0$.

The index of CCP is the smallest p such that

$$(d/dt)^p m(t) = m_0 \delta(t) + m_1(t)$$

where $m_1(t)$ has no atom at $t = 0$.

Application: impact of a viscoelastic rod



$$
\begin{aligned}
\rho \frac{\partial^2 u}{\partial t^2} &= E \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^3 u}{\partial t\, \partial x^2} \qquad \text{in } (0, L), \\
-E \frac{\partial u}{\partial x}(t, 0) &= -N(t), \\
+E \frac{\partial u}{\partial x}(t, L) &= 0 \qquad (\text{no force}), \\
0 \leq u(t, 0) &\perp N(t) \geq 0.
\end{aligned}
$$

Solution:

$$\begin{aligned} u(t,0) &= q(t) + \int_0^t G(t-\tau)\, N(\tau)\, d\tau \qquad \text{with} \\ G(t) &\sim \text{const}\, t^{1/2} \qquad \text{for } t > 0 \text{ and } t \approx 0. \end{aligned}$$

Fractional index

The index of a CCP can be fractional. In particular, if $\psi_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ then its Laplace transform is

$$\mathcal{L}\psi_\alpha(s) \quad = \quad s^{-\alpha}.$$

Note that $\mathcal{L}\delta(s) = 1$ and $\mathcal{L}H(s) = s^{-1}$ where $H(t) = 1$ for $t > 0$ and $H(t) = 0$ for $t < 0$; if $f(t) = 0$ for $t < 0$ then $\mathcal{L}[f'](s) = s\,\mathcal{L}f(s)$. Since $\mathcal{L}[f * g](s) = \mathcal{L}f(s)\,\mathcal{L}g(s)$. Convolution with ψ_α can be considered to be order α integration.

If $m(t) \sim m_0\, t^{\alpha-1}/\Gamma(\alpha)$, then we say that the CCP

$$\begin{aligned} 0 \leq \mathbf{z}(t) \quad &\perp \quad (m * \mathbf{z})(t) + \mathbf{q}(t) \geq 0 \qquad \text{for all } t \geq 0, \text{ where} \\ (m * \mathbf{z})(t) \quad &= \quad \int_0^t m(t-\tau)\,\mathbf{z}(\tau)\,d\tau. \end{aligned}$$

has index α.

Fourier analysis and positive definite operators

Note that $\mathcal{F}\psi_\alpha(\omega) = (i\omega)^{-\alpha}$, but we have to use the principal branch of $(\cdot)^{-\alpha}$; that is, $(e^{i\theta})^\beta = e^{i\theta\beta}$ for $-\pi < \theta < +\pi$.

So $\mathcal{F}[\psi_\alpha * z](\omega) = \mathcal{F}\psi_\alpha(\omega)\,\mathcal{F}z(\omega)$; so $\mathcal{F}w(\omega) = \mathcal{F}\psi_\alpha(\omega)\,\mathcal{F}z(\omega) + \mathcal{F}q(\omega)$. The inner product condition

$$\int_0^\infty z(t)\,w(t)\,dt = 0$$

implies

$$\operatorname{Re}\int_{-\infty}^{+\infty} \overline{\mathcal{F}z(\omega)}\,\left[\mathcal{F}\psi_\alpha(\omega)\,\mathcal{F}z(\omega) + \mathcal{F}q(\omega)\right]\,d\omega = 0.$$

Then

$$\begin{aligned}\mathrm{Re}\int_{-\infty}^{+\infty}\mathcal{F}\psi_\alpha(\omega)\,|\mathcal{F}z(\omega)|^2\,d\omega &= -\mathrm{Re}\int_{-\infty}^{+\infty}\overline{\mathcal{F}z(\omega)}\,\mathcal{F}q(\omega)\,d\omega\\ &\le \|\mathcal{F}z\|_{H^{-\alpha/2}}\,\|\mathcal{F}q\|_{H^{\alpha/2}}\end{aligned}$$

That is,

$$\begin{aligned}\cos(\pi\alpha/2)\int_{-\infty}^{+\infty}|\omega|^{-\alpha}\,|\mathcal{F}z(\omega)|^2\,d\omega &\le \|z\|_{H^{-\alpha/2}}\,\|q\|_{H^{\alpha/2}} \qquad \text{or}\\ \cos(\pi\alpha/2)\,\|z\|^2_{H^{-\alpha/2}} &\le \|z\|_{H^{-\alpha/2}}\,\|q\|_{H^{\alpha/2}}\end{aligned}$$

Convolution with ψ_α is a positive operator if $0 < \alpha < 1$, but *not* if $1 < \alpha < 2$.

Uniqueness and positivity

Suppose we have two solutions to generalized complementarity problems: $w_i = \mathcal{L} z_i + q$ where $\mathcal{L}$ is a linear operator for $i = 1, 2$ where $\mathcal{K}^* \ni w_i \perp z_i \in \mathcal{K}$ for all t.

Then

$$\langle w_2 - w_1, z_2 - z_1 \rangle = \langle w_2, z_2 \rangle - \langle w_1, z_2 \rangle - \langle w_2, z_1 \rangle + \langle w_2, z_2 \rangle \leq 0.$$

If $\mathcal{L}$ is an elliptic or positive definite operator, then

$$\begin{aligned} 0 &\geq \langle w_2 - w_1, z_2 - z_1 \rangle \\ &= \langle \mathcal{L}(z_1 - z_1), z_2 - z_1 \rangle \end{aligned}$$

which implies that $z_2 = z_1$. (Allowing different q's shows the solution operator is Lipschitz if $\mathcal{L}$ elliptic or positive definite.)

Corollary: If $\alpha \in (0, 1)$ then solutions $z(\cdot)$ to the CCP $0 \leq \psi_\alpha * z(t) + q(t) \perp z(t) \geq 0$ are unique.

Existence for $\alpha \in (1, 2)$

Existence for $\alpha \in (1, 2)$ can be shown via index reduction and a differentiation lemma: consider the CCP below for $\epsilon > 0$:

$$0 \leq (\psi_\alpha + \epsilon H) * z_\epsilon(t) + q(t) \quad \perp \quad z_\epsilon(t) \geq 0.$$

This is index one and solutions $z_\epsilon(\cdot)$ exist (and are unique). But if

$$0 \leq z(t) \quad \perp \quad w(t) \geq 0 \qquad \text{for all } t$$

with $z(\cdot) \in L^p$ and $w'(\cdot) \in L^q$, $p^{-1} + q^{-1} = 1$, then $z(t)\, w'(t) = 0$ for almost all t.

So $(\psi'_\alpha * z_\epsilon(t) + \epsilon z_\epsilon(t) + q'(t))z_\epsilon(t) = 0$ for almost all t. As $\psi'_\alpha = \psi_{\alpha-1}$, we can integrate & re-arrange to get

$$\int (\psi_{\alpha-1} * z_\epsilon) z_\epsilon \, dt \leq \int (-q') z_\epsilon \, dt \leq \left\| q' \right\|_{H^{(\alpha-1)/2}} \left\| z_\epsilon \right\|_{H^{-(\alpha-1)/2}}$$

so z_ϵ uniformly bounded in $H^{-(\alpha-1)/2}$ as $\epsilon \downarrow 0$.

So we get a weakly converging subsequence $z_\epsilon(\cdot) \rightharpoonup \widehat{z}(\cdot)$ in $H^{-(\alpha-1)/2}$, but $\psi_\alpha * z_\epsilon \to \psi_\alpha * \widehat{z}$ strongly in $H^{+(\alpha-1)/2}$ (thanks to compactness of $H^{(\alpha+1)/2} \subset H^{(\alpha-1)/2}$) so limit(s) satisfy complementarity.

But uniqueness does not follow from these arguments as $z \mapsto \psi_\alpha * z$ is not a positive operator for $\alpha \in (1, 2)$.

Uniqueness for $\alpha \in (1,2)$?

Pros:

- kernel function is positive: $\psi_\alpha(t) > 0$ for $t > 0$.
  - for analytic $q(\cdot)$, $z(\cdot)$, there is uniqueness.
- short-time behavior looks a bit like a δ-function.

Cons:

- Fourier transform $\mathcal{F}\psi_\alpha(\omega)$ has negative real part.
- non-uniqueness clear for index $\alpha = 2$.

And the verdict is . . . No!

There is a counter-example to uniqueness, but it is very far from being analytic, but it can be C^p for any finite p.

How to construct?

(1) Use version of **Mandelbaum's DCP non-uniqueness theorem**: $0 \leq (m * z)(t) + q(t) \perp z(t) \geq 0$ for all t has no solutions or multiple solutions if and only if there is $\zeta(\cdot)$ where $(m * \zeta)(t)\,\zeta(t) \leq 0$ for all $t \geq 0$ with $\zeta(0) = 0$.

(2) Use **self-similarity.** Suppose $\theta(s) = 1$ for $|s| \le 1$, $\theta(s) = 0$ for $|s| \ge 2$, $s\,\theta'(s) \ge 0$ for all s, θ is C^∞ and $\int_{-2}^{+2} \theta(s)\,ds = 1$. Look for

$$\zeta(t;\eta) = \sum_{k\in\mathbb{Z}} (-1)^k \mu^{-k}\,\zeta_1(\gamma^k t;\eta), \qquad \zeta_1(s;\eta) = \eta^{-1}\,\theta(\eta^{-1}(s-\widehat{s}))$$

where $\widehat{s} = \frac{1}{2}(1+\gamma)$.

(3) **Choose** μ, $\gamma > 0$ so that $\sum_{k=1}^{\infty} (-1)^k (\mu\gamma)^{-k}(1-\gamma^{-k})^\alpha < 0$; thus $\lim_{\eta\downarrow 0}(\psi_\alpha * \zeta)(\widehat{s};\eta) < 0$. Then **make** $\eta > 0$ **sufficiently small** so that $(\psi_\alpha * \zeta)(s;\eta) < 0$ for all $|s-\widehat{s}| \le 2\eta$. **Note:** We need $\mu\gamma > 1$ for the infinite series to converge.

Conclusions

- Soutions are exist for index in range $[0, 2]$
- Solutions are unique (supposing dominant term in $m(t)$ for $t \approx 0$ is symmetric and positive definite) for index in range $[0, 1]$
- Non-uniqueness of solution to Kelvin–Voigt 1-D viscoelastic rod in impact at end
- *Probably*, non-uniqueness for general K–V viscoelastic bodies in impact